

# Convergence to Gibbs Equilibrium — Unveiling the Mystery

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**Abstract.** We consider general hamiltonian systems with quadratic interaction potential and  $N < \infty$  degrees of freedom, only  $m$  of which have contact with external world, that is subjected to damping and random stationary external forces. We show that, as  $t \rightarrow \infty$ , already for  $m = 1$ , the unique limiting distribution exists for almost all interactions. Moreover, it is Gibbs if the external force is the white noise, but typically not Gibbs for gaussian processes with smooth trajectories. This conclusion survives also in the thermodynamic limit  $N \rightarrow \infty$ .

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## 1. Introduction

One of the most important, hard and long-standing problems in non-equilibrium classical statistical physics is the convergence to Gibbs equilibrium. One can say even that the mathematical status of this problem had always been a bit mysterious. No mathematical argument, for many-particle systems, appeared to justify convergence to equilibrium for closed deterministic systems. On the contrary, there were many examples (linear systems, completely integrable systems and their non-linear perturbations within KAM theory) showing exactly the contrary. Moreover, for finite quantum systems with unitary dynamics it is obvious that there cannot be any convergence to equilibrium — contact with external world is absolutely necessary.

In spite of this, since Boltzmann and Gibbs, it has often been believed that non-linear effects (particle collisions), inside the closed system, could provide

this convergence. Closed linear hamiltonian systems were always considered as annoying, thus rare and uninteresting exception, where the abundance of invariant subspaces and invariant tori prevents the dynamic emergence of limiting Gibbs states.

Sometimes, this difficulty has been overcome by artificially introducing specially chosen stochastic internal dynamics throughout the closed system. Then sometimes it became possible to prove convergence to Gibbs invariant measure.

We cannot and even do not intend to disprove the common belief. Our goal is much more modest — we want to show that there can be an alternative approach to the convergence problem. Namely, let us assume that completely closed system is an idealization, and there is always some, even the smallest possible, contact with the external media. Then, as we show here, for general systems with quadratic interaction, the situation changes drastically — invariant subspaces and tori become dynamically intermixed — and linear systems become not an exception, but a legal member of the model community. “Very small” means for us that, for example, only one (of  $N$ ) degree of freedom contacts external world. There were a series of papers by J. Lebowitz and colleagues (see for example [1–3] and references therein), devoted to non-equilibrium models of one-dimensional crystals with different assumptions and a different goal.

We consider general linear hamiltonian system with  $N$  degrees of freedom and assume that one (or more) fixed degree of freedom is subjected to damping and random stationary external force. We prove that if the external force is the white noise then there is convergence to a Gibbs state. However, if the external force is a stationary random process with smooth trajectories then “typically” the system converges to equilibrium but this equilibrium will not be Gibbs.<sup>1</sup> This leads to the conclusion that the absence of memory in the external force may be crucial for the convergence to Gibbs equilibrium. More interesting (and more difficult to prove) is that this assertion holds also in the thermodynamic limit, that is, for the degrees of freedom far away from the contacts with external world.

Our paper puts also another question: why a closed deterministic system feels even the smallest influence from the boundary so sharply. We think that the same should hold also for non-linear systems — collisions can only either accelerate or relax this influence. However we cannot prove it now. Such sharp feeling of the boundary is rarely possible for stochastic dynamics. Possibly this is the reason why the fundamental physical laws are deterministic, not stochastic. We do not claim that our scheme for the convergence is the only possible but we do not know other possibilities.

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<sup>1</sup>“Typically” means generic situation in a common sense and is accurately explained in the text.

## 2. Necessary definitions

We consider the phase space

$$L = L_{2N} = \mathbb{R}^{2N} = \left\{ \psi = \begin{pmatrix} q \\ p \end{pmatrix}, q = (q_1, \dots, q_N)^T, p = (p_1, \dots, p_N)^T \in \mathbb{R}^N \right\},$$

( $T$  denotes transposition, thus  $q, p, \psi$  are the column vectors) with the scalar product

$$(\psi, \psi')_2 = \sum_{i=1}^N (q_i q'_i + p_i p'_i).$$

It can be presented as the direct sum

$$L = l_N^{(q)} \oplus l_N^{(p)} \tag{2.1}$$

of orthogonal coordinate and momentum subspaces, with induced scalar products  $(q, q')_2$  and  $(p, p')_2$  correspondingly. We distinguish several degrees of freedom, say,

$$\Lambda^{(m)} = \Lambda^{(N,m)} = \{N - m + 1, \dots, N\} \subset \Lambda = \{1, \dots, N\}, \quad 1 \leq m \leq N,$$

(we shall call the set  $\Lambda^{(m)}$  the *boundary* of  $\Lambda$ ) and consider the dynamics defined by the system of  $2N$  stochastic differential equations

$$\begin{aligned} \frac{dq_k}{dt} &= p_k, \\ \frac{dp_k}{dt} &= - \sum_{l=1}^N V(k, l) q_l - \alpha \delta_k^{(N,m)} p_k + F_{t, N+k} \end{aligned} \tag{2.2}$$

where  $k = 1, \dots, N$ ,  $V = (V(k, l))$  is a positive definite  $(N \times N)$ -matrix,  $\delta_k^{(N,m)} = 1$  if  $k > N - m$  and zero otherwise. It is convenient to define the  $2N$ -vector  $F_t$  with the components:  $F_{t,k} = 0$ ,  $k \leq 2N - m$ , and  $F_{t,k}$ ,  $k > 2N - m$  are independent copies of a gaussian stochastic stationary process  $f_t$ . This means that only degrees of freedom from the set  $\Lambda^{(m)}$  are subjected to damping (defined by the factor  $\alpha > 0$ ) and to the external forces  $F_{t,k}$ .

If  $\alpha = 0$ ,  $f_t = 0$ , then the system is the linear hamiltonian system with the quadratic hamiltonian

$$H(\psi) = \frac{1}{2} \sum_{i=1}^N p_i^2 + \frac{1}{2} \sum_{i,j} V(j, i) q_i q_j = \frac{1}{2} \left( \begin{pmatrix} V & 0 \\ 0 & E \end{pmatrix} \psi, \psi \right)_2. \tag{2.3}$$

Note that the Gibbs distribution

$$Z^{-1} \exp(-\beta H) = Z^{-1} \exp\left(-\frac{1}{2} (C_{G, \beta}^{-1} \psi, \psi)_2\right), \tag{2.4}$$

corresponding to the hamiltonian (2.3), is gaussian, and

$$C_{G,\beta} = C_{Gibbs} = \frac{1}{\beta} \begin{pmatrix} V^{-1} & 0 \\ 0 & E \end{pmatrix} \quad (2.5)$$

is its covariance matrix.

One can rewrite system (2.2) in the vector notation

$$\frac{d\psi}{dt} = A\psi + F_t, \quad (2.6)$$

where

$$A = \begin{pmatrix} 0 & E \\ -V & -\alpha D \end{pmatrix}, \quad (2.7)$$

$E$  is the unit  $(N \times N)$ -matrix, and  $D$  is the diagonal  $(N \times N)$ -matrix with all zeroes on the diagonal except  $D_{k,k} = 1$ ,  $k = N - m + 1, \dots, N$ .

## 2.1. Classes of hamiltonians

For any  $N$  let  $\mathbf{H}_N$  denote the set of all hamiltonians (2.3) with positive definite  $V$ . Note that the dimension of this set is  $\dim \mathbf{H}_N = (N(N+1))/2$ , that coincides with the dimension of the set of symmetric  $V$ . In fact, take some symmetric positive definite  $V$ , for example diagonal one, then for any symmetric  $V_1$  with sufficiently small elements,  $V + V_1$  will be symmetric and positive definite.

More generally, let  $\Gamma = \Gamma_N$  be connected graph with  $N$  vertices  $i = 1, \dots, N$ , and not more than one edge per each (unordered) pair of vertices  $(i, j)$ . It is assumed that all loops  $(i, i)$  are the edges of  $\Gamma$ . Denote by  $\mathbf{H}_\Gamma$  the set of (positive definite)  $V$  such that  $V(i, j) = 0$  if  $(i, j)$  is not an edge of  $\Gamma$ . The same argument shows that the dimension of  $\mathbf{H}_\Gamma$  is equal to the number of edges of  $\Gamma$ . Note that  $\mathbf{H}_N = \mathbf{H}_\Gamma$  for the complete graph  $\Gamma$  with  $N$  vertices.

In particular, we can consider the  $d$ -dimensional integer lattice  $Z^d$  and the graph  $\Gamma = \Gamma(d, \Lambda)$ , the set of vertices of which is the cube

$$\Lambda = \Lambda(d, M) = \{(x_1, \dots, x_d) \in Z^d : |x_i| \leq M, i = 1, \dots, d\} \subset Z^d$$

and the edges are  $(i, j)$ ,  $|i - j| \leq 1$ .

In general,  $V$  is called  $\gamma$ -local on  $\Gamma$  if  $V(i, j) = 0$  for all pairs of vertices  $i, j$  such that the distance  $r(i, j)$  between  $i$  and  $j$  is greater than  $\gamma$  (the distance  $r(i, j)$  between two vertices  $i, j$  on a graph is defined as the minimal length (number of edges) of paths connecting them).

We shall say that some property holds for almost any hamiltonian from the set  $\mathbf{H}_\Gamma$  if the set  $\mathbf{H}_\Gamma^{(+)}$ , where the property holds, is open and everywhere dense. One can prove in fact that the dimension of the set  $\mathbf{H}_\Gamma^{(-)} = \mathbf{H}_\Gamma \setminus \mathbf{H}_\Gamma^{(+)}$  is less than the dimension of  $\mathbf{H}_\Gamma$  itself.

**2.2. Invariant subspaces**

Consider the following subset of the phase space  $L$

$$L_- = \{\psi \in L : H(e^{tA}\psi) \rightarrow 0, t \rightarrow \infty\} \subset L.$$

We will need the following result. Let  $e_i, i = 1, \dots, N$ , be column  $N$ -vectors with zero components except the  $i$ th component which is equal to 1.

**Lemma 2.1.**  $L_-$  is a linear subspace of  $L$  and

$$L_- = \left\{ \begin{pmatrix} q \\ p \end{pmatrix} \in L : q \in l_V, p \in l_V \right\},$$

where  $l_V$  is the subspace of  $R^N$  spanned by the vectors  $V^k e_i, i = N - m + 1, \dots, N; k = 0, 1, \dots$ . Moreover,  $L_-$  and its orthogonal complement, denoted by  $L_0$ , are invariant with respect to the operator  $A$ .

The proof is identical to the proof of Theorem 2.1 in [6].

**Lemma 2.2.** The spectrum of the restriction  $A_-$  of  $A$  on the subspace  $L_-$  belongs to the left half-plane, and

$$\|e^{tA_-}\|_2 \rightarrow 0$$

exponentially fast as  $t \rightarrow \infty$ .

*Proof.* By the definition of  $L_-$  and boundedness of  $H$  from below, we have  $\exp\{tA\}\psi \rightarrow 0$  for any  $\psi \in L_-$ . □

**Lemma 2.3.** For almost any  $H \in \mathbf{H}_\Gamma$  we have  $\dim L_0 = 0$ .

*Proof.* For given  $V$ , the subspace  $L_- = L_-(m)$  depends on  $\Lambda^{(m)}$ . If  $m_1 < m_2$  then  $L_-(m_1) \subseteq L_-(m_2)$ . That is why it is sufficient to prove the lemma in case of one-point subset  $\Lambda^{(1)}$ . If  $l_V$  is spanned by the vectors  $V^k e_N, k = 0, 1, \dots$ , then it is spanned by  $N$  vectors  $V^k e_N, k = 1, \dots, N$ , and obviously vice-versa. Let  $\Sigma(V)$  be the  $(N \times N)$ -matrix the columns of which are the vectors  $V^k e_N, k = 1, \dots, N$ . Then the relation  $\det(\Sigma(V)) \neq 0$  for a matrix  $V \in \mathbf{H}_\Gamma$  is equivalent to the statement that the vectors  $V^k e_N, k = 1, \dots, N$  are linearly independent, that is,  $\dim l_V = N$ . Then the set  $\mathbf{H}_\Gamma^{(-)}$  of hamiltonians for which  $\dim L_0 > 0$  is

$$\mathbf{H}_\Gamma^{(-)} = \{V : \dim(l_V) < N\} = \{V : \det(\Sigma(V)) = 0\}.$$

Thus,  $\mathbf{H}_\Gamma^{(-)}$  is the set of zeros of a polynomial function on a smooth manifold  $\mathbf{H}_\Gamma$ . Hence, its dimension is less than the dimension of  $\mathbf{H}_\Gamma$ . □

### 2.3. Covariances

In our model, all the external forces  $f_t$  will be gaussian stationary processes with zero mean. Among them there is the white noise — the generalized stationary gaussian process with covariance  $C_f(s) = \sigma^2\delta(s)$ , it is sometimes called a process with independent values (without memory). All other stationary gaussian processes, which we consider here, are processes with memory. We will assume that they have continuous trajectories and integrable (short memory) covariance

$$C_f(s) = \langle f_t f_{t+s} \rangle.$$

Then the solution of (2.6) with arbitrary initial vector  $\psi(0)$  is unique and is equal to (for the white noise case see for example [4], section 12.4)

$$\psi(t) = e^{tA} \left( \int_0^t e^{-sA} F_s ds + \psi(0) \right). \quad (2.8)$$

Our goal is to show that even weak memory, in the generic situation, prevents the limiting invariant measure (which always exists and is unique) from being Gibbs. To formulate more readable results we assume more:  $C_f$  belongs to the Schwartz space  $S = S(\mathbb{R})$ . Then also the spectral density

$$a(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-it\lambda} C_f(t) dt$$

belongs to the space  $S$ .

We shall say that some property (for given  $V$ ) holds for almost all  $C_f$  from the space  $S$  if the set  $S^{(+)} \subset S$  where this property holds is open and everywhere dense in  $S$ .

## 3. Main results

### 3.1. Finite system

Further on we denote, using (2.5),  $C_G = C_{G,2\alpha}$ . Fix some connected graph  $\Gamma$  with  $N$  vertices.

**Theorem 3.1.** *Let  $f_t$  be either white noise or a process with continuous trajectories and integrable  $C_f$ . Then for any hamiltonian  $H \in \mathbf{H}_\Gamma$  with  $L_0 = L_0(H) = \{0\}$  the following holds:*

- (1) *there exists gaussian random  $(2N)$ -vector  $\psi(\infty)$  such that for any initial condition  $\psi(0)$  the distribution of  $\psi(t)$  converges, as  $t \rightarrow \infty$ , to that of  $\psi(\infty)$ ;*

(2) for the covariance of the process  $\psi(t)$  we have  $E\psi(t) \rightarrow 0$  and

$$\begin{aligned} C_{\psi(\infty)}(s) &= \lim_{t \rightarrow \infty} \langle \psi(t)\psi^T(t+s) \rangle \\ &= \lim_{t \rightarrow \infty} C_\psi(t, t+s) = W(s)C_G + C_GW(-s)^T, \end{aligned} \tag{3.1}$$

where

$$W(s) = \int_0^{+\infty} e^{\tau A} C_f(\tau+s) d\tau. \tag{3.2}$$

**Corollary 3.1.** For the white noise with variance  $\sigma^2$  the vector  $\psi(\infty)$  has Gibbs distribution (2.4) with the temperature

$$\beta^{-1} = \frac{\sigma^2}{2\alpha}.$$

For  $m = 1$  this corollary was proved in [8]. Denote  $C_{\psi(\infty)}(0)$  by  $C_\psi$ . Further on we denote the matrix elements  $C_\mu(i, N+j)$  of the matrix  $C_\mu$  by  $C_\mu(q_i, p_j)$  (and analogously for the other  $(2N) \times (2N)$ -matrices).

**Theorem 3.2.** Let  $N \geq 2$ , fix some graph  $\Gamma$  and any  $H \in \mathbf{H}_\Gamma$  with  $L_0 = L_0(H) = \{0\}$ . Then the following assertions hold:

- (1) for any  $C_f \in S$  in the limiting distribution there are no correlations between coordinates and velocities, that is,  $C_\psi(q_i, p_j) = 0$  for any  $i, j$ ;
- (2) for almost any  $C_f \in S$  there are nonzero correlations between velocities, that is,  $C_\psi(p_i, p_j) \neq 0$  for some  $i \neq j$ . Thus, the limiting distribution cannot be Gibbs.

### 3.2. Large $N$

It is more interesting, however, that the convergence to Gibbs is impossible even in the points of  $\Lambda$  far away from the boundary, in the thermodynamic limit  $N \rightarrow \infty$ .

The following result reduces (for large  $N$ ) calculation of the matrix  $C_\psi$  to that of the simpler matrix

$$C_V = \frac{\pi}{\alpha} \begin{pmatrix} a(\sqrt{V})V^{-1} & 0 \\ 0 & a(\sqrt{V}) \end{pmatrix}$$

where  $\sqrt{V}$  is the unique positive root of  $V$ .

*Remark 3.1.* It is interesting to note that: 1)  $C_V$  also defines an invariant measure with respect to pure (that is with  $\alpha = 0, F_t = 0$ ) hamiltonian dynamics; 2) for the white noise case  $C_V$  corresponds to the Gibbs distribution.

We assume that some graph  $\Gamma$  is given with the set of vertices  $\Lambda, |\Lambda| = N$ , and the boundary set  $\Lambda^{(m)}$ . For any  $V \in \mathbf{H}_\Gamma$  such that  $L_0(V) = \{0\}$ , the following representation of the limiting covariance matrix appears to be crucial

$$C_\psi = C_V + Y_V$$

where  $Y_V$  is some remainder term. The following theorem gives the estimates for  $Y_V$ . We define the norm  $\|V\|_\infty$  of a matrix  $V$  as follows:

$$\|V\|_\infty = \max_i \sum_j |V(i, j)|.$$

**Theorem 3.3.** *Assume that  $V$  is  $\gamma$ -local and  $\|V\|_\infty < B$  for some  $B > 0$ . Fix a number  $\eta = \eta(N) \geq \gamma$ . The following assertions hold.*

- (1) *Let  $C_f \in S$  and have a bounded support, that is,  $C_f(t) = 0$  if  $|t| > b$  for some  $b > 0$ . Then for any pair  $i, j$  far away from the boundary (that is, such that  $r(i, \Lambda^{(m)}) > \eta(N), r(j, \Lambda^{(m)}) > \eta(N)$ ), there is the following estimate*

$$|Y_V(q_i, q_j)|, |Y_V(p_i, p_j)| < |\Lambda^{(m)}| K_0 \left(\frac{K}{\eta}\right)^{\eta\gamma^{-1}}$$

*for some constants  $K_0 = K_0(C_f, B, b, \alpha, \gamma)$  and  $K = K(C_f, B, b, \alpha, \gamma)$ , not depending on  $N$ .*

- (2) *For arbitrary  $C_f \in S$  the estimate is*

$$|Y_V(q_i, q_j)|, |Y_V(p_i, p_j)| < |\Lambda^{(m)}| C(k)\eta^{-k},$$

*for any  $k > 0$ , where the constant  $C(k) = C(C_f, k, B, \alpha, \gamma)$  does not depend on  $N$ .*

This theorem allows to make various conclusions concerning the thermodynamic limit. Below we give an example.

Fix some  $C_f(t) \in S$  and some connected countable graph  $\Gamma_\infty$  with the set of vertices  $\Lambda_\infty$ , and an increasing sequence of subsets  $\Lambda_1 \subset \Lambda_2 \subset \dots \subset \Lambda_n \subset \dots$  such that  $\Lambda = \cup \Lambda_n$ . Let  $\Gamma_n$  be the subgraph of  $\Gamma_\infty$  with the set of vertices  $\Lambda_n$ , that is,  $\Gamma_n$  inherits all edges between vertices of  $\Lambda_n$  from  $\Gamma$ . Denote  $|\Lambda_n|$  by  $N_n$  and assume that the boundaries  $\Lambda_n^{(m)}$  are given with  $m = m(n)$  such that the following conditions hold:

- (1) *there exists  $d > 0$  such that for any  $i \in \Lambda_\infty$  there exists  $n(i)$  such that for any  $n > n(i)$  the following inequality holds*

$$r_n(i, \Lambda_n^{(m)}) > \max\{m(n)^{1/d}, \gamma\},$$

*where  $r_n(i, \Lambda_n^{(m)})$  is the distance from vertex  $i$  to the boundary  $\Lambda_n^{(m)}$  on the graph  $\Gamma_n$ ;*



- (2) for any  $i \in \Lambda_\infty$  we have  $r_n(i, \Lambda_n^{(m)}) \rightarrow \infty$  as  $n \rightarrow \infty$  (that is, the boundary runs to infinity with  $n$ ).

Let  $l^\infty(\Gamma_\infty)$  be the complex Banach space of bounded functions on the set of vertices of  $\Gamma_\infty$ :

$$l^\infty(\Gamma_\infty) = \{(x_i)_{i \in \Gamma_\infty} : \sup_{i \in \Gamma_\infty} |x_i| < \infty, x_i \in \mathbb{C}\}.$$

Fix some  $\gamma$ -local infinite matrix  $V$  on this space such that  $\|V\|_\infty \leq B$ . It is clear that  $V$  defines a bounded linear operator on  $l^\infty(\Gamma_\infty)$ . Denote by  $\sigma(V)$  the spectrum of this operator. Let  $V_n = (V(i, j))_{i, j \in \Lambda_n}$  be the restriction of  $V$  on  $\Lambda_n$ , it is a matrix of the order  $N_n$ . Assume that for all  $n = 1, 2, \dots$  the matrices  $V_n$  are positive-definite. Note that the condition  $L_-(V_n) = L$  may not hold for some  $n$ . However, one can choose a sequence of positive-definite matrices  $V'_n \in \mathbf{H}_{\Lambda_n}$  such that  $\|V_n - V'_n\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$  with  $L_0(V'_n) = \{0\}$ . Moreover, the convergence of  $V'_n$  to  $V_n$  can be chosen arbitrarily fast. Denote by  $C_\psi^{(n)}$  the limiting covariance matrices corresponding to  $V'_n$ .

**Corollary 3.2.** *The following assertions hold:*

- (1) for any  $i, j \in \Lambda_\infty$  there exists the thermodynamic limit

$$\lim_{n \rightarrow \infty} C_\psi^{(n)}(p_i, p_j) = C_\psi^{(\infty), p}(i, j),$$

that is for distribution of velocities;

- (2) if for any  $i, j \in \Lambda_\infty$  there exist finite limits:

$$U(i, j) \doteq \lim_{n \rightarrow \infty} V_n^{-1}(i, j), \tag{3.3}$$

then for the coordinates we have

$$\lim_{n \rightarrow \infty} C_\psi^{(n)}(q_i, q_j) = C_\psi^{(\infty), q}(i, j);$$

- (3) assume that the spectral density  $a(\sqrt{\lambda})$  is analytic on the open set containing the spectrum  $\sigma(V)$ . Then

$$C_\psi^{(\infty), p}(i, j) = a(\sqrt{V}),$$

where  $a(\sqrt{V})$  is defined in terms of the operator calculus on  $l^\infty(\Gamma_\infty)$  [7, p. 568].

Let us add some comments to this corollary. Firstly, we want to emphasize that item 2 does not contain any restrictions on  $U(i, j)$ . Secondly, it is easy to see that the condition of item 3 is fulfilled if  $C_f$  has bounded support (in this case the spectral is an entire function). And finally, the thermodynamic limit typically is not Gibbs, more exactly  $C_\xi^{(\infty), p}(i, j) \neq 0$  for any  $i \neq j$  in  $\Lambda_\infty$  such that  $a(\sqrt{V})(i, j) \neq 0$ .

#### 4. Proof of Theorem 3.1

The process  $\psi(t)$  is not stationary. However, the following calculation shows that it is asymptotically stationary.

Let  $D^{(2)}$  be the diagonal  $(2N \times 2N)$ -matrix with all zero elements on the diagonal except  $D_{k,k}^{(2)} = 1, k = 2N - m + 1, \dots, 2N$ . Obviously  $D^{(2)} = (D^{(2)})^T = D^{(2)}(D^{(2)})^T$ .

Then

$$\begin{aligned} C_\psi(t, t+s) &= E \int_0^t dt_1 \exp\{(t-t_1)A\} F_{t_1} \int_0^{t+s} F_{t_2}^T \exp\{(t+s-t_2)A^T\} dt_2 \\ &= e^{tA} \int_0^t dt_1 \exp\{-t_1 A\} D_2 D_2^T \int_0^{t+s} dt_2 \exp\{-t_2 A^T\} C_f(t_1 - t_2) \\ &\quad \times \exp\{(t+s)A^T\}. \end{aligned} \quad (4.1)$$

For better understanding of the following calculations, it is useful to start with the white noise case, i.e. when

$$C_f(s) = \sigma^2 \delta(s).$$

It is a generalized function but the calculation follows the same line. For  $s = 0$ , (4.1) becomes

$$\sigma^2 \int_0^t dt_1 \exp\{(t-t_1)A\} D^{(2)} \exp\{(t-t_2)A^T\}.$$

We use a straightforward algebraic calculation with  $(2 \times 2)$ -block matrices (2.5) and (2.7) to get

$$AC_G + C_G A^T = -D^{(2)} \quad (4.2)$$

where  $C_G$  is given by (2.5) with  $\beta = 2\alpha$ . Then

$$\frac{d}{dt} (e^{-tA} C_G e^{-tA^T}) = e^{-tA} D^{(2)} e^{-tA^T} \quad (4.3)$$

and thus

$$\begin{aligned} C_\psi(t, t) &= \sigma^2 e^{tA} \left( \int_0^t dt_1 \exp\{-t_1 A\} D^{(2)} \exp\{-t_1 A^T\} \right) \exp\{tA^T\} \\ &= \sigma^2 e^{tA} (e^{-tA} C_G e^{-tA^T} - C_G) e^{tA^T} \end{aligned}$$

and  $C_\psi(t, t) \rightarrow \sigma^2 C_G$  as  $t \rightarrow \infty$ . This proves Corollary 3.1. Similarly one can show that  $W(s) = 0, s > 0, W(0) = (\sigma^2/2)E$  and

$$W(s) = \frac{1}{2}\sigma^2 e^{-sA}, \quad s < 0.$$

In the general case define the new variables  $t'_i = t - t_i, i = 1, 2$ . Then the integral can be rewritten as

$$\int_0^t dt'_1 \exp\{t'_1 A\} D^{(2)} \int_{-s}^t dt'_2 \exp\{t'_2 A^T\} C_f(t'_1 - t'_2) \exp\{sA^T\}.$$

Now we see that the limit  $t \rightarrow \infty$  exists (first assertion of Theorem 3.1) and we can write it, using Lemma 2.2, as

$$\int_0^\infty dt'_1 \exp\{t'_1 A\} D^{(2)} \int_{-s}^\infty dt'_2 \exp\{t'_2 A^T\} C_f(t'_1 - t'_2) \exp\{sA^T\}.$$

First consider the case  $s = 0$ . We integrate over the quarter plane  $t'_1 \geq 0, t'_2 \geq 0$ . Put  $t'_1 = t'_2 + \tau$ . Consider two cones,  $\tau > 0$  and  $\tau < 0$ . Integration over the first (lower one), gives

$$\begin{aligned} & \int_{\tau > 0} d\tau \int_{\tau}^\infty dt'_1 \exp\{t'_1 A\} D^{(2)} \exp\{t'_1 A^T\} \exp\{-\tau A^T\} C_f(\tau) \\ &= \int_{\tau > 0} e^{\tau A} C_G e^{\tau A^T} e^{-\tau A^T} C_f(\tau) d\tau \\ &= \int_{\tau > 0} e^{\tau A} C_G C_f(\tau) d\tau. \end{aligned}$$

Symmetrically, integration over the upper angle gives

$$\int_{\tau > 0} C_G e^{\tau A^T} C_f(\tau) d\tau.$$

The case  $s > 0$  is similar. We have

$$\int_0^\infty dt'_1 \exp\{t'_1 A\} D^{(2)} \int_{-s}^\infty dt'_2 \exp\{t'_2 A^T\} C_f(t'_1 - t'_2) \exp\{sA^T\}.$$

We integrate over the quarter plane  $t'_1 \geq 0, t'_2 \geq -s$ . Put  $t'_1 = t'_2 + \tau$ . The domain of integration  $(\tau, t'_1)$  consists of two non-intersecting subdomains: the

first one is a “shifted” quarter-plane  $\Omega_1 = \{(\tau, t'_1) : \tau < s, t'_1 > 0\}$ , the second is the cone  $\Omega_2 = \{(\tau, t'_1) : \tau > s, t'_1 > \tau - s\}$ . For the integral over  $\Omega_1$  we have

$$\begin{aligned} & \int_{\tau < s} d\tau \int_0^{\infty} dt'_1 \exp\{t'_1 A\} D^{(2)} \exp\{t'_1 A^T\} \exp\{-\tau A^T\} C_f(\tau) \exp\{s A^T\} \\ &= \int_{\tau < s} C_G \exp\{-\tau A^T\} C_f(\tau) d\tau \exp\{s A^T\} \\ &= C_G \int_{\tau < s} e^{-\tau A^T} C_f(\tau) d\tau e^{s A^T}. \end{aligned}$$

Changing variables  $\tau' = s - \tau$  we have

$$\int_{\tau < s} e^{-\tau A^T} C_f(\tau) d\tau e^{s A^T} = \int_0^{+\infty} e^{\tau' A^T} C_f(\tau' - s) d\tau' = W^T(-s).$$

The integral over the cone gives

$$\begin{aligned} & \int_{\tau > s} d\tau \int_{\tau-s}^{+\infty} dt'_1 \exp\{t'_1 A\} D^{(2)} \exp\{t'_1 A^T\} \exp\{-\tau A^T\} C_f(\tau) e^{s A^T} \\ &= \int_{\tau > s} e^{(\tau-s)A} C_G e^{(\tau-s)A^T} e^{-\tau A^T} C_f(\tau) d\tau e^{s A^T} \\ &= \int_{\tau > s} e^{(\tau-s)A} C_G C_f(\tau) d\tau \\ &= \int_{\tau > 0} e^{\tau A} C_f(\tau + s) d\tau C_G = W(s)C_G. \end{aligned}$$

## 5. Proof of Theorem 3.2

We will need another expression for  $C_\psi$  in terms of the spectral density of the process  $f_t$  and the resolvent of  $A$ :

$$R_A(z) = (A - z)^{-1}.$$

**Lemma 5.1.** *Fix any  $C_f \in S$ . Then for almost any  $H \in \mathbf{H}_\Gamma$  the following assertion holds:*

$$C_\psi = - \int_{-\infty}^{+\infty} a(\lambda) (R_A(i\lambda)C_G + C_G R_A^T(i\lambda)) d\lambda; \quad (5.1)$$

To prove this, we just express  $W$  in terms of the spectral density  $a(\lambda)$  and the resolvent of  $A$ ,

$$\begin{aligned} W &= \int_0^{+\infty} C_f(s)e^{sA} ds = \int_{-\infty}^{+\infty} d\lambda \int_0^{+\infty} ds a(\lambda)e^{is\lambda}e^{sA} \\ &= - \int_{-\infty}^{+\infty} a(\lambda)(A + i\lambda)^{-1} d\lambda = - \int_{-\infty}^{+\infty} a(\lambda)R_A(i\lambda)d\lambda, \end{aligned}$$

where the symmetry of the spectral density  $a(\lambda) = a(-\lambda)$  is used.

Explicit expressions for the matrix elements  $C_\psi(q_i, p_j)$  of  $C_{\psi(\infty)}(0)$  seem to be ugly. Instead we will write the matrix  $C_\psi$  in the two-block form. For example,

$$R_A(z)C_G = \frac{1}{2\alpha} \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}, \tag{5.2}$$

where the  $(N \times N)$ -blocks  $Q_{11}, Q_{22}, Q_{12} + Q_{21}^T$  give, after integration, the matrix elements  $C_\psi(q_i, q_j), C_\psi(p_i, p_j), C_\psi(q_i, p_j)$  correspondingly.

To get explicit expression for  $Q_{ij}$  we need some notation. Define the following  $N \times N$  rational matrices:

$$\rho(z) = (V + z^2)^{-1}, \quad \theta(z) = \rho(z)T(z)\rho(z), \quad T(z) = \alpha\hat{e}\tau^{-1}(z)\hat{e}^T,$$

where

$$\tau(z) = E + \alpha z\kappa(z)$$

is  $m \times m$ -matrix,  $E = E^{(m)}$  is the unit  $m \times m$ -matrix,  $\hat{e}$  is the  $(N \times m)$ -matrix with the only non-zero entries  $\hat{e}_{N-m+i, i} = 1, i = 1, \dots, m$ , and

$$\kappa(z) = (\rho(z)_{i,j})_{i,j=N-m+1, \dots, N}$$

is the restriction of  $\rho$  on  $\Lambda^{(m)}$ . It is clear that  $\kappa(z) = \hat{e}^T \rho(z) \hat{e}$ .

**Lemma 5.2.** *The block matrices  $Q_{ij}$  are given by*

$$\begin{aligned} Q_{11} &= -z\rho(z)V^{-1} - \theta(z), & Q_{12} &= -\rho(z) + z\theta(z), \\ Q_{21} &= zQ_{11} + V^{-1}, & Q_{22} &= zQ_{12}. \end{aligned} \tag{5.3}$$

*Proof.* Multiplying both sides of (5.2) by  $A - zE$ , we get 4 equations for  $(N \times N)$ -matrices:

$$V^{-1} = Q_{21} - zQ_{11}, \tag{5.4}$$

$$0 = VQ_{11} + (\alpha D + zE)Q_{21}, \tag{5.5}$$

$$0 = Q_{22} - zQ_{12}, \tag{5.6}$$

$$-E = VQ_{12} + (\alpha D + zE)Q_{22}. \tag{5.7}$$

It is clear that (5.4) and (5.6) are equivalent to the first and second equalities of (5.3) correspondingly. Note also the following simple equality

$$z\alpha D\rho T = \alpha^2 z\hat{e} (\hat{e}^T \rho \hat{e}) \tau^{-1} \hat{e}^T = \alpha^2 z\hat{e}\kappa(z)\tau^{-1} \hat{e}^T = \alpha\hat{e}(\tau - E)\tau^{-1} \hat{e}^T = \alpha D - T, \quad (5.8)$$

where  $D$  is the diagonal  $(N \times N)$ -matrix introduced above as the projection onto the subspace generated by the vectors  $e_{N-m+1}, \dots, e_N$ . We get (5.7), expressing  $Q_{22}$  through  $Q_{12}$ , by using the second equality of (5.3),

$$\begin{aligned} VQ_{12} + (\alpha D + zE)Q_{22} &= (V + z^2)Q_{12} + z\alpha DQ_{12} \\ &= -E + zT\rho - z\alpha D\rho + z(z\alpha D\rho T)\rho \\ &= -E + zT\rho - z\alpha D\rho + z(\alpha D - T)\rho = -E. \end{aligned}$$

Thus, we have proved (5.7). Note that the following equality holds

$$z^2\rho V^{-1} = (z^2 + V)\rho V^{-1} - V\rho V^{-1} = V^{-1} - \rho.$$

Similarly, we check (5.5), expressing  $Q_{21}$  through  $Q_{11}$ , by using (5.3):

$$\begin{aligned} VQ_{11} + (\alpha D + zE)Q_{21} &= (V + z^2)Q_{11} + z\alpha DQ_{11} + (\alpha D + zE)V^{-1} \\ &= -zV^{-1} - (V + z^2)\theta + z\alpha D(-z\rho V^{-1} - \theta) \\ &\quad + (\alpha D + zE)V^{-1} \\ &= -(V + z^2)\theta - z^2\alpha D\rho V^{-1} - z\alpha D\theta + \alpha DV^{-1} \\ &= -(V + z^2)\theta - \alpha D(V^{-1} - \rho) - z\alpha D\theta + \alpha DV^{-1} \\ &= -(V + z^2)\theta + \alpha D\rho - z\alpha D\theta \\ &= -T\rho + \alpha D\rho - (z\alpha D\rho T)\rho \\ &= -T\rho + \alpha D\rho - (\alpha D - T)\rho = 0. \end{aligned}$$

Lemma 5.2 is proved. □

Now we will prove Theorem 3.2. To prove the first part it is sufficient to take the sum of  $R_A(z)C_G$  and its transposition, that is, to verify that  $Q_{12} + Q_{21} = 0$ . But it is a simple calculation using Lemma 5.2.

As for the second part of Theorem 3.2, we should show that the  $(N \times N)$ -matrix equation for lower diagonal block

$$-\frac{1}{\alpha} \int_{-\infty}^{+\infty} a(\lambda) Q_{22}(i\lambda) d\lambda = E \quad (5.9)$$

is rarely fulfilled. One can see that the matrix elements of  $Q_{22}$  are bounded since the matrix elements of the resolvent  $R_A(i\lambda)$  are bounded, and moreover

have no poles by Lemma 2.2. This is not clear from the explicit expression

$$\begin{aligned} Q_{22}(z) &= -z\rho(z) + z^2\rho(z)T(z)\rho(z) \\ &= z\rho(z)(-E + zT(z)\rho(z)) \\ &= z\rho(z)(-1 + \alpha z\hat{e}(E + \alpha z\kappa(z))^{-1}\hat{e}^T\rho(z)). \end{aligned}$$

Equation (5.9) is equivalent to  $N^2$  equations with respect to the function  $a(\lambda)$ , given  $V$ . Each of these equations is of the type

$$\int_{-\infty}^{\infty} a(\lambda)\varphi(V(i, j), \lambda)d\lambda = 0 \tag{5.10}$$

for some bounded function  $\varphi$ . The set of solutions is a closed subset of the Schwartz space and in any small neighborhood of any solution (of at least one of the equations) there is an open set of points which do not satisfy this equation. Thus, the complement of the set of solutions of (5.9) is an everywhere dense subset. It is an open subset because if some  $a(\lambda)$  does not satisfy the equation then its small neighborhood also does not.

**6. Large  $N$  — proof of Theorem 3.3**

We will find now the main term of  $C_\psi$  for large  $N$ . Decompose the matrix  $A$  as follows:

$$\begin{aligned} A &= A_V + A_D, \\ A_V &= \begin{pmatrix} 0 & E \\ -V & 0 \end{pmatrix}, \\ A_D &= \begin{pmatrix} 0 & 0 \\ 0 & -\alpha D \end{pmatrix} \end{aligned}$$

and use the formula

$$\exp\{t(A_V + A_D)\} = \exp\{tA_V\} + Y_t, \tag{6.1}$$

where

$$Y_t = \int_0^t \exp\{(t-s)A_V\}A_D \exp\{sA\} ds. \tag{6.2}$$

Then by Theorem 3.1 for any  $b > 0$  we can write

$$C_\psi = W(0)C_G + C_GW(0)^T = C_V + Y_{V,b} + Y_{V,\infty}, \tag{6.3}$$

where

$$\begin{aligned}
 C_V &= \int_0^{+\infty} C_f(s) \exp\{sA_V\} ds C_G + C_G \int_0^{+\infty} C_f(s) \exp\{sA_V^T\} ds, \\
 Y_{V,b} &= \int_0^b C_f(s) Y_s ds C_G + C_G \int_0^b C_f(s) Y_s^T ds, \\
 Y_{V,\infty} &= \int_b^\infty C_f(s) Y_s ds C_G + C_G \int_b^\infty C_f(s) Y_s^T ds.
 \end{aligned}$$

First we will find  $C_V$ .

**Lemma 6.1.** *We have*

$$C_V = \frac{\pi}{\alpha} \begin{pmatrix} a(\sqrt{V})V^{-1} & 0 \\ 0 & a(\sqrt{V}) \end{pmatrix},$$

*Proof.* Using the formula (see, for example [5], section II.3)

$$\exp(tA_V) = \begin{pmatrix} \cos(\sqrt{V}t) & (\sqrt{V})^{-1} \sin(\sqrt{V}t) \\ -\sqrt{V} \sin(\sqrt{V}t) & \cos(\sqrt{V}t) \end{pmatrix} \tag{6.4}$$

one can get

$$C_V = \frac{1}{\alpha} \int_0^{+\infty} C_f(s) \begin{pmatrix} V^{-1} \cos(\sqrt{V}s) & 0 \\ 0 & \cos(\sqrt{V}s) \end{pmatrix} ds.$$

Let  $dE_\lambda$  be the spectral presentation for  $V$ , then

$$V = \int \lambda dE_\lambda, \quad \cos(\sqrt{V}s) = \int_{-\infty}^{+\infty} \cos(\sqrt{\lambda}s) dE_\lambda,$$

where the integral is taken only over positive half-axis because of the spectrum of  $V$ . Thus

$$\begin{aligned}
 \int_0^{+\infty} C_f(s) \cos(\sqrt{V}s) ds &= \int_{-\infty}^{+\infty} \left( \int_0^{+\infty} C_f(s) \cos(\sqrt{\lambda}s) ds \right) dE_\lambda \\
 &= \pi \int_{-\infty}^{+\infty} a(\sqrt{\lambda}) dE_\lambda = \pi a(\sqrt{V}).
 \end{aligned}$$

Lemma 6.1 is proved. □



Now we will prove Theorem 3.3 for the case when  $a(\lambda)$  has bounded support  $[-b, b]$ . Let us estimate matrix elements of

$$Y_{V,b} = \int_0^b C_f(s) Y_s ds C_G + C_G \int_0^b C_f(s) Y_s^T ds,$$

where

$$Y_t = \int_0^t e^{(t-s)A_V} A_D e^{sA} ds.$$

Denote  $U_{i,j}(s, t) = (\exp\{(t-s)A_V\} A_D \exp\{sA\} C_G)(p_i, p_j)$ . Then

$$U_{i,j}(s, t) = \sum_{k_1, k_2, k_3} \sum_{x_{k_1}, x_{k_2}, x_{k_3}} e^{(t-s)A_V}(p_i, x_{k_1}) A_D(x_{k_1}, x_{k_2}) \times e^{sA}(x_{k_2}, x_{k_3}) C_G(x_{k_3}, p_j),$$

where  $x_k$  can be either  $q_k$  or  $p_k$ . It is clear that the terms of this sum can be non-zero only if  $x_{k_3} = p_j$  and  $x_{k_1} = x_{k_2} = p_k$ , where  $k \in \Lambda^{(m)}$ . Thus

$$U_{i,j}(s, t) = -\frac{1}{2} \sum_{k \in \Lambda^{(m)}} \exp\{(t-s)A_V\}(p_i, p_k) \exp\{sA\}(p_k, p_j). \quad (6.5)$$

**Lemma 6.2.** For any  $k \in \Lambda^{(m)}$  we have

$$|e^{(t-s)A_V}(p_i, p_k)| \leq \frac{(\sqrt{B}(t-s))^{r(i)}}{r(i)!} e^{\sqrt{B}(t-s)}$$

where  $r(i) = 2[\gamma^{-1}r(i, \Lambda^{(m)})]$  is an integer.

*Proof.* We have

$$e^{(t-s)A_V}(p_i, p_k) = \cos(\sqrt{V}(t-s))(i, k) = \sum_{n=0}^{\infty} (-1)^n \frac{(t-s)^{2n}}{(2n)!} (V^n)(i, k). \quad (6.6)$$

By locality of the hamiltonian  $V$  we have  $V^n(i, k) = 0$  if  $n < \gamma^{-1}r(i, \Lambda^{(m)})$ . Then (6.6) can be estimated as follows:

$$\begin{aligned} |e^{(t-s)A_V}(p_i, p_k)| &= \left| \sum_{n=[\gamma^{-1}r(i, \Lambda^{(m)})]}^{\infty} (-1)^n \frac{(t-s)^{2n}}{(2n)!} (V^n)(i, k) \right| \\ &\leq \sum_{n=[\gamma^{-1}r(i, \Lambda^{(m)})]}^{\infty} \frac{(t-s)^{2n}}{(2n)!} B^n \\ &\leq \frac{(\sqrt{B}(t-s))^{r(i)}}{(r(i))!} e^{\sqrt{B}(t-s)}. \end{aligned}$$

□

**Lemma 6.3.** For any  $k \in \Lambda^{(m)}$ ,

$$|e^{sA}(p_k, p_j)| \leq \frac{(cs)^{r(j)}}{r(j)!} e^{cs},$$

where  $r(j) = 2[\gamma^{-1}r(j, \Lambda^{(m)})]$  and  $c = B + \alpha$ .

*Proof.* Consider the following expansion:

$$e^{sA} = \sum_{n=0}^{\infty} \frac{s^n}{n!} A^n.$$

We have

$$|A^n(p_k, p_j)| \leq \|A^n\| \leq \|A\|^n \leq c^n,$$

where  $c = B + \alpha$ . Moreover, let us prove that  $A^n(p_k, p_j) = 0$  for any  $n$  such that

$$n < 2\gamma^{-1}r(j, \Lambda^{(m)}).$$

It is easy to see that  $A^n(p_k, p_j) = (A_V + A_D)^n(p_k, p_j)$  is the sum of the terms

$$(-\alpha)^{u_0} A_V^{u_1}(p_{k_1}, p_{k_2}) A_V^{u_2}(p_{k_1}, p_{k_2}) \dots A_V^{u_q}(p_{k_q}, p_j), \quad k_1 = k,$$

where  $k_1, \dots, k_q \in \Lambda^{(m)}$ ,  $u_0 + u_1 + \dots + u_q = n$  and  $u_l \geq 0$  for all  $l = 0, 1, \dots, q$ . For the last factor we get, using (6.6),

$$A_V^{u_q}(p_{k_q}, p_j) = \begin{cases} (-1)^u V^u(k_q, j), & u_q = 2u, \\ 0, & \text{otherwise.} \end{cases}$$

By locality of  $V$  we get that  $A_V^{u_q}(p_{k_q}, p_j) = 0$  if  $u < \gamma^{-1}r(j, \Lambda^{(m)})$ . Since  $n \geq 2u$ , we have that  $A^n(p_k, p_j) = 0$  for all  $n < 2\gamma^{-1}r(j, \Lambda^{(m)})$ . Then

$$|e^{sA}(p_k, p_j)| \leq \sum_{n=2[\gamma^{-1}r(j, \Lambda^{(m)})]}^{\infty} \frac{s^n}{n!} c^n \leq \frac{(cs)^{r(j)}}{r(j)!} e^{cs}.$$

□

Returning to (6.5) we have

$$\begin{aligned} |U_{i,j}(s, t)| &\leq \frac{1}{2} |\Lambda^{(m)}| \frac{(B'(t-s))^{r(i)} (cs)^{r(j)}}{r(i)! r(j)!} e^{B'(t-s)+cs} \\ &\leq \frac{1}{2} |\Lambda^{(m)}| \frac{c_1^{r(i)+r(j)}}{r(i)! r(j)!} (t-s)^{r(i)} s^{r(j)} e^{c_1 t}, \end{aligned}$$

where  $c_1 = \sqrt{B} + c$ . For the integral,

$$\begin{aligned} |(Y_i C_G)(p_i, p_j)| &\leq \frac{1}{2} |\Lambda^{(m)}| \frac{c_1^{r(i)+r(j)}}{r(i)!r(j)!} \exp\{c_1 t\} \int_0^t (t-s)^{r(i)} s^{r(j)} ds \\ &= \frac{1}{2} |\Lambda^{(m)}| \frac{c_1^{r(i)+r(j)}}{(r(i) + r(j) + 1)!} \exp\{c_1 t\} t^{r(i)+r(j)+1} \\ &= \frac{1}{2c_1} |\Lambda^{(m)}| \frac{(c_1 t)^{r(i)+r(j)+1}}{(r(i) + r(j) + 1)!} e^{c_1 t}, \end{aligned}$$

and finally,

$$|Y_{V,b}(p_i, p_j)| \leq |\Lambda^{(m)}| K_0 \frac{(c_1 b)^{r(i)+r(j)+1}}{(r(i) + r(j) + 1)!} \leq |\Lambda^{(m)}| K_0 \left(\frac{K}{\eta}\right)^{\gamma^{-1}\eta}$$

where

$$K = c_1 b \gamma e, \quad K_0 = \frac{1}{c_1} C_f(0) \exp\{2c_1 b\}. \tag{6.7}$$

For  $Y_{V,b}(q_i, q_j)$  the proof and the estimates are quite similar and we omit the proof. The constant  $K$  is the same as in (6.7) and the new constant  $K_0$  is

$$\tilde{K}_0 = \frac{1}{\sqrt{B}c_1} C_f(0) \exp\{2c_1 \omega\}.$$

For an arbitrary  $C_f \in S$  the proof is as follows. Put  $b = \sqrt{\eta}$  and estimate the integral over  $(0, b)$  as above. Then we get:

$$\begin{aligned} K_0 &= \frac{1}{c_1} C_f(0) \exp\{2c_1 \sqrt{\eta}\} \leq \frac{1}{c_1} C_f(0) (\exp\{2c_1 \gamma\})^{\gamma^{-1}\eta}, \\ |Y_{V,\omega}(p_i, p_j)| &\leq |\Lambda^{(m)}| \frac{1}{c_1} C_f(0) \left(\frac{c_1 \gamma \exp\{2c_1 \gamma + 1\}}{\sqrt{\eta}}\right)^{\gamma^{-1}\eta}, \\ |Y_{V,\omega}(q_i, q_j)| &\leq |\Lambda^{(m)}| \frac{1}{\sqrt{B}c_1} C_f(0) \left(\frac{c_1 \gamma \exp\{2c_1 \gamma + 1\}}{\sqrt{\eta}}\right)^{\gamma^{-1}\eta}. \end{aligned}$$

Then it is easy to see that for all  $k = 0, 1, 2, \dots$  there exists a constant  $C = C(c_1, B, \gamma, k)$  such that for any  $\eta > 0$  we have

$$|Y_{V,\omega}(p_i, p_j)| \leq |\Lambda^{(m)}| C \eta^{-k}, \quad |Y_{V,\omega}(q_i, q_j)| \leq |\Lambda^{(m)}| C \eta^{-k}.$$

To estimate the integral over  $(b, \infty)$ , we need the following lemma.

**Lemma 6.4.** *For any  $i, j \in \Lambda$  the following inequalities hold:*

$$\begin{aligned} |e^{tA}(p_i, p_j)| &\leq 1, \quad |e^{tA}(q_i, p_j)| \leq t, \\ |e^{tA_V}(p_i, p_j)| &\leq 1, \quad \alpha |e^{tA_V} C_G(p_i, q_j)| \leq \frac{t}{2}. \end{aligned}$$

*Proof.* Denote  $\exp\{tA\}g_j = (q(t), p(t))^T$ , where  $p(t) = (p_1(t), \dots, p_N(t))^T$ ,  $q(t) = (q_1(t), \dots, q_N(t))^T$  and the "initial" vector  $g_j = (0, e_j)^T \in l^{(p)}$ . From the definition of the matrix exponent we have:

$$\begin{aligned} e^{tA}(p_i, p_j) &= p_i(t), \\ e^{tA}(q_i, p_j) &= \int_0^t p_i(s) ds. \end{aligned}$$

Therefore the bound for  $e^{tA}(q_i, p_j)$  in the lemma follows from the inequality  $|e^{tA}(p_i, p_j)| \leq 1$ . Since the energy along  $e^{tA}g_j$  cannot increase, we get:

$$\begin{aligned} |e^{tA}(p_i, p_j)|^2 &= p_i^2(t) \leq 2H((q(t), p(t))^T) \\ &\leq 2H((q(0), p(0))^T) = 1. \end{aligned} \quad (6.8)$$

Thus, the inequalities for the matrix elements of  $e^{tA}$  have been proven. The estimate for  $|e^{tAv}(p_i, p_j)|$  is obtained similarly to (6.8). Let us check the last inequality. From (6.4) we have:

$$e^{tAv}C_G = \frac{1}{2\alpha} \begin{pmatrix} V^{-1} \cos(tR) & R^{-1} \sin(tR) \\ -R^{-1} \sin(tR) & \cos(tR) \end{pmatrix},$$

where we have put  $R = \sqrt{V}$ . Thus,

$$\alpha(e^{tAv}C_G)(p_i, q_j) = -\frac{1}{2}(R^{-1} \sin(tR))(i, j) \quad (6.9)$$

and

$$\alpha|(e^{tAv}C_G)(p_i, q_j)| \leq \frac{1}{2}\|R^{-1} \sin(tR)\|_2. \quad (6.10)$$

Since  $R$  is selfadjoint,

$$\|R^{-1} \sin(tR)\|_2 = \max\left\{\left|\frac{\sin t\lambda}{\lambda}\right| : \lambda \in \sigma(R)\right\}.$$

For any  $\lambda \in \mathbb{R}$  and  $t \geq 0$  we have

$$\left|\frac{\sin t\lambda}{\lambda}\right| \leq t,$$

and it follows that  $\|R^{-1} \sin(tR)\|_2 \leq t$ . Applying this estimate for the norm to (6.10) we obtain the final estimate for  $\alpha|(e^{tAv}C_G)(p_i, q_j)|$ . Lemma 6.4 is proved.  $\square$

From Lemma 6.4 and (6.5) we get the estimate

$$|U_{ij}(s, t)| \leq \frac{1}{2} |\Lambda^{(m)}|$$

for any  $0 \leq s \leq t$  and any  $i, j$ . Then

$$|Y_{V, \infty}(p_i, p_j)| \leq |\Lambda^{(m)}| \int_{\sqrt{\eta}}^{+\infty} s |C_f(s)| ds.$$

By definition of the space  $S$ , it is clear from the last inequality that for any  $k > 0$ , the following inequality holds:

$$Y_{V, \infty}(p_i, p_j) \leq |\Lambda^{(m)}| C(k) \eta^{-k}, \quad C(k) = C(k, \alpha, B, \gamma).$$

The estimate for coordinates can be proved similarly. Theorem 3.3 is proved.

*Proof of Corollary 3.2.* In what follows we write  $V_n$  for  $V'_n$ . From Theorem 3.3 it follows that

$$\lim_{n \rightarrow \infty} |C_\xi^{(n)}(q_i, q_j) - C_{V_n}(q_i, q_j)| = 0, \tag{6.11}$$

$$\lim_{n \rightarrow \infty} |C_\xi^{(n)}(p_i, p_j) - C_{V_n}(p_i, p_j)| = 0. \tag{6.12}$$

Thus it is sufficient to show that the elements of the matrix  $C_{V_n}$  have finite limits as  $n \rightarrow \infty$ .

**Lemma 6.5.** *Let  $P$  be an arbitrary polynomial of degree  $d$ , then for any  $i, j \in \Gamma_\infty$ ,*

$$\lim_{n \rightarrow \infty} P(V_n)(i, j) = P(V)(i, j).$$

*Proof.* For  $d = 2$ ,

$$|V^2(i, j) - V_n^2(i, j)| = \left| \sum_{k \notin \Lambda_n} V(i, k) V(k, j) \right| \leq \|V\|_\infty \sum_{k \notin \Lambda_n} |V(i, k)| \rightarrow 0$$

as  $n \rightarrow \infty$ . For an arbitrary degree the proof is similar. □

**Lemma 6.6.** *For any  $i, j \in \Lambda_\infty$  there exists the finite limit*

$$\lim_{n \rightarrow \infty} a(\sqrt{V_n})(i, j) = C_\xi^{(\infty), p}(i, j).$$

*Proof.* The function  $a(\sqrt{x})$  is continuous on the segment  $[0, B]$ , hence there exists a sequence of real polynomials  $P_k(x)$ ,  $k = 1, 2, \dots$  uniformly converging

to  $a(\sqrt{x})$  on  $[0, B]$  as  $k \rightarrow \infty$ . Note that the spectrum of  $V_n$  belongs to  $[0, B]$  for any  $n = 1, 2, \dots$ . Then the following inequalities hold:

$$|P_k(V_n)(i, j) - a(\sqrt{V_n})(i, j)| \leq \|P_k(V_n) - a(\sqrt{V_n})\|_2 \leq \sup_{x \in [0, B]} |P_k(x) - a(\sqrt{x})|, \quad (6.13)$$

the second one follows from the spectral mapping theorem ([7, p. 569]). From (6.13) it follows that  $P_k(\sqrt{V_n})(i, j) \rightarrow a(\sqrt{V_n})(i, j)$  as  $k \rightarrow \infty$ , uniformly in  $n = 1, 2, \dots$ . Then by Lemma 6.5 we have the assertion of Lemma 6.6.  $\square$

Returning to the proof of Corollary 3.2 we see that the first item follows immediately from the equality (6.12) and Lemma 6.6. To prove the second assertion of Corollary 3.2, we use equality (6.11). Rewrite the elements  $C_{V_n}(q_i, q_j)$  as follows:

$$\begin{aligned} C_{V_n}(q_i, q_j) &= \left( a(\sqrt{V_n})V_n^{-1} \right) (i, j) \\ &= \left( \left( a(\sqrt{V_n}) - a(0) \right) V_n^{-1} \right) (i, j) + a(0)V_n^{-1}(i, j) \\ &= f(V_n)(i, j) + a(0)V_n^{-1}(i, j), \end{aligned}$$

where we introduced the function  $f(x) = (a(\sqrt{x}) - a(0))x^{-1}$ . The spectral density  $a(x)$  is even, hence  $f(x)$  is continuous on  $\mathbb{R}_{\geq 0}$ . The arguments, similar to those in the proof of Lemma 6.6, show that for any  $i, j \in \Lambda_\infty$  there exists the limit

$$\lim_{n \rightarrow \infty} f(V_n)(i, j).$$

Since  $V_n^{-1}(i, j) \rightarrow U(i, j)$  as  $n \rightarrow \infty$ , the first two assertions of Corollary 3.2 are proved.

The proof of the last assertion is similar to the proof of Lemma 6.6 and Lemma 13 in [7, p. 571], if applied to the sequence  $P_k(V)$ .  $\square$

## 7. Comments

1. For concrete (even simply looking)  $V$  it may be rather difficult to find  $\dim L_0$ , and moreover, mostly it is not 0. As an example we mention the one-dimensional harmonic chain

$$\sum_{i=-N}^N \omega_0 q_i^2 + \omega_1 \sum_{i=-N}^{N-1} (q_i - q_{i+1})^2, \quad \omega_0, \omega_1 > 0,$$

where the calculation of  $\dim L_0$  leads to problems in number theory. However, this dimension in most cases is much less than the dimension of  $L$  itself (more exactly, is  $o(N)$ ), see [6]. However, one can always use instability of the integer  $\dim L_0$ : even a smallest generic perturbation of  $V$  leads to the desired zero dimension effect.

2. All questions concerning the alternative Gibbs–non-Gibbs lead to equations of the type (5.10). In Theorem 3.2 we considered (5.10) as equation for  $a(\lambda)$  with given  $V$ . However, one can ask also the question dual to Theorem 3.22. Namely, fix an arbitrary  $a(\lambda) \in S$ . Is it true that for almost any  $H \in \mathbf{H}_G$  there is a pair  $i \neq j$  such that  $C_\psi(p_i, p_j) \neq 0$ ? It is more or less clear that the answer will be ‘yes’. We do not prove it carefully here. For example, consider the famous Ornstein–Uhlenbeck process with the spectral density

$$a(\lambda) = \frac{c}{\mu^2 + \lambda^2}, \quad (7.1)$$

so that the limiting covariance has inter-velocity correlations for a class of  $V_n$  with  $L_0 = \emptyset$ . It is easy to get such examples. Assume that in (7.1)  $\mu$  is sufficiently large. Put  $V = 1 + V_1$  where  $V_1$  has sufficiently small  $l_\infty$ -norm, then

$$a(\sqrt{V}) = \frac{c}{\mu^2 + V} = \frac{c}{\mu^2} \left( 1 - \frac{1}{\mu^2} V_1 + o\left(\frac{1}{\mu^2}\right) \right)$$

and the linear in  $V_1$  term provides non-zero correlations  $\langle p_i p_j \rangle, i \neq j$ , if  $V_1(i, j) \neq 0$ .

3. As a rare exception, one can construct, using (5.10), even for  $N = 1, 2$ , examples of  $H \in \mathbf{H}_G$  and  $C_f \in S$  with Gibbs limiting distribution. We do not know whether such kind of examples have physical sense.
4. We did not consider here other generalized processes with independent values — derivatives of the white noise and of the (non-gaussian) Levy processes. It is an open question what limiting distribution will be for these “no-memory” cases. It seems that the white noise is the only stationary gaussian process, providing convergence of the system to Gibbs states for almost any  $V$ .

## References

- [1] Z. RIEDER, J. LEBOWITZ AND E. LIEB (1967) Properties of a harmonic crystal in a stationary nonequilibrium state. *J. Math. Phys.* **8** (5), 1073–1078.
- [2] H. SPOHN AND J. LEBOWITZ (1977) Stationary non-equilibrium states of infinite harmonic systems. *Commun. Math. Phys.* **54**, 97–120.
- [3] F. BONETTO, J. LEBOWITZ AND J. LUKKARINEN (2004) Fourier’s law for a harmonic crystal with self-consistent stochastic reservoirs. *J. Stat. Phys.* **116** (1/4), 783–813.
- [4] A.D. WENTZELL (A.D. VENTSEL) (1981) *A Course in the Theory of Stochastic Processes*. McGraw-Hill.
- [5] YU. L. DALETSKIJ AND M.G. KREJN (1970) Stability of solutions of differential equations in Banach space. Nauka, Moscow.

- [6] A.A. LYKOV AND V.A. MALYSHEV (2012) Harmonic chain with weak dissipation. *Markov Processes and Relat. Fields* **18** (4), 721–729.
- [7] N. DUNFORD AND J. SCHWARTZ (1958) *Linear Operators. Part 1*. Interscience.
- [8] A.A. LYKOV, V.A. MALYSHEV AND S.A. MUZYCHKA (2012) Linear hamiltonian systems under microscopic random influence. *Theory of Probab. and its Appl.* **57** (4), 794–799.