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Dynamical Triangulation Models with Matter: High Temperature Region

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Abstract: We consider a canonical ensemble with a fixed number N of triangles for planar dynamical triangulation models with compact spin in the high temperature region. We find the asymptotics of the partition function $Z(N)$ and reveal the analytic properties of the generating function $U(x) = \sum Z(N)x^N$. New cluster expansion techniques are developed for this case. For fixed triangulation it would be quite standard but for random triangulations one has to deal with the non-zero entropy of the space between clusters. It is a multiscale expansion, where the role of scale is played by a topological parameter – the maximal length of chains of imbedded not simply connected clusters.

1. Definitions and Main Results

1.1. Introduction. We consider a model, related to quantum gravity, called the planar dynamical triangulation model with matter fields in the high temperature region (see the exact definitions below). Planar models without matter have an extensive history and are sufficiently well understood, both on physical and mathematical levels. In the physical literature there is a powerful random matrix method, in mathematics earlier combinatorial results by Tutte (see [4, 7]) solve the problem without spin. Random matrix methods in physics give some information about the Ising model on dynamical triangulations, see reviews [2, 1]. On the contrary, there are almost no rigorous mathematical results for models with matter fields.

We develop new cluster expansion techniques. For a fixed triangulation it would be quite a standard exercise. But for dynamical triangulations the space outside clusters has non-zero entropy. It is related to diffeomorphism invariance for original continuous models. Note that for the lattice case we would have only a translation group conserving the form and the distances between clusters. Our cluster expansion is a kind of multiscale expansion where the induction parameter $n = 1, 2, \dots,$ has a topological nature. Each n corresponds to the summation over all configuration with clusters of level not greater than n. The level of a cluster is defined by induction: a cluster has level n if inside it all clusters have level less than n.

One can roughly describe the situation as follows. In the absence of spin the asymptotics of the partition function is defined by an algebraic singularity of the generating function at some point x_+ on the positive halfaxis. If the perturbation is imposed, an infinite number of new algebraic singularities $x_n, n = 1, 2, \ldots$, positive numbers close to x_+, appear. They have an accumulation point $x_{acc} = \lim x_n$. If there exists n such that $x_n < x_{acc}$, then the asymptotics is canonical with the critical exponent $-\frac{7}{2}$. Otherwise, for example when $x_n < x_{n-1}$ for all n, then the asymptotics is not canonical.

The proof consists of two parts. The first part presents an inductive formal cluster expansion. The second part uses complex analysis to get inductive estimates.

1.2. Triangulations and partition function. Graphs here can have multiple edges but no loops. Whenever necessary the graphs are considered as 1-dimensional complexes. In this paper we call triangulation a pair (G, ϕ), where G is a graph and ϕ is an imbedding of G in a closed two-dimensional sphere with a hole, that is a closed disk D. The following conditions are assumed to hold: if l is an edge of G then $\phi(l)$ is a smooth curve in D, and each of the open components of $D \setminus \phi(G)$ is homeomorphic to an open disk, the closure of each open component contains 3 different vertices and 3 different edges, that is 3 smooth curves $\phi(l_i)$, where l_i are edges of G. Note that two vertices of G can be connected by more than one edge. Triangulation is called rooted if an edge (the root) on the boundary ∂D is specified together with its, say clockwise, direction (orientation). Two triangulations are called equivalent if there is a homeomorphism $D \to D$ which respects orientation, vertices, edges and the root. Let $\mathcal{T}_0(N, m)$ be the set of all equivalence classes (called further on also triangulations for brevity) of rooted triangulations with N triangles and m boundary edges. Let $C_0(N, m) = |\mathcal{T}_0(N, m)|$.

It is very convenient to assume the following conditions which will be the boundary conditions for the systems of equations below:

$$C_0(N, 0) = C_0(N, 1) = 0, C_0(0, m) = \delta_{m,2}, C_0(1, m) = \delta_{m,3}.$$

Only the case $N = 0, m = 2$ needs commentaries: this corresponds to a degenerate disk, an edge with two vertices.

Let $V(T), L(T), F(T), B(T)$ be correspondingly the sets of all vertices, edges, triangles and boundary edges of T.

We denote T^* the dual graph of the triangulation T, its vertices $v \in V(T^*)$ correspond to triangles of T, edges $l \in L(T^*)$ – to pairs of adjacent triangles. All vertices of T^* have degree 3 except vertices corresponding to the triangles (there are not more than $m = |B(T)|$ of such triangles), incident to at least one boundary edge.

In each triangle of T, or in each vertex v of the dual graph T^*, there is a spin σ_v with values in the set S, this set is assumed finite for simplicity.

Partition function for the canonical ensemble (with fixed number $N \geq 0$ of triangles and fixed number $m \geq 2$ of boundary edges) is defined as

$$Z(N, m) = Z_\beta(N, m) = \sum_{T: F(T)=N, B(T)=m} Z(T),$$

where the partition function $Z(T)$ for a given triangulation $T \in \mathcal{T}_0(N, m)$ is

$$Z(T) = |S|^{-N} \sum_{\{\sigma_v : v \in V(T^*)\}} \exp(-\beta \sum_{<v,v'>} \Phi(\sigma_v, \sigma_{v'})), N = |F(T)| = \left|V(T^*)\right|,$$

where $< v, v' >$ means a pair of nearest neighbor vertices (that is of adjacent triangles) $v, v' \in V(T^*)$, $\Phi(s, s')$ is a real function on $S \times S$, $\beta > 0$ – inverse temperature.

The set of all symmetric interactions Φ for given S is the Euclidean space R^d of dimension $d = |S| + \frac{|S|(|S|-1)}{2}$. We call a set of interactions generic if its complement has measure 0 in R^d.

Further on, for technical reasons only, we shall consider the ensemble with boundary conditions empty on the internal boundary, that is there are no spins on the triangles of $F(T)$ adjacent to the boundary of the disk. Somewhere we shall say how to treat more general boundary conditions.

1.3. Main results. It is known (see [4,7]) that for fixed m as $N \to \infty$ so that $N + m$ is even (we always assume this condition in the sequel)

$$Z_0(N, m) = C_0(N, m) \sim \phi(m) N^{-\frac{5}{2}} c^N, c = \sqrt{\frac{27}{2}}, \phi(m) > 0.$$

Note that $Z_0(N, m) = 0$ if $N + m$ is odd. We want to stress that to get the asymptotics of the partition function itself is certainly more difficult than to get the asymptotics of its logarithm. The asymptotics $cN^{-\frac{5}{2}} c_1^N$ is called canonical, and the critical exponent $\alpha = -\frac{5}{2}$ is also called canonical.

Our goal is to prove similar results in the situation with spins. We prove that in many cases the partition function has canonical asymptotics (with the canonical critical exponent). In general, there is a constant $c = c(\Phi, \beta)$ such that

$$Z(N, m) \sim \phi(m, \Phi, \beta) N^{-\frac{5}{2}} c^N.$$

For example, we have the following result.

Theorem 1. *Let*

$$k = \sum_{\sigma, \sigma'} [\exp(-\beta \Phi(\sigma, \sigma')) - 1] < 0.$$

Then for β sufficiently small $Z(N, m)$ has canonical asymptotics.

We shall see below that this asymptotics, that is the constant $c(\Phi, \beta)$, is defined by the level 1 of the multiscale expansion. However, there are exceptions.

Theorem 2. *If $\Phi \leq 0$ is not identically constant, then the asymptotics is not canonical.*

Non-rooted triangulations. The easy corollary is that for non-rooted triangulations of the sphere (thus there is no boundary) we have $Z_0(N) \sim \phi(\beta) N^{-\frac{7}{2}} c(\beta)^N$ in the canonical case, and for $\Phi \leq 0$ the asymptotics is not canonical. The canonical critical exponent is defined for non-rooted triangulations to be $-\frac{7}{2}$.

Example: scaling transformation. Introduce the constant nearest-neighbor interaction $\Psi_\mu(\sigma, \sigma') \equiv \mu$. For non-rooted triangulations the term Ψ gives an overall factor $\exp(-\beta \mu L^*) = \exp(-\frac{3}{2} \beta \mu N)$. However, for the ensembles where m is not fixed, such interaction Ψ is a nontrivial interaction; it leads to an interesting phase transition in μ. Appending Ψ_μ to some interaction Φ results in a scaling transformation of the generating functions (see below). Otherwise speaking, appending such interaction changes only the constant $c(\beta)$ in the asymptotics, and does not change the canonical exponent.

2. Formal Expansion

2.1. Cluster representation for a fixed triangulation. Assume that the triangulation T is fixed. Expanding the exponent as

$$\exp(-\beta\Phi(\sigma_v, \sigma_{v'})) = 1 + \exp(-\beta\Phi(\sigma_v, \sigma_{v'})) - 1$$

one can write

$$Z(T) = \sum_{L^*} z(L^*),$$

where $\sum_{L^*}$ is over all subsets L^* of edges of the dual graph T^*, and

$$z(L^*) = |S|^{-N} \sum_{\{\sigma_v : v \in V(T^*)\}} \prod_{l=(v,v') \in L^*} k_l, k_l = k_l(\sigma_v, \sigma_{v'}) = \exp(-\beta\Phi(\sigma_v, \sigma_{v'})) - 1.$$

For each pair (T, L^*) a triangle δ of T is called colored if it corresponds to a vertex of some edge of L^*, and blank otherwise. Denote the set of coloured triangles as $V(L^*) = V(T, L^*)$.

Recall that the distance between triangles in dynamical triangulation models is the distance between the corresponding vertices in the dual graph, that is the length (number of edges) of the shortest path between them in T^*. A set Δ of triangles is called 1-connected (or connected) if between each pair of triangles $t, s \in \Delta$ there is a path, belonging to Δ, in which any pair of consecutive triangles are on the distance not greater than $d = 1$. For each set Δ define the external boundary $\partial_e\Delta$ as the set of triangles on distance 1 from Δ, and the internal boundary $\partial_i\Delta$ as the set of triangles in Δ on the distance 1 from the complement of Δ.

For each pair (T, L^*) there is a unique decomposition of the closure of $V(L^*)$,

$$cl(V(L^*)) =^{\text{def}} V(L^*) \cup \partial_e V(L^*) = \cup V_i,$$

where $V_i = V_i(T, L^*)$ are maximal connected subsets of $cl(V(L^*))$. Finally

$$Z(T) = 1 + \sum_{p=1}^{\infty} \sum_{\{V_1,\dots,V_p\}} k(V_1)\dots k(V_p), \tag{1}$$

where $\Sigma = \{V_1, \dots, V_p\}$ is any system (called configuration) of connected subsets of $V(T^*)$ such that $\text{dist}(V_i, V_j) > 1$ for any $i \neq j$, and for any such V,

$$k(V) = \sum_{L^* : cl(V(L^*)) = V} |S|^{-|V(L^*)|} \sum_{\{\sigma_v : v \in V(T^*)\}} \prod_{l \in L^*} k_l. \tag{2}$$

We call V_i clusters for given (T, Σ), or (T, Σ)-clusters. That is for a given triangulation T and the subset $\Sigma \subset F(T)$, (T, Σ)-clusters are maximal connected components of Σ. Thus in (1) for any i there exists at least one nonempty L_i^* such that $cl(V(L_i^*)) = V_i$.

2.2. Hierarchy of clusters and generating functions. For any set $V \subset F(T)$ the complement $F(T) \setminus V$ consists of two parts: the exterior part $\mathrm{Ext}(F(T) \setminus V)$, consisting of all triangles of $F(T) \setminus V$, which can be connected with the boundary by a connected path, belonging to $F(T) \setminus V$, and the interior part $\mathrm{Int}(F(T) \setminus V)$, containing all other triangles.

Let V be one of the (T, Σ)-clusters. Then the interior part of its complement $F(T) \setminus V$ consists of some number r of connected components $V_1, \dots, V_r$.

For given T a set $V \subset F(T) = V(T^*)$ of triangles is called simple if it is connected (that is connected) and its interior part is empty. We say that (T, Σ)-cluster has level 1 if it is simple.

We define (T, Σ)-clusters of level $n > 1$ by induction: cluster V has level n if n is the minimal number such that in its interior part there are only clusters of level less than n. Thus the (T, Σ)-clusters form a forest (a set of connected trees), where clusters are vertices of this forest. Two vertices of the tree are connected by an edge if one of the corresponding clusters is in the interior part of the other one, and their levels differ by 1.

For given T a configuration Σ is said to be of level 1 if either there are no clusters at all or all (T, Σ)-clusters are simple. For given T a configuration Σ is said to be of level $n > 1$ if all (T, Σ)-clusters have level not greater than n and at least one of them has level n. For given T denote for $n \geq 1$,

$$Z^{(n)}(T) = 1 + \sum_{p=1}^{\infty} \sum_{\Sigma=\{V_1,\dots,V_p\}}^{(\leq n)} k(V_1)\cdots k(V_p),\ \Theta^{(n)}(T)$$

$$= \delta_{n1} + \sum_{p=1}^{\infty} \sum_{\Sigma=\{V_1,\dots,V_p\}}^{(n)} k(V_1)\cdots k(V_p),$$

where in the first case the sum with index $(\leq n)$ is over all configurations $\Sigma = \{V_1, \dots, V_p\}$ of level at most n. The sum with index (n) is over all configurations $\Sigma = \{V_1, \dots, V_p\}$ of level n. Let us put

$$Z^{(0)}(T) = 1,\ \Theta^{(0)}(T) = 0,$$

and for $n \geq 0$,

$$Z^{(n)}(N, m) = \sum_{T \in \mathcal{T}_0(N,m)} Z^{(n)}(T),\ \Theta^{(n)}(N, m) = \sum_{T \in \mathcal{T}_0(N,m)} \Theta^{(n)}(T).$$

For $n \geq 0$ let us call

$$U^{(n)}(x, y) = \sum_{N=0}^{\infty} \sum_{m=2}^{\infty} Z^{(n)}(N, m) x^N y^m,\ Y^{(n)}(x, y) = \sum_{N=0}^{\infty} \sum_{m=2}^{\infty} \Theta^{(n)}(N, m) x^N y^m$$

the generating function of level at most n and of level n correspondingly. Then obviously

$$U^{(n)}(x, y) = \sum_{1 \leq k \leq n} Y^{(k)}(x, y),$$

$$Y^{(1)}(x, y) = U^{(1)}(x, y),\ Y^{(n)}(x, y) = U^{(n)}(x, y) - U^{(n-1)}(x, y),\ n \geq 2.$$

Lemma 1. *There exists $\delta > 0$ such that the functions $U^{(n)}(x, y)$, $Y^{(n)}(x, y)$ and*

$$U(x, y) =^{def} \sum_{N=0}^{\infty} \sum_{m=2}^{\infty} Z(N, m) x^N y^m = \lim_{n \to \infty} U^{(n)}(x, y)$$

are analytic for $|x|, |y| < \delta$.

Proof. Note first that the limit

$$Z(N, m) = \lim_{n \to \infty} Z^{(n)}(N, m)$$

exists (in fact, for fixed N, m, the sequence $Z^{(n)}(N, m) \leq Z^{(n+1)}(N, m)$ stabilizes as $n \to \infty$) and is the partition function for the ensemble with the boundary conditions defined above. Moreover, there are a priori exponential bounds, easy to prove,

$$Z(N, m) < C^{N+m}$$

for some $C > 0$ depending only on Φ and β. □

We shall study properties of the functions $U^{(n)}$ by induction in n. If $\beta\Phi \leq 0$ we call the model with the partition function $Z^{(n)}$ the random cluster model with clusters of the level not greater than n. For example, the level 0 partition function is the case when there is no spin at all, and the level 1 corresponds to a special random cluster model, where only simple clusters are taken into account.

2.3. Level 1 *cluster expansion.* As it is standard in cluster expansions, we have started with a resummation formula (polymer expansion, or cluster representation, see [3,6]) for a given triangulation T. After this in the standard theory some kind of correlation equations are used. However, here we will have to follow a different way, by incorporating our expansion into the recurrent formulae of Tutte, that allow to censor all possible triangulations together with spin configurations on them. The reason is that the "empty space", outside the clusters of the expansion, can vary considerably. In other words, the empty space has nonzero entropy.

Our cluster expansion is inductive: it consists of steps $n = 1, 2, \ldots$. On each step a new cluster expansion has to be done. It is a kind of multi-scale cluster expansion, where the role of scale is played by a topological parameter n, the length of maximal chains of imbedded non-simply connected clusters.

2.3.1. Level 1 *cluster function.* The nonempty (T, Σ)-cluster V is called complete if it contains all triangles of T. It is obviously simple and thus Σ consists only of this cluster. The complete (T, Σ)-cluster V is unique and we put

$$K(T) = k(V).$$

Then the cluster function is defined as

$$W(x, y) = W^{(1)}(x, y) = \sum_{N=3}^{\infty} \sum_{m=2}^{\infty} W_{N,m} x^N y^m, \; W_{N,m} = W^{(1)}_{N,m} = \sum_{T: T \in \mathcal{T}_0(N,m)} K(T).$$

Lemma 2. *There exist constants $C_2 > 0$ such that for any β sufficiently small*

$$K(T) \leq (C_2\beta)^{\frac{N}{6}}, N = \left|V(T^*)\right|. \tag{3}$$

It follows that the function $W(x, y)$ is analytic in $|x|, |y| < C\beta^{-a}$ for some $C > 0, a > 0$, and $W(x, y)$ together with its first partial derivatives are $O(\beta\,|x|^3\,|y|^2)$ for $|x|, |y| \leq 1$.

Proof. We have obviously for small β

$$\prod_{l \in L^*} k_l < (2\beta \max|\Phi|)^{|L^*|}.$$

At the same time, the dual graph is 3-regular and thus $2\,|L^*| \geq |V(L^*)| \geq \frac{|L^*|}{3}$. As the number of L^* such that $V(L^*) = V$ is bounded by $2^{3|V|}$ then from (2) it follows that

$$k(V) \leq (C_1\beta)^{\frac{|V|}{2}}$$

for some $C_1 > 0$. For any triangulation T and any L^*, given a complete (T, Σ)-cluster, we have $|V(L^*)| \geq \frac{|V(T^*)|}{3}$. The number of V giving the complete (T, Σ)-cluster for given T is not greater than $2^{|V(T^*)|}$. Thus we have the cluster estimate (3). From the exponential estimate for the number $C_0(N; m)$ of rooted triangulations we have the result if we note also that $m \leq 2N$ for $N > 1$.

As clusters are assumed to be nonempty, one can factor out $\beta x^3 y^2$, for any $c > 0$ and $|x|, |y| < c$ we have by the cluster estimate

$$W(x, y) = \beta\,|x|^3\,|y|^2\,O(1). \qquad \square$$

2.3.2. Recurrent equations. Now we shall give a procedure to construct all configurations with simple clusters only from complete clusters and the degenerate disk, that is the configuration with $N = 0, m = 2$. It is of primary importance that all complete clusters have blank triangles on their internal boundary.

The canonical functional equation is the following equation in a small neighbourhood $\Omega \in C^2$ of $x = y = 0$:

$$F(x, y) = F(x, y)xy^{-1} + F^2(x, y)xy^{-1} + y^2 + J(x, y) - xyF_2(x), \tag{4}$$

where we denote

$$F(x, y) = \sum_{N=0}^{\infty} \sum_{m=2}^{\infty} F_{N,m} x^N y^m,\ F_m(x) = \sum_{N=0}^{\infty} F_{N,m} x^N.$$

F and F_2 are unknown functions, $J = J(x, y) = \sum_{N=3}^{\infty} \sum_{m=2}^{\infty} J_{N,m} x^N y^m$ is given and analytic in Ω.

Lemma 3. *The level 1 generating function $U^{(1)} = U_1$ satisfies the canonical functional equation*

$$U_1(x, y) = U_1(x, y)xy^{-1} + U_1^2(x, y)xy^{-1} + y^2 + W^{(1)}(x, y) - xyS(x) \tag{5}$$

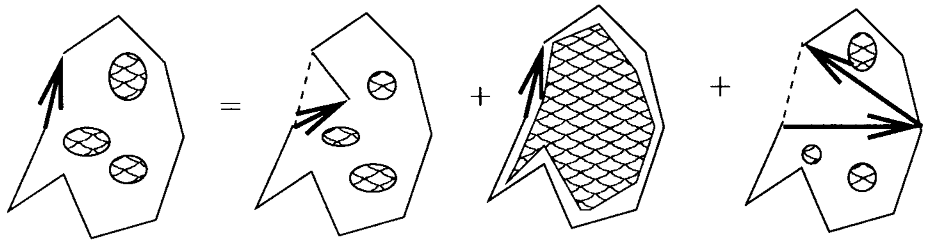

Fig. 1. Recurrent relation

with

$$S(x) = S_2^{(1)}(x) = \sum_{N=0}^{\infty} Z^{(1)}(N,2)x^N,$$

where it is convenient to use notation S instead of $(U_1)_2$.

Proof. We use the idea of Tutte's algorithm for censoring all triangulations. We have

$$\begin{aligned} Z^{(1)}(N,m) =& Z^{(1)}(N-1,m+1) + \delta_{N,0}\delta_{m,2} + W_{N,m} \\ &+ \sum_{N_1+N_2=N-1, m_1+m_2=m+1} Z^{(1)}(N_1,m_1)Z^{(1)}(N_2,m_2) \end{aligned} \tag{6}$$

for $m \geq 2$, $N \geq 0$ and

$$Z^{(1)}(-1,m)) = Z^{(1)}(N,0)) = Z^{(1)}(N,1)) = 0.$$

It follows that

$$Z^{(1)}(0,m) = \delta_{m,2}, Z^{(1)}(1,m) = \delta_{m,3}.$$

Figure 1 shows the meaning of this recurrent relation. The irst term on the right-hand of (6) side corresponds to appending a triangle, the next two terms – taking the degenerate triangulation with $N = 0, m = 2$ (omitted on the picture) and a complete cluster with the chosen root edge, the last term – joining together two already constructed triangulations by appending a triangle. Fat edges denote the root edges and show the rule to choose the root.

Multiplying (6) on $x^N y^m$ and summing $\sum_{N=0}^{\infty}\sum_{m=2}^{\infty}$ we get the result. □

2.4. Inductive resummation formula. The (T, Σ)-cluster V is called a boundary cluster if it contains all boundary triangles. Let V_{co} denote all coloured triangles of V, and let V' be V_{co} together with the blank triangles, adjacent to the boundary. The complement of the closure $cl(F(T) \setminus V) = F(T) \setminus V'$ consists of some number $r \geq 0$ of nonempty maximal connected components $R_1, \dots, R_r$, isomorphic to a disk, and such that for each i there is at least one triangle in R_i not belonging to V. Note that if $r = 0$ then the boundary cluster V is a complete cluster. Denote $m_i = m_i(V)$ the number of edges on the boundary of R_i. In other words, the boundary of $cl(F(T) \setminus V)$ can be uniquely subdivided on r circles (they can intersect in some vertices of T), and m_i are the lengths of these circles.

A configuration of level n is called basic if there is only one cluster of level n and it is a boundary cluster. We shall construct all configurations of level n from the basic configurations of level n. Introduce the basic generating function of level n

$$W^{(n)}(x, y) = \sum_{N,m} x^N y^m W_{N,m}^{(n)},$$

$$W_{N,m}^{(n)} = \sum_{T \in \mathcal{T}_0(N,m)} \sum_{p=0}^{\infty} \sum_{\{V_0, V_1, \dots, V_p\}} k(V_0)k(V_1)\cdots k(V_p),$$

where the sum $\sum_{\{V_0, V_1, \dots, V_p\}}$ is over all configurations $\Sigma = \{V_0, V_1, \dots, V_p\}$ (for given T) with a boundary cluster of level n, we denote it V_0.

Assume now that we know the generating functions $U^{(k)}(x, y)$ for $k < n$ and thus we know all $S_m^{(k)}(x)$, $k < n$, $m = 2, 3, \dots$, defined similarly to the functions F_m in the canonical equation. Put

$$Y_m^{(0)}(x) = S_m^{(0)}(x),\ Y_m^{(1)}(x) = S_m^{(1)}(x),\ Y_m^{(k)}(x) = S_m^{(k)}(x) - S_m^{(k-1)}(x),\ k \geq 2$$

or

$$S_m^{(k)}(x) = \sum_N Z^{(k)}(N, m)x^N = \sum_{j \leq k} Y_m^{(j)}(x).$$

Let the boundary cluster V_0 have m edges on its exterior boundary, and m_i, $i = 1, \dots, r$, edges on the "interior" boundaries of V_0. Then resummation of the latter formula gives

$$W^{(n)}(x, y) = \sum_m y^m \sum_{V_0} k(V_0) x^{N(V_0')} \sum_{k_1, \dots, k_r} \prod_{i=1}^{r} Y_{m_i}^{(k_i)}(x), \tag{7}$$

where the sum $\sum_{V_0}$ is over all boundary clusters, having m edges on their exterior boundary, $N(V_0')$ is the number of triangles in V_0', m_i, $i = 1, \dots, r$, is the number of edges on the i^{th} component of the interior boundary of V_0'. And moreover, it is assumed that in $\sum_{k_1, \dots, k_p}$ any $k_i \leq n - 1$ and at least one k_i equals $n - 1$.

2.4.1. Equation for the n^{th} level generating function.

Lemma 4. *The function $U^{(n)}$ satisfies the canonical functional equation (4) with $F = U^{(n)}$, $F_2 = S_2^{(n)}(x)$, $J = J^{(n)} = W^{(1)} + \cdots + W^{(n)}$.*

We will call such an equation the n^{th} level equation.

Proof is quite similar to the proof for $n = 1$. The function J in the canonical recurrent equation has n new (comparatively to the case without spins) terms in the right-hand side appear, terms corresponding to the basic functions $W^{(i)}$, $i = 1, \dots, n$. On Fig. 2 each of them is represented as a generic (third in the right-hand side) term with a boundary cluster.

This corresponds to the recurrent equation (6) where instead of $W_{N,m}$ we substituted $W_{N,m}^{(n)}$. One can check that in the consecutive iterations of this recurrent relation only

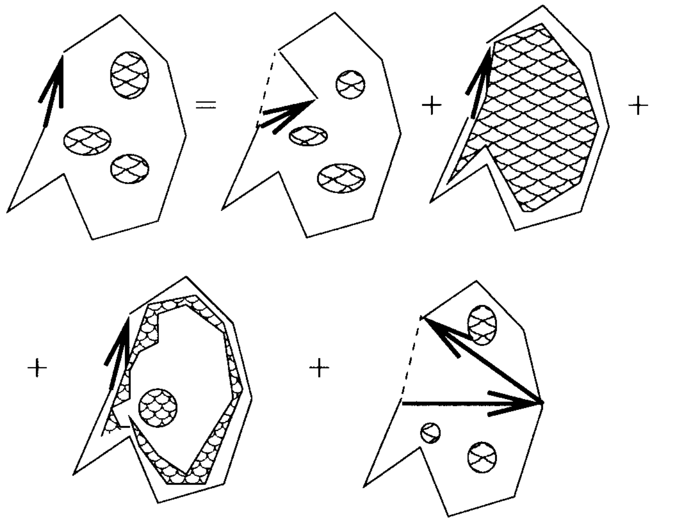

Fig. 2. Additional term with boundary cluster

configurations of level not more than n appear. And each configuration of level not more than n can be constructed with this recurrent relation.

Note that all $U^{(k)}$ are analytic in some neighborhood of $x = y = 0$, because $|Z^{(k)}(N,m)| \leq \tilde{Z}(N,m) \leq C^{N+m}$, where $\tilde{Z}(N,m)$ is the partition function for the interaction $-|\Phi|$. Thus the generating functions and the recurrent equations will be well-defined in some neighborhood of $x = y = 0$.

3. Analytic Part

3.1. Level 1.

3.1.1. Algebraic functions. Here we solve the functional equation (5) for the case when there is no spin, that is when $W = 0$. We rewrite it in the following form:

$$(2xU_1(x,y) + x - y)^2 = 4x^2y^2S^{(1)}(x) + (x-y)^2 - 4xy^3 - 4xyW(x,y) \tag{8}$$

and denote D its right-hand side. Consider the analytic set $\{(x,y) : 2xU_1 + x - y = 0\}$ in a small neighbourhood of $x = y = 0$. Note that it is not empty, $(0,0)$ belongs to this set and it defines a function $y(x) = x + O(x^2)$ in a neighbourhood of $x = 0$. In particular, it will be shown that $y(x)$ and $S(x)$ are algebraic functions if $W = 0$. We have two equations valid at the points of this analytic set

$$D = 0, \frac{\partial D}{\partial y} = 0$$

or

$$4x^2y^2S(x) + (x-y)^2 - 4xy^3 - 4xyW(x,y) = 0, \tag{9}$$

$$8x^2yS(x) - 2(x - y) - 12xy^2 - 4xW(x, y) - 4xyW'_y(x, y) = 0$$

from where one can exclude the function $S(x)$ by multiplying the second equation (9) by $\frac{y}{2}$ and subtracting it from the first equation. Then

$$y = x + 2y^3 - 2yW + 2y^2W'_y \tag{10}$$

or

$$y = \frac{x}{1 - 2y^2 - 2(yW'_y - W)}. \tag{11}$$

Here the functions $W = W(x, y, \beta)$, $S = S(x, \beta)$, $y = y(x, \beta)$ are also functions of the parameter β. By the theorem on implicit functions this equation gives the unique function $y(x, \beta)$, analytic for small x with $y(0, \beta) = 0$. It is evident from (11) that the convergence radius of $y(x, \beta)$ is finite. Note that $y(x; \beta)$ is odd and $S(x; \beta)$ is even, because for any triangulation $N - m$ is even, and thus the coefficients of monomials $x^i y^j$ of $yW'_y - W$ have even $i + j$. Such symmetry will also hold in all future constructions.

Now we consider the case $\beta = 0$ (or $W = 0$) in more detail. We rededuce here Tutte results in a different way, suitable for further generalizations. $y(x) = y(x, 0)$ is an algebraic function satisfying the equation $y^3 + py + q = 0$ with $p = -\frac{1}{2}, q = \frac{x}{2}$. The polynomial $f(y) = y^3 + py + q$ can have multiple roots only when $f = f'_y = 0$, which gives $x_\pm = \pm\sqrt{\frac{2}{27}}$. These roots are double roots because $f''_y \neq 0$ at these points. For $x_+ = \sqrt{\frac{2}{27}}$ we have $y_+ = y(x_+) = \frac{1}{\sqrt{6}}$, that can be seen from $f'_y = 3y^2 - \frac{1}{2} = 0$ and $f = 0$. From (11) it follows that $x(-y) = -x(y)$ and thus $y(x)$ is odd. It follows that $y(x)$ has both $x_\pm = \pm\sqrt{\frac{2}{27}}$ as its singular points.

From (9) we know $S(x) = S(x, 0)$, after that $U_1(x, y)$ is explicit from Eq. (8). The unique branch $y(x)$, defined by Eq. (11), is related to the unique branch of $S(x)$ by the equation

$$S = \frac{(1 - 3y^2(x))}{(1 - 2y^2(x))^2} = x^{-2}y^2(1 - 3y^2)$$

that is obtained by substituting $x = y - 2y^3$ to the first equation (11).

We know that $S(x)$ has positive coefficients, that is why $x = \sqrt{\frac{2}{27}}$ should be among its first singularities. Then $x = -\sqrt{\frac{2}{27}}$ should also be a singularity of both $y(x)$ and $S(x)$.

The principal part of the singularity at the double root x_+ is $y(x) = A(x - x_+)^{d+\frac{1}{2}}$ for some integer d. As $y_+ = y(x_+)$ is finite then $d \geq 0$. At the same time $y'(x) = \frac{1}{1-6y^2(x)}$ that is ∞ for $x = x_+$. It follows that $d = 0$. For S we have the same type of singularity $A(x - x_+)^{d+\frac{1}{2}}$ but here $d = 1$ as $S(x_+)$ and $S'(x_+)$ are finite but $S''(x_+)$ is infinite.

If we introduce the Riemann surface $\mathcal{S}_0$ of the algebraic function $y(x)$, then $x(s)$ and $y(s)$, $s \in \mathcal{S}_0$, are analytic on $\mathcal{S}_0$, except only the points where $x = y = \infty$. The function $S(x(s))$ is meromorphic on $\mathcal{S}_0$ with poles in the points s where $x(s) = 0$, $y(s) = \pm\frac{1}{\sqrt{2}}$. Thus, $S(x(s))$ does not have poles if, for example, $|x| \leq \frac{1}{2}$. Denote

$$\max_{|x| \leq \frac{1}{2}} |y(x(s))| = \bar{y} < \infty.$$

3.1.2. The perturbation. Now let β be sufficiently small. To find the first positive singularity $x_+(\beta)$ of $y(x, \beta)$ put

$$f(x, y; \beta) = -y + x + 2y^3 + Z(x, y), Z(x, y) = -2yW + 2y^2 W'_y, \quad (12)$$

$$f'_y = -1 + 6y^2 + Z'_y.$$

We can rewrite Eqs. (12) as

$$y^2 = \frac{1}{6} - \frac{1}{6} Z'_y,$$

$$x = \frac{2}{3} y + \frac{y}{3} Z'_y - Z.$$

Consider first the case when we allow only simple clusters with the size $N \leq N_0$ for some N_0, that is when $W(x, y)$ is a polynomial. The same argument as for the level 0 case gives that $y^{(1)}(x, \beta)$, $S_m^{(1)}(x, \beta)$ are algebraic functions, as well as $U^{(1)}(x, y)$. However, the first singularity $x_+(\beta)$ will be different. The fixed point $(x, y) = (\sqrt{\frac{2}{27}}, \sqrt{\frac{1}{6}})$ is perturbed, and the first singularity $x_+^{(1)}(\beta) = x_+(\beta)$ will be an analytic function of β for small β.

Denote now $y^{(0)}(x) = y(x, 0)$, $y^{(1)}(x) = y(x, \beta)$. We will need the following bounds: for some $C > 0$,

$$\left| x_+^{(1)}(\beta) - x_+ \right| < C\beta, \quad (13)$$

and, for example, if x is inside the circle of convergence of both one-valued branches $y^{(0)}(x)$ and $y^{(1)}(x)$,

$$\left| y^{(1)}(x) - y^{(0)}(x) \right| < C\beta. \quad (14)$$

However this bound holds also for any x, $|x| \leq \frac{1}{2}$, if $y^{(0)}(x)$ and $y^{(1)}(x)$ are corresponding branches. Correspondence between branches is established uniquely by analytic continuation if it is fixed for one point $x = 0$. It is convenient to use for this continuation the Riemann surfaces $\mathcal{S}_0$, $\mathcal{S}_1$ of these two functions correspondingly.

In fact, we have for any x, $|x| \leq \frac{1}{2}$,

$$f(x, y, \beta) - f(x, y, 0) = O(\beta), f'_y(x, y, \beta) - f'_y(x, y, 0) = O(\beta).$$

Fix some δ such that $0 < \beta \ll \delta \ll 1$. Then there is such $c > 0$ that for all pairs $(x, y(x))$, outside some $c\delta$-neighborhoods $O(x_\pm)$ of the branching points $x_\pm$, one can choose closed countours $\Gamma(x, y(x))$ in the y-plane around $y = y(x)$ so that $|f(x, y, 0)| = \delta$ for all $y \in \Gamma(x, y(x))$. Then the bound (14) is readily obtained by the Cauchy formula,

$$y^{(1)}(x) - y^{(0)}(x) = \frac{1}{2\pi i} \int_{\Gamma(x, y(x))} y\left(\frac{f'_y(x, y, \beta)}{f(x, y, \beta)} - \frac{f'_y(x, y, 0)}{f(x, y, 0)}\right) dy.$$

To get (14) inside $O(x_\pm)$ it could be more convenient to use the convergent Puiseux series (see for example [8]) at the branching points.

The bound (13) follows from the graphs of two functions y and $x + 2y^3 + Z(x, y)$ for fixed $0 < x < \frac{1}{2}$. They have a common tangent at the point $x = x_+(\beta) = x_+^{(1)}(\beta)$. If the sign of the $O(\beta)$ term in $Z(x, y)$ is plus then $x_+^{(1)}(\beta) < x_+(\beta)$, if the sign is minus then $x_+^{(1)}(\beta) > x_+(\beta)$. If the $O(\beta)$ term in $Z(x, y)$ is zero then one should consider the $O(\beta^2)$ term.

In the double root $x_+(\beta)$ the function $y(x)$ has the main part of the singularity equal to $A(\beta)(x - x_+(\beta))^{d+\frac{1}{2}}$ with $d = 0$ because $y(x_+(\beta))$ is finite, but

$$y'(x_+(\beta)) = \frac{1 + Z'_x}{1 - 6y^2 - Z'_y} = -\frac{1 + Z'_x}{f'_y} = \infty.$$

At the same time from the first equation (9) it follows that

$$xS = -\frac{x}{4y^2} + \frac{1}{2y} - \frac{1}{4x} + y + \frac{W}{y} \tag{15}$$

and

$$(xS)'_x = 2y^3 y' f(x, y, \beta) + \frac{1}{x^2} + \frac{W'_x}{y}$$

are finite at $x = x_+(\beta)$, but $S''_{xx}(x_+(\beta)) = \infty$. We have also that

$$\bar{S} = \max_{|x| \le \frac{1}{2}} |S| < \infty.$$

3.1.3. Asymptotics. In our case $S(x, \beta)$ has two singular points $x_+(\beta)$ and $-x_+(\beta)$ on the radius of convergence. The main parts of the singularities are $c_+(1 - \frac{x}{x_+(\beta)})^{\frac{3}{2}}$ and $c_+(1 + \frac{x}{x_+(\beta)})^{\frac{3}{2}}$ correspondingly for some positive constant $c_+ = c_+(\beta)$. Thus the asymptotics of the coefficients of $S(x, \beta)$, by Darboux theorem, see [5]) is

$$c_+\Gamma^{-1}(-\frac{3}{2})N^{-\frac{5}{2}}(x_+(\beta))^{-N} + c_+\Gamma^{-1}(-\frac{3}{2})N^{-\frac{5}{2}}(-x_+(\beta))^{-N}.$$

Thus for even N

$$Z^{(1)}(N, 2) \sim 2c_+\Gamma^{-1}(-\frac{3}{2})N^{-\frac{5}{2}}(x_+(\beta))^{-N}.$$

3.1.4. Arbitrary $m > 2$. The generating functions $S_m^{(1)}(x) = \sum_N Z^{(1)}(N, m)x^N$ for $m > 2$ can be obtained easily by the following recurrent procedure. Put $U = y^2 R$, then

$$R(x, y) = \sum_{m=2}^{\infty} S_{m-2}^{(1)}(x) y^{m-2}$$

and we can rewrite the functional equation as

$$yR = y + xy^2R^2 + x(R - S^{(1)}) + y^{-1}J(x, y), S^{(1)}(x) = S_2^{(1)}(x) = R(x, 0). \tag{16}$$

Then all $S_m^{(1)}$ are defined recursively by

$$xS_m^{(1)}(x) = S_{m-1}^{(1)}(x) + x \sum_{j+k=m-2} S_j^{(1)}(x)S_k^{(1)}(x) + J_{m+1}(x),$$

where $J(x, y) = \sum_m J_{m+1}(x)y^m$. We see from this that $S_m^{(1)}(x)$ have similar singularities as $S_0^{(1)}(x)$ and for $|x| \leq \frac{1}{2}$ and some $C > 0$,

$$\left|S_m^{(0)}(x)\right| < c^m, \left|S_m^{(1)}(x, \beta)\right| < (c + C\beta)^m.$$

Thus we have proved

Lemma 5. *The asymptotics for the level* 1 *partition function is*

$$Z^{(1)}(N, m) \sim \phi^{(1)}(m, \beta)N^{-\frac{5}{2}}(x_+(\beta))^{-N}.$$

3.2. Inductive estimates. The scheme of the induction is the following. We use the functional equation in Lemma 4, and for fixed n denote the solutions of the n^{th} level equation as $y^{(n)}(x, \beta)$, $S_2^{(n)}(x, \beta)$, $U^{(n)}(x, y)$. These functions will be initially defined as one-valued and analytic functions in $|x|, |y| < \varepsilon$ for sufficiently small $\varepsilon > 0$. However, they are branches of some multivalued functions. It is convenient to consider these multi-valued functions as functions on $\mathcal{S}_n$ or $\mathcal{S}_n \times \mathcal{C}$, where $\mathcal{S}_n$ is the Riemann surface of $y^{(n)}(x, \beta)$ and $\mathcal{C}$ is the complex plane. Thus we shall define a sequence of Riemann surfaces $\mathcal{S}_n$, $n = 2, \ldots$, and analytic covering maps $\phi^{(n)} : \mathcal{S}_n \to \mathcal{S}_{n-1}$, that is

$$\ldots \to \mathcal{S}_n \to^{\phi^{(n)}} \mathcal{S}_{n-1} \to \cdots \to \mathcal{S}_1 \to^{\phi^{(1)}} \mathcal{C}$$

In fact, we will not need the complete Riemann surface $\mathcal{S}_n$, but only some open part $\mathcal{D}_n$ of it. $\mathcal{D}_n$ will be defined inductively.

The induction procedure depends on the case. We consider first the case ofTheorem 1.

Definition of $\mathcal{D}_1 \subset \mathcal{S}_1$. *Let $O_{\delta\beta}(x_\pm^{(1)}(\beta))$ be the $\delta\beta$-neighborhood of the points $x_\pm^{(1)}(\beta) \in \mathcal{C}$. In the complex plane $\mathcal{C}$ the function $y^{(1)}(x, \beta)$ has the unique analytic continuation to the set*

$$A_{\delta\beta} = \left\{|x| < x_+^{(1)}(\beta) + \delta\beta\right\} \setminus (O_{\delta\beta}(x_+^{(1)}(\beta)) \cup O_{\delta\beta}(-x_+^{(1)}(\beta)))$$

of the x-plane $\mathcal{C}$. If $y^{(1)}(s^{(1)}) = y^{(1)}(x(s^{(1)}, \beta)$ on $\mathcal{S}_1$, then put

$$\mathcal{D}_1 = \left\{s : y^{(1)}(s) = y^{(1)}(x(s), \beta), x \in A_{\delta\beta}\right\} \cup (\phi^{(1)})^{-1}(O_{\delta\beta}(x_+^{(1)}(\beta)) \cup O_{\delta\beta}(x_-^{(1)}(\beta))).$$

It is instructive to start with the case $n = 2$. Before studying the 2-level equation the domain of analyticity the function $W^{(2)}(x, y)$ should be established. There exist constants $C, a > 0$ such that the function $W^{(2)}(x, y)$ is analytic on $\mathcal{D}_1 \times \mathcal{D}(\bar{y})$, $\mathcal{D}(\bar{y}) \subset C$, and

$$\left|W^{(2)}(s^{(1)}, y)\right| \leq (C\beta)^{2a}, (s^{(1)}, y) \in \mathcal{D}_1 \times \mathcal{D}(\bar{y})$$

This will be proved below. Note that it is no longer true that $W^{(i)}$ for $i > 1$ have radius of convergence of order β^{-a}. The functions

$$y^{(2)}(x, \beta), S_m^{(2)}(x, \beta), Y_m^{(2)}(x, \beta), U^{(2)}(x, y), Y^{(2)}(x, y)$$

are defined initially in a neighborhood of $x = y = 0$ as the solutions of the 2-level functional equation. All these functions have an analytic continuation to $\mathcal{D}_2 \subset \mathcal{S}_2$ defined as

$$\begin{aligned}\mathcal{D}_2 = &(\phi^{(2)})^{-1}(\mathcal{D}_1)\\ &\cap \Big\{s^{(2)} : y^{(2)}(s^{(2)}) = y^{(2)}(x(s^{(2)}), \beta), x \in A_{\delta\beta} \cup (O_{\delta\beta}(x_+^{(1)}(\beta))\\ &\cup O_{\delta\beta}(-x_+^{(1)}(\beta)))\Big\}.\end{aligned}$$

Then the functions $x(s^{(2)}, \beta)$, $y^{(2)}(s^{(2)}, \beta)$, $S_m^{(2)}(s^{(2)}, \beta)$, $Y_m^{(2)}(s^{(2)}, \beta)$ are defined as one-valued analytic functions on $\mathcal{D}_2$. Moreover, $U^{(2)}(s^{(2)}, y)$, $Y^{(2)}(s^{(2)}, y)$ are analytic functions on $\mathcal{D}_2 \times \mathcal{D}(\bar{y})$.

Now we can formulate inductive assumptions, definitions and estimates for

$$W^{(n)}, y^{(n)}(x, \beta), r^{(n)}(\beta), Y_2^{(n)}(x, \beta), Y_m^{(n)}(x, \beta),$$

where $r^{(n)}(\beta)$ is the convergence radius of $y^{(n)}(x, \beta)$. Using the functional equation and resummation formula (7) we prove the inductive assumptions for

$$W^{(n+1)}, y^{(n+1)}(x, \beta), x_+^{(n+1)}(\beta), Y_2^{(n+1)}(x, \beta), Y_m^{(n+1)}(x, \beta)$$

in this order.

1. Let $s^{(n)} \in \mathcal{S}_n$ and assume that $\mathcal{D}_n$ is already defined. Then the functions $x(s^{(n)})$, $y^{(n)}(s^{(n)}, \beta)$, $S_2^{(n)}(s^{(n)}, \beta)$, $S_m^{(n)}(s^{(n)}, \beta)$, $Y_m^{(nb)}(s^{(n)}, \beta)$ are defined as analytic functions on $\mathcal{D}_n$. The functions $U^{(n)}(s^{(n)}, y)$, $Y^{(n)}(s^{(n)}, y)$, $W^{(n+1)}(s^{(n)}, y)$ will be one-valued analytic functions on $\mathcal{D}_n \times \mathcal{D}(\bar{y})$. Then $\mathcal{D}_{n+1} \subset \mathcal{S}_{n+1}$ is defined as

$$\begin{aligned}\mathcal{D}_{n+1} = &(\phi^{(n+1)})^{-1}(\mathcal{D}_n)\\ &\cap \Big\{s^{(n+1)} : y^{(n+1)}(s^{(n+1)}) = y^{(n+1)}(x(s^{(n+1)}), \beta), x \in A_{\delta\beta}\\ &\cup (O_{\delta\beta}(x_+^{(1)}(\beta)) \cup O_{\delta\beta}(-x_+^{(1)}(\beta)))\Big\}.\end{aligned}$$

2. The function $W^{(n)}(x, y)$ is analytic on $\mathcal{D}_{n-1} \times \mathcal{D}(c)$ and

$$\left|W^{(n)}(s^{(n-1)}, y)\right| \leq (C\beta)^{an}, (s^{(n-1)}, y) \in \mathcal{S}_{n-1} \times \mathcal{D}(\bar{y}).$$

3. $(n+1)$-level functional equation is defined on $\mathcal{D}_n \times \mathcal{D}(\bar{y})$, because $W^{(n+1)}(s^{(n)}, y)$ is analytic on $\mathcal{S}_n \times \mathcal{D}(\bar{y})$. However, its unknown functions $U^{(n+1)}(s^{(n)}, y)$ and $Y^{(n+1)}(s^{(n)}, y)$ will have branching points on $\mathcal{S}_n \times \mathcal{D}(\bar{y})$, and thus it is reasonable to consider them as functions $\mathcal{S}_{n+1} \times \mathcal{D}(\bar{y})$ without branching points.
4. $x_+^{(n)}(\beta)$ is defined as the positive singularity of the curve
$$f(x, y; \beta) = -y + x + 2y^3 + Z^{(n)}(x, y) = 0$$
in $\mathcal{S}_n \times \mathcal{D}(\bar{y})$. We will prove that $x_+^{(n)}(\beta) > x_+^{(1)}(\beta)$ for all $n > 1$ and $r^{(n)}(\beta) = x_+^{(1)}(\beta)$.
5. For any $s^{(n)} \in \mathcal{D}_n$,
$$\left| y^{(n)}(s^{(n)}) - y^{(n-1)}(\phi^{(n)}(s^{(n)})) \right| < (C\beta)^{an}.$$
6. The functions $Y_m^{(n)}$, $m = 2, 3, \ldots$ are analytic on $\mathcal{D}_n$ and
$$\left| Y_m^{(n)}(s^{(n)}) \right| \le (C\beta)^{an}, s^{(n)} \in \mathcal{D}_n$$
and have the same (canonical) main singularities at $x_\pm^{(1)}(\beta)$.

Lemma 6. *Assume that inductive assumptions hold for all $k \le n$. Then they hold also for $k = n+1$.*

3.2.1. Bounds on the cluster function. To prove the inductive assumptions we need the following cluster estimate for the cluster functions of levels $j = 2, 3, \ldots$.

Lemma 7. *Consider the sum $\sum_{V_0}$ over all boundary clusters V_0 with fixed $r, m, m_1, \ldots, m_r$ and $N_0 = N(V_0')$. Then there exist $C, a > 0$ such that*
$$\sum_{V_0} |k(V_0)| \le (C\beta)^{aN_0} \le (C\beta)^{\frac{a}{2}(N_0 + (m + m_1 + \cdots + m_r))},$$
where m is the length of the boundary and m_i are the lengths of the interior boundaries of V_0'.

Proof. As in the proof of (3) we have
$$|k(V_0)| \le (C\beta)^{\frac{N_0}{6}}$$
and $m + m_1 + \cdots + m_r \le 3N_0$. This gives the result. □

Consider for example the case $n = 1$. Then the function $W^{(2)}(x, y)$ is analytic in $\mathcal{S}_1 \times \{y : |y| < \bar{y}\}$ and, by resummation formula, is bounded as
$$\left| W^{(2)}(x, y) \right| \le \sum_{N,m} y^m \sum_{N_0, r, m_1, \ldots, m_r} (C\beta)^{\frac{a}{2}(N_0 + (m + m_1 + \cdots + m_r))} x^{N_0} \prod_{i=1}^{r} \left| S_{m_i}^{(1)}(x) \right|$$
$$\le \sum_{N,m} y^m \sum_{N_0, r, m_1, \ldots, m_r} (C\beta)^{\frac{a}{2}(N_0 + (m + m_1 + \cdots + m_r))} x^{N_0} c^{m_1 + \cdots + m_r} = 0(\beta^{2a}).$$

In general the estimation of $W^{(n+1)}(x, y)$ is quite similar. It follows that the function $W^{(n+1)}(x, y)$ is analytic, and has the same main singularities at $x_\pm^{(k)}(\beta)$, $k = 1, \ldots, n$, as $U^{(n)}$.

3.2.2. Singular points.

Lemma 8. *The convergence radius of $y^{(2)}$ is equal to $x_+^{(1)}(\beta)$.*

Proof. Note that the sign of k or of

$$\beta \sum_{\sigma,\sigma'} \Phi(\sigma,\sigma')$$

defines whether $x_+^{(1)}(\beta)$ is less or greater than $x_+(0)$. The fact that $x_+^{(2)}(\beta) > x_+^{(1)}(\beta)$ is also defined by k together with two other facts. First one is the existence of boundary clusters of level 2 of the first order in β. This will give a first order terms in $W^{(2)}$. An example of triangulation which gives first order term is shown in Fig. 3. Here the triangles 2 and 3 are the colored triangles of the boundary cluster of level 1 (an interaction between them is shown by the fat horizontal line), the triangles 1 and 4 are blank triangles of the cluster adjacent to the "exterior" boundary with $m = 4$, region 5 (not made precise) is the interior part of the cluster, containing blank triangles of the cluster itself and possibly other blank triangles but no other clusters.

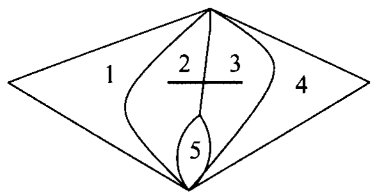

Fig. 3. First order boundary cluster term of level 2

We have the following equation:

$$\begin{aligned} y^{(2)} &= x + 2(y^{(2)})^3 + Z(x, y^{(2)}),\ Z(x, y^{(2)}) \\ &= -2y(W^{(1)} + W^{(2)}) + 2y^2(W^{(1)} + W^{(2)})'_y. \end{aligned}$$

We have

$$-W^{(2)}(x, y) + yW_y^{(2)} = 3ky^4x^4S_2^{(1)}(x) + \dots .$$

Note that $y'_x = -\frac{f'_x}{f'_y}$ is never zero for x, y of order one because f'_x is of order one for such x, y. We have from $k < 0$ that $x_+^{(2)}(\beta) - x_+^{(1)}(\beta) > \varepsilon\beta$ for some $\varepsilon > 0$.

The second fact is the absence of first order terms for boundary clusters of level greater than 2. It follows that all higher perturbations give that $x_+^{(2)} - x_+^{(k)} = O((\beta)^2)$ for any $k > 2$. $\square$

3.2.3. Bounds on $y^{(n+1)}$. Denote the right hand side of (10)

$$f^{(n+1)}(s^{(n)}, y, \beta) = x + 2y^3 - 2yJ^{(n+1)} + 2y^2(J^{(n+1)})'_y.$$

We want to find the difference $\Delta^{(n+1)}(s^{(n)}, \beta) = y^{(n+1)}(s^{(n)}, \beta) - y^{(n)}(s^{(n)}, \beta)$. We have

$$\left|f^{(n+1)}(s^{(n)}, y, \beta) - f^{(n)}(s^{(n)}, y, \beta)\right|$$
$$< (C\beta)^{a(n+1)}, \left|\frac{df^{(n+1)}}{dy}(s^{(n)}, y, \beta) - \frac{df^{(n)}}{dy}(s^{(n)}, y, \beta)\right| = (C\beta)^{a(n+1)}.$$

Outside a vicinity of the branching points we have by Cauchy formula

$$\Delta^{(n+1)}(s^{(n)}, \beta) = \frac{1}{2\pi i}\int_\Gamma y(\frac{\frac{d(f^{(n+1)})}{dy}(\Delta^{(n+1)}(s^{(n)}, \beta), y, \beta)}{f^{(n+1)}(\Delta^{(n+1)}(s^{(n)}, \beta), y, \beta)}$$
$$- \frac{\frac{d(f^{(n)})}{dy}(\Delta^{(n+1)}(s^{(n)}, \beta), y, \beta)}{f^{(n)}(\Delta^{(n+1)}(s^{(n)}, \beta), y, \beta)})dy,$$

where $\Gamma = \Gamma(x(s^{(n)}, y^{(n)}(s^{(n)}, \beta)))$ are the same family of contours, which were used in step 1 of the cluster expansion. Thus

$$\left|\Delta^{(n+1)}(s^{(n)}, \beta)\right| < (C\beta)^{a(n+1)}.$$

In the vicinity of the branching points one can again use Puiseux series.

To prove the bound

$$\left|x_+^{(n+1)}(\beta) - x_+^{(n)}(\beta)\right| < (C\beta)^{a(n+1)}$$

one could use the same trick for f'_y instead of f, or simply to consider graphs of $f^{(n+1)}(x, y)$ for fixed $0 < x < \frac{1}{2}$, as in step 1. There exists $x_0(\beta)$ such that for fixed $x = x_0(\beta)$ these functions have common tangent. Then put $x_+^{(n+1)}(\beta) = x_0(\beta)$.

3.2.4. Bounds on the functions $Y_m^{(n+1)}$. Then from

$$xS_2^{(n+1)}(s^{(n+1)}, \beta) = -\frac{x}{4(y^{(n+1)})^2} + \frac{1}{2y^{(n+1)}} - \frac{1}{4x} + y^{(n+1)} + \frac{J^{(n+1)}}{y^{(n+1)}},$$

we get

$$\left|Y_2^{(n+1)}(s^{(n+1)})\right| = \left|S_2^{(n+1)}(s^{(n+1)}) - S_2^{(n)}(\phi^{(n+1)}s^{(n)})\right| < (C\beta)^{a(n+1)}.$$

For $Y_m^{(n+1)}(s^{(n+1)}) = S_m^{(n+1)}(s^{(n+1)}) - S_m^{(n+1)}(\phi^{(n+1)}s^{(n)})$ the derivation is similar to step 1.

3.3. Proof of Theorem 2. Let now $\beta\Phi \leq 0$. Induction procedure for this case should be done in a different way. But the details are quite similar, and we give only main points for shortness. Iterating equation (11) we get that the expansion of $y(x, \beta)$ at $x = 0$ has all coefficients positive, because $yW'_y - W$ has positive coefficients. By the same reason, the coefficients of $y(x, \beta)$ increase when β increases, and the radius of convergence $x_+^{(n)}(\beta)$ of $y^{(n)}(x, \beta)$ decreases as $n \to \infty$. Note also that

$$\max_{|x|\leq\sqrt{\frac{2}{27}}} |y(x, 0)| = \frac{1}{\sqrt{6}}$$

and is attained at $|x| = \sqrt{\frac{2}{27}}$, because the power series has positive coefficients. Note that $y(x, \beta)$ also has its maximum on the circle of convergence on the positive half-axis. From the first equation we see also that this maximum $y(x_+(\beta), \beta)$ decreases when β increases. It is interesting to remark that it follows that cut-off models of level not greater than n that is the random cluster models, defined at the end of the Sect. 2.2, have the canonical behaviour. However the complete model has not. The domains $\mathcal{D}_n$ should be chosen as

$$\mathcal{D}_n = \left\{s^{(n)} : y^{(n)}(s^{(n)}) = y^{(n)}(x(s^{(n)}), \beta), x \in \mathcal{D}(\delta c_n)\right\},$$

where $c_n = x_+^{(n-1)}(\beta) - x_+^{(n)}(\beta)$ and $\delta > 0$ does not depend on β and sufficiently small. Here $y^{(n)}(x, \beta)$ is considered as a two valued function on $\mathcal{D}(\delta c_n)$.

Assume that the asymptotics is canonical, that is $\phi(m)N^{-\frac{5}{2}}c^N$. Then c^{-1} is the convergence radius and thus equals $\lim_{n\to\infty} x_+^{(n)}(\beta)$, because if $\beta\Phi \leq 0$ then $x_+^{(n+1)}(\beta) < x_+^{(n)}(\beta)$. As above one can prove that

$$\Theta^{(n)}(N, m) \sim \phi^{(n)}(m, \beta)N^{-\frac{5}{2}}(x_+^{(n)}(\beta))^{-N}.$$

Then the asymptotics is bounded above by the sum

$$\sum_n \phi^{(n)}(m, \beta)N^{-\frac{5}{2}}(x_+^{(n)}(\beta))^{-N}.$$

As $\sum_n \phi^{(n)}(m, \beta)$ converges, then $\phi(m) < \varepsilon$ for any $\varepsilon > 0$.

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