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# Dynamics of Phase Boundary with Particle Annihilation

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We are happy to dedicate this paper to the memory of R.L. Dobrushin. He was a very curious person and had wide scope in probability. Obviously he could bring his own vision for statistical physics of economic phenomena which is now at its very beginning.

**Abstract.** On the real axis there are point particles of two kinds: plus particles on the positive part moving to the left, and minus particles on the negative part moving to the right. When plus particle and minus particle collide both disappear. It is clear that two phases are always separated. For one particular model of such kind we study the movement of the boundary between phases in the large density limit.

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## 1. Introduction

On one-dimensional lattice $\mathbf{Z}_\varepsilon = \varepsilon\mathbf{Z} = \{\varepsilon m : m \in \mathbf{Z}\}$, $\varepsilon > 0$, there are particles of two types — "plus particles" and "minus particles". Denote by $\nu_m^{\pm}(t)$ the number of plus(minus)-particles at site $\varepsilon m$ at time $t$. We define a continuous time Markov process on $[0, \infty)$ by the following conditions:

(1) at time 0 all plus particles have positive coordinates, all minus particles have negative coordinates;

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(2a) any plus particle, independently of other particles, performs a simple random walk: that is it jumps from $\varepsilon m$ to $\varepsilon(m+1)$ with rate $\mu_+$ and from $\varepsilon m$ to $\varepsilon(m-1)$ with rate $\lambda_+$;

(2b) any minus particle, independently of other particles, performs a simple random walk, that is it jumps from $\varepsilon m$ to $\varepsilon(m+1)$ with rate $\lambda_-$ and from $\varepsilon m$ to $\varepsilon(m-1)$ with rate $\mu_-$;

(3a) if a plus particle jumps to a site where there are minus particles, it immediately annihilates with one of the minus particles at this site;

(3b) if a minus particle jumps to a site where there are plus particles, it immediately annihilates with one of the plus particles at this site.

At any time $t > 0$ the state of the process is the vector $(\nu_m^{\pm}(t), m \in \mathbf{Z})$. However, it follows from (3a) and (3b) that for any $m$ and $t$

$$\nu_m^+(t)\nu_m^-(t) = 0.$$

Moreover, all minus particles are always to the left of the leftmost plus particle. It will be convenient to define $\beta_\varepsilon(t) \in \mathbf{Z}_\varepsilon$ as the point where the last annihilation before time $t$ happened. We call it the phase boundary.

Note that there are no problems with the existence of this process.

Besides the interpretation related to annihilation of particles there is another one — the microdynamics of the price formation, where the market contains many players and is formed by their behaviour. Namely, $\beta_\varepsilon(t)$ is the price of some product at time $t$. Plus particles (bears) want to lower the price of this product (we assume further on that $\alpha_+ = \lambda_+ - \mu_+ > 0$), minus particles (bulls) want to increase the price (we assume $\alpha_- = \lambda_- - \mu_- > 0$). Annihilation is a bargain which is performed when the demand and offer prices meet together. Recent models of price formation [2–5] have much in common with our model, however they are closer in spirit to queueing models. Our model is closer to statistical physics models. Anyway, all such models cannot pretend on practical implementation, mainly because external influence on the action of players is not taken into account.

We consider the large density limit $\varepsilon \to 0$ under the time scaling $t = \tau\varepsilon^{-1}$, where $t$ is microtime and $\tau$ is macrotime. Our goal is to find asymptotic behaviour of the price $\beta(\tau) = \lim_{\varepsilon \to 0} \beta_\varepsilon(t)$ as the result of many micro-bargains.

## 2. Main result

*Initial distribution of particles*

We assume for simplicity that at time $t = 0$ the distribution of plus particles is (inhomogeneous) Poisson with density $\rho_+(\varepsilon m)$, where $\rho_+(x)$ is some strictly positive continuous function on $(0,\infty)$. This means that the random variables $\nu_m^+(0)$ are independent and have Poisson distribution with rate $\rho_+(\varepsilon m)$. Similarly, the distribution of minus particles is (inhomogeneous) Poisson with density $\rho_-(\varepsilon m)$, where $\rho_-(x)$ is some strictly positive continuous function on $(0,\infty)$.

*Notation*

Next we introduce main definitions and give their intuitive interpretation. Our interpretation concerns the situation of large density limit when, instead of point particles on a lattice, there is a continuous media of infinitesimally small particles of two types, the particles move with fixed velocities $-\alpha_+ < 0$, $\alpha_- > 0$ and have initial densities $\rho_\pm(x)$ correspondingly. That is there are no fluctuations. Define the functions

$$M_-(r) = \int_{-r}^{0} \rho_-(y)\,dy \qquad \text{and} \qquad M_+(r) = \int_{0}^{r} \rho_+(y)\,dy \quad \text{for} \quad r \geq 0. \tag{2.1}$$

We interprete $M_\pm(r)$ as the cumulative mass of plus (minus) particles on the distance less than $r$ from zero. Under above assumptions on $\rho_\pm$ we see that the functions $M_\pm(r)$ are strictly increasing on $(0,+\infty)$ and, therefore, the inverse functions $r_\pm(M)$, defined by the equation

$$M_\pm(r_\pm(M)) = M,$$

exist and are strictly increasing. For example, the function $r_+ = r_+(M)$ defines the interval $(0, r_+)$ where the mass of plus particles equals $M$. Then the function

$$T(M) := \frac{r_-(M) + r_+(M)}{\alpha_- + \alpha_+} \tag{2.2}$$

defines the time interval $(0, T(M))$ during which mass $M$ of plus and mass $M$ of minus particles annihilate. The function $T(M)$ is also strictly increasing on $[0,+\infty)$ and is invertible. Denote its inverse function by $M(T)$. The place where the latter of these particles meet

$$r_+(M(T)) - \alpha_+ T = -r_-(M(T)) + \alpha_- T = \beta(T) \tag{2.3}$$

is the coordinate of the boundary at time $T$. Excluding from the system (2.3) the terms that are linear in $T$, we get

$$\beta(T) = r_+(M(T))\frac{\alpha_-}{\alpha_- + \alpha_+} - r_-(M(T))\frac{\alpha_+}{\alpha_- + \alpha_+}\,.$$

*Scaling limit for the stochastic model*

Here we return to the stochastic particle model and formulate the main result.

**Theorem 2.1.** *For any fixed $\tau \geq 0$ the following convergence in probability holds*

$$\beta_\varepsilon(\varepsilon^{-1}\tau) \to \beta(\tau) \qquad (\varepsilon \to 0),$$

*where the function $\beta : \mathbf{R}_+ \to \mathbf{R}$ is deterministic and has the following explicit form*

$$\beta(\tau) = \frac{-\alpha_+ r_-(M(\tau)) + \alpha_- r_+(M(\tau))}{\alpha_- + \alpha_+}.$$

**Corollary 2.1.** *Consider the homogeneous case $\rho_-(y) \equiv \rho_-$, $y < 0$, $\rho_+(y) \equiv \rho_+$, $y > 0$. All functions defined above are linear: $M_-(r) = \rho_- r$, $M_+(r) = \rho_+ r$, $r_\pm(M) = M/\rho_\pm$,*

$$T(M) = M\,\frac{\rho_-^{-1} + \rho_+^{-1}}{\alpha_- + \alpha_+},$$

$$M(T) = T\frac{\alpha_- + \alpha_+}{\rho_-^{-1} + \rho_+^{-1}},$$

*and, hence, the phase boundary $\beta_\varepsilon(\tau\varepsilon^{-1})$ moves with an asymptotically constant velocity:*

$$\begin{aligned}\beta_\varepsilon(\tau\varepsilon^{-1}) \to \beta(\tau) &= \tau\,\frac{-\alpha_+\rho_-^{-1} + \alpha_-\rho_+^{-1}}{\rho_-^{-1} + \rho_+^{-1}} \\ &= \tau\,\frac{-\alpha_+\rho_+ + \alpha_-\rho_-}{\rho_+ + \rho_-}.\end{aligned}$$

## 3. Proof

Our plan is to show that the limiting behavior of $\beta_\varepsilon(t)$ in the stochastic model corresponds to the deterministic evolution described in (2.1)–(2.3). To do this we need some control over the random fluctuations in the limit $\varepsilon \to 0$. Now we fix $M$ and consider the following random variables $A_{i,\pm} = A_{i,\pm}(M)$, $i = 0, 1, 2$, (we will prove that they are of the order $o(\varepsilon^{-1})$):

1. Denote by $Q_{\varepsilon,\pm} = Q_{\varepsilon,\pm}(M)$ the number of $(\pm)$-particles which were at time $t = 0$ correspondingly in the intervals

   $$I_+^\circ = (0, r_+(M)) \cap \mathbf{Z}_\varepsilon \quad \text{and} \quad I_-^\circ = (-r_-(M), 0) \cap \mathbf{Z}_\varepsilon. \tag{3.1}$$

   Define $A_{0,\pm} = Q_{\varepsilon,\pm} - M\varepsilon^{-1}$.

2. All particles among them will be annihilated during time $t = T(M)\varepsilon^{-1}$ except of the number $A_{1,\pm}$ of them.

3. Define $A_{2,\pm}$ as the number of plus and minus particles which were not at time $t$ in the intervals (3.1) but were annihilated during time $t = T(M)\varepsilon^{-1}$.

This control can be achieved by use of exponential bounds for some families of events. The proof uses some ideas from [1].

**Definition 3.1.** We say that a family of events $\mathcal{A} = \{A_\varepsilon\}_{\varepsilon>0}$ has a property of exponential asymptotic sureness (e.a.s.) if there exist constants $\mathcal{K}_\mathcal{A} > 0$, $q_\mathcal{A} > 0$, $\varepsilon_\mathcal{A} > 0$ such that for all $\varepsilon < \varepsilon_\mathcal{A}$ the following inequality holds

$$\mathsf{P}\left(A_\varepsilon\right) \geq 1 - \mathcal{K}_\mathcal{A} \exp\left(-q_\mathcal{A}\varepsilon^{-1}\right).$$

In the sequel, for breavity, we say sometimes that the event $A_\varepsilon$ has probability exponentially close to one. We will use the following fact: if two sequences $\mathcal{A} = \{A_\varepsilon\}_{\varepsilon>0}$ and $\mathcal{B} = \{B_\varepsilon\}_{\varepsilon>0}$ have the property e.a.s., then this property holds also for the sequence $\mathcal{C} = \{A_\varepsilon \cap B_\varepsilon\}_{\varepsilon>0}$.

It is helpful to enumerate the particles at time 0 somehow with the only condition that

$$\cdots \leq x_3^-(0) \leq x_2^-(0) \leq x_1^-(0) < 0 < x_1^+(0) \leq x_2^+(0) \leq x_3^+(0) \leq \cdots$$

Denote by $q_-(1)$ and $q_+(1)$ the indices of plus and minus particles of the first annihilating pair. One can assume that if some plus (minus) particle jumps to a site where there are several minus (plus) particles then it annihilates with the minus (plus) particle having minimal index. Let $\sigma_1$ be the time moment when the first annihilation occurs. Since particles move independently, their order can change in time, so, in general, $x^-_{q_-(1)}(0) \neq x_1^-(0)$ and $x^+_{q_+(1)}(0) \neq x_1^+(0)$. Similarly, we define $q_-(m)$ and $q_+(m)$ as the indices of the particles of the $m$-th annihilating pair and $\sigma_m$ as the time moment of the $m$-th annihilation.

Fix some $M > 0$. Let $N_\varepsilon = [M\varepsilon^{-1}]$. Consider the $N_\varepsilon$-th pair of annihilating particles, $x^-_{q_-(N_\varepsilon)}$ and $x^+_{q_+(N_\varepsilon)}$. The main idea is to prove that for small $\varepsilon$ the random time $\sigma_{N_\varepsilon}$ is close to the value $T(M)\varepsilon^{-1}$ and the random coordinate $x^-_{q_-(N_\varepsilon)}(0) \in \mathbf{Z}_\varepsilon$ is close to $-r_-(M)$. In more precise terms, it is sufficient to prove that for any small fixed positive numbers $\varkappa_0, \varkappa_1, \zeta_-, \zeta_+$ *with probability exponentially close to one* (as $\varepsilon \to 0$) the following holds:

(a) the moment $\sigma_{N_\varepsilon}$ belongs to the time interval $(t_0(M,\varepsilon), t_1(M,\varepsilon))$, where

$$\begin{aligned} t_0(M,\varepsilon) &= (T(M) - \varkappa_0)\varepsilon^{-1}, \\ t_1(M,\varepsilon) &= (T(M) + \varkappa_1)\varepsilon^{-1}; \end{aligned} \tag{3.2}$$

(b) the starting point of the minus particle $x^-_{q_-(N_\varepsilon)}(0)$ belongs to the set $(-r_-(M) - \zeta_-, -r_-(M) + \zeta_-) \cap \mathbf{Z}_\varepsilon$;

(c) similarly, the starting point of the plus particle $x^+_{q_+(N_\varepsilon)}(0)$ belongs to the set $(r_+(M) - \zeta_+, r_+(M) + \zeta_+) \cap \mathbf{Z}_\varepsilon$.

Let us prove the theorem assuming that the above statements (a)–(c) are proved. Recall that $\beta_\varepsilon(\sigma_{N_\varepsilon}+0) = x^-_{q_-(N_\varepsilon)}(\sigma_{N_\varepsilon}) = x^+_{q_+(N_\varepsilon)}(\sigma_{N_\varepsilon})$. Individual motion of a minus particle is a simple random walk on $\mathbf{Z}_\varepsilon$ with the mean drift $\alpha_-\varepsilon = (\lambda_- - \mu_-)\,\varepsilon$, so applying the upper bound of the large deviation theory, we get that for any fixed $i \in \mathbf{N}$, $s > 0$ and $\delta_0 > 0$ with probability exponentially close to one

$$x_i^-(s\varepsilon^{-1}) - x_i^-(0) \in ((\alpha_- - \delta_0)\,s, (\alpha_- + \delta_0)\,s)\,.$$

In fact, even stronger result holds: for fixed $s_2 > s_1 > 0$ and $\delta_0 > 0$ the family of events $\{D_\varepsilon\}$, where

$$D_\varepsilon = \left\{\, x_i^-(s\varepsilon^{-1}) - x_i^-(0) \in ((\alpha_- - \delta_0)\,s, (\alpha_- + \delta_0)\,s)\,,\ \forall s \in [s_1, s_2]\,\right\},$$

has a property of e.a.s. Together with (a) this gives

$$x^-_{q_-(N_\varepsilon)}(\sigma_{N_\varepsilon}) - x^-_{q_-(N_\varepsilon)}(0) \in ((\alpha_- - \delta_0)\,(T(M) - \varkappa_0)\,, (\alpha_- + \delta_0)\,(T(M) + \varkappa_1))$$

with probability exponentially close to one. Combining the latter statement with the statement (b) we conclude that with probability exponentially close to one

$$x^-_{q_-(N_\varepsilon)}(\sigma_{N_\varepsilon}) \in (\alpha_- T(M) - r_-(M) - \gamma, \alpha_- T(M) - r_-(M) + \gamma)$$

where $\gamma = \gamma(\delta_0, \varkappa_0, \varkappa_1, \zeta_-) > 0$ can be made arbitrary small, i.e.,

$$\gamma(\delta_0, \varkappa_0, \varkappa_1, \zeta_-) \to 0 \quad \text{as } \max(\delta_0, \varkappa_0, \varkappa_1, \zeta_-) \to 0.$$

Using (2.2) we see that

$$\begin{aligned} -r_-(M) + \alpha_- T(M) &= -r_-(M) + \alpha_- \frac{r_-(M) + r_+(M)}{\alpha_- + \alpha_+} \\ &= -r_-(M)\frac{\alpha_+}{\alpha_- + \alpha_+} + r_+(M)\frac{\alpha_-}{\alpha_- + \alpha_+} \end{aligned}$$

and, hence,

$$\beta_\varepsilon(\sigma_{N_\varepsilon}) \to -r_-(M)\frac{\alpha_+}{\alpha_- + \alpha_+} + r_+(M)\frac{\alpha_-}{\alpha_- + \alpha_+}$$

in probability as $\varepsilon \to 0$.

To finish the proof of theorem we need only to check that

$$\beta_\varepsilon(\sigma_{N_\varepsilon}) - \beta_\varepsilon(\varepsilon^{-1}T(M)) \to 0 \quad \text{as } \varepsilon \to 0.$$

This corresponds to continuity property of the border on the macroscopic time scale $\tau = T$. To establish this fact we should take into account that: 1) due to the drift assumption ($\alpha_\pm > 0$) the random sequence $\{\sigma_{m+1} - \sigma_m,\ m \in \mathbf{N}\}$ admits uniform exponential estimates for the tails of the distribution functions of $\sigma_{m+1} - \sigma_m$ (we refer the reader to [6] for the corresponding techniques); 2) in finite microtime $t$ the displacements of walking particles have the order $O(\varepsilon)$ while in finite macrotime $\tau$ their displacements have the order $O(1)$. We omit the details.

To prove the statement (a) we need the following main lemma. Denote by $\mathcal{N}_-(0, t_m(M,\varepsilon))$ the set of minus particles that collide with plus particles on the time interval $(0, t_m(M,\varepsilon))$.

**Lemma 3.1.** *For any sufficiently small $\varkappa_2, \varkappa_3 > 0$ the following events*

$$F_\varepsilon = \left\{|\mathcal{N}_-(0, t_0(M,\varepsilon))| < (M - \varkappa_2)\varepsilon^{-1}\right\},$$
$$G_\varepsilon = \left\{|\mathcal{N}_-(0, t_1(M,\varepsilon))| > (M + \varkappa_3)\varepsilon^{-1}\right\}$$

*have the probabilities exponentially close to one.*

Lemma 3.1 follows from Lemmas 3.2 and 3.3. Lemma 3.2 deals with the initial distribution of particles and Lemma 3.3 controls the deplacements of minus and plus particles.

**Lemma 3.2.** *Let $y_1 < y_2 \le 0$ and $0 \le z_1 < z_2$. Then for any $\delta > 0$ the following families of events have probabilities exponentially close to one:*

$L_\varepsilon = \{$*the number of minus particles sitting at time $t = 0$ in the set $(y_1, y_2) \cap \mathbf{Z}_\varepsilon$ is between* $\left(\int_{y_1}^{y_2} \rho_-(y)\,dy - \delta\right)\varepsilon^{-1}$ *and* $\left(\int_{y_1}^{y_2} \rho_-(y)\,dy + \delta\right)\varepsilon^{-1}\}$,

$R_\varepsilon = \{$*the number of plus particles sitting at time $t = 0$ in the set $(z_1, z_2) \cap \mathbf{Z}_\varepsilon$ is between* $\left(\int_{z_1}^{z_2} \rho_+(y)\,dy - \delta\right)\varepsilon^{-1}$ *and* $\left(\int_{z_1}^{z_2} \rho_+(y)\,dy + \delta\right)\varepsilon^{-1}\}$.

**Lemma 3.3.** *For any $\delta_1 > 0$ each family of events*

$A_\varepsilon = \{$*all particles $x_k^\pm(0) \in (-r_-(M) + \delta_1,\ r_+(M) - \delta_1) \cap \mathbf{Z}_\varepsilon$ collide with particles of opposite sign till the time moment $t(M)\varepsilon^{-1}$*$\}$,

$B_\varepsilon = \{$*on the time interval $t \in (0, s\varepsilon^{-1})$ none of minus particles, started at $t = 0$ from the set $(-\infty, -r_-(M) - \delta_1) \cap \mathbf{Z}_\varepsilon$, collides with any plus particle, started at $t = 0$ from the set $(r_+(M) + \delta_1, +\infty) \cap \mathbf{Z}_\varepsilon$*$\}$,

*satisfies the e.a.s. property. Moreover, fix any $y, \varkappa > 0$ and consider the following subsets of $\mathbf{Z}_\varepsilon$:*

$$S_2^- = (-\infty, -y - \varkappa), \quad S_1^- = (-y, 0), \quad S_1^+ = (0, y), \quad S_2^+ = (y + \varkappa, +\infty).$$

*Define the events*

$V_\varepsilon = \{$*on the time interval* $t \in (0, s\varepsilon^{-1})$ *none of minus particles, started at* $t = 0$ *from* $S_2^-$, *will meet some minus particle, started from the set* $S_1^-\}$;

$U_\varepsilon = \{$*on the time interval* $t \in (0, s\varepsilon^{-1})$ *none of plus particles, started at* $t = 0$ *from* $S_2^+$, *will meet some minus particle, started from the set* $S_1^+\}$.

*Then the families of events* $\{V_\varepsilon\}$ *and* $\{U_\varepsilon\}$ *have the e.a.s. property.*

Proofs of Lemmas 3.2 and 3.3 are based on standard probabilistic methods [6] and are omitted. Let us explain now how using these two lemmas one can get, for example, the upper bound for $|\mathcal{N}_-(0, t_0(M,\varepsilon))|$ in Lemma 3.1.

Firstly, we include in this bound all minus particles starting at $t = 0$ from the set $(-r_-\,(\,M(T(M) - \varkappa_0)\,) - \delta_5, 0)$ where $\delta_5 > 0$ is small and will be fixed later. By Lemma 3.2 there is no more than $(M_-\,(r_-\,(\,M(T(M) - \varkappa_0)\,) + \delta_5) + \delta_6)\,\varepsilon^{-1}$ of such particles e.a.s. for small $\delta_6 > 0$.

We should add to this bound all minus particles that started at $t = 0$ from the set $(-\infty, -r_-\,(\,M(T(M) - \varkappa_0)\,) - \delta_5)$ and annihilated in the time interval $(0, t_0(M,\varepsilon))$ with some plus particles. We will show now that with probability exponentially close to one the number $N^\circ(0, t_0(M,\varepsilon))$ of such minus particles can be estimated as $c\varepsilon^{-1}$ where $c > 0$ is any prefixed small constant. Indeed, by Lemma 3.3 (again in the sense of e.a.s.) the minus particles in question can annihilate only by colliding with some plus particles, started at $t = 0$ from the set $(0, r_+\,(\,M(T(M) - \varkappa_0)\,) + \delta_5)$. By Lemma 3.2, the number of the plus particles in this set is bounded by $(M_+\,(r_+\,(\,M(T(M) - \varkappa_0)\,) + \delta_5) + \delta_6)\,\varepsilon^{-1}$. From this bound we should exclude plus particles which was annihilated in collisions with minus particles started from the set $(-r_-\,(\,M(T(M) - \varkappa_0)\,) + \delta_5, 0)$, since by the part "$V_\varepsilon$" of Lemma 3.3 during the time interval $(0, t_0(M,\varepsilon))$ the latter minus particles will go ahead of the minus particles started from $(-\infty, -r_-\,(\,M(T(M) - \varkappa_0)\,) - \delta_5)$. By Lemma 3.2, initially there was no less than $(M_-\,(r_-\,(\,M(T(M) - \varkappa_0)\,) - \delta_5) - \delta_6)\,\varepsilon^{-1}$ particles in the set

$$(-r_-\,(\,M(T(M) - \varkappa_0)\,) + \delta_5, 0)\,.$$

So using the mean value theorem from analysis we get

$$\begin{aligned}
\varepsilon \cdot N^\circ(0, t_0(M,\varepsilon)) &\le (M_+\,(r_+\,(\,M(T(M) - \varkappa_0)\,) + \delta_5) + \delta_6) \\
&\quad - (M_-\,(r_-\,(\,M(T(M) - \varkappa_0)\,) - \delta_5) - \delta_6) \\
&= M(T(M) - \varkappa_0) + M_+'(\theta_1)\delta_5 + \delta_6 \\
&\quad - \big((\,M(T(M) - \varkappa_0)\,) - M_-'(\theta_2)\delta_5 - \delta_6\big) \\
&\le \delta_5(\|M_-'\|_C + \|M_+'\|_C) + 2\delta_6.
\end{aligned}$$

Hence, in the sense of e.a.s.,

$$\begin{aligned}\varepsilon \cdot |\mathcal{N}_-(0, t_0(M,\varepsilon))| &\le (M_-(r_-(M(T(M) - \varkappa_0)) \\ &\qquad + \delta_5) + \delta_6) + \delta_5(\|M'_-\|_C + \|M'_+\|_C) + 2\delta_6 \\ &= M(T(M) - \varkappa_0) + M'_-(\theta_3)\delta_5 \\ &\qquad + \delta_5(\|M'_-\|_C + \|M'_+\|_C) + 3\delta_6 \\ &\le M(T(M) - \varkappa_0) + \delta_5(2\|M'_-\|_C + \|M'_+\|_C) + 3\delta_6 .\end{aligned}$$

It follows from (2.2) and assumptions on $\rho_\pm$ that $M'(t) \ge k$ for some $k > 0$. Therefore, $M(T(M) - \varkappa_0) \le M - k\varkappa_0$. Given $\varkappa_0 > 0$ we are allowed to chose positive constants $\delta_5$ and $\delta_6$ as small as we like. So, finally, we get that with probability exponentially close to one the following estimate holds

$$|\mathcal{N}_-(0, t_0(M,\varepsilon))| \le \Big(M - \frac{k\varkappa_0}{2}\Big)\varepsilon^{-1} .$$

The lower bound for $|\mathcal{N}_-(0, t_1(M,\varepsilon))|$ can be obtained in a similar way.

To get the proof of statement (b) one should combine Lemma 3.3 with the following lemma.

**Lemma 3.4.** *For any* $\varkappa_5 > 0$, *the event*

$$H_\varepsilon = \big\{x_i^-(0) \in (-(1+\varkappa_5)r_-(M), 0) \quad \forall i \in \mathcal{N}_-(0, t_1(M,\varepsilon))\big\}$$

*has the property of e.a.s.*

Proof of this lemma uses arguments similar to the proof of Lemma 3.1. The statement (c) is just a symmetric modification of the statement (b).

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