Mathematics and Computer Science

Algorithms, Trees, Combinatorics and Probabilities

Danièle Gardy
Abdelkader Mokkadem
Editors

Birkhäuser Verlag
Basel · Boston · Berlin

Editors' addresses:

Danièle Gardy
Université de Versailles-St-Quentin
PRISM
Bâtiment Descartes
45 avenue des Etats-Unis
78035 Versailles Cedex
France

e-mail: gardy@prism.uvsq.fr

Abdelkader Mokkadem
Université de Versailles-St-Quentin
Département de Mathématiques
Bâtiment Fermat
45 avenue des Etats-Unis
78035 Versailles Cedex
France

e-mail: mokkadem@math.uvsq.fr

2000 Mathematical Subject Classification 68Q25; 68Rxx

A CIP catalogue record for this book is available from
the Library of Congress, Washington D.C., USA

Deutsche Bibliothek Cataloging-in-Publication Data

Mathematics and computer science : algorithms, trees, combinatorics and probabilities /
Danièle Gardy ; Abdelkader Mokkadem ed.. - Basel ; Boston ; Berlin : Birkhäuser, 2000
(Trends in mathematics)
ISBN 3-7643-6430-0

ISBN 3-7643-6430-0 Birkhäuser Verlag, Basel – Boston – Berlin

Printed on acid-free paper produced from chlorine-free pulp. TCF ∞
Printed in Germany
ISBN 3-7643-6430-0

9 8 7 6 5 4 3 2 1

Contents

V Other Topics 243

Foreword

This book is based on the proceedings of the Colloquium on Mathematics and Computer Science held in the University of Versailles-St-Quentin on September 18-20, 2000. It contains articles on topics that are relevant both to Mathematics and to Computer Science. These articles have been selected throughout a rigourous process; each of them has been submitted to two referees at least. This selection process, which took place under the supervision of the scientific committee, ensures the high level and the originality of the contributions.

We thank the invited speakers, the authors, the participants to the colloquium and, of course, the Scientific Committee whose members were D. Aldous, F. Baccelli, P. Cartier, B. Chauvin, P. Flajolet, J.M. Fourneau, D. Gardy, D. Gouyou-Beauchamps, R. Kenyon, J.F. Le Gall, C. Lemaréchal, R. Lyons, A. Mokkadem, A. Rouault and C. Roucairol.

We also take this opportunity to thank the University of Versailles-St-Quentin, the University of Paris-Sud (Orsay) and the Centre National de la Recherche Scientifique (CNRS), whose financial and material support contributed to the success of the Colloquium.

The Organisation Committee

Trends in Mathematics, © 2000 Birkhäuser Verlag Basel/Switzerland

Gibbs Families

V. MALYSHEV *I.N.R.I.A. Rocquencourt, Domaine de Voluceau, 78153 Le Chesnay Cedex, France*

Abstract. *Gibbs random field is now one of the central objects in probability theory. We define a generalization of Gibbs distribution, when the space (lattice, graph) is not fixed but random. Moreover, randomness of the space is not given independently of the configuration but both depend on each other. We call such objects Gibbs families because they appear to parametrize sets of ordinary Gibbs distributions. Moreover, they are well suited to study local probability structures on graphs with random topology. First results of this theory are presented here.*

Gibbs random field [2, 3] is now one of the central objects in probability theory. Here we define a natural generalization of Gibbs distribution, when the space (lattice, graph) is not fixed but random. Moreover, randomness of the space is not given apriori and independently of the configuration but both depend on each other. We call the introduced objects Gibbs families because, as it will be shown, they parametrize sets of Gibbs distributions on fixed graphs.

We present here the foundations of such theory. Two central definitions are given. The first one is Gibbs family on the set of countable spingraphs (spingraph is a graph with a function on its vertices) with the origin (a specified vertex). To deal with countable graphs where no vertex is specified we define the notion of empirical distribution. The conditional measure on the configurations of a given graph G gives standard Gibbs field on G with the same potential. We discuss some examples of Gibbs families.

This paper is selfcontained, but it is a part of the series of papers by the author starting with [5] concerning new connections between computer science and mathematical physics.

1 Definitions

Spingraphs We consider non-directed connected (unless otherwise stated) graphs G (finite or countable) with the set $V = V(G)$ of vertices and the set $L = L(G)$ of links. The following properties are always assumed: between each pair of vertices there is 1 or 0 edges; each node (vertex) has finite degree (the number of edges incident to it).

A *subgraph* of G is a subset $V_1 \subset V$ of vertices together with some links connecting pairs of vertices from V_1 and inherited from L. A *regular subgraph* $G(V_1)$ of G is a subset $V_1 \subset V$ of vertices together with ALL links connecting pairs of vertices from V_1 and inherited from L.

The set V of nodes is a metric space with the following metrics: the distance $d(x, y)$ between vertices $x, y \in V$ is the minimum of the lengths of paths connecting these vertices. The lengths of all edges are assumed to be equal, say to some constant, assumed equal to 1.

Spingraph $\alpha = (G, s)$ is a graph G together with a function $s : V \to S$, where S is some set of spin values. Spingraphs are always considered up to isomorphisms. Isomorphism here is an isomorphism of graphs respecting spins. Denote $\mathcal{G}_N$ ($\mathcal{A}_N$) - the set of equivalence classes (with respect to isomorphisms) of connected finite graphs (spingraphs) with N vertices.

Spingraphs with the origin Define the annular neighbourhood $\gamma(\alpha, v; a, b)$ to be the regular subgraph of α defined by the set $V(\gamma(\alpha, v; a, b)) = \{v' : a \leq d(v, v') \leq b\}$. $O_d(v) = \gamma(\alpha, v; 0, d)$ is the d-neighborhood of $v, O(v) = O_1(v)$. Let $R_0(G) = \max_{v \in V(G)} d(0, v)$ be the radius of graph G with respect to vertex 0. Thus $R_v(O(v)) = 1$. Radius of a graph is $R(G) = \min_{v \in V(G)} R_v(G)$.

We will also consider graphs with one specified vertex, the origin. In this case isomorphisms are also assumed to respect the origin. When we want to emphasize that the graphs are given together with an origin v we shall write it as $G^{(v)}, \mathcal{G}^{(v)}$ etc.

Let $\mathcal{G}_N^{(0)}$ ($\mathcal{A}_N^{(0)}$) - the set of equivalence classes of finite graphs (spingraphs) with respect to isomorphisms, with origin 0 and having radius N with respect to 0. Further on, spingraph will mean its equivalence class. Let $\mathcal{A}^{(0)} = \cup \mathcal{A}_N^{(0)}$ be the set of all finite spingraphs with the origin 0.

σ-algebra and free measure Let $\mathcal{A}$ be an arbitrary set of finite spingraphs $\alpha = (G, s)$. Let $\mathcal{G} = \mathcal{G}(\mathcal{A})$ be the set of all graphs G such that there exists s with $\alpha = (G, s) \in \mathcal{A}$. We assume always that if $\alpha = (G, s) \in \mathcal{A}$ then for any s' all (G, s') also belong to $\mathcal{A}$.

$\mathcal{A}$ is a topological space which is a discrete (finite or countable) union of topological spaces $T_G = S^{V(G)}, G \in \mathcal{G}(\mathcal{A})$. We consider the Borel σ-algebra on $\mathcal{A}$. It is generated by cylindrical subsets $A(G, B_v, v \in V(G))$, $G \in \mathcal{G}$, where B_v are some Borel subsets of S. $A(G, B_v, v \in V(G))$ is the set of all $\alpha = (G, s)$ such that G is fixed and functions (configurations) $s(v) : V(G) \to S$ are such that $s(v) \in B_v$ for all $v \in V(G)$.

Let some nonnegative measure λ_0 on S be given. The following nonnegative measure $\lambda_{\mathcal{A}}$ (not necessarily a probability measure) on $\mathcal{A}$

$$\lambda_{\mathcal{A}}(A(G, B_v, v \in V(G))) = \prod_{v \in V(G)} \lambda_0(B_v) \tag{1}$$

is called a free measure.

Regular potentials Potential is a function $\Phi : \cup_N \mathcal{A}_N \to R \cup \{+\infty\}$, that is a function on the set of finite spingraphs invariant with respect to isomorphisms of spingraphs. We say that Φ has a finite radius if $\Phi(\alpha) = 0$ for all α with radius greater than some $r_0 < \infty$. Unless otherwise stated we shall consider only radius 1 case, that is $r_0 = 1$. Fix potential Φ and measure λ_0 on S. The energy of spingraph α is defined as

$$H(\alpha) = \sum_{B \subset V(G)} \Phi(\Gamma(B)), \alpha = (G, s) \tag{2}$$

where the sum is over all subsets B of $V(G)$, $\Gamma(B)$ is the regular subgraph of α with $V(\Gamma(B)) = B$. Gibbs $\mathcal{A}$-family with potential Φ is the following probability measure $\mu_{\mathcal{A}}$ on $\mathcal{A}$

$$\frac{d\mu_{\mathcal{A}}}{d\lambda_{\mathcal{A}}}(\alpha) = Z^{-1}\exp(-\beta H(\alpha)), \alpha \in \mathcal{A} \tag{3}$$

assuming $Z \neq 0$ and the stability condition

$$Z = Z(\mathcal{A}) = \int_{\mathcal{A}} \exp(-\beta H(\alpha))d\lambda_{\mathcal{A}} < \infty \tag{4}$$

Here $\beta \geq 0$ is the inverse temperature.

Markov property The set $\mathcal{A}_{\infty}^{(0)}$ of countable spingraphs with the origin is a topological space. The basis of its open subsets is defined as follows. Let C_N be an arbitrary open subset of the set $\mathcal{A}_N^{(0)}$ of all spingraphs with radius N with respect to the origin. Then the basis consists of sets of all countable spingraphs such that (for some N and some C_N) their N-neighborhood of the origin belongs to C_N. We again take the Borel σ-algebra as the basic σ-algebra.

For notational convenience we give definition only for potential of radius 1. For any spingraph α with the origin 0 we call $\gamma(\alpha, 0; N, N)$ its N-slice, $\gamma(\alpha, 0; 0, N)$ - the N-interior, $\gamma(\alpha, 0; N, \infty)$ - the N-exterior parts. Note that N-slice and the exterior part may be non-connected. There are no links connecting $(N-1)$-interior and $(N+1)$-exterior parts of the graph: this will give us an analog of the Markov property. For given N and given some finite graph γ (it may be non-connected) let $\mathcal{A}(\leq N, \gamma) \subset \mathcal{A}_N^{(0)}$ be the set of all finite connected spingraphs α having radius N with respect to the origin and such that γ is the N-slice of α.

Definition 1 *We call measure μ on $\mathcal{A}_{\infty}^{(0)}$ a Gibbs family with regular potential Φ if, for any γ and any N-exterior part with N-slice γ, the conditional distribution on the set of spingraphs α having fixed exterior part (that is on N-neighborhoods with N-slice γ) coincides a.s. with the Gibbs family with potential Φ on $\mathcal{A}(\leq N, \gamma)$.*

In particular, it depends only on the spingraph γ but not on the whole N-exterior part.

Boundary conditions Boundary condition is a sequence $\nu_N(\gamma)$ of measures on the set of finite (not necessary connected) spingraphs γ. Intuitively - on N-slices. Gibbs family on $\mathcal{A}_N^{(0)}$ with boundary conditions $\nu_{N+1}(\gamma)$ is defined as

$$\mu_N(\alpha) = Z^{-1}(N, \nu_{N+1}) \sum_{\xi \in A(\alpha,\gamma)} \int \exp(-\beta H(\xi))d\nu_{N+1}(\gamma), \alpha \in \mathcal{A}_N^{(0)} \tag{5}$$

where $A(\alpha, \gamma)$ is the set of all spingraphs from $\mathcal{A}_{N+1}^{(0)}$ with $(N+1)$-slice γ and N-interior part α. Note that $A(\alpha, \gamma)$ is finite for any α, γ.

Non-regular potentials Gibbs family with non-regular potential Φ on $\mathcal{A}$ here is defined similarly but with the energy of α defined by the formula

$$H(\alpha) = \sum_{\gamma \subset \alpha} \Phi(\gamma), \alpha = (G, s) \tag{6}$$

where the sum is over all (not necessary regular) subspingraphs γ of α.

A particular case is the "chemical" potential. It is given by a function Φ equal to a constant z_0 for each vertex (independently of spins in this vertex), equal to a constant z_1 for each link (independently of spins in both vertices of the link) and equal 0 otherwise. Then

$$H_N(\alpha) = z_0 V(\alpha) + z_1 L(\alpha) \tag{7}$$

where $V(\alpha), L(\alpha)$ are the numbers of vertices and links in α.

2 Limiting correlation functions

Compactness Gibbs families on the set $\mathcal{A}_{\infty}^{(0)}$ of connected countable spingraphs with fixed vertex 0 can be obtained as weak limits of Gibbs families on finite graphs. There are 3 sources of non-existence of limiting Gibbs families: non-existence of finite Gibbs families (see examples below), non-compactness of S, the distribution for a finite Gibbs family can be concentratred on the graphs with large degrees of vertices (ultraviolet problem). The assumptions of the following propostion correspond to this list, but the compactness could be proven under weaker conditions.

Let $\mathcal{A}_{\infty}^{(0)}(r) \subset \mathcal{A}_{\infty}^{(0)}$ be the set of countable spingraphs where each vertex has degree not greater than r.

Proposition 1 *Assume S to be compact. Assume that for all N there exists a Gibbs family with potential Φ on $\mathcal{A}_N^{(0)}$. Assume that Φ has radius 1 and $\Phi(O(v)) = \infty$ if the degree of v is greater than some constant r. Then there exists a Gibbs family with potential Φ, with support on $\mathcal{A}_{\infty}(r)$.*

Proof. Consider the Gibbs family μ_N on $\mathcal{A}_N^{(0)}$. Let $\tilde{\mu}_N$ be an arbitrary probability measure on $\mathcal{A}_{\infty}^{(0)}$ such that its factor measure on $\mathcal{A}_N^{(0)}$ coincides with μ_N. Then the sequence $\tilde{\mu}_N$ of measures on the set $\mathcal{A}_{\infty}^{(0)}$ is compact, that can be proved by the standard diagonal process by enumerating all possible N-neighborhoods. Moreover, any limiting point of $\tilde{\mu}_N$ is a Gibbs family with potential Φ.

Generators To give examples of Gibbs families it is useful to introduce the following classes of graphs. Consider a set of "small" graphs $G_1, ..., G_k$ of the same radius r with respect to a specified vertex 0. Graph G is said to be generated by $G_1, ..., G_k$ if each vertex of G has a neighborhood (of radius r) isomorphic to one of $G_1, ..., G_k$. Let $\mathcal{G}(G_1, ..., G_k)$ be the set of all graphs generated by $G_1, ..., G_k$.

With each system $\mathcal{G}(G_1, ..., G_k)$ and fixed S we associate the following class $\mathcal{F}(G_1, ..., G_k)$ of potentials $\Phi = \Phi(G, s)$ if $G = G_i$ for some $i = 1, ..., k$, $\Phi(G, s) = \infty$ for all other graphs with the same radius r and $\Phi(G, s) = 0$ for all graphs with $R_0(G) \neq r$.

Nonexistence This class of potentials satisfies the previous compactness criteria if there exist Gibbs families on finite graphs. However this is not always the case.

For example, consider the case $k = 1, r = 1$. If G_1 is a finite complete graph then it is "selfgenerating" but there are no countable graphs generated by it. Introduce some graphs with radius 1: g_k with $k + 1$ vertices $0, 1, 2, 3, ..., k$ and k links $01, 02, ..., 0k$, and $g_{k,k}$ with $k + 1$ vertices $0, 1, 2, 3, ..., k$ and $2k$ links $01, 02, ..., 0k, 12, 23, ..., k1$. One can see after several iterations that the pentagon $G_1 = g_{5,5}$ cannot generate any countable graph.

Gibbs families and Gibbs fields ¿From the following simple result it becomes clear why the introduced distributions are called Gibbs families. Let μ be a Gibbs family with potential Φ. Assume that the conditions of the previous proposition hold. Consider the measurable partition of the space $\mathcal{A}_\infty^{(0)}$ of spin graphs: any element S_G of this partition is defined by a fixed graph G and consists of all configurations s_G on G.

Theorem 1 *For any given graph G the conditional measure on the set of configurations s_G is a.s. a Gibbs measure with the same potential Φ.*

Thus, any Gibbs family is a convex combination (of a very special nature) of Gibbs fields (measures) on fixed graphs, with the same potential, with respect to the measure $\nu = \nu(\mu)$ on the factor space $\mathcal{A}_\infty^{(0)} / \{S_G\}$ induced by μ.

The following lemma reduces this result to the corresponding result for Gibbs families on finite graphs where it is a straightforward calculation.

Lemma 1 *Let μ be a Gibbs family on $\mathcal{A}_\infty^{(0)}$ with potential Φ. Let $\nu_N(\gamma)$ be the measure on N-slices induced by μ. Then μ is a weak limit of Gibbs families on $\mathcal{A}_N^{(0)}$ with potential Φ and boundary conditions $\{\nu_N(\gamma)\}$.*

Topological phase transition We say that Gibbs family with potential Φ is a pure Gibbs family if it is not a convex combination of Gibbs families with the same potential. If there are more than one pure Gibbs family with potential Φ we say that Gibbs family with potential Φ is not unique. Intuitively, nonuniqueness of Gibbs families can be of two kinds: due to the structure of configurations (inherited from usual Gibbs fields) and due to topology of the graph. The following example (using Cayley graphs of abelian groups) gives an example of a "topological phase transition".

Let $U_{d,2}$ be the graph isomorphic to the 2-neighborhood of 0 on the lattice Z^d. Then the following graphs belong to $\mathcal{G}(U_{d,2})$. This is Z^d itself and any of its factorgroups $Z(k_1, ..., k_d)$ with respect to some subgroup of Z^d, generated by some vector $(k_1, ..., k_d), k_i \geq 1$.

Consider now Ising Gibbs fields on $Z(k_1, ..., k_d)$ with some potential from $\mathcal{F}(G_1)$. More exactly, take $S = \{-1, 1\}$ and introduce the following potential Φ. $\Phi(U_{d,2}, s_{U_{d,2}})$ is equal to the energy of the configuration $s_{U_{d,2}}$ corresponding to the nearest neighbor Ising model on $U_{d,2}$. For all other spingraphs α of radius 2 we put $\Phi(\alpha) = \infty$, we put $\Phi(\alpha) = 0$ if the radius of α is different from 2.

We know from statistical physics that for any such graph there is only one Gibbs field with potential Φ for β sufficiently small. The following theorem shows that the Gibbs family with potential Φ is not unique, however, each pure Gibbs family we found is a Gibbs field on the fixed graph.

Theorem 2 *For any β sufficiently small there are countable number of pure Gibbs families with potential Φ.*

Proof. An example (for $d = 2$) of a pure Gibbs family with potential Φ is the unique Gibbs field on $Z \times Z_n$ for any $n \geq 5$. Consider the (unique) Gibbs random field $\mu(n)$ with potential Φ on this infinite cylinder. Take the induced measure $\nu_N(\mu(n))$ on N-slices, that is on non-connected union of two circular strips of length n. Using Lemma 1 take Gibbs families with potential Φ and boundary conditions $\nu_N(\mu(n))$. Then the graph is uniquely defined (one can construct slice $N-1, N-2, ...$ by induction). Thus, this Gibbs family coincides the with the pure Gibbs field on the correponding graph. Another interesting possibility are Gibbs fileds on twisted cylinders, that is on the sets obtained from the strip $Z \times \{1, 2, ..., k\}$ if for each n we identify the points $(n, 1)$ and $(n + j, k)$.

3 Empirical correlation functions

We constructed probability measures on $\mathcal{A}_\infty^{(0)}$ in quite a standard way, using standard Kolmogorov approach with cylindrical subsets. However, one cannot use similar approach to define a probability measure on the set $\mathcal{A}_\infty$ of equivalence classes of countable connected spingraphs. The problem is that it is not at all clear how to introduce finite-dimensional distributions here, because the vertices are not enumerated (there is no coordinate system). Thus all Kolmogorov machinery fails. However one can propose an analog of finite-dimensional distributions. We call the resulting system an empirical distribution.

Assume S to be finite or countable. Let us consider systems of numbers

$$\pi = \left\{p(\Gamma), \Gamma \in \mathcal{A}^{(0)}\right\}, 0 \leq p(\Gamma) \leq 1, \tag{8}$$

i.e. Γ is an arbitrary finite spingraph with origin 0. We assume the following compatibility condition: for any $k = 0, 1, 2, ...$ and any fixed graph Γ_k of radius k we have

$$\sum_{\Gamma_{k+1}} p(\Gamma_{k+1}) = p(\Gamma_k), k = 0, 1, 2, ... \tag{9}$$

where the sum is over all Γ_{k+1} of radius $k+1$ such that $O_k(0, \Gamma_{k+1})$ is isomorphic to Γ_k. It is assumed that the summation is over equivalence classes of spingraphs.

We assume also the following normalization condition

$$\sum p(\Gamma_0) = 1 \tag{10}$$

where Γ_0 is the vertex 0 with any possible spin on it.

Definition 2 *Any such system π is called an empirical distribution.*

One can also rewrite compatibility conditions in terms of conditional probabilities

$$\sum_{\Gamma_{k+1}} p(\Gamma_{k+1} \mid \Gamma_k) = 1 \tag{11}$$

where the summation is as above and

$$p(\Gamma_{k+1} \mid \Gamma_k) = \frac{p(\Gamma_{k+1})}{p(\Gamma_k)} \tag{12}$$

Thus we consider $\mathcal{A}^{(0)}$ as a tree where vertices are spingraphs and a link between Γ_{k+1} and Γ_k exists iff Γ_k is isomorphic to the k-neighborhood of 0 in Γ_{k+1}.

Examples of empirical distributions can be obtained via the following limiting procedure. Let μ_N - probability measure on $\mathcal{A}_N$. For any N and any $\Gamma \in \mathcal{A}_k^{(0)}, \alpha \in \mathcal{A}_N$ put

$$p^N(\Gamma) = \left\langle \frac{n^N(\alpha, \Gamma)}{N} \right\rangle_{\mu_N} \tag{13}$$

where $n^N(\alpha, \Gamma)$ is the number of vertices in α having their k-neighborhoods isomorphic to Γ. Denote $\pi_N = \{p^N(\Gamma)\}$.

Lemma 2 *Assume that S is finite. Assume that for any r*

$$\mu_N(\min_{\alpha \in \mathcal{A}_N} R(\alpha) \leq r) \to 0 \tag{14}$$

as $N \to \infty$. Then any weak limiting point of π_N is an empirical distribution.

Proof. Note that for any n_0 there exists such $N_0 = N_0(n_0)$ such that for any $N > N_0$ numbers $p_N(\Gamma)$ satisfy compatibility conditions for all Γ with radius $n < n_0$. Finiteness of S implies compactness.

Interesting question is how one could characterize empirical distributions which can be obtained via this limiting procedure.

Empirical Gibbs families Note that for a given π the numbers $p(\Gamma)$ for all Γ with $R_0(\Gamma) = N$ define a probability distribution π_N on $A_N^{(0)}$. For any $n < N$ let $\pi_{N,n}(\gamma)$ be the probability that n-slice of Γ is equal to γ, and $\pi_{N,n}(\Gamma_n)$ be the probability that the neighborhood $O_n(\Gamma)$ of 0 is isomorphic to Γ_n.

For any γ and any Γ_n with n-slice γ introduce conditional probabilities

$$\pi_{N,n}(\Gamma_n|\gamma) = \frac{\pi_{N,n}(\Gamma_n)}{\pi_{N,n}(\gamma)} \tag{15}$$

An empirical distribution π is an empirical Gibbs family with potential Φ if for any $n < N, \gamma, \Gamma_n$ the distribution $\pi_{N,n}(\Gamma_n|\gamma)$ is the Gibbs family with potential Φ on the set spingraphs Γ_n with n-slice γ, that is

$$\pi_{N,n}(\Gamma_n|\gamma) = Z_n^{-1}(\gamma)\exp(-H(\Gamma_n)) \tag{16}$$

We see that in this case $\pi_{N,n}(\Gamma_n|\gamma)$ do not depend on N.

Two-dimensional quantum gravity (or string in zero dimension) In the following examples there is no spin, the free measue is trivial, and the partition function contains summation over the corresponding graphs.

We consider graphs corresponding to the set of all triangulations $\mathcal{T}_N$ of two-dimensional sphere with N triangles, where combinatorially isomorphic triangulations are identified. Another way to specify such graphs is to use the potential Φ such that $\Phi(O_1(v)) = \infty$ if $O_1(v) \neq g_{k,k}$ for some $,k \geq 2.$. Put also $\Phi(K_5) = \Phi(K_{3,3}) = \infty$, where K_5 is the complete graph with 5 vertices, $K_{3,3}$ is the graph with six vertices $1,2,3,4,5,6$ and all links $(i,j), i = 1,2,3, j = 4,5,6$ (by Pontriagin-Kuratowski theorem this singles out planar graphs). In all other cases put $\Phi = 0$. This model is called in physics pure two-dimensional gravity or quantum string in zero dimensions.

Theorem 3 *There exists a unique limit* $p(\Gamma) = \lim_{N\to\infty} p_N(\Gamma)$. *Moreover,* $p(\Gamma)$ *is an empirical Gibbs family with potential* Φ.

Proof. We shall prove this by giving explicit expressions. Denote the number of triangulations with specified vertex 0 and fixed $O_d(0) = \Gamma_d$, let Γ_d have u triangles and k boundary links (with both vertices on the d-slice). The external part is the triangulation of the disk with $N-u$ triangles and also with k boundary links. The number $D(N-u,k)$ of such triangulations (this follows from results by Tutte, see [4], pp.21, 25) has the following asymptotics as $N \to \infty$

$$D(N-u,k) \sim k^{-1}\phi(k)N^{-\frac{5}{2}}c^{N-u} \tag{17}$$

Thus the probability of Γ (at vertex 0) is

$$P^N(\Gamma) \sim k^{-1}\phi(k)N^{-\frac{7}{2}}c^{N-u}C_0^{-1}(N) \tag{18}$$

where

$$C_0(N) \sim aN^{-\frac{5}{2}}c^N \tag{19}$$

is the number of triangulations with N triangles. The function ϕ and constants c, a are known explicitely but we do not need this. It can be proved (see similar statements in [4]) that probabilities for the case of specified origin and without it are asymptotically the same as $N \to \infty$.

Trees Trees cannot be characterized locally. That is they cannot be singled out from the set of all graphs via restrictions on subgraphs having radius not greater than some constant. It is interesting, however, that a local probabilistic characterization of trees can be given using Gibbs families. More exactly, one can define empirical Gibbs families with radius 1 potential and such that the distribution has its support on the set of trees. We give an example with p-regular trees.

Consider the set $\mathcal{A}_{N,p}$ of all p-regular graphs with N vertices, that is graphs with all vertices having degree p. Consider Gibbs family on $\mathcal{A}_{N,p}$ with potential $\Phi \equiv 0$. Otherwise speaking, we define Gibbs family on $\mathcal{A}_N$ with the following potential Φ : $\Phi = 0$ on all radius 1 graphs, where 0 has degree p, and $\Phi = \infty$ for all other radius 1 graphs.

Theorem 4 *$p^N(\Gamma_k)$ have limits $p(\Gamma_k) = 1$ for p-ary tree of height k, and 0 otherwise.*

Proof. The case $\Phi = 0$ can be reduced to combinatorics. To prove the theorem we need some techniques from graph enumeration.

Lemma 3 *The asymptotics of the number $C_p(N)$ of connected p-regular graphs with N vertices*

$$C_p(N) \sim \frac{(pN-1)(pN-3)...}{N!(p!)^N} \tag{20}$$

Proof. We use a combinatorial method to prove this lemma. Such method is known and was applied to other situations. It consists of several statements, see [1], section 9.4. We call graph labelled if both vertices and legs are labelled (enumerated). The scheme of the proof is the following:

1. Number of labelled p-regular graphs is $(pN-1)(pN-3)....$

2. Almost all labelled p-regular graphs are connected.

3. Almost all p-regular graphs are asymmetric.

4. It follows from 1-3 that almost all p-regular graphs are connected.

In the same way on can show that the number of p-regular graphs with no cycles of length q at vertex 0, is equal asymptotically to the number of all graphs. It follows that the probability that in a vertex there are no cycles tends to 1 as $N \to \infty$.

An interesting problem is to generalize this result for the case with $S = \{-1, 1\}$ and Ising type interaction.

References

[1] F. Harary, E. Palmer. Graphical Enumeration. Academic Press. 1973.

[2] R. Dobrushin. The description of random field by means of conditional probabilities and conditions of its regularity. Theory of Probability and Appl., 1968, v. 13, 197-224.

[3] O. Lanford, D. Ruelle. Observables at infinity and states with short range correlations in statistical mechanics. Comm. Math. Phys., 1969, v. 13, 174-215.

[4] V. Malyshev. Probability around Quantum Gravity. Russian Math. Surveys, 1999, v. 54, pp. 685-728.

[5] V. Malyshev. Random Grammars. Russian Math. Surveys, 1998, v. 53, pp. 345-370.