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EXPONENTIAL CONVERGENCE OF ONE-DIMENSIONAL TOOM'S PROBABILISTIC CELLULAR AUTOMATA

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by

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ABSTRACT

We obtain cluster expansions for small random perturbations of deterministic Toom's automata in the one-dimensional case. Exponential convergence follows.

Key words: Probabilistic Cellular Automata (PCA); Toom's condition; exponential convergence; cluster expansions.

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Sur le convergence exponentielle d' automates probabilistiques cellulaires de Toom en dimension un

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Résumé

On obtient une serie convergente pour les perturbations aléatoires d' une classe d'automates cèllulaires en dimension 1.

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EXPONENTIAL CONVERGENCE OF ONE-DIMENSIONAL TOOM'S PROBABILISTIC CELLULAR AUTOMATA

SECTION 1. INTRODUCTION

Processes with local interaction in the "high temperature" region (i.e. when the interaction is weak) are sufficiently well understood (see review [1], [7]). One can reasonably assert that the low temperature region for Gibbs random fields corresponds to small perturbations of determinstic processes with local interaction. One of the deepest results is the proof of stability by Toom [6] for his class of deterministic processes. For the simplest processes of this type (e.g. for Stavskaya's model) cluster expansion techniques and as a consequence stability, exponential convergence and an analytic property were known earlier ([3], [4], [5]). In this paper we solve this problem completely (mainly the one of exponential convergence) for the general Toom's model in the one-dimensional case. For more dimensions only very special cases can be treated with our ideas. The general case now seems to be beyond our reach.

PCA formalism

We consider PCA's with memory which describe the stochastic discrete time evolution of spin variables on the lattice Z. We denote the value of the spin at the point $x \in Z$ at time $t \in Z$ by $\sigma_{(x,t)} = \pm 1$ and write $\underline{\sigma}_t$ for the configuration at time t; σ_A will denote the configuration on the space-time set $A \subset Z^2$. We assume that the past of the PCA is fixed: $\sigma_{(x,t)} = +1$ if $t < 0$. The PCA evolves by simultaneous updating of spins. That is the spin configurations $\underline{\sigma}_{t-1}, \cdots, \underline{\sigma}_{t-T}$, $t \geq 0$, determine the probabilities $P(\sigma_{(x,t)} | \underline{\sigma}_{t-1}, \cdots, \underline{\sigma}_{t-T})$, $\sigma_{(x,t)} = \pm 1$ of the spin values at each point x at time t. The natural number T is called the depth of memory. The conditional probability distribution of $\underline{\sigma}_t$ is a product measure given by

$$\prod_{x \in Z} P(\sigma_{(x,t)} | \underline{\sigma}_{t-1}, \ldots, \underline{\sigma}_{t-T}). \tag{1.1}$$

The transition probabilities satisfy the normalization condition

$$\sum_{\sigma_{(x,t)}=\pm 1} P(\sigma_{(x,t)}|\underline{\sigma}_{t-1},\ldots,\underline{\sigma}_{t-T}) = 1, \tag{1.2}$$

which is taken into account by writing

$$P(\sigma_{(x,t)}|\underline{\sigma}_{t-1},\ldots,\underline{\sigma}_{t-T}) = \frac{1}{2}(1+\sigma_{(x,t)}h_{(x,t)}(\underline{\sigma}_{t-1},\ldots,\underline{\sigma}_{t-T})) \tag{1.3}$$

with $\left|h_{(x,t)}(\underline{\sigma}_{t-1},\ldots,\underline{\sigma}_{t-T})\right| \leq 1$. We assume that $h_{(x,t)}$ is translation invariant, time homogeneous and of finite range, that is

$$h_{\underline{0}}(\underline{\sigma}_{-1},\ldots,\underline{\sigma}_{-T}) = h_{\underline{0}}(\sigma_U),$$

where $U \subset Z \times \{-1,\ldots,-T\}$ is a fixed finite set, which we call the basic set of the origin $\underline{0}$ of the lattice Z^2. We shall consider nearly deterministic PCA's with

$$h_{\underline{0}} = \Phi_{\underline{0}} \cdot (1-2f_{\underline{0}}) \ , \tag{1.4}$$

where $|\Phi_{\underline{0}}| = 1$ and $|f_{\underline{0}}| \leq \epsilon$ for a small parameter $\epsilon > 0$. If $f_{\underline{0}} \equiv 0$ we obtain a deterministic PCA with deterministic function $\Phi_{\underline{0}}$. The existence of a small perturbation $f_{\underline{0}}$ implies that for each point $(x,t) \in Z^2$

$$P(\sigma_{(x,t)} \neq \Phi_{(x,t)}(\underline{\sigma}_{t-1},\ldots,\underline{\sigma}_{t-T})) \leq \epsilon \ .$$

The transition rates (1.1)-(1.4) define a distribution on the space of spin configurations on Z^2. We shall investigate the limiting behaviour of finite dimensional probabilities $P(\sigma_A)$, $A \subset Z^2$. Thus we shall say that PCA exponentially converges to the stationary state if the probabilities $P(\sigma_{T^\tau A})$ tend to some limit $\mu(\sigma_A)$ as $\tau \in \mathbb{N}$ tends to infinity and this convergence is exponential: that is, there exist constants $C(A)$ dependent on A and $\gamma < 1$ independent of A such that

$$|\mu(\sigma_A) - P(\sigma_{T^\tau A})| < C(A) \,.\, \gamma^\tau \ .$$

Here, by $T^\tau A$ we denote time shift of A - the set $\{(0,\tau)+a|a \in A\}$.

A PCA with the deterministic function $\Phi_{\underline{0}}$ is called stable if for any $\delta > 0$ there is an $\epsilon > 0$, such that for any $z \in Z^2$ we have $P(\sigma_z = -1) < \delta$ uniformly in $f_{\underline{0}}$, provided $|f_{\underline{0}}| < \epsilon$.

Now consider Toom's criteria of stability for PCA's under consideration given in [6]. Toom calls a subset $Q \subset U$ a plus set, if $\Phi_{\underline{0}}(\sigma_U) = +1$ for any configuration σ_U equal to $+1$ for all $z \in Q$; Q is a minimal plus set if it contains no other plus sets. From now on we shall consider Z^2 as a subset of real space $\mathbb{R}^2$; for any $A \subset Z^2$ we denote by Conv(A) the convex hull of A in $\mathbb{R}^2$. Toom stated his criteria for monotone deterministic functions $\Phi_{\underline{0}}$, satisfying $\Phi_{\underline{0}}(\sigma'_U) \leq \Phi_{\underline{0}}(\sigma''_U)$ if $\sigma'_U \leq \sigma''_U$, where the last unequality is considered at every point of U.

For monotone PCA's satisfying conditions (1.1)-(1.4) Toom's criteria is stated as follows: "a PCA is stable if and only if there is no ray from the origin, intersecting the convex hull of any minimal plus set" (Toom's condition).

We shall prove that Toom's condition is sufficient for exponential convergence of the PCA satisfying (1.1)-(1.4) to a stationary state. The requirement of monotonicity may be omitted.

Certain constructions used in our investigation are rather geometrical, so we need to introduce some, mostly well known, geometrical notions.

For any set $G \subset \mathbb{R}^2$ by $z + G$ we denote a shift $\{z + g | g \in G\}$ of the set G, and we designate $U(z) = z + U$ - the basic set of the point $z \in Z^2$. The set $O = \{\pm v | v \in U\} \cup \{v - w | v, w \in U\}$ is called a neighbourhood of zero and the set $O(z) = z + O$ is the neighbourhood of the point z.

The notation $|A|$ will be used for the cardinality of any set $A \subset Z^2$. For arbitrary $A \subset Z^2$ we define its height up$A = \max_t\{t | (x,t) \in A\}$ and its top layer sup $A = \{(x,t) \in A | t = \text{up}A\}$.

By Z_t^2 and $\mathbb{R}_t^2$ we denote the sets $Z \times \{t, t-1, \ldots\}$ and $\mathbb{R} \times [0, t]$ respectively. We shall write $R^t(A)$ for the projection of the set $A \subset Z^2$ into Z_t^2, so $R^t(A) = A \cap Z_t^2$; $L^t(A) = \{(x, \tau) \in A | \tau = t\}$ is a t-cut of the set A.

The set $A \subset Z^2$ is called connected, if for any $u, v \in A$ there exists $z_1 = u, z_2, \ldots, z_m = v$ such that $z_i \in A$ and $z_{i+1} \in O(z_i), i = 1, \ldots, m-1$. For points u and v of this sort we shall write $u \overset{A}{\sim} v$.

The designation σ_A^- will be used for the configuration of -1 on A and σ_A^+ for the configuration of $+1$. For convenience we shall write $P(A_1^+, A_2^+, \ldots, B_1^-, B_2^-, \ldots, \sigma_{C_1}, \sigma_{C_2}, \ldots)$ for the probability of the configuration, coinciding with $\sigma_{A_i}^+$ on A_i, $\sigma_{B_i}^-$ on B_i and configuration σ_{C_i} on C_i, $i = 1, 2, \ldots$. The number of sets in $P(\cdot)$ is arbitrary.
A notation $P(A_1^+, A_2^+, \ldots, B_1^-, B_2^-, \ldots, \sigma_{C_1}, \sigma_{C_2}, \ldots, | D_1^+, D_2^+, \ldots, E_1^-, E_2^-, \ldots, \sigma_{F_1}, \sigma_{F_2}, \ldots)$ will be used for the conditional probability of the configuration specified above given configuration which agrees with $\sigma_{D_i}^+, \sigma_{E_i}^-, \sigma_{F_i}$, $i = 1, 2, \ldots$.
Having given these notions we are able to proceed to the meaningful part of our work.

SECTION 2. STATEMENTS AND PROOFS

We shall prove the theorem under the condition, which is wider than that given by Toom, stated as follows:

> "There exist two plus sets Q_l and Q_r and a line l, passing through the origin and separating these sets in $\mathbb{R}^2$". (Modified Toom's condition)

Obviously any stable PCA has at least one plus set, because if $\Phi_0(\sigma_U^+) = -1$ the PCA is unstable. Thus we reformulated Toom's condition in a wider version, since we don't require monotonicity of the deterministic function.

Theorem

Consider the PCA defined by (1.1)-(1.4) and satisfying the modified Toom's condition. Then for sufficiently small $\epsilon > 0$ the PCA converges exponentially to the stationary state.

We shall give the proof for the case $T = 1$ only. The case $T > 1$ is similar to rather annoying technicalities (see Berezner [8]). Besides, our method allows us to consider a few modifications of the considered models. For example, we can allow the perturbation $f_{\underline{0}}$ from (1.4) not to be bounded by ϵ for configurations σ_U where $\Phi_{\underline{0}}(\sigma_U) = -1$. We can apply our constructions for more values of $\sigma_{(x,t)}$ (see [8]).

Remark A

To prove the theorem we have to prove the exponential convergence of probability $P(\sigma^-_{T^\tau A})$, $\tau \to \infty$.

Our goal is to obtain the exponentially convergent (uniformly in τ) series $\sum_n C_n^\tau(A)$ for the probability $P(\sigma^-_{T^\tau A})$, where the coefficients of the two series $\sum C_n^{\tau_1}(A)$ and $\sum C_n^{\tau_2}(A)$ can differ only for $n \geq \min(\tau_1, \tau_2)$. This will immediately imply the exponential convergence of $P(\sigma^-_{T^\tau A})$ as $\tau \to \infty$.

For this purpose we shall introduce the special construction, which we shall call cluster expansion.

The Construction of the Cluster

For convenience we assume that the basic set U of the origin is the set of points $(-\nu,-1),\ldots,(\nu,-1),\nu\in\mathbb{N}$, adding points of fictitious dependence.

Consider arbitrary finite sets $A, F\subset Z^2$, such that $\text{up}F<\text{up}A=t$, and fix an arbitrary configuration $\underline{\sigma}$ on Z_t^2, coinciding with σ_A^- on A and some fixed configuration σ_F on F.

Remark

The necessity of considering set F with fixed configuration on it will become clear later. Such a set will represent a set of points, where the configuration will be fixed as a result of using multistep cluster expansion.

For any such configuration we obtain a partition of Z_t^2 into the maximal connected sets $\{G_\omega^-\}$, $\omega\in\Omega^-$, where $\underline{\sigma}$ is equal to -1, and sets $\{G_\omega^+\}$, $\omega\in\Omega^+$, where $\underline{\sigma}$ is equal to $+1$.

Let us choose sets $G_1^-,\ldots,G_m^-$ satisfying

$$\sup A\subset\bigcup_{i=1}^m G_i^- \quad\text{and}\quad \sup A\cap G_i\neq\emptyset\quad i=1,\ldots,m\,. \tag{2.1}$$

We shall call such sets carrier sets of the cluster. To any carrier set we assign a component of the cluster in the same way. So, to avoid unnecessary designations we shall consider in detail the case of the unique carrier set G_1^-, which we shall denote by omitting suffixes. The case of the multicomponent carrier will then be obtained by simply taking the union of corresponding components of the cluster.

Taking the set G we shall assign to it a set $\Gamma \subset Z_t^2$, which we shall call the cluster, a corresponding oriented contour $\overline{\Gamma}$ in $\mathbb{R}_t^2$, and the set $\partial\Gamma \subset Z_t^2$, called the boundary of Γ.

Remark

We shall consider only such configurations $\underline{\sigma}$ on Z_t^2 for which carrier sets are finite, because it is easy to prove that the probability of the infinite carrier set for a finite set A is zero. To do this let us take a sequence of lines $l_k = l + (0, k(2\nu + 1))$ in $\mathbb{R}^2, k \in Z$ (remember that l is a separating line from the modified Toom's coindition). By J_k we denote a set $\{(x,\tau) \in Z_t^2, \tau \geq 0|$ either (x,τ) lies in the band between l_k and l_{k+1} or between l_{-k} and $l_{-k-1}\}$.

As a result of the choice of J_k and the properties of separating lines we have $P(J_{2k}^+) \geq (1-\epsilon)^{|J_{2k}|}$, and the configurations $\sigma_{J_{2k}}^+, k \in \mathbb{N}$ are independent. As $\sum_{k=1}^{\infty} P(J_{2k}^+) = \infty$ we obtain that with the probability one, there exist two bands of +1 points, separating any set inside from infinity, which implies that any carrier set of the cluster is finite with probability one.

Now denote $\overline{G} = \bigcup_{u,v \in G: u \in O(v)} [u,v] \subset \mathbb{R}^2$, where any $[u,v]$ is a segment from $\mathbb{R}^2$.

There exists a closed oriented anticlockwise contour $\overline{\Gamma}$ formed by the ordered set of segments $[z_0, z_1], [z_1, z_2], \ldots, [z_{n-1}, z_n], [z_n, z_0]$ oriented from the first to the second point and satisfying the following conditions:

(a) $z_{i+1} \in O(z_i), [z_i, z_{i+1}] \cap G = \{z_i, z_{i+1}\}, i = 1, \ldots, n$.

(b) $O(z_i) \cap \overline{G} \cap \text{Ang}(z_{i-1}, z_i, z_{i+1}) = \emptyset$, $i = 1, \ldots, n$, where $\text{Ang}(u, z, v)$ is the angle in $\mathbb{R}^2$ formed by the rays $[z, u)$ and $[z, v)$, turning from the first to the second ray anticlockwise.

(c) The set $\overline{G} \backslash \overline{\Gamma}$ belongs to the internal domain $\hat{\Gamma} \subset \mathbb{R}_t^2$ of the contour $\overline{\Gamma}$, which lies to the left of the contour $\overline{\Gamma}$, while traversing $\overline{\Gamma}$ anticlockwise.

(d) There does not exist any closed subcontour, formed by the segments of $\overline{\Gamma}$ and oriented clockwise.

The conditions (a)-(d), and especially the condition (b), are very constructive, and one can easily check the existence of the contour mentioned by straightforward construction of the contour. To do this, one can start from the point $z_0 \in \sup G$ with the minimal x-coordinate and then determine uniquely at every step the next segment of $\overline{\Gamma}$ in keeping with the conditions in (a)-(b). Using loose but descriptive geometric terminology we can say that $\overline{\Gamma}$ is an external oriented geometrical boundary of the set $\overline{G}$.

The cluster Γ corresponding to the set G (and to the configuration $\underline{\sigma}$ coinciding with σ_A^- on A and σ_F on F) is the set $\Gamma = \overline{\Gamma} \cap G$. The set

$$\partial\Gamma = \left\{ \bigcup_{z\in\Gamma} O(z) \right\} \backslash (\hat{\Gamma} \cup \Gamma) \cap Z_t^2 \tag{2.2}$$

is called the boundary of the cluster Γ.

In the case where the carrier of the cluster consists of few components $G_1^-, \ldots, G_m^-$ we assign to every component G_i^- the corresponding component Γ^i of the cluster, contour $\overline{\Gamma}^i$, and boundary $\partial\Gamma^i$ and internal domain $\hat{\Gamma}^i$ and then set $\Gamma = \cup\Gamma^i$; $\hat{\Gamma} = \cup\hat{\Gamma}^i$; $\partial\Gamma = \cup\partial\Gamma^i$; $\overline{\Gamma} = \cup\overline{\Gamma}^i$, $i = 1, \ldots, m$.

Thus to any configuration $\underline{\sigma}$, coinciding with σ_A^- on A and σ_F on F, we assign the cluster Γ, which we shall call the cluster Γ with kernel A and fixed configuration σ_F.

The immediate consequence of the construction of the cluster is the fact that $\underline{\sigma}$ coincides with σ_Γ^- on Γ and $\sigma_{\partial\Gamma}^+$ on $\partial\Gamma^+$. Moreover, if any other configuration σ agrees with $\sigma_A^-, \sigma_\Gamma^-, \sigma_{\partial\Gamma}^+$ and σ_F then its cluster is the same.

Now we shall formulate one important property of clusters, which will be proved at the end of this section.

The point $z \in \Gamma$ is called an error point if at least one of the plus sets $z + Q_r$ or $z + Q_l$ belongs to $\partial\Gamma$; this will imply $\Phi_z(\sigma_{U(z)}) = +1$. The set of error points is denoted by $\epsilon(\Gamma)$.

Lemma

There exists sufficiently small $\alpha > 0$ such that for any Γ

$$|\epsilon(\Gamma)| > \alpha \cdot |\Gamma| \,. \tag{2.3}$$

This lemma will be proved at the end of this section.

Now we are ready to give the cluster expansion for the probability $P(\sigma^-_{A_0})$ of the configuration on an arbitrary finite set $A_0 \subset Z^2_t$ (we introduce "0" to show that we are at the starting point of our expansion). Taking $t = \mathrm{up} A_0$ we consider all configurations $\underline{\sigma}$ in Z^2_t coinciding with $\sigma^-_{A_0}$ on A_0, and assign to each such configuration the cluster $\Gamma_0 = \Gamma_0(\underline{\sigma})$ with kernel A_0 and corresponding sets $\partial\Gamma_0$, $\overline{\Gamma}_0$, $\hat{\Gamma}_0$. Taking into account the relation between $\underline{\sigma}$ and the cluster Γ_0 we can give the expansion

$$P(\sigma^-_{A_0}) = \sum_{\Gamma_0} \sum_{\underline{\sigma}:\Gamma_0(\underline{\sigma})=\Gamma_0} P(\sigma) = \sum_{\Gamma_0} P(A^-_0, \Gamma^-_0, \partial\Gamma^+_0), \tag{2.4}$$

where the sum is over all possible clusters Γ_0.

Let us consider the set D_0 from $Z \times \{0, 1, 2, \ldots\}$ defined by

$$D_0 = \left\{ \bigcup_{z \in \sup \partial\Gamma_0} U(z) \right\} \setminus (A_0 \cup \partial\Gamma_0 \cup \hat{\Gamma}_0 \cup \Gamma_0). \tag{2.5}$$

So the points of D_0 belong to the basic set of $\sup \partial\Gamma_0$ and lie outside the interior of $\overline{\Gamma}_0$ and the values of the configuration $\underline{\sigma}$ at these points are not fixed by the algorithm for constructing the cluster at the previous step. We can rewrite the probabilities on the right hand side of (2.4) as

$$P(A^-_0, \Gamma^-_0, \partial\Gamma^+_0) = \sum_{\sigma_{D_0}} P(A^-_0, \Gamma^-_0, \partial\Gamma^+_0, \sigma_{D_0}) \,. \tag{2.6}$$

We shall write $P_t(A^-, B^+, \sigma_C, \ldots)$ for the probability of the corresponding configuration on the projection of the sets in the parenthesis in Z^2_t.

Then the probability on the right side of (2.6) can be written as

$$\begin{aligned} P(A_0^-,\Gamma_0^-,\partial\Gamma_0^+,\sigma_{D_0}) &= P_t(A_0^-,\Gamma_0^-,\partial\Gamma_0^+,\sigma_{D_0}) \\ &= P_{t-1}(A_0^-,\Gamma_0^-,\partial\Gamma_0^+,\sigma_{D_0}) \\ &\times P(L^t(A_0\cup\Gamma_0)^-,L^t(\partial\Gamma_0)^+|R^{t-1}(A_0\cup\Gamma_0)^-,R^{t-1}(\partial\Gamma_0)^+,\sigma_{D_0}). \end{aligned} \tag{2.7}$$

The second probability on the right-hand side is the conditional probability of the configuration on the t-cut L^t $(A_0\cup\Gamma_0\cup\partial\Gamma_0)$ under the condition of fixed configuration on the projection of the set $A_0\cup\Gamma_0\cup\partial\Gamma_0\cup D_0$ in Z^2_{t-1}.

Taking into account that any configuration σ_{D_0} can be written as $\sigma^-_{B_0}\cup\sigma^+_{D_0\backslash B_0}$ for some $B_0\subset D_0$ we write

$$P(\sigma_{D_0}) = P(B_0^-\cup(D_0\backslash B_0)^+) = \sum_{W_0:B_0\subset W_0\subset D_0} (-1)^{|W_0\backslash B_0|}\cdot P(W_0^-)\ . \tag{2.8}$$

Using the argument similar to what is used in the deriving of (2.8) and expression (2.7) the expansion (2.6) could be written as

$$P(A_0^-,\Gamma_0^-,\partial\Gamma_0^+) = \sum_{W_0\subset D_0} Q^t_{\rm cond}(A_0,\Gamma_0,\partial\Gamma_0,W_0)\cdot P_{t-1}(A_0^-,\Gamma_0^-,\partial\Gamma_0^+,W_o^-) \tag{2.9}$$

with

$$\begin{aligned} &Q^t_{\rm cond}(A_0,\Gamma_0,\partial\Gamma_0,W_0) \\ = &\sum_{B_0\subset W_0}(-1)^{|W_0\backslash B_0|}\cdot P(L^t(A_0\cup\Gamma_0)^-,L^t(\partial\Gamma_0)^+|R^{t-1}(A_0\cup\Gamma_0\cup B_0)^-,R^{t-1}(\partial\Gamma_0\cup D_0\backslash B_0)^+). \end{aligned} \tag{2.10}$$

The 4-tuple of sets $A_0,\Gamma_0,\partial\Gamma_0,W_0$ is denoted by H_0. A very important consequence of the definition (2.5) of the set D_0 and property (2.2) of the boundary $\partial\Gamma_0$ is the fact that conditional probabilities in (2.10) do not depend on the time shift $T^\tau,\tau\in\mathbb{N}$, of all constructed sets. Thus if we apply our construction to the probabilities $P(\sigma_{T^{\tau_i}A_0})$, $i=1,2$, we obtain

$$Q^{t+\tau_1}_{\rm cond}(T^{\tau_1}H_0) = Q^{t+\tau_2}_{\rm cond}(T^{\tau_2}H_0)\ . \tag{2.11}$$

This fact together with the property (2.3) and with the expansion (2.9) is the corner-stone of the expansion mentioned in Remark A. We need only to show how the probabilities in the right hand side of (2.9) can be expressed through the probabilities in Z^2_{t-2} in a similar way. Considering the probability $P_{t-1}(A_0^-, \Gamma_0^-, \partial\Gamma_0^+, W_0^-)$ in (2.9) we determine $A_1 = W_0 \cup (A_0 \backslash (\Gamma_0 \cup \hat{\Gamma}_0))$ and take an arbitrary configuration $\underline{\sigma}$ on Z^2_{t-1} coinciding with $\sigma^-_{A_1}$ on A_1 and σ_{F_1} on the set $F_1 = R^{t-1}(H_0) \backslash A_1$, where σ_{F_1} has values fixed on the set F_1 by our algorithm at previous steps. Here we write $R^{t-1}(H_0)$ for the union of the sets $R^{t-1}(A_0)$, $R^{t-1}(\Gamma_0)$, $R^{t-1}(\partial\Gamma_0)$, $R^{t-1}(W_0)$. For each such configuration $\underline{\sigma}$ on Z^2_{t-1} with fixed configuration on $A_1 \cup B_1$ we construct cluster Γ_1 with the kernel A_1 (see the section "The construction of the cluster"). Thus

$$P_{t-1}(A_0^-, \Gamma_0^-, \partial\Gamma_0^+, W_0^-) = \sum_{\Gamma_1} P_{t-1}(\sigma_{F_1}, A_1^-, \Gamma_1^-, \partial\Gamma_1^+) \, . \tag{2.12}$$

Defining D_1 in the same way as we have defined D_0 (see general definition (2.13)) we can get the expansion for the probability on the left hand side of (2.12) through the probabilities in Z^2_{t-2}. Thus the expansion of the type of (2.9) can be iterated many times. The difference is only in the growing number of points where the value of the configuration is fixed at the previous stages. Remember that at every stage we move down the t-axis by a unit step. Assuming that we have constructed the chain of 4-tuples $H_k = \{A_0, \Gamma_0, \partial\Gamma_0, W_0\}, \ldots, \{A_k, \Gamma_k, \partial\Gamma_k, W_k\}$ let us determine the 4-tuple of the $(k+1)$-st step. We denote $A_{k+1} = A_k \backslash (\Gamma_k \cup \hat{\Gamma}_k) \cup W_k$ and $F_{k+1} = R^{t-k-1}(H_k) \backslash A_{k+1}$. The cluster Γ_{k+1} with the kernel A_{k+1} is then constructed for the configuration $\underline{\sigma}$ in Z^2_{t-k-1} with fixed values at the points of $R^{t-k-1}(H_k)$. The set D_{k+1} is the set from $Z \times \{0, 1, 2 \ldots\}$ defined by

$$D_{k+1} = U\left(\bigcup_{j=0}^{k+1} (\partial\Gamma_j \cap Z^2_{t-k-1})\right) \backslash (A_{k+1} \cup \partial\Gamma_0 \cup \cdots \cup \partial\Gamma_{k+1} \cup \hat{\Gamma}_0 \cup \cdots \cup \hat{\Gamma}_{k+1}) \ , \tag{2.13}$$

where $U(G) = \bigcup_{z \in G} U(z)$ is the basic set of the set G.

The expansion (2.9) of the $(k+1)$-st step is then

$$P_{t-k-1}(A_0^-,\Gamma_0^-,\partial\Gamma_0^+,\dots,A_{k+1}^-,\Gamma_{k+1}^-,\partial\Gamma_{k+1}^+) = P_{t-k-1}(\sigma_{F_{k+1}},A_{k+1}^-,\Gamma_{k+1}^-,\partial\Gamma_{k+1}^+) \tag{2.14}$$

$$= \sum_{\Gamma_{k+1},W_{k+1}\subset D_{k+1}} Q_{\text{cond}}^{t-k-1}(H_{k+1})P_{t-k-2}(\sigma_{F_{k+1}},A_{k+1}^-,\Gamma_{k+1}^-,\partial\Gamma_{k+1}^+,W_{k+1}^-)\ ,$$

where

$$Q_{\text{cond}}^{t-m}(H_m) = \sum_{B_m\subset W_m}(-1)^{|W_m\setminus B_m|}\times$$

$$P\left(L^{t-m}(A_m\cup\Gamma_m)^-, L^{t-m}(\bigcup_{i=1}^m \partial\Gamma_i)^+ | R^{t-m-1}(A_m\cup\Gamma_m\cup B_m)^-, R^{t-m-1}(\sigma_{F_m}),(D_m\setminus B_m)^+\right).$$

At the step j when for the first time

$$\emptyset = A_{j+1} = W_j = R^{t-j}(\partial\Gamma_0\cup\dots\cup\partial\Gamma_j)\cap Z\times\{0,1,2\cdots\} \tag{2.15}$$

we say that the chain H_j is truncated. The contribution $S(H_j)$ of the truncated chain H_j in the expansion of probability $P(\sigma_{A_0}^-)$ is

$$S(H_j) = \prod_{k=0}^{j} Q_{\text{cond}}^{t-k}(H_k), \tag{2.16}$$

where $Q_{\text{cond}}^{t-j}(H_j)$ is equal to 1.

Thus we can write the complete expansion for $P(\sigma_{A_0}^-)$ as the sum of contributions over all truncated chains H_j

$$P(A_0^-) = \sum_{j=0}^{t}\sum_{H_j} S(H_j)\ . \tag{2.17}$$

To each truncated chain H_j we assign the value $|\Gamma| = |\Gamma_0| + \dots + |\Gamma_j|$. Then we have

$$P(A_0^-) = \sum_{n=1}^{\infty}\sum_{H_j:|\Gamma|=n} S(H_j) = \sum_{n=1}^{\infty} C_n(A). \tag{2.18}$$

Let us estimate the terms of this series. Consider H_j with $|\Gamma| = n$. It follows from (2.3) that

$$|S(H_j)| = \left|\prod_{k=0}^{j} Q_{\text{cond}}^{t-k}(H_k)\right| < 2^{|D|}\cdot\epsilon^{\alpha n}\ , \tag{2.19}$$

where $|D| = |D_0| + \cdots + |D_j|$. Now we shall show that the number of H_j with $|\Gamma| = n$ is bounded by C^n for some $C > 0$. We have four obvious estimates. The first one is

$$|\{\Gamma_i| \, |\Gamma_i| = k\}| < d_1^k \text{ for } d_1 > 0 , \tag{2.20}$$

the second is

$$|\partial\Gamma| = |\partial\Gamma_0| + \cdots + |\partial\Gamma_j| < d_2 \cdot |\Gamma| \text{ for some } d_2 > 0 , \tag{2.21}$$

the third

$$|D| < d_3 \cdot |\partial\Gamma| \text{ for some } d_3 > 0 \tag{2.22}$$

and the fourth, which claims that the number $||W||$ of choices of the sets $W_0, \ldots, W_j$ from the sets $D_0, \ldots, D_j$ is bounded by

$$||W|| \leq 2^{|W_0|+\cdots+|W_j|} \leq 2^{|D|} . \tag{2.23}$$

¿From these estimates we obtain

$$|\{H_j| \, |\Gamma| = n\}| \leq C^n \text{ for some } C > 0 . \tag{2.24}$$

Combining (2.19)-(2.24) we get that for some constant $\Psi > 0$

$$|C_n(A)| < (\Psi \cdot \epsilon^\alpha)^n . \tag{2.25}$$

This proves that for sufficiently small $\epsilon > 0$ the series (2.18) converges exponentially. Considering a similar expansion for $P(\sigma^-_{T^\tau A})$

$$P(\sigma^-_{T^\tau A}) = \sum_{n=1}^{\infty} C_n^\tau(A) \tag{2.26}$$

we note that the coefficients $C_n^\tau(A)$ and $C_n(A)$ for $n < t$ are the sums over congruent sets $T^\tau H_j$ and H_j. Taking into account the equality (2.11) we find that these coefficients coincide for $n < t$. Having proved this fact we refer the reader to Remark A, which completes the proof of the theorem.

Proof of Lemma

Obviously it is enough to prove (2.3) in the case when Γ consists of one component.

Let n_+ be the number of plus vectors in the oriented contour $\overline{\Gamma} = [z_0, z_1], \ldots, [z_{n-1}, z_n], [z_n, z_0]$ where the oriented segment $\overrightarrow{z_i z_{i+1}}$ is called a plus vector, if z_{i+1} lies to the right of the line $z_i + l$. Consider an arbitrary plus vector $\overrightarrow{z_i z_{i+1}}$. We claim that either z_i or z_{i+1} is an error point. Certainly, if $\mathrm{pr}_{\vec{e}}\ \overrightarrow{z_i z_{i+1}} > 0$ (where $\vec{e}$ is a unit vector $\overrightarrow{\underline{0}e}, e = (0,1)$ and $\mathrm{pr}_{\vec{a}}\ \vec{b}$ is the orthogonal projection of the vector $\vec{b}$ on the vector $\vec{a}$) then $\mathrm{pr}_{\vec{e}}\ \overrightarrow{z_{i+1} z_{i+2}} \geq 0$ (otherwise z_i and z_{i+2} will be connected by the segment $[z_i, z_{i+2}]$, but this contradicts the construction of the contour $\overline{\Gamma}$). This fact implies that $z_{i+1} + Q_r(z_{i+1}) \subset \partial\Gamma$ and, correspondingly, $z_{i+1} \in \epsilon(\Gamma)$. If $\mathrm{pr}_{\vec{e}}\ \overrightarrow{z_i z_{i+1}} < 0$ then by the same argument $\mathrm{pr}_{\vec{e}}\ \overrightarrow{z_{i-1} z_i} \leq 0$ and $z_i \in \epsilon(\Gamma)$. If $\mathrm{pr}_{\vec{e}}\ \overrightarrow{z_i z_{i+1}} = 0$ then either z_{i-1} lies to the left of $z_i + l$ (and this implies $z_i \in \epsilon(\Gamma)$) or z_{i+2} lies to the right of $z_{i+1} + l$ – which implies that $z_{i+1} \in \epsilon(\Gamma)$. Thus we assign the error point to an arbitrary plus vector.

Let us note that each error point corresponds to no more than $2|O|$ plus vectors, so that

$$|\epsilon(\Gamma)| > \alpha_1 \cdot n_+ \quad , \quad \alpha_1 = 1/2|O| \ . \tag{2.27}$$

As the contour $\overline{\Gamma}$ is closed there exists a constant $\alpha_2 > 0$ such that

$$n_+ > \alpha_2(n + 1 - n_+) \ . \tag{2.28}$$

Since $|\overline{\Gamma}| = n + 1 > \alpha_3|\Gamma|$ for some $\alpha_3 > 0$ then summing (2.27) and (2.28) we obtain

$$|\epsilon(\Gamma)| > \alpha|\Gamma| \text{ for some } \alpha > 0 \ .$$

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