Markov Processes Relat. Fields **28**, 179–184 (2022)

Markov Processes and Related Fields

Stochastic Growth Models without Classical Branching Processes

V.A. Malyshev

Mechanics and Mathematics Faculty, Lomonosov Moscow State University, Leninskie Gory 1, Moscow, 119991, Russia

Received January 30, 2022

Abstract. All mathematical journals, including MPRF, now look like random collections of papers in one of the immense domains of mathematics like analysis, algebra, probability etc. But there are other classifications of mathematical domains (theories), where techniques from all mathematics can be used, but all activity is concentrated around some very important applied problem. One of such applied problems we consider here, and it leads us, in particular, to one of many possible projects in probability theory. One of our goals is to get the journal out of the state, where it serves as a stock of random papers. Submitted to MPRF papers, which appear to be in the framework of this project, will be reviewed and published as fast as possible.

Keywords: stochastic growth models, accumulation of particles, formation of clusters, large random systems

AMS Subject Classification: 60-XX

Random branching processes, a still popular part of probability theory, includes beautiful mathematics and seems to have analogs in the common life. For example, dynamics and growth of populations, tree and bush branches, branching of transport routes. However, as always, the life appears to be more complicated. And the question arises: is life a branching or/and accumulation process ? For example, cell division process can be described with different degrees of detailing. In particular, physics has the law of mass conservation. And due to this law each cell division provides cells with smaller masses. But it is clear that the cells live in some environment, which somehow appends necessary particles or components to new cells. Moreover, not only to increase their mass but also to conserve their structure. To obtain more complete picture, it is helpful to firstly to consider simplest but rigorous mathematical models of such appending, even without getting closer to the biological point. And it can be encouraging that immediately one sees many different examples of random processes of accumulation and growth.

It is not clear yet how deep and wide can be the structure of such models. But a good point is that such models are available for pure mathematician where he will see pure mathematical problems without any connection with reality.

That is why it is natural to start a project – to describe mathematical models how micro particles can accumulate, form clusters and complex macro structures. There are many applied examples – growth of the organism, tumor, warehouse, pile of rubbish, city and country. How to start:

1) One of the possibilities to start was to find in the existing books or papers some problem and to solve it. But I saw immediately that there is a lot of books where even in the title there were words “Mathematics of Growth”. And I started to think – to read all these books or to read them in random sequence until finding at least one interesting mathematical problem.

2) Another possibility for anyone is, without looking at these books, to invent some solvable but non-trivial models and then compare them with existing state of art. This I did here.

1. Particle accumulation in one point

First of all, as always, it is natural to start with one-dimensional models. Initially, at time $t = 0$, in each point $k \in Z_+$ there is a particle with probability p (independently of other points) and no particles with probability q, where $p + q = 1$. Each particle starts simple discrete time random walk with jump probabilities p_-, p_+ on 1 correspondingly to the left and to the right. All these random walks are mutually independent and particles do not see each other. When a particle hits point 0 it stays in it forever. Let $N(t)$ be the number of particles at point 0 at time t. The problem is to find asymptotics of the distribution of random variable $N(t)$ for large t.

Cases with increasing complexity.

1.1. $p_- = 1, p_+ = 0$. At time t to point 0 will come those and only those particles which at time $t = 0$ are in the segment $[0, t]$, hence this case is evidently reduced to CLT for the sum of $N + 1$ i.i.d. random variables $N(t) = \xi_0 + \xi_1 + \xi_2 + ... + \xi_N$, equal to 1 or 0 with probabilities p, q correspondingly.

1.2. $p_- > p_+$,

1.3. $p_- = p_+$,

1.4. $p_- < p_+$.

1.5. Essential complication – particles wander independently but cannot jump to the point (except point 0 of course) where there is already or will be simultaneously another particle. Here it could be more convenient to consider continuous time random walk.

1.6. On the plane lattice or in any Z^d, particles are scattered initially and perform simple random walk similarly, and stop when they hit the origin. Much more complicated cases – if, instead of the whole plane, we consider areas with corners, for example a quarter plane.

1.7. It is also interesting to consider accumulation of particles not in one point but in bounded or even unbounded domains. Note that in many cases it is possible to restrict oneself to discrete random walks. Continuous time and space could seem to be only technical generalization. But these generalizations can be important to simplify the problems using Brownian motion and stochastic differential equations. For example, assume there is a random point field of particles on R^2. They start to move via some stochastic process. Initially there is a circle of radius one. When some particle touches this circle, its radius increases on some $\varepsilon > 0$ – then the same problems as above.

2. Clusters

Here there is one more complication – accumulation warehouse is not fixed but can change its volume and even its form.

Increasing volume of the simplest cluster

2.1. The same situation as in one-dimensional case on Z_+. When the first particle stays at point 0, but the next incoming stays at the point 1, the third one – at point 2. And so on – if at time t the volume (the length of this chain of accumulated particles) is $V(t)$, then the first particle getting to the point $V(t)+2$ increases the volume on 1: $V(t) \to V(t)+1$.

The problem is the same – find asymptotics of the distribution of $V(t)$ for big times.

2.2. In two-dimensional case, as in point 1.6, but only the first particle gets stuck at point $(0, 0)$. When some particle hits one of 4 points at the distance 1 from the origin, it stays at the same point, etc. Here it is interesting not only the rate of volume increase but also the form of its domain.

2.3. Another example is when particles are initially scattered in the half-plane $Z_+ \times Z$. And their fixation occurs when they hit the vertical line $\{0\} \times Z$. But also other particles which come to a distance 1 from any of already fixed particle. Here random fluctuations of the boundary of growing domain is interesting to understand.

Association of particles into clusters

3.1. Let finite number of particles be initially on Z. They wander independently but with strong restriction – when the distance between some two particles becomes equal to 1, they stay immobilized. Problem – how many clusters appear and of what sizes. This depends of course on the initial conditions. The same problem for larger dimension.

3.2. Another variant – when two particles are at a distance of 1, this cluster of two particles does not stay immobilized, but starts its own random walk as a whole according to some rules to be exactly defined. The same for clusters of k particles. Also the question is the distribution of volumes and forms of the clusters for large times. Example – what is the mean volume of appeared clusters.

3.3. Models where there are restrictions on the form of clusters could be even more interesting. The simplest example is the following. When the distance between two particles becomes 1, connect them conditionally by an edge on the lattice. One can imagine that some physical laws connect two small molecules into one. Afterwards, some third particle can join this pair. It can be that these physical laws are such that any particle is allowed to be at distance 1 with not more than two particles. Then any growing clusters will be "long molecules" like "worms". One of the problems – the mean length of such "worms".

3.4. Many other interesting problems appear if we assume that the worms can stir somehow. For example, assume that any edge of the worm can rotate on 90 degrees, but without touching other edges of this worm. Then closed worms can appear. Again the question concerning their number and forms. This is sufficiently complicated problem, there are no any analytic methods for similar problems. And only qualitative rough estimates can be obtained.

3.5. And also the branching can appear, if we assume that clusters can divide (like cells). The simplest model here is the model 3.3 with worms. Assume additionally that one of the particles can randomly escape from the warm. Then instead of one worm we get two worms and one particle, which again take part in the process. It is even more appealing to consider, instead of warms, more complicated graphs, which resemble picture and even structure of the biological cells.

3. What science, called "mathematics of growth", now exists and what else can be

There are many papers and books (see references below), where already the title contains words "mathematics of growth". I will mention some directions of this research:

1) After collecting applied statistical information, one can find some parameters of the growth process. For example, the increase or decrease percentage (for some time interval) of several population groups depending on the age, region, etc. Then, assuming constant growth rates (that is the derivatives do not change) during this time period, one can write down linear differential equations, which define change of densities, for example of various age groups. Classical

branching process was an interesting generalization. The most known practical application of such models is the population dynamics.

2) In some works [4, 8, 9] the biological structure is considered as soft solid state and its growth is understood as the change of its volume (tension or compression). Here a lot of equations of continuum mechanics can be used.

3) The models considered in sections 1 and 2 can also be considered as cluster growth models, or even as formation of large particle systems with local interaction. In many references of [2–15] there are applied examples for some models of stochastic processes above. For example in section 1.1.2 of [4] it is said that the tooth growth occurs when particles from environment join the surface of the tooth, as in section 2.3 above.

4) Applied problems of economical growth [15–17] require development of mathematical models which are completely different from the existing game like models. It is believed that probability theory appeared from card games, and afterwards was generalized to currency exchange games. Other (in particular, multiscale) mathematical problems concerning finance also exist but are not so popular.

5) The question how growth on macro scale is related to micro scale (growth of smaller parts of the organism) is related in fact to all mathematical biophysics, which demands even more attention than mathematical physics. Concerning similar problems for the whole world, see [1].

References

[1] Geoffrey West (2017) *SCALE. The universal laws of Growth, Innovation, Sustainability, and the pace of life, in Organisms, Cities, Economies and Companies.* Penguin Press.

[2] Davide Ambrosi, Krishna Garikipati, Ellen Kuhl (organisers) (2008) Mini-Workshop: The Mathematics of growth and remodeling of soft biological tissues. *Oberwolfach Reports* **5** (3), 2219–2260.

[3] G. Jones, A. Chapman (2012) Modeling growth in biological materials. *SIAM Review* **54** (1), 52–118.

[4] A. Goriely (2017) *The Mathematics and Mechanics of Biological Growth.* Springer.

[5] A. Zaidi (2014) *Mathematics of Cell Growth.* Thesis. Massey University. New Zealand.

[6] Alexander Erlich, Gareth W. Jones, Françoise Tisseur, Derek E. Moulton, Alain Goriely The role of topology and mechanics in uniaxially growing cell networks. `rspa.royalsocietypublishing.org`

[7] J. King and S. Franks (2004) Mathematical analysis of some multi-dimensional tissue-growth models. *Euro. Journal of Applied Math.* **15**, 273–295.

[8] A. Goriely and V. Klika (2012) Current challenges in the mechanobiology of growth. In: Report on BIRS workshop *Mathematical Foundations in Mechanical Biology.*

[9] A. Chauvière, L. Preziosi, C. Verdier (2010) *Cell Mechanics. From Single Scale-Based Models to Multiscale Modeling.* Chapman and Hall.

[10] N. Segev, A. Alfonso, G.S. Payne, J. Donaldson (2009) *Trafficking Inside Cells. Pathways, Mechanisms and Regulation.* Springer.

[11] P. Jagers (2000) The deterministic evolution of general branching populations. In: *State of the Art in Statistics and Probability Theory. Festschrift for Willem R. van Zwet.* M. de Gunst, C. Klaassen, and A. van der Vaart (eds.), IMS Lecture Notes, 384–398.

[12] A. Lasota, M. Mackey (1999) Cell division and the stability of cellular populations. *J. Math. Biol.* 38, 241–261.

[13] Po-Yi Ho, Jie Lin, and Ariel Amir (2018) Modeling cell size regulation: From single-cell level statistics to molecular mechanisms and population level effects. *Annual Review of Biophysics* **47**, 251–271.

[14] A.S. Bratus, A.S. Novozhilov, A.P. Platonov (2010) *Dynamical Systems and Biological Models.* Fizmatlit. In Russian.

[15] Germinal G. Van (2020) A Mathematical Theory of Economic Growth. Preprint MPRA, Paper No. 99940.

[16] Daron Acemoglu (2009) *Introduction to Modern Economic Growth.* Princeton University Press, Princeton and Oxford.

[17] B. Jensen (1994) *The Dynamic Systems of Basic Economic Growth Models.* Springer.

[18] S.F. Gilbert (2000) *Developmental Biology.* Sunderland (MA): Sinauer Associates.

[19] O.A. Nev, H.A. van den Berg (2017) Mathematical models of microbial growth and metabolism: a whole-organism perspective. *Science Progress* **100**, 343–362.

[20] A. Lindenmayer (1968) Mathematical Models for Cellular Interactions in Development II. Simple and Branching Filaments with Two-sided Inputs. *J. Theoret. Biol.* **18**, 300–315.