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Phase Transitions in the One-Dimensional Coulomb Medium

V. A. Malyshev

Laboratory of Large Random Systems, Faculty of Mathematics and Mechanics, Lomonosov Moscow State University, Moscow, Russia
e-mail: 2malyshev@mail.ru

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Abstract—A Coulomb medium is a system of N charged particles of equal charge on an interval with nearest-neighbor Coulomb interaction and constant external electric field. We show that, asymptotically as $N \to \infty$, stable configurations have four possible phases of the particle density depending on the external field, which is assumed to be a function of N. Moreover, we find these phases explicitly.

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1. INTRODUCTION

The problem of finding N-point particle configurations on a manifold having minimal energy (or even fixed configurations) was regarded to be important long ago [1]. Therefore, we should say some words about the history of this question. First of all, we consider systems of particles with equal charges and with the Coulomb interaction. Immediately, the problem is separated into the case of small N, where the problem is finding such configurations explicitly, and the case of large N, where asymptotics is of the main interest. Already J.J. Thomson (who discovered the electron) suggested the problem of finding such configurations on a sphere, and an answer for $N = 2, 3, 4$ has been known for more than 100 years, but for $N = 5$ the solution was obtained quite recently [2]. In the one-dimensional case, already T.J. Stieltjes studied the problem on an interval with logarithmic interaction and found its connection with zeros of orthogonal polynomials on the corresponding interval; see [3, 4]. However, the problem of finding minimal energy configurations on the two-dimensional sphere for any N and for power interaction (sometimes, this is called the seventh S. Smale's problem; it is also connected with the names of F. Risz and M. Fekete) was completely solved for quadratic interaction only (see [5–7] and review [8]). For more general compact manifolds, see a survey [9].

Here we follow an alternative direction: namely, we study how a configuration could change in the presence of a weak or strong external force. It appears that even in a simplified one-dimensional model with nearest-neighbor interaction there is an interesting structure of fixed points (more exactly, fixed configurations), rich both in the number and in the charge distribution. For the constant force case, we find four phases of the charge density, with respect to a parameter which is the ratio of the interaction strength constants and the value of the external force.

By a Coulomb medium we call the space of configurations

$$-L \leq x_N < \ldots < x_1 < x_0 \leq 0$$

of $N + 1$ point particles with equal charges on the segment $[-L, 0]$. Here N is assumed to be sufficiently large; however, some results are valid for any $N \geq 2$. We assume a repulsive Coulomb interaction of the nearest neighbors and an external force $\alpha_{\rm ext}F_0(x)$; i.e., the potential energy is

$$U = \sum_{i=1}^{N} V(x_{i-1} - x_i) - \sum_{i=0}^{N} \int_{-L}^{x_i} \alpha_{\rm ext} F_0(x)\,dx, \quad V(x) = \frac{\alpha_{\rm int}}{|x|}, \tag{1}$$

where $\alpha_{\rm ext}$ and $\alpha_{\rm int}$ are positive constants. This determines the dynamics of a system of charges if one defines exactly what happens with particles 0 and N in the points 0 and $-L$, respectively; namely, we assume completely inelastic boundary conditions. More precisely, when a particle $x_0(t)$ at time t reaches the point 0 having some velocity $v_0(t-0) \geq 0$, then its velocity $v_0(t)$ immediately becomes zero and the particle itself stays at 0 until the force acting on it (which varies accordingly to the motion of other particles) becomes negative. Similarly for the particle $x_N(t)$ at the point $-L$.

To discover phase transitions, it is common to consider asymptotics as $N \to \infty$, with the parameters L and $F(x)$ being fixed. Then the fixed points depends only on the "renormalized force" $F = \frac{\alpha_{\rm ext}}{\alpha_{\rm int}} F_0$, and we assume that the renormalized constant $\alpha_{\rm ren} = \frac{\alpha_{\rm ext}}{\alpha_{\rm int}}$ (or the renormalized force) can tend to infinity together with N, namely as $\alpha_{\rm ren} = cN^\gamma$, where $c, \gamma > 0$. It is obvious that if $F_0 \equiv 0$, then for a unique fixed point and for all $k = 1, \ldots, N$ we have

$$\delta_k = x_{k-1} - x_k = \frac{L}{N}. \tag{2}$$

The case where $\alpha_{\rm ren}$ does not depend on N was discussed in detail in [10]; there are no phase transitions but it is discovered that the structure of the fixed configuration differs from (2) only on a submicroscale of the order N^{-2}.

The necessity to consider cases where $\alpha_{\rm ren}$ depends on N issues from particular examples where $\alpha_{\rm ren} \gg N$. For instance, the linear density of electrons in some conductors (see [11]) is of the order $N \approx 10^9\,\text{m}^{-1}$, $\alpha_{\rm int} = \frac{e^2}{\varepsilon_0} \approx 10^{-28}$ (in the SI system), and thus $\alpha_{\rm ext} = 220\frac{\text{eV}}{\text{m}} \approx 352 \times 10^{-19}$. Thus, $\alpha_{\rm ren}$ is of the order 10^{11}. This is close to a critical point of our model, which, as will be shown below, is asymptotically $c_{\rm cr}N$, i.e., is close to 4×10^9 in our case.

We study the density $\rho(x)$ (proving its existence) defined so that for any subintervals $I \subset [-L, 0]$ the limits

$$\rho(I) = \int_I \rho(x)\,dx = \lim_{N\to\infty} \frac{\#\{i:\ x_i \in I\}}{N}$$

exist. We find four phases: (1) uniform (constant) density, (2) nonuniform but positive smooth density, (3) continuous density vanishing on some subinterval, (4) density of a δ function type.

The one-dimensional case shows what can be expected in the multidimensional case, which is more complicated but has great interest in connection with static charge distributions in the atmosphere or in a living organism. For example, case (4) of Theorem 2 is related to the discharge possibility, since after disappearance of an external force a large concentration of charged particles can produce strong discharge.

2. MAIN RESULTS

Lemma. *Assume that $F_0(x)$ is continuous, nonnegative, and nonincreasing, i.e., $F(x) \leq F(y)$ if $x > y$. Then for any N, L, and $\alpha_{\rm ren}$ a fixed point exists and is unique. If y is such that $F(x) = 0$, $x \geq y$, and $F(x) > 0$, $x < y$, then $\delta_{k+1} > \delta_k$ if $x_{k+1} < y$.*

In what follows we assume for simplicity that $F_0 > 0$ is uniform (constant in x).

Theorem 1 (critical force). *For any N and L there exists $F_{\rm cr} = F_{\rm cr}(N, L)$ such that for the fixed point we have $x_N > -L$ for $F > F_{\rm cr}$ and $x_N = -L$ for $F \le F_{\rm cr}$. If $F = cN^\gamma$, $\gamma > 1$, then we have $x_N \to 0$ as $N \to \infty$ for any $c > 0$. Furthermore, $F_{\rm cr} \sim_{N\to\infty} c_{\rm cr}N$, where*

$$c_{\rm cr} = \frac{4}{L^2}. \tag{3}$$

Theorem 2 (four phases). (1) *If $F = o(N)$, then the density exists and is strictly uniform; i.e., for all $k = 1, \ldots, N$ as $N \to \infty$*

$$\max_k \Big|(x_{k-1} - x_k) - \frac{L}{N}\Big| = o\Big(\frac{1}{N}\Big); \tag{4}$$

(2) *If $F = cN$ and $0 < c \le c_{\rm cr}$, then $x_N = -L$ and the density of particles exists, is nowhere zero, but is not uniform (not constant in x);*

(3) *If $F = cN$ and $c > c_{\rm cr}$, then as $N \to \infty$*

$$-L < x_N \to -\frac{2}{\sqrt{c}}, \tag{5}$$

and the density on the interval $(-\frac{2}{\sqrt{c}}, 0)$ is not uniform;

(4) *If $F = cN^\gamma$, $\gamma > 1$, then the density $\rho(x) \to \delta(x)$ in the sense of distributions.*

3. PROOFS

Uniqueness: Proof of Lemma 1. Put

$$f_k = \delta_k^{-2}, \quad k = 1, \ldots, N.$$

At least one fixed point exists because the minimum of U obviously exists. Any fixed point satisfies the following conditions:

$$x_0 = 0$$

and

$$f_{k+1} + F(x_k) = f_k, \quad k = 1, \ldots, N-1. \tag{6}$$

However, for the particle N there are two possibilities:

$$f_N \ge F(x_N) \tag{7}$$

if $x_N = -L$, and

$$f_N = F(x_N) \tag{8}$$

if $x_N > -L$.

Forgetting for a while about fixed points, we will consider equations (6) as equations uniquely defining (by induction on k) the functions f_k of δ_1, and thus $\delta_k = \dfrac{1}{\sqrt{f_k}}$, and also $x_k = -(\delta_1 + \ldots + \delta_k)$. It is obvious that f_k and x_k are decreasing and δ_k are increasing functions of δ_1. Moreover, if $\delta_1 \to 0$, then all $f_k \to \infty$, and δ_k and x_k tend to zero, and therefore, for δ_1 sufficiently small, inequality (7) holds. Thus, if δ_1 increases, two cases are possible: (a) There exists $\delta_{1,\rm final}$ such that

$$F(x_N) = f_N, \qquad x_N > -L.$$

Furthermore, if $\delta_1 > \delta_{1,\rm final}$, then $F(x_N)$ and δ_N increase as functions of δ_1, and f_N decreases, and therefore $F(x_N) > f_N$. It follows that in this case there are no other fixed points; (b) Such δ_1 does not exist, but then for some δ_1 we have

$$x_N = -L, \qquad F(x_N) \le f_N.$$

This defines the unique fixed point.

Remark about nonuniqueness. The monotonicity assumption in the uniqueness lemma is very essential. One can give an example of nonuniqueness, for a function $F_0(x)$ with a single maximum, where the number of fixed points is of the order of at least N. Namely, on the interval $[-1, 1]$ put for $b > a > 0$

$$F_0(x) = a - 2ax, \quad x \geq 0,$$
$$F_0(x) = a + 2bx, \quad x \leq 0.$$

Then there exists $C_{\rm cr} > 0$ such that for all sufficiently large N and $\alpha_{\rm ren} = cN$, $c > C_{\rm cr}$, one can similarly show that for any odd $N_1 < N$ there exists a fixed point such that

$$-1 = x_N < \ldots < x_{N_1} < 0 < x_{N_1 - 1} < \ldots < x_{\frac{N_1+1}{2}} = \frac{1}{2} < \ldots < x_0 < 1.$$

Moreover, any such point will give a local minimum of the energy.

Critical force: Proof of Theorem 1 and case (4) of Theorem 2. In the case of a constant positive force it follows from (6) that

$$f_i > f_{i+1} \iff \delta_i < \delta_{i+1}, \quad i = 1, \ldots, N-1, \tag{9}$$

i.e., the lengths δ_i of intervals strictly increase with i. Therefore,

$$\delta_1 < \frac{L}{N}. \tag{10}$$

Summation of equalities (6) over $i = 1, \ldots, k-1$ gives that for any $k = 1, \ldots, N$

$$f_k = f_1 - (k-1)F, \quad k = 1, \ldots, N. \tag{11}$$

Similarly to (11), summing over $i = N-1, \ldots, k-1$, we obtain

$$f_k = f_N + (N-k)F. \tag{12}$$

Then from (11) we get

$$\delta_k = \left(\delta_1^{-2} - (k-1)F\right)^{-\frac{1}{2}} = \delta_1\left(1 - \delta_1^2(k-1)F\right)^{-\frac{1}{2}} \tag{13}$$

and, since a fixed point exists,

$$1 - \delta_1^2(k-1)F > 0, \tag{14}$$

or

$$\delta_1 < \left(\frac{1}{(N-1)F}\right)^{\frac{1}{2}}. \tag{15}$$

To prove Theorem 1, consider a simpler auxiliary model with $L = \infty$. In other words, we assume that the particles are located on the interval $(-\infty, 0]$, and the force F is constant on the whole $(-\infty, 0]$. In this model for any $F > 0$ there is a unique fixed point, given explicitly by

$$f_N = F, \qquad f_k = (N-k+1)F, \quad k = N-1, \ldots, 1,$$

which follows from (12). Hence we obtain

$$\delta_N = F^{-\frac{1}{2}}, \qquad \delta_k = \frac{1}{\sqrt{(N-k+1)F}} \tag{16}$$

and

$$-x_N = \sum_{k=1}^{N} \delta_k = \frac{1}{\sqrt{F}} \sum_{k=1}^{N} \frac{1}{\sqrt{k}}.$$

The relation of this model to the initial one is quite simple. If $S \le L$, then the fixed points for both models coincide. If $S \ge L$, then $x_N = -L$. In fact, assuming that $x_N > -L$ for the critical point in the main model, we obtain a contradiction with the auxiliary model. Therefore, the critical force can be found from the condition that $x_N = -L$ in the auxiliary model; i.e.,

$$F = F_{\rm cr} = \left(\frac{1}{L} \sum_{k=1}^{N} \frac{1}{\sqrt{k}}\right)^2 \sim_{N\to\infty} \left(\frac{2}{L}\right)^2 N.$$

One can also say that for any $x < 0$ there exists a unique $F = F_x$ such that $x_N = x$.

For $F = cN$, $c > c_{\rm cr}$, we have

$$-x_N = L = \sum_{k=1}^{N} \delta_k = \frac{1}{\sqrt{F}} \sum_{k=1}^{N} \frac{1}{\sqrt{k}} \sim \frac{2}{\sqrt{c}},$$

whence (3) follows, and this gives Theorem 1. Case (4) of Theorem 2 follows similarly, since $x_N \to 0$ for $F = cN^{\gamma}$, $\gamma > 1$.

Nonuniform density: Proof of cases (2) and (3) of Theorem 2. First, consider the case $F = cN$, $c > c_{\rm cr}$. Then for $k = aN$ we have from (16)

$$\delta_k = \frac{1}{\sqrt{(N-k+1)F}} \sim \frac{1}{\sqrt{(1-a)c}}\frac{1}{N}.$$

Therefore, the density exists; moreover, it equals zero on $[-L, x_N]$ and is nonuniform on $[x_N, 0]$.

Now let $c \le c_{\rm cr}$. One can assume that $\delta_1 = b\dfrac{L}{N}$, $0 < b = b(N) \le 1$. Then for $k = aN$, $a < 1$, from (13) we obtain

$$\delta_k = b\frac{L}{N}\Big(1 - L^2 c b^2 \frac{k-1}{N}\Big)^{-\frac{1}{2}} \sim b\frac{L}{N}(1 - b^2 c L^2 a)^{-\frac{1}{2}}.$$

Thus, the density is not uniform.

Uniform density: Proof of case (1) of Theorem 2. From (10) we have

$$\delta_1^2 (k-1)F \le \delta_1^2 (N-1)F = o(1). \tag{17}$$

Then

$$\begin{aligned} L = \sum_{k=1}^{N} \delta_k &= \delta_1 \sum_{k=1}^{N} (1 - \delta_1^2 (k-1)F)^{-\frac{1}{2}} \\ &= \delta_1 \sum_{k=1}^{N} \Big(1 + \frac{1}{2}\delta_1^2 (k-1)F + O\big((\delta_1^2 (k-1)F)^2\big)\Big) \\ &= N\delta_1 + \frac{1}{4}\delta_1^3 F N^2 + o(\delta_1^3 F N^2). \end{aligned} \tag{18}$$

But by (17) we have

$$\delta_1^3 F N^2 = o(N\delta_1),$$

and therefore

$$\delta_1 = \frac{L}{N} + o\Big(\frac{L}{N}\Big).$$

The result for all k follows from (13).

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