ADVANCES IN SOVIET MATHEMATICS
Volume 5, 1991

Asymptotic Completeness and All That for an Infinite Number of Fermions

D. D. BOTVICH AND V. A. MALYSHEV

Introduction

The mathematical statistical physics of equilibrium systems is now a comprehensive science with a variety of ramifications, methods, and results (see [MM4], [GJ], and [R]).

The situation in the study of dynamics of near-equilibrium systems is already radically different. Until recently, the investigation of dynamics was pursued in the following three directions:

(1) general results concerning the existence of dynamics of both equilibrium and nonequilibrium type [Ro1], [Ro2], [T], [GJ], [DF];

(2) the investigation of free systems [SuS2], [Ar1], [Ar2] or those that may be reduced to them by simple transformations (like the hard rod model and the XY-model) [SuS1], [ArW]; in such cases all quantities may be computed explicitly, and the system can be fully controlled;

(3) the investigation of the spectrum in the ground state for the massive case [GJ], [MM3].

Here we should also note the axiomatic results [BF] about massless theories for the ground state.

Lately, papers on locally perturbed free classical systems [PSS], [MNT], [Te] have appeared. By a local perturbation we mean either the case of an infinite-particle system with interaction inside a bounded region and free motion of particles outside the region, or the case of a trial particle interacting with other particles, introduced into a free system of particles. Thus the other particles interact as the result of its presence, and we may assume that the region of interaction moves through space together with the particle. Although there exist results concerning asymptotic completeness even in the case of translation-invariance [BM2], there is still a lot of ground to cover.

In this survey, we shall concentrate on a vital part of this problem, where the situation is understood in the main and well in hand.

1991 *Mathematics Subject Classification*. Primary 82C10; Secondary 81T18.

First of all, our technique consists in using perturbation theory series similar to Dyson-Schwinger series as well as their formal generalizations as given in the classical papers by Friedrichs [Fr] and Hepp [H]. The key restrictions that have been imposed in the proofs of convergence of such series [BM1], [M2] are the following:

(1) the interaction is small and is restricted to a bounded region or vanishes sufficiently fast at infinity (i.e., there exists a spatial cut-off);

(2) Fermi systems are treated because their creation-annihilation operators are bounded;

(3) the system has no ultraviolet divergences.

Due to the above restrictions, the perturbation is a small bounded operator.

The main unsolved problem consists in the transition to a translation-invariant interaction, which implies no spatial cut-off. In our opinion, this is the only practical way to prove asymptotic completeness.

Until the paper [BM1], there did not exist a single proof of asymptotic completeness, but there were other weaker results in some other papers [Fr], [R1], [R2], [H1]–[H6], [Ch], [Ro3], [BR2], [E1], where a scattering theory was constructed in Fock space and in C^*-algebras for different theories of this type. However, no proof of asymptotic completeness was obtained. The asymptotic completeness in the temperature state, with no vacuum polarization for the ground state, follows from [BM1]. By means of a standard technique of scattering theory, completeness was proved in [Ai] for the case of a mass slit with vacuum polarization. The convergence of the expansions in the linked cluster theorem (see the classical books [Fr], [H]) was proved in [M2]. In particular, this leads to asymptotic completeness in the case of vacuum polarization without mass slit. By means of similar methods, this result was later transferred by Aizenstadt to the case of an arbitrary chemical potential with parity interaction.

Chapter 1 is devoted to the so-called *Fock spectral representation* of free dynamics in the temperature state, i.e., the representation in the form $\Gamma(e^{it(h\oplus h^*)})$, where h and h^* are one-particle Hamiltonians for the particle and antiparticle. The techniques related to such representations involve Wick brackets with respect to the corresponding state.

In Chapter 2, by following [Ro3], [E1], we first show how a scattering theory in C^*-algebras is constructed. In §2.3, asymptotic completeness in the C^*-algebra CAR, similar to the one in [BM1], is proved. In §2.4 and 2.5, this result is used to prove unitary equivalence in the ground and KMS state. In §2.6, a result concerning unbounded perturbations is formulated. In §2.7, we compute a unitarily equivalent representation of the Hamiltonian H_{GNS} for a system with interaction in the β-KMS-state in terms of the creation-annihilation operators in Fock space.

In Chapter 3, the key result is proved, namely convergence in the famous "linked cluster theorem", from which asymptotic completeness under the condition of vacuum polarization is derived. This requires much more complex techniques in cluster expansions, namely fermion cancellations in mode-time cells. For the benefit of the reader, we present the definitions of the Friedrichs diagrams and the formal renormalization theory suitable for our case. This chapter can be read independently of Chapter 2.

Chapter 4 is devoted to the application of the techniques used for proving the convergence of series of Dyson-Schwinger type to the investigation of an important type of kinetic limit, namely the weak interaction limit. We prove a result of Domnenkov [Do2], then we discuss some earlier results of Davis and others [D1]–[D8], [GK1], [FG1], [FG2], [P], [Du1], [Du2] concerning the weak interaction limit and explain why in Davis' stronger assumptions there were no difficulties connected with the estimation of the number of diagrams.

Some related results are not touched upon in this paper, namely asymptotic completeness in the spin system in an ideal Fermi system (see the survey [BDM]), the translation-invariance case (see [DM], [BM2], [BDM]), and some other results about to be published.

CHAPTER 0. BASIC DEFINITIONS AND NOTATION

§0.1. Fock space. Second quantization

Fock space is a Hilbert space with an additional structure.

Let the separable Hilbert space $\mathscr{H}$ be given. Unless stated otherwise, it will be further treated as complex with scalar product antilinear in the second argument.

We shall define antisymmetric (fermion) Fock space $\mathscr{F}_a = \mathscr{F}_a(\mathscr{H})$ over $\mathscr{H}$ and symmetric (boson) Fock space $\mathscr{F}_s = \mathscr{F}_s(\mathscr{H})$.

It is convenient to begin by introducing a more general Fock space

$$\mathscr{F} = \mathscr{F}(\mathscr{H}) = \bigoplus_{n=0}^{\infty} \mathscr{F}^{(n)}, \tag{0.1.1}$$

where $\mathscr{F}^{(0)} = \mathbb{C}$ (space of constants), $\mathscr{F}^{(n)} \equiv \mathscr{F}^{(n)}(\mathscr{H}) = \mathscr{H}^{\otimes n} = \mathscr{H} \otimes \mathscr{H} \otimes \cdots \otimes \mathscr{H}$ ($\otimes$ denotes the tensor product of Hilbert spaces [BR1]), and $\mathscr{F}$ denotes the direct sum of Hilbert spaces, i.e., the space of sequences

$$\Phi = (\varphi_0, \varphi_1, \dots, \varphi_n, \dots), \quad \varphi_n \in \mathscr{F}^{(n)}, \tag{0.1.2}$$

with finite norm

$$(\Phi, \Phi) = \|\Phi\|^2 = \sum_{n=0}^{\infty} \|\varphi_n\|^2. \tag{0.1.3}$$

Observe that if $\mathscr{H} = L_2(X, \Sigma, \mu)$, then $\mathscr{F}^{(n)}$ is the space of square-integrable functions $\varphi_n(x_1, \dots, x_n)$, $x_i \in X$, with respect to the measure $\mu^n \equiv \mu \otimes \cdots \otimes \mu$ (n times).

In the space $\mathscr{F}$ the symmetric group S_n acts in a natural way by interchanging factors in the tensor products $f_1 \otimes \cdots \otimes f_n$, $f_i \in \mathscr{H}$. The subspace $\mathscr{F}^{(n)}$ invariant with respect to symmetric (antisymmetric) permutations of elements will be denoted by $\mathscr{F}_s^{(n)}$ ($\mathscr{F}_a^{(n)}$); by an "antisymmetric" permutation we mean a permutation followed by multiplication by ± 1 depending on its parity.

DEFINITION 0.1. Let

$$\mathscr{F}_s = \bigoplus_{n=0}^{\infty} \mathscr{F}_s^{(n)}, \qquad \mathscr{F}_a = \bigoplus_{n=0}^{\infty} \mathscr{F}_a^{(n)}. \tag{0.1.4}$$

In the case mentioned above where $\mathscr{H}$ is a space of functions, $\mathscr{F}_s^{(n)}$ ($\mathscr{F}_a^{(n)}$) consists of symmetric (antisymmetric) functions. We call $\mathscr{F}_s^{(n)}$ and $\mathscr{F}_a^{(n)}$ *n-particle subspaces.*

If a bounded operator U, $\|U\| \leqslant 1$, is given in $\mathscr{H}$, we denote by $\Gamma(U)$ the operator in $\mathscr{F}$ acting in each component by

$$\Gamma(U)(f_1 \otimes \cdots \otimes f_n) = (Uf_1) \otimes \cdots \otimes (Uf_n) \tag{0.1.5}$$

and then extended to the entire space $\mathscr{F}$ by linearity and continuity. Note that $\mathscr{F}_s$ and $\mathscr{F}_a$ are invariant with respect to $\Gamma(U)$, and therefore the restrictions of $\Gamma(U)$ to these subspaces will be denoted by the same symbol. If U is unitary in $\mathscr{H}$, then $\Gamma(U)$ is unitary in $\mathscr{F}_s$ and $\mathscr{F}_a$.

Suppose that h is a selfadjoint operator with domain $\mathscr{D} \subseteq \mathscr{H}$. We let $d\Gamma(h)$ denote the selfadjoint operator which is the closure of the symmetric operator in $\mathscr{F}$ acting on any $f = f_1 \otimes \cdots \otimes f_n \in \mathscr{D} \otimes \cdots \otimes \mathscr{D} \subseteq \mathscr{F}^{(n)}$ via

$$d\Gamma(h)(f_1 \otimes \cdots \otimes f_n) = hf_1 \otimes f_2 \otimes \cdots \otimes f_n + \cdots + f_1 \otimes f_2 \otimes \cdots \otimes hf_n \tag{0.1.6}$$

(this formula is obtained by formal differentiation of $\Gamma(e^{ith})$ at the point $t = 0$). Thus, $d\Gamma(h)$ may be viewed as acting in $\mathscr{F}_s$ and $\mathscr{F}_a$, where it is also a selfadjoint operator.

§0.2. Creation and annihilation operators. C^*-algebra CAR

For all $f \in \mathscr{H}$, we shall define the creation and annihilation operators $a^*(f)$ and $a(f)$ in $\mathscr{F}(\mathscr{H})$:

$$\begin{aligned} &a(f)\Omega = 0, \qquad a^*(f)\Omega = (0, f, 0, \ldots), \\ &a(f)(f_1 \otimes \cdots \otimes f_n) = \sqrt{n}(f, f_1)(f_2 \otimes \cdots \otimes f_n), \\ &a^*(f)(f_1 \otimes \cdots \otimes f_n) = \sqrt{n+1} f \otimes f_1 \otimes \cdots \otimes f_n, \end{aligned} \tag{0.2.1}$$

where the vector $\Omega = (1, 0, 0, \ldots) \in \mathscr{F}$ is called *vacuum.*

It is readily seen that

$$(a(f))^* = a^*(f).$$

These operators can be extended by linearity to operators with dense domain $\mathscr{F}_0 \equiv \mathscr{F}_0(\mathscr{H}) \subseteq \mathscr{F}$ consisting of finitary sequences (0.1.2), i.e. such that all $\varphi_n \equiv 0$ starting with some n, $\varphi_k \in \mathscr{F}^{(k)}$.

We define linear operators $P_\pm$ in $\mathscr{F}$ by

$$P_\pm(f_1 \oplus \cdots \oplus f_n) = \frac{1}{n!} \sum_{\pi \in S_n} (\pm 1)^{|\pi|} (f_{\pi(1)} \oplus \cdots \oplus f_{\pi(n)}), \tag{0.2.2}$$

where $|\pi|$ is the parity of the permutation π. The $P_\pm$ are orthogonal operators in $\mathscr{F}_s^{(n)}$ and $\mathscr{F}_a^{(n)}$, respectively.

DEFINITION 0.2. We set

$$a_\pm(f) = P_\pm a(f), \qquad a_\pm^*(f) = P_\pm a^*(f) \tag{0.2.3}$$

on $\mathscr{F}_{s,0}$ and $\mathscr{F}_{a,0}$, respectively. From here on, we often omit the subscripts $\pm$ in the notation of operators if it is clear which of the spaces $\mathscr{F}_s$ or $\mathscr{F}_a$ is considered. In the case when Fock space consists of sequences of symmetric (antisymmetric) functions

$$\Phi = (\varphi_0, \varphi_1(x_1), \dots, \varphi_n(x_1, \dots, x_n), \dots),$$

the creation and annihilation operators act according to the formulas

$$(a(f)\Phi)_n(x_1, \dots, x_n) = \sqrt{n+1} \int_{\mathbb{R}^\nu} \varphi_{n+1}(x, x_1, \dots, x_n) \bar{f}(x)\, d\mu(x),$$
$$(a^*(f)\Phi)_n(x_1, \dots, x_n) = \frac{1}{\sqrt{n}} \sum_{\pi \in S_n} (\pm 1)^{|\pi|} \varphi_{n-1}(x_1, \dots, \check{x}_i, \dots, x_n) f(x_i), \tag{0.2.4}$$

where the sign "ˇ" means that the corresponding variable is omitted.

REMARK. If the norm (0.1.3) is replaced by the norm

$$\|\Phi\|^2 = \frac{1}{n!} \sum_{n=0}^{\infty} \|\varphi_n\|^2, \tag{0.2.5}$$

then the operators $a^*(f)$ and $a(f)$ will have the form (0.2.4) but without the factors $1/\sqrt{n}$ and $\sqrt{n+1}$, which are difficult to remember.

In the antisymmetric (fermion) case, the canonical anticommutation relations (CAR) hold

$$\begin{gathered} a^*(f)a(g) + a(g)a^*(f) = (f, g)I, \\ \{a(f), a(g)\} = \{a^*(f), a^*(g)\} = a^*(f)a^*(g) + a^*(g)a^*(f) = 0, \end{gathered} \tag{0.2.6}$$

where 0 and I are, respectively, the zero and identity operators in $\mathscr{F}_a$.

It is readily seen that on $\mathscr{F}_a$

$$\|a(f)\| = \|a^*(f)\| = \|f\|, \tag{0.2.7}$$

i.e., in the fermion case the creation and annihilation operators are bounded.

Furthermore,

$$a(f)\colon \mathscr{F}_s^{(n)} \to \mathscr{F}_s^{(n-1)} \quad \text{and} \quad a^*(f)\colon \mathscr{F}_s^{(n)} \to \mathscr{F}_s^{(n+1)}$$

have the norm

$$\|a(f)\|_{n,n-1} = \|a^*(f)\|_{n-1,n} = \sqrt{n}\|f\|. \tag{0.2.8}$$

It follows that $\mathscr{F}_{s,0}$ forms a dense set of analytic vectors for $a^{\#}(f)$, where $a^{\#} = a^*$ or $a^{\#} = a$, i.e., for all $\Phi \in \mathscr{F}_{s,0}$ and all $z \in \mathbb{C}$,

$$\sum_{n=0}^{\infty} \frac{|z|^n}{n!} \|(a^{\#}(f))^n \Phi\| < \infty.$$

We shall be interested in the C^*-algebra CAR generated by the creation and annihilation operators acting in Fock space $\mathscr{F}(\mathscr{H})$.

DEFINITION 0.3. A C^*-algebra $\mathfrak{A}(\mathscr{H})$ (or one isomorphic to it) generated by the operators $a(f)$ and $a^*(f)$, $f \in \mathscr{H}$, is called the *C^*-algebra of canonical anticommutation relations* (*C^*-algebra CAR*) *over* $\mathscr{H}$, where $\mathscr{H}$ is a Hilbert space.

§0.3. Free dynamics and quasifree state on the C^*-algebra $\mathfrak{A}(\mathscr{H})$. The GNS-representation

Let $h = h^*$ be a selfadjoint operator in $\mathscr{H}$ with dense domain $\mathscr{D} \subseteq \mathscr{H}$. This operator is called a *one-particle Hamiltonian*.

DEFINITION 0.4. Any strongly continuous group of $*$-automorphisms on the C^*-algebra $\mathfrak{A}$ is called a *dynamics* τ_t on $\mathfrak{A}$.

DEFINITION 0.5. A pair $(\mathfrak{A}, \tau_t)$, where $\mathfrak{A}$ is a C^*-algebra and τ_t is a dynamics on $\mathfrak{A}$, is called a *C^*-dynamical system*.

DEFINITION 0.6. Let $(\mathfrak{A}_1, \tau_t^1)$ and $(\mathfrak{A}_2, \tau_t^2)$ be two C^*-dynamical systems. They are said to be *equivalent* if there exists a $*$-isomorphism of C^*-algebras $\gamma\colon \mathfrak{A}_1 \to \mathfrak{A}_2$ such that

$$\tau_t^2 = \gamma \tau_t^1 \gamma^{-1} \quad \text{for all } t \in \mathbb{R}.$$

DEFINITION 0.7. A one-parameter strongly continuous group τ_t^0 on the C^*-algebra CAR $\mathfrak{A}(\mathscr{H})$ acting on the creation and annihilation operators by

$$\tau_t^0(a^{\#}(f)) \overset{\text{def}}{=} e^{itd\Gamma(h)} a^{\#}(f) e^{-itd\Gamma(h)} = a^{\#}(e^{ith} f) \tag{0.3.1}$$

is called a *free dynamics on $\mathfrak{A}(\mathscr{H})$ generated by the one-particle Hamiltonian h*.

The last equality in (0.3.1) must be proved. Suppose that $\{e_x,\ x \in \mathbb{Z}^\nu\}$ is a basis in $\mathscr{H}$ and $(he_x, e_y) = c_{xy} = \bar{c}_{yx}$, then

$$H_0 \equiv d\Gamma(h) = \sum_{x,y} c_{xy} a_x^* a_y,$$

$$i[H_0, a(e_z)] = ia\left(-\sum_y c_{zy} a_z\right) = -ia\left(\sum_y c_{yz} e_z\right)$$
$$= -ia(he_z) = \frac{d}{dt} a(e^{ith} e_z)\bigg|_{t=0}.$$

In a similar way, we can easily obtain

$$i[H_0, a^*(e_z)] = \frac{d}{dt} a^*(e^{ith} e_z)\bigg|_{t=0}. \tag{0.3.2}$$

These equalities yield the last equality in (0.3.1).

Further, let $\mathfrak{A}$ always be a C^*-algebra with unit.

DEFINITION 0.8. A continuous positive normalized linear functional on $\mathfrak{A}$, i.e., a continuous linear functional satisfying

$$\langle A^* A\rangle \geqslant 0 \quad \text{for all } A \in \mathfrak{A}$$

and

$$\langle I\rangle = 1,$$

$I \in \mathfrak{A}$ being the unit in the C^*-algebra $\mathfrak{A}$, is called a *state* $\langle\,\cdot\,\rangle$ *on the* C^**-algebra* $\mathfrak{A}$.

DEFINITION 0.9. *A quasifree state* $\langle\,\cdot\,\rangle_0$ *on* $\mathfrak{A}(\mathscr{H})$ is a positive linear functional such that $\langle I\rangle_0 = 1$ and

$$\langle a^{\#}(f_1)\cdots a^{\#}(f_{2k})\rangle_0 = \sum_{\pi}(-1)^{|\pi|}\prod_{s=1}^{k}\langle a^{\#}(f_{i_s})a^{\#}(f_{j_s})\rangle_0, \tag{0.3.3}$$

where the sum is taken over all partitions of the set of indices $(1, \dots, 2k)$ into k pairs $(i_1, j_1), \dots, (i_k, j_k)$ with $i_s < j_s$, and $|\pi|$ is the parity of the permutation $\pi = (i_1, j_1, \dots, i_k, j_k)$.

DEFINITION 0.10. A quasifree state $\langle\,\cdot\,\rangle_0$ on $\mathfrak{A}(\mathscr{H})$ is called *gauge-invariant* if the gauge-invariance condition is fulfilled:

$$\langle a(f)a(g)\rangle_0 = \langle a^*(f)a^*(g)\rangle_0 = 0. \tag{0.3.4}$$

REMARK. In order that the functional defined above be a state, i.e., be positive definite, some supplementary conditions are necessary. It is clear that if $\langle\,\cdot\,\rangle_0$ is a state, then the following equality holds:

$$\langle a^*(f)a(g)\rangle_0 = (f, Bg). \tag{0.3.5}$$

Here B is a positive linear operator in $\mathscr{H}$ such that

$$0_{\mathscr{H}} \leqslant B \leqslant I_{\mathscr{H}}, \tag{0.3.6}$$

where $0_{\mathscr{H}}$ is zero operator, and $I_{\mathscr{H}}$ is identity operator in $\mathscr{H}$.

It can be proved that condition (0.3.6) is not only necessary but also sufficient in order that $\langle\,\cdot\,\rangle_0$ be a quasifree gauge-invariant state (see [SS]).

By $\langle\,\cdot\,\rangle_0^B$, we denote a quasifree gauge-invariant state specified by the operator B in (0.3.5) and by $\langle\,\cdot\,\rangle_0$, an arbitrary quasifree state on the C^*-algebra $\mathfrak{A}(\mathscr{H})$.

Observe that the case $B \equiv 0_{\mathscr{H}}$ corresponds to the so-called *Fock state* on $\mathfrak{A}(\mathscr{H})$.

The definition of the quasifree gauge-invariant state $\langle\,\cdot\,\rangle_0^B$ and (0.3.5) yield the following useful formula

$$\langle a^*(f_1)\cdots a^*(f_m)a(g_n)\cdots a(g_1)\rangle_0^B = \delta_{mn}\det\{(f_i, Bg_j)\}. \tag{0.3.7}$$

DEFINITION 0.11. A state $\langle \cdot \rangle$ is called *invariant with respect to the dynamics* τ_t if

$$\langle \tau(A) \rangle = \langle A \rangle \tag{0.3.8}$$

for all $A \in \mathfrak{A}(\mathscr{H})$, $t \in \mathbb{R}$.

If (0.3.8) holds, then the dynamics τ_t is said to be *invariant with respect to the state* $\langle \cdot \rangle$.

The assertion that follows provides a necessary and sufficient condition for the free dynamics to be invariant with respect to a quasifree gauge-invariant state.

PROPOSITION 0.1. *The quasifree state* $\langle \cdot \rangle_0^B$ *generated by the operator* B *is invariant with respect to the free dynamics* τ_t^0 *if and only if the operators* B *and* h *commute, i.e., if*

$$Be^{ith} = e^{ith}B, \quad \forall t \in \mathbb{R}. \tag{0.3.9}$$

The proof of Proposition 0.1 is self-evident.

We now define the so-called *physical Hilbert space* $\mathscr{H}_{\text{phys}}$ and the physical Hamiltonian H_{phys}.

Let $(\mathscr{H}_{\langle \cdot \rangle}, \pi_{\langle \cdot \rangle}, \Omega_{\langle \cdot \rangle})$ be a cyclic GNS-representation of the C^*-algebra $\mathfrak{A}$ with respect to the state $\langle \cdot \rangle$. By definition, set

$$\mathscr{H}_{\text{phys}} \equiv \mathscr{H}_{\langle \cdot \rangle}.$$

Suppose that the dynamics τ_t on the C^*-algebra is invariant with respect to the state, in which case we may define a strongly continuous one-parameter group U_t in $\mathscr{H}_{\text{phys}}$

$$U_t(\pi_{\langle \cdot \rangle}(A)\Omega_{\langle \cdot \rangle}) = \pi_{\langle \cdot \rangle}(\tau_t(A)\Omega_{\langle \cdot \rangle}), \tag{0.3.10}$$

which by Stone's theorem has an infinitesimal generator H_{GNS} such that

$$e^{itH_{\text{GNS}}}(\pi_{\langle \cdot \rangle}(A)\Omega_{\langle \cdot \rangle}) = \pi_{\langle \cdot \rangle}(\tau_t(A)\Omega_{\langle \cdot \rangle}). \tag{0.3.11}$$

We set

$$H_{\text{phys}} \equiv H_{\text{GNS}}.$$

Further, a unitary one-parameter group U_t will also be called a *dynamics*. It will be clear from the context whether we are dealing with a dynamics in a C^*-algebra or in a physical Hilbert space.

§0.4. A free Fermi gas. KMS-states

DEFINITION 0.12 [BR1]. Suppose that on the C^*-algebra $\mathfrak{A}$ a dynamics τ and a state $\langle \cdot \rangle$ are given. The state $\langle \cdot \rangle$ is called a (τ, β)*-KMS-state* ($\beta \in \overline{\mathbb{R}} \cup \{\pm\infty\}$) if for all $A, B \in \mathfrak{A}$ there exists a function $F_{A,B}$ such that

(1) $F_{A,B}$ is analytic in $\mathbb{D}_\beta$;

(2) $F_{A,B}$ is continuous and bounded in $\overline{\mathbb{D}}_\beta$;

(3)

$$F_{A,B}(t) = \langle A\tau_t(B)\rangle, \quad \forall t \in \mathbb{R},$$
$$F_{A,B}(t+i\beta) = \langle \tau_t(B)A\rangle, \quad \forall t \in \mathbb{R}, \tag{0.4.1}$$

where

$$\mathbb{D}_\beta = \{z : 0 < \Im z < \beta\} \quad \text{for } \beta > 0,$$
$$\mathbb{D}_\beta = \{z : \beta < \Im z < 0\} \quad \text{for } \beta < 0.$$

If $\beta = +\infty$, then the (τ, β)-KMS-state is called the *ground state*.

We define a KMS-state with respect to the free dynamics τ_t^0 generated by the one-particle Hamiltonian h.

PROPOSITION 0.2. *For each $\beta \in \mathbb{R}$, there exists a unique state $\langle \cdot \rangle$ that is a (τ^0, β)-KMS-state. Moreover, the state $\langle \cdot \rangle$ is a quasifree gauge-invariant state, with*

$$B = \exp(-\beta h)(I_{\mathscr{H}} + \exp(-\beta h))^{-1}, \tag{0.4.2}$$

i.e., $\langle \cdot \rangle = \langle \cdot \rangle_0^B$.

PROOF. By the definition of a β-KMS-state, we have

$$\langle a(g)a^*(f)\rangle = \langle a^*(e^{-\beta h}f)a(g)\rangle = (e^{-\beta h}f, g) - \langle a^*(f)a(g)\rangle$$

if f is an analytic vector for h, in which case

$$\langle a^*((I_{\mathscr{H}} + e^{-\beta h})f)a(g)\rangle = (e^{-\beta h}f, g),$$

i.e., the relation (0.4.2) holds. For a monomial of general type, these computations must be iterated (see [BR1]); hence it follows that $\langle \cdot \rangle$ is quasifree and B is of the form (0.4.2). ■

CHAPTER 1. SPECTRAL PROPERTIES OF FREE DYNAMICS IN A QUASIFREE STATE

§1.1. The Wick brackets

We define a special mapping of the C^*-algebra CAR $\mathfrak{A}(\mathscr{H})$ into itself which will be called the *Wick bracket*. It will be defined with respect to a quasifree state $\langle \cdot \rangle_0$ which is not necessarily gauge-invariant. It will become clear from the context which state we have in mind.

Let us fix a certain quasifree state $\langle \cdot \rangle \equiv \langle \cdot \rangle_0$ on $\mathfrak{A}(\mathscr{H})$. We shall define Wick monomials, i.e., the result of the application of Wick brackets to a monomial of the form

$$w = a^\#(f_1)\cdots a^\#(f_m), \quad f_i \in \mathscr{H},$$

by means of an induction formula which is similar to the one used to determine Wick monomials for Gaussian systems (see [MM3]).

Let us first define Wick brackets $::\equiv::_{\langle\cdot\rangle}$ on monomials and then extend them by linearity to the entire C^*-algebra.

DEFINITION 1.1. (Wick monomials on the C^*-algebra CAR $\mathfrak{A}(\mathscr{H})$.) We set

$$a_T \equiv a_1^{\#}\cdots a_1^{\#} = \sum_{T'} :a_T^{\#}:\langle a_{T\setminus T'}^{\#}\rangle(-1)^{\pi(T,T')}, \tag{1.1.1}$$

where $T = (1, \dots, n)$, T' is a subsequence of T preserving the order, $T\setminus T'$ is the remaining subsequence, and $\pi(T, T')$ is the parity of the permutation $(1, \dots, n) \to (T\setminus T', T')$, $a_i^{\#} = a^{\#}(f_i)$, $f_i \in \mathscr{H}$.

PROPOSITION 1.1. *The inversion formula holds*:

$$:a_T: = \sum_{T'} a_{T'}^{\#}\langle a_{T\setminus T'}^{\#}\rangle(-1)^{\pi(T,T')} + |T\setminus T'|/2. \tag{1.1.2}$$

The proof of this formula is quite similar to that for Gaussian systems ([MM3]).

In what follows, we shall often denote by $:\cdot:_B$ the Wick brackets with respect to a quasifree gauge-invariant state defined by the operator B.

REMARK. From definition 1.1, we can easily obtain the following formula, used earlier to introduce the Wick brackets ([B], [E3]) $:\cdot:\equiv:\cdot:_B$ with respect to a quasifree gauge-invariant state ($\langle\cdot\rangle\equiv\langle\cdot\rangle_0^B$):

$$\begin{aligned} &:a^*(f_1)\cdots a^*(f_m)a(f_{m+1})\cdots a(f_{m+n}): \\ &\quad= \sum_{k=0}^{\min(m,n)}(-1)^k\sum_\pi(-1)^{|\pi|}\prod_{s=1}^{k}\langle a^*(f_{i_s})a(f_{j_s})\rangle \\ &\qquad\qquad \times a^*(f_1)\cdots\check{a}^*(f_{i_1})\cdots\check{a}^*(f_{i_k})\cdots a^*(f_m) \\ &\qquad\qquad \times a(f_{m+1})\cdots\check{a}(f_{j_1})\cdots\check{a}(f_{j_k})\dots a(f_{m+n}), \end{aligned} \tag{1.1.3}$$

where the sum $\sum_\pi$ is taken over all permutations $\pi \in S_{m+n}$ such that

$$\pi = \begin{pmatrix} 1 & 2 & \dots & 2k-1 & 2k & 2k+1 & \dots & m+n \\ i_1 & j_1 & \dots & i_k & j_k & r_1 & \dots & r_{m+n-2k}\end{pmatrix}$$

for any sequence $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant m$, $m+1 \leqslant j_s \leqslant m+n$, $s = 1, \dots, k$, and any increasing sequence $r_1, \dots, r_{m+n-2k}$ of the numbers $1, 2, \dots, m+n$ lacking $i_1, \dots, i_k, j_1, \dots, j_k$. The symbol "ˇ" means that there is no corresponding creation-annihilation operator in the product.

In the following theorem we present the basic properties of Wick brackets.

THEOREM 1.2 ([B]). *Let* $:\cdot:\equiv:\cdot:_{\langle\cdot\rangle}$ *be the Wick brackets with respect to a quasifree state* $\langle\cdot\rangle\equiv\langle\cdot\rangle_0$. *Then*

(1) $(:A_1A_2:)^* = :A_2^*A_1^*:$,

(1′) $(:a^{\#}(f_1)\cdots a^{\#}(f_m):)^* = :a^{i(\#)}(f_m)\cdots a^{i(\#)}(f_1):$, *where* $a^{i(*)} = a$, $a^{i(\,)} = a^*$;

(2) $\langle :a^{\#}(f_1)\cdots a^{\#}(f_m):\rangle = 0$ *if* $m > 0$,

(3)

$$\langle : a^{\#}(f_m)\cdots a^{\#}(f_1) : : a^{\#}(g_1)\cdots a^{\#}(g_n) : \rangle = \begin{cases} 0, & m \neq n, \\ \sum (-1)^{|\pi|} \langle a^{\#}(f_1) a^{\#}(g_{\pi(1)})\rangle \cdots \langle a^{\#}(f_n) a^{\#}(g_{\pi(n)})\rangle, & m = n, \end{cases} \tag{1.1.4}$$

where the sum is taken over all substitutions $\pi = \binom{1 \; \dots \; n}{\pi(1) \dots \pi(n)}$;

(3′) *If the state* $\langle \cdot \rangle_0$ *is gauge-invariant, i.e.,* $\langle \cdot \rangle \equiv \langle \cdot \rangle_0^B$, *then*

$$\begin{aligned} &\langle : a^*(f_m)\cdots a^*(f_1) a(g_1)\cdots a(g_n) : : a^*(h_k)\cdots a^*(h_1) a(u_1)\cdots a(u_s) : \rangle \\ &= \langle a^*(f_m)\cdots a^*(f_1) a(u_1)\cdots a(u_s)\rangle \langle a(g_1)\cdots a(g_n) a^*(h_k)\cdots a^*(h_1)\rangle \\ &= \delta_{ms}\delta_{kn} \det\{(Bf_i, u_j)\} \det\{((I_{\mathscr{H}} - B)h_i, h_j)\} \end{aligned} \tag{1.1.5}$$

for all $f_i, g_i, u_i, h_i \in \mathscr{H}$.

PROOF. Properties (1) and (1′) follow immediately from the definition of Wick brackets, and properties (2) and (3) follow from (3′).

Let us prove property (3′). Note that both sides of equality (1.1.5) are linear or antilinear with respect to the same arguments. Therefore it is sufficient to verify this equality for the case when all the f_i, g_i, u_i, h_i, $i \in \mathbb{N}$, are elements of some basis in $\mathscr{H}$.

It is readily seen that if equality (1.1.5) is satisfied for the Wick brackets $: \cdot :_{B_k}$ with respect to quasifree gauge-invariant states $\langle \cdot \rangle_0^{B_k}$ specified by the operators B_k, where

$$0_{\mathscr{H}} \leqslant B_k \leqslant I_{\mathscr{H}}$$

and

$$(x, B_k y) \to (x, By), \quad \forall x, y \in \mathscr{H},$$

then it is satisfied also for the Wick brackets $: \cdot :_B$ with respect to the quasifree gauge-invariant state specified by the operator B.

For any operator B, $0_{\mathscr{H}} \leqslant B \leqslant I_{\mathscr{H}}$, we may, of course, select a sequence of operators B_k such that $0_{\mathscr{H}} \leqslant B_k \leqslant I_{\mathscr{H}}$, B_k has a discrete spectrum, and

$$(x, B_k y) \to (x, B_y), \quad \forall x, y \in \mathscr{H}.$$

We shall prove equality (1.1.5) for the operator B_k with discrete spectrum $\{\lambda_i\}$, where $0 \leqslant \lambda_i \leqslant 1$ and $i \in \mathbb{N}$. Let e_i be a normalized eigenvector of the operator B_k corresponding to an eigenvalue λ_i, the set of vectors $\{e_i, i \in \mathbb{N}\}$ being an orthonormal basis. Making use of

$$(e_i, e_j) = \delta_{ij}, \qquad \langle a^*(e_i) a(e_j)\rangle_0^{B_k} = \delta_{ij}\lambda_i,$$

equality (1.1.5) can be easily verified for the basis elements. ■

§1.2. The Fock representation of free dynamics in a quasifree state

For brevity of notation, we set $\langle \cdot \rangle_0^B \equiv \langle \cdot \rangle$ in this section. We shall consider the GNS-representation $(\mathscr{H}_{\langle \cdot \rangle}, \pi_{\langle \cdot \rangle}, \Omega_{\langle \cdot \rangle})$ of the C^*-algebra $\mathfrak{A}(\mathscr{H})$ with respect to a quasifree gauge-invariant state $\langle \cdot \rangle_0^B$.

Suppose that the free dynamics τ_t^0 generated by a one-particle Hamiltonian $h = h^*$ with dense domain $\mathscr{D}_0 \subset \mathscr{H}$, does not affect the invariance property of $\langle \cdot \rangle_0^B$, i.e., B and h commute.

Consider the operator H_{GNS} generated by the free dynamics τ_t^0 in the space $\mathscr{H}_{\text{GNS}} \equiv \mathscr{H}_{\langle \cdot \rangle}$.

For Hilbert space $\mathscr{H}$ with scalar product $(\cdot, \cdot)_{\mathscr{H}}$, we define the conjugate space $\mathscr{H}^\imath$ with scalar product $(\cdot, \cdot)_{\mathscr{H}}^\imath$, where $\mathscr{H}^\imath$ coincides with $\mathscr{H}$ as a set, and $\imath: \mathscr{H} \to \mathscr{H}$ is a mapping with the following properties

$$\imath(\lambda f) = \bar{\lambda}\imath(f), \quad f \in \mathscr{H},$$
$$(\imath(f), \imath(g))_{\mathscr{H}^\imath} = (g, f)_{\mathscr{H}}, \quad f, g \in \mathscr{H}.$$

For any operator $D: \mathscr{H} \to \mathscr{H}$, we set

$$D^\imath: \mathscr{H}_\imath \to \mathscr{H}^\imath, \quad D^\imath(\imath(f)) \stackrel{\text{def}}{=} \imath(Df), \quad f \in \mathscr{H}.$$

In what follows, we use the notation $B_1 = (I_{\mathscr{H}} - B)^{1/2}$, $B_2 = B^{1/2}$.

THEOREM 1.3 ([B]). *Suppose that the quasifree state $\langle \cdot \rangle_0^B$ is invariant with respect to the free dynamics τ_t^0 generated by h. Set $\mathscr{H}_1 = B_1\mathscr{H}$, $\mathscr{H}_2^\imath = B_2^\imath\mathscr{H}^\imath$. Let*

$$\begin{aligned} \widehat{\mathscr{H}}_{m,n} &= (\underbrace{\mathscr{H}_1 \otimes \cdots \otimes \mathscr{H}_1}_{m \text{ times}})_{\text{as}} \otimes (\underbrace{\mathscr{H}_2^\imath \otimes \cdots \otimes \mathscr{H}_2^\imath}_{n \text{ times}})_{\text{as}}, \\ \widehat{\mathscr{H}} &\equiv \bigoplus_{m,n=0}^{\infty} \widehat{\mathscr{H}}_{m,n}, \end{aligned} \tag{1.2.1}$$

where $(\otimes \cdots \otimes)_{\text{as}}$ denotes the antisymmetrized tensor product of Hilbert spaces. Then there exist a decomposition of $\mathscr{H}_{\text{GNS}}$ into orthogonal and invariant subspaces of the operator H_{GNS}

$$\mathscr{H}_{\text{GNS}} \equiv \bigoplus_{m,n=0}^{\infty} \mathscr{H}_{m,n}$$

and a unitary operator $U: \mathscr{H}_{\text{GNS}} \to \widehat{\mathscr{H}}_{\text{GNS}}$ such that

$$U_{m,n} \stackrel{\text{def}}{=} U|_{\mathscr{H}_{m,n}} : \mathscr{H}_{m,n} \to \widehat{\mathscr{H}}_{m,n}.$$

Besides,

$$U_{m,n} H_{\mathrm{GNS}} U_{m,n}^{-1} = \sum_{i=1}^{m} I_{\mathscr{K}} \otimes \cdots \otimes I_{\mathscr{K}} \otimes \overset{i}{\check{h}} \otimes I_{\mathscr{K}} \otimes \cdots \otimes I_{\mathscr{K}} - \sum_{i=m+1}^{m+n} I_{\mathscr{K}} \otimes \cdots \otimes I_{\mathscr{K}} \otimes \overset{i}{\check{h}^{\iota}} \otimes I_{\mathscr{K}} \otimes \cdots \otimes I_{\mathscr{K}} \quad (1.2.2)$$

on a dense domain $\widehat{\mathscr{D}}_{m,n} \subset \widehat{\mathscr{H}}_{m,n}$, *where*

$$\widehat{\mathscr{D}}_{m,n} = \Big(\underbrace{\mathscr{D}_1 \otimes \cdots \otimes \mathscr{D}_1}_{m \text{ times}}\Big)_{\mathrm{as}} \otimes \Big(\underbrace{\mathscr{D}_2^{\iota} \otimes \cdots \otimes \mathscr{D}_2^{\iota}}_{n \text{ times}}\Big)_{\mathrm{as}},$$

and $\mathscr{D}_1 = B_1 \mathscr{D} \subset \mathscr{H}_1$, $\mathscr{D}_2^{\iota} = B_2 \mathscr{D}^{\iota} \subset \mathscr{H}_2^{\iota}$.

PROOF OF THEOREM 1.3. For brevity of notation, set $\Omega \equiv \Omega_{\langle\,\cdot\,\rangle}$, $\pi \equiv \pi_{\langle\,\cdot\,\rangle}$. We shall consider the following subspaces $\mathscr{H}_{\mathrm{GNS}}$:

$$\mathscr{H}_{m,n} = \{\pi(:a^*(f_1)\cdots a^*(f_m)a(f_{m+1})\cdots a(f_{m,n}):)\Omega,\ f_i \in \mathscr{K}\}.$$

PROPOSITION 1.4. *The subspaces* $\mathscr{H}_{m,n}$ *are pairwise orthogonal and invariant with respect to the free dynamics*

$$\mathscr{H}^0 = \bigoplus_{m,n=0}^{\infty} \mathscr{H}_{m,n}. \quad (1.2.3)$$

PROOF. The orthogonality of $\mathscr{H}_{m,n}$ follows from the properties of Wick monomials (Theorem 1.2).

From the definition of Wick brackets and the fact that h and B commute, it follows that

$$\tau_t(:a^*(f_1)\cdots a(f_{m,n}):) = :\tau_t(a^*(f_1)\cdots a(f_{m,n})):, \quad (1.2.4)$$

in which case the subspaces $\mathscr{H}_{m,n}$ are invariant with respect to the operator H_{GNS}. ■

Let the spaces $\widehat{\mathscr{H}}_{m,n}$ be defined by (1.2.1). Define the operators $U_{m,n}$: $\mathscr{H}_{m,n} \to \widehat{\mathscr{H}}_{m,n}$ in the following way:

$$U_{m,n}\pi(:a^*(f_1)\cdots a^*(f_m)a(g_1)\cdots a(g_n):)\Omega = (B_1 f_1 \otimes \cdots \otimes B_1 f_m)_{\mathrm{as}} \otimes (B_2^{\iota} g_1 \otimes \cdots \otimes B_2^{\iota} g_n)_{\mathrm{as}}. \quad (1.2.5)$$

From the properties of Wick monomials and the definition of the operator $U_{m,n}$, one can readily verify that $U_{m,n}$ is unitary.

We shall prove that relation (1.2.2) holds. We have

$$\begin{aligned}
&\{U_{m,n}H_{\text{GNS}}U_{m,n}^{-1}(B_1f_1\otimes\cdots\otimes B_1f_m)_{\text{as}}\otimes(B_2^lg_n\otimes\cdots\otimes B_2^lg_1)_{\text{as}},\\
&\qquad (B_1u_1\otimes\cdots\otimes B_1u_m)_{\text{as}}\otimes(B_2^lv_n\otimes\cdots\otimes B_2^lv_1)_{\text{as}}\}\\
&= i\frac{d}{dt}\{\pi(:a^*(e^{ith}f_1)\cdots a^*(e^{ith}f_m)a(e^{ith}g_n)\cdots a(e^{ith}g_1):)\Omega,\\
&\qquad \pi(:a^*(u_1)\cdots a^*(u_m)a(v_n)\cdots a(v_1):)\Omega\}\\
&= \Bigg\{\sum_{k=1}^{m}(B_1f_1\otimes\cdots\otimes hB_1f_k\otimes\cdots\otimes B_1f_m)_{\text{as}}\otimes(B_2^lg_n\otimes\cdots\otimes B_2^lg_1)_{\text{as}}\\
&\qquad -\sum_{k=m+1}^{m+n}(B_2f_1\otimes\cdots\otimes B_2f_m)_{\text{as}}\otimes(B_2^lg_n\otimes\cdots\otimes h^lB_2^lf_k\otimes\cdots\otimes B_2^lg_1)_{\text{as}},\\
&\qquad\qquad (B_1f_1\otimes\cdots\otimes B_1f_m)_{\text{as}}\otimes(B_2^lg_n\otimes\cdots\otimes B_2^lg_1)_{\text{as}}\Bigg\}.
\end{aligned}$$

This implies equality (1.2.2). It is readily seen that the subset $\mathscr{D}_{m,n}$ is dense in $\mathscr{H}_{m,n}$. The proof of Theorem 1.3 is complete. ■

§1.3. The Fock representation of free dynamics in the KMS-state

We shall consider a free Fermi gas in equilibrium temperature state at temperature $1/\beta$, which is a quasifree gauge-invariant state with

$$B=\exp(-\beta h)(I_{\mathscr{H}}+\exp(-\beta h))^{-1}, \tag{1.3.1}$$

where $h=-\Delta+\mu$ is the operator in $\mathscr{H}=L_2(\mathbb{R}^\nu, dx)$ and $\mu\in\mathbb{R}$ is the chemical potential, Δ being the Laplace operator.

Set

$$\mathscr{F}_a^l \overset{\text{def}}{=} \bigoplus_{n=0}^{\infty}\mathscr{F}_a^{(n)}(\mathscr{H}^l).$$

THEOREM 1.5. *In the case of a free Fermi gas in equilibrium state at temperature* $1/\beta$, $\beta\in\mathbb{R}$, *the operator* H_{GNS} *acts in the space*

$$\mathscr{H}_{\text{GNS}}=\mathscr{F}_a\otimes\mathscr{F}_a^l=\bigoplus_{m,n=0}^{\infty}\mathscr{H}_{m,n}, \tag{1.3.2}$$

where $\mathscr{H}_{m,n} = L_2^{\text{as}}((\mathbb{R}^\nu)^m, dx)\otimes L_2^{\text{as}}((\mathbb{R}^\nu)^n, dx)$ *is the subspace in* $L_2((\mathbb{R}^\nu)^{m+n}, dx)$ *of all functions that are antisymmetric with respect to the first* m *and the last* n *arguments. The operator* H_{GNS} *leaves these subspaces invariant, and its restriction to* $\mathscr{H}_{m,n}$ *coincides with the operator*

$$-\Delta_1-\cdots-\Delta_m+\Delta_{m+1}+\cdots+\Delta_{m+n}+\mu(m-n), \tag{1.3.3}$$

where $-\Delta_i$ *is the Laplace operator acting on the* ith *variable.*

PROOF. Theorem 1.5 follows from Theorem 1.3. ■

COROLLARY. *The dynamics* $e^{itH_{\text{GNS}}}$ *in the Hilbert space* $\mathscr{H}_{\text{GNS}} = \mathscr{F}_a \otimes \mathscr{F}_a^{\prime}$ *is determined by*

$$\Gamma(e^{ith}) \otimes \Gamma(e^{-ith'}). \tag{1.3.4}$$

REMARK. It is easy to see that

$$\mathscr{H}_{\text{GNS}} = \mathscr{F}(\mathscr{H}_1 \oplus \mathscr{H}_2), \qquad H_{\text{GNS}} = d\Gamma(h), \tag{1.3.5}$$

where

$$h = h \otimes 1 - 1 \otimes h'. \tag{1.3.6}$$

Theorem 1.5 allows us to apply the operator H_{GNS} on invariant subspaces as a multiplication operator. Note that such a representation of the operator does not depend on $\beta \in \mathbb{R}$. But if $\beta = +\infty$ (the ground state), then $B \equiv 0$, and for it $\mathscr{H}_{\text{GNS}} = \mathscr{F}_a$ is one Fock space and not a tensor product of Fock spaces. In the ground state, the spectrum of the operator H_{GNS} is bounded from below, but in the temperature state (KMS-state) it is not bounded either from below or from above.

§1.4. The ground state for free dynamics

The following proposition is a necessary and sufficient condition on the operator B in order that the quasifree gauge-invariant state $\langle \cdot \rangle_0^B$ be the leading one.

PROPOSITION 1.6. *In order that the quasifree state* $\langle \cdot \rangle_0^B$ *be the leading one, it is necessary and sufficient to have*

$$I_{\mathscr{H}} - B = E_{(0,+\infty)} + c(E_{[0,+\infty)} - E_{(0,+\infty)}), \tag{1.4.1}$$

where $0 \leqslant c \leqslant 1$, *and* E_Δ *is the spectral measure of the operator* h.

PROOF. The proof will be carried out for the case $\dim \mathscr{H} = n < \infty$. We may assume that $d\Gamma(h)$ may be represented in the form

$$d\Gamma(h) = \sum_{i=1}^{n} h_i a_i^* a_i, \tag{1.4.2}$$

where $\{e_i,\ i = 1, \dots, n\}$ is a basis in $\mathscr{H}$ of normalized eigenvectors of the operator h, $he_i = h_i e_i$, with $a_i = a(e_i)$, $a_i^* = a^*(e_i)$, and $h_i = (e_i, he_i)$. Due to the commutativity of the operators h and B, each vector e_i is also an eigenvector of the operator B, $Be_i = b_i e_i$.

Let $T = \{i_1, \dots, i_k\}$ and set

$$a_T^* = a^*(e_{i_1}) \cdots a^*(e_{i_k}), \qquad a_T = a(e_{i_1}) \cdots a(e_{i_k}).$$

Observe that the elements

$$: a_T^* a_{T'} : \tag{1.4.3}$$

for $T, T' \subseteq \{1, \dots, n\}$ are orthogonal with respect to the scalar product $(A, A') = \langle (A')^* A \rangle_0^B$ and have zero norm if there are zero eigenvalues of the operator B in T or zero eigenvalues of the operator $I_{\mathscr{H}} - B$ in T'. It is the elements of the form (1.4.3) with nonzero norm that generate $\mathscr{H}_{\mathrm{GNS}}$.

Since

$$\left[a_i^* a_i, : a_T^* a_{T'} :\right] = r_i(T, T') : a_T a_{T'} :, \tag{1.4.4}$$

where

$$r_i(T, T') = \begin{cases} -1, & i \in T', \quad i \notin T, \\ 1, & i \notin T', \quad i \in T, \\ 0, & \text{in other cases}, \end{cases}$$

it follows that the eigenvalue of H_{GNS} for the eigenvector $: a_T^* a_{T'} :$ is equal to

$$\sum_{i \in T} h_i - \sum_{j \in T'} h_j. \tag{1.4.5}$$

Proposition 1.6 follows from (1.4.5) and the definition of the ground state.

COROLLARY TO PROPOSITION 1.6. *The ground state of a free Fermi gas is a quasifree gauge-invariant state with*

$$B \equiv 0, \tag{1.4.6}$$

where $h = -\Delta$ *is an operator in* $\mathscr{H} = L_2(\mathbb{R}^\nu, dx)$, Δ *being the Laplace operator.*

CHAPTER 2. THE FERMI SYSTEM WITH LOCAL INTERACTION

§2.1. Local perturbations of free dynamics. Møller morphisms. The Cook criterion

In previous chapters, we considered as the dynamics of the C^*-algebra $\mathfrak{A}(\mathscr{H})$ the so-called *free dynamics* τ_t^0, constructed by means of the one-particle Hamiltonian h.

There exists a certain standard procedure of constructing a new C^*-dynamical system $(\mathfrak{A}, \tau_t^V)$ using an arbitrary C^*-dynamical system $(\mathfrak{A}, \tau_t)$ and $V = V^* \in \mathfrak{A}$.

This procedure can be explained as follows. Let τ_t^0 be the free dynamics on $\mathfrak{A}(\mathscr{H})$ generated by the one-particle Hamiltonian h. If $V = V^* \in \mathfrak{A}(\mathscr{H})$, we shall determine the dynamics τ_t^V, obtained by a local perturbation of the free dynamics τ_t^0, in the following way

$$\begin{aligned} \tau_t^V(A) &\stackrel{\text{def}}{=} e^{itH} A e^{-itH} \\ &= \tau_t(A) + \sum_{n=1}^{\infty} i^n \int_{\Delta_n^t} [\tau_{s_1}^0(V), [\tau_{s_2}^0(V), [\dots [\tau_{s_n}^0(V), \tau_t^0(A)] \dots]]] \\ &\qquad \times ds_1 \cdots ds_n, \quad A \in \mathfrak{A}, \end{aligned} \tag{2.1.1}$$

where $\Delta_n^t = \{0 < s_1 < \dots < s_n < t\}$, and $H = H_0 + V$, $H_0 = d\Gamma(h)$, with the series on the right-hand side of (2.1.1) converging in the norm for all $t \in \mathbb{R}$.

Indeed, since

$$\|\tau_t^0(A)\| = \|A\|, \quad \forall A \in \mathfrak{A}(\mathscr{H}),$$

then

$$\| [\tau_{s_1}^0(V), [\tau_{s_2}^0(V), [\ldots [\tau_{s_n}^0(V), \tau_t^0(A)] \ldots]]] \| \leqslant (2\|V\|)^n \|A\|$$

and, consequently,

$$\|\tau_T^V(A)\| \leqslant 1 + \sum_{n=1}^{\infty} \frac{(2\|V\|)^n \|A\| \, |t|^n}{n!} = \exp(2\,|t|\,\|V\|)\,\|A\|.$$

We shall prove that τ_t^V is indeed a dynamics, i.e., a one-parameter group of $*$-automorphisms of the algebra $\mathfrak{A}(\mathscr{H})$. For this, it is sufficient to prove the equality (2.1.1), since by the definition of the mapping τ_t^V (see the first line of (2.1.1)) and selfadjointness of the operator $H = d\Gamma(h) + V$, τ_t^V is a group of $*$-automorphisms in a wider algebra $\mathfrak{B}(\mathscr{F}_a)$ of bounded operators in $\mathscr{F}_a$, and the equality means that τ_t^V leaves $\mathfrak{A}(\mathscr{H}) \subseteq \mathfrak{B}(\mathscr{F}_a)$ invariant.

Indeed, the expression

$$B_t = i\tau_t^V \tau_{-t}^0(B) \stackrel{\text{def}}{=} e^{it(H_0+V)} e^{-itH_0} B e^{itH_0} e^{-it(H_0+V)}$$

as is readily verified by differentiating B_t with respect to t, satisfies the equation

$$\frac{dB_t}{dt} = i\tau_t^V \tau_{-t}^0([\tau_t^0(V), B]). \tag{2.1.2}$$

Hence

$$B_t = B_0 + i \int_0^t \tau_s^V \tau_{-s}^0([\tau_s(V), B])\, ds. \tag{2.1.3}$$

If in the right-hand side of (2.1.3) we substitute the expression for $B_s^1 = \tau_s^V \tau_{-s}^0(B^1)$, where $B^1 = [\tau_t^0(V), B]$, in the form (2.1.3), then we shall obtain the first two terms of the series (2.1.1) for $A = \tau_{-t}^0(B)$. By repeating this procedure several times, we obtain the required result.

DEFINITION 2.1. The interaction $V = V^*$ is called *local* (or *bounded*) if $V \in \mathfrak{A}$.

PROPOSITION 2.1. *Let $\mathfrak{A}$ be an arbitrary C^*-algebra and τ_t be a dynamics on it, $V = V^* \in \mathfrak{A}$. Then τ_t^V is also a dynamics on $\mathfrak{A}$ and*

$$\tau_t^V(A) = \tau_t(A) + \sum_{n=1}^{\infty} i^n \int_{\Delta_n^t} [\tau_{s_1}(V), [\tau_{s_2}(V), [\ldots [\tau_{s_n}(V), \tau_t(A)] \ldots]]] \times ds_1 \cdots ds_n, \quad A \in \mathfrak{A}.$$

PROOF. For the proof, see, for example, [BR1].

Thus by means of the series (2.1.1) we can determine a local perturbation of any dynamics on an arbitrary C^*-algebra $\mathfrak{A}$.

Now let us define the Møller morphisms in the C^*-algebra $\mathfrak{A}$; they are analogs of wave operators used in scattering theory to prove asymptotic completeness. By means of the Møller morphisms, we shall also prove asymptotic completeness in the case of a Fermi gas with small bounded interaction.

Let two dynamics τ_t^1 and τ_t^2 be defined on the C^*-algebra $\mathfrak{A}$.

DEFINITION 2.2 [Ro3]. The mappings $\gamma_\pm^{\tau^1,\tau^2}: \mathfrak{A} \to \mathfrak{A}$ defined as the limits

$$\gamma_\pm^{\tau^1,\tau^2}(A) = \lim_{t\to\pm\infty} \tau_{-t}^2\tau_t^1(A), \quad A \in \mathfrak{A}, \tag{2.1.4}$$

are called the *Møller morphisms* for the ordered pair of dynamics (τ^1, τ^2) if the limits exist.

PROPOSITION 2.2 [Ro3] (The intertwining property of Møller morphisms). *Let $\mathfrak{A}$ be a C^*-algebra, and τ^1, τ^2 be two dynamics on $\mathfrak{A}$. Then*

(1) *if $\gamma_\pm^{\tau^1,\tau^2}$ and $\gamma_\pm^{\tau^2,\tau^1}$ exist, then*

$$\gamma_\pm^{\tau^1,\tau^2} \equiv (\gamma_\pm^{\tau^2,\tau^1})^{-1}; \tag{2.1.5}$$

(2) *if $\gamma_\pm^{\tau^1,\tau^2}$ exist, then*

$$\tau_t^2\gamma_\pm^{\tau^1,\tau^2} = \gamma_\pm^{\tau^1,\tau^2}\tau_t^1, \quad \forall t \in \mathbb{R}; \tag{2.1.6}$$

(3) *if $\gamma_\pm^{\tau^1,\tau^2}$ and $\gamma_\pm^{\tau^2,\tau^1}$ exist, then*

$$\tau^2 = \gamma_\pm^{\tau^1,\tau^2}\tau_t^1\gamma_\pm^{\tau^2,\tau^1}, \quad \forall t \in \mathbb{R}. \tag{2.1.7}$$

PROOF. It is readily seen that (1) follows from the definition of Møller morphisms.

To prove (2), note that for all $A \in \mathfrak{A}$ and $t \in \mathbb{R}$, we have

$$\begin{aligned}\tau_t^2\gamma_\pm^{\tau^1,\tau^2}(A) &= \lim_{s\to\pm\infty} \tau_t^2\tau_{-s}^2\tau_s^1(A) \\ &= \lim_{s\to\pm\infty} \tau_{-(s-t)}^2\tau_{s-t}^1\tau_t^1(A) = \gamma_\pm^{\tau^1,\tau^2}\tau_t^1(A).\end{aligned}$$

(3) follows from (2) and the definition of Møller morphisms for two dynamics τ^1 and τ^2. ■

The case in which the dynamics τ_t^1 and τ_t^2 coincide with τ_t^0 or τ_t^V will be of the greatest interest to us. For this special case we introduce the definition that follows.

DEFINITION 2.3 [Ro3]. The Møller morphisms for the pair of dynamics (τ^0, τ^V), i.e., the mappings $\gamma_\pm: \mathfrak{A}(\mathscr{H}) \to \mathfrak{A}(\mathscr{H})$ defined as the limits

$$\gamma_\pm(A) = \lim_{t\to\pm\infty} \tau_{-t}^V\tau_t^0(A), \quad A \in \mathfrak{A}(\mathscr{H}), \tag{2.1.8}$$

are called the *direct Møller morphisms* if the limits exist.

The Møller morphisms for (τ^V, τ^0), i.e., the mappings $\widehat{\gamma}_\pm : \mathfrak{A}(\mathscr{H}) \to \mathfrak{A}(\mathscr{H})$ defined as the limits

$$\widehat{\gamma}_\pm(A) = \lim_{t\to\pm\infty} \tau^0_{-t}\tau^V_t(A), \quad A \in \mathfrak{A}(\mathscr{H}), \tag{2.1.9}$$

are called the *inverse Møller morphisms* if the limits exist.

PROPOSITION 2.3 [Ro3] (Cook method). *If there exists a dense (in the norm) subset $\mathfrak{A}^0 \subseteq \mathfrak{A}$ such that for all $A \in \mathfrak{A}^0$ there is an $R \geqslant 0$ such that*

$$\|[\tau^1_t(V), A]\| \in L_1((-\infty, -R] \cup [R, \infty)), \tag{2.1.10}$$

then $\gamma^{\tau^1, \tau^2}_\pm(A)$ exists for all $A \in \mathfrak{A}$.

PROOF. Reversing time and integrating both sides of (2.1.2) from t_1 to t_2 yields

$$\tau^2_{-t_2}\tau^1_{t_2}(B) - \tau^2_{-t_1}\tau^1_{t_1}(B) = i\int_{t_1}^{t_2} \tau^2_{-t}\tau^1_t([\tau^1_{-t}(V), B])\,dt.$$

Hence

$$\|\tau^2_{-t_2}\tau^1_{t_2}(B) - \tau^2_{-t_1}\tau^1_{t_1}(B)\| \leqslant \int_{t_1}^{t_2} \|[\tau^1_{-t}(V), B]\|\,dt < \infty,$$

since τ^2 and τ^1 preserve the norm.

Therefore for all $B \in \mathfrak{A}^0$, $\gamma^{\tau^1, \tau^2}_\pm(B)$ exist. Since

$$\|\gamma_\pm(B)\| = \|B\|, \quad \forall B \in \mathfrak{A}^0,$$

it is easily seen that the limits for $\gamma_\pm$ exist on the entire C^*-algebra $\mathfrak{A}$. ■

§2.2. The existence of direct Møller morphisms under bounded perturbations of the free dynamics

In $\mathfrak{A}(\mathscr{H})$, the set $\mathfrak{A}^0(\mathscr{H})$ of finite linear combinations of monomials in creation-annihilation operators of functions whose Fourier transforms belong to $C^\infty(\mathbb{R}^\nu)$ is a dense $*$-subalgebra.

In the algebra $\mathfrak{A}(\mathscr{H})$, we shall consider the C^*-subalgebra $\mathfrak{A}_e(\mathscr{H})$ formed by polynomials in even number of creation-annihilation operators. The elements of the C^*-subalgebra $\mathfrak{A}_e(\mathscr{H})$ will be henceforth called *even.* In $\mathfrak{A}_e(\mathscr{H})$, the set $\mathfrak{A}^0_e(\mathscr{H}) = \mathfrak{A}^0(\mathscr{H}) \cap \mathfrak{A}_e(\mathscr{H})$ is a dense $*$-subalgebra.

In this section, we prove that there exists a dense subset in the set of even selfadjoint elements of the C^*-algebra $\mathfrak{A}(\mathscr{H})$ for which direct Møller morphisms exist.

We introduce the following classes of smooth bounded interactions. Suppose that $V = V^*$ is of the form

$$V = \sum_{i=1}^{d} V_i, \qquad V_i = \sum_{k=1}^{M_i} c_k a^*(f_{i,1}^{(k)}) \cdots a^*(f_{i,m_i}^{(k)}) a(f_{i,m_i+1}^{(k)}) \cdots a(f_{i,m_i+n_i}^{(k)}), \tag{2.2.1}$$

where d and M_i are bounded, $m_i + n_i > 0$, and $\widehat{f}_{i,j}^{(k)} \in C_0^\infty(\mathbb{R}^\nu)$ for all i, j, k.

We say that $V \in \mathscr{A}_e^0$ if in (2.2.1) $m_i + n_i$ are even for all i, and $V \in \mathscr{A}^0$ if $m_i > 0$, $n_i > 0$ for all i.

It is readily seen that both classes belong to $\mathfrak{A}^0(\mathscr{H})$, with $\mathscr{A}_e^0 \subseteq \mathfrak{A}_e^0(\mathscr{H})$.

REMARK. All further results will also hold for wider classes $\mathscr{A}$, $\mathscr{A}_e$ of smooth bounded interactions. In this case, $V = V^*$ is of the form

$$V = \sum_{i=1}^{d} \int \mathscr{V}_i(x_1, \dots, x_{m_i}, x_{m_i+1}, \dots, x_{m_i+n_i}) a^*(x_1) \cdots a^*(x_{m_i}) a(x_{m_i+1}) \cdots a(x_{m_i+k_i})\, dx_1 \cdots dx_{m_i+n_i},$$

$$\mathscr{V}_i \in S(\mathbb{R}^{\nu(m_i+n_i)}), \quad m_i + n_i > 0. \tag{2.2.1'}$$

If $m_i + n_i$ are even for all i, then $V \in \mathscr{A}_e$. If $m_i > 0$ and $n_i > 0$, then $V \in \mathscr{A}$.

Generally, we shall prove all the theorems of this and succeeding chapters only for interactions from $\mathscr{A}_e^0$, $\mathscr{A}^0$ and will note only briefly the essential features of the proofs for interactions from $\mathscr{A}_e$ and $\mathscr{A}$.

We shall use the following notation

$$m_{\max} = \max_i m_i = \max_i n_i,$$
$$m_{\min} = \min_i m_i = \min_i n_i$$

in (2.2.1) and (2.2.1′), respectively.

We shall also introduce the class $\mathbb{H}$ of one-particle Hamiltonians. We say that $h \in \mathbb{H}$ if, in the Fourier representation, h is the operator of multiplication by the function $\mathscr{h}(k)$, $\mathscr{h} \in C_0^\infty(\mathbb{R}^\nu)$, $k \in \mathbb{R}^\nu$, the function and its first two derivatives being bounded by a polynomial in $\mathbb{R}^\nu$, and

$$\operatorname{dist}(S_{\mathscr{h}}, G_{\mathscr{h}}) > 0, \qquad \operatorname{meas} S_{\mathscr{h}} = \operatorname{meas} G_{\mathscr{h}} = 0,$$

where $S_{\mathscr{h}} = \{k : \nabla \mathscr{h}(k) = 0\}$ is the set of stationary points of the function $\mathscr{h}$, and $G_{\mathscr{h}}$ the set of the $k \in \mathbb{R}^\nu$ for which the matrix of second derivatives of the function is singular.

Observe that one-particle nonrelativistic Hamiltonians $-\Delta + \mu$, $\mu \in \mathbb{R}$, in $\mathbb{R}^\nu$ and on the lattice $\mathbb{Z}^\nu$ as well as the relativistic Hamiltonian $\sqrt{-\Delta + m^2}$, $m > 0$, belong to the class $\mathbb{H}$.

Henceforth, we shall treat the free dynamics τ_t^0 as being generated by a one-particle Hamiltonian h belonging to the class $\mathbb{H}$.

THEOREM 2.4 (The existence of direct Møller morphisms). *Let $\nu \geqslant 1$, then for $V = V^* \in \mathscr{A}_e$ there exist morphisms*

$$\gamma_\pm(A) = \lim_{t\to\pm\infty} \tau_{-t}^V \tau_t^0(A), \quad \forall A \in \mathfrak{A}(\mathscr{H}).$$

REMARK. The existence of Møller morphisms was first proved by Robinson in [Ro3] and generalized by Evans [E1].

PROOF. The case in which $\nu = 1, 2$ is somewhat different from that of $\nu \geqslant 3$.

Let $\nu \geqslant 3$. By virtue of the Cook criterion, it is sufficient to choose a dense subset $\mathfrak{A}^0 \subseteq \mathfrak{A}(\mathscr{H})$ for which condition (2.1.10) is fulfilled.

Let $\mathfrak{A}^0 \equiv \mathfrak{A}^0(\mathscr{H})$, i.e.,

$$\mathfrak{A}^0 = \{a^*(f_1)\cdots a^*(f_m)a(g_1)\cdots a(g_n),\ m, n \geqslant 0,\ \widehat{f}_i, \widehat{g}_j \in C_0^\infty(\mathbb{R}^\nu)\}.$$

If we prove that condition (2.1.10) is satisfied for $A = a^\#(f)$, where $\widehat{f} \in C_0^\infty(\mathbb{R}^\nu)$ is an arbitrary function, then since $\tau_{-t}^V \tau_t^0$ is the $*$-automorphism for any fixed $t \in \mathbb{R}$, we shall have shown the existence of $\gamma_\pm(A)$ for all $A \in \mathfrak{A}^0$.

Let $A = a(f)$. We have

$$\begin{aligned}\|[\tau_t^0(V), A]\| &= \|[\tau_{-t}^0(A), V]\| \leqslant \|[a(e^{-ith} f), V]\| \\ &= \Bigg\| \sum_{i=1}^{d} \sum_{k=1}^{M_i} \sum_{j=1}^{m_i} (-1)^{j-1} (f_{i,j}^{(k)}, e^{-ith} f) a^*(f_{i,1}^{(k)}) \cdots a^*(\check{f}_{i,j}^{(k)}) \cdots \\ &\qquad\qquad a^*(f_{i,m_i}^{(k)}) a(f_{i,m_i+1}^{(k)}) \cdots a(f_{i,m_i+n_i}^{(k)}) \Bigg\| \\ &\leqslant \frac{C(V, f)}{(1+|t|)^{\nu/2}},\end{aligned} \tag{2.2.2}$$

since by the stationary phase theorem (see, for example, [RS2])

$$|(f_{i,j}^{(k)}, e^{-ith} f)| \leqslant \frac{C(f, f_{i,j}^{(k)})}{(1+|t|)^{\nu/2}}, \tag{2.2.3}$$

where $C(f, f_{i,j}^{(k)})$ is a constant and

$$C(V, f) = \sum_{i=1}^{d} \sum_{k=1}^{M_1} \sum_{j=1}^{m_i} |c_k| C(f, f_{i,j}^{(k)}) \prod_{1\neq j}^{m_i+n_i} \|f_{i,l}^{(k)}\| < \infty. \tag{2.2.4}$$

Since on the right-hand side of (2.2.2) there is a function belonging to $L_1(\mathbb{R})$, it follows from the Cook criterion that $\gamma_\pm(a(f))$ exists. The case $a = a^*(f)$ is treated likewise.

For $\nu = 1, 2$, we choose the dense subset $\mathfrak{A}^0$ in $\mathfrak{A}(\mathscr{H})$ in the following way:

$$\mathfrak{A}^0 = \{a^*(f_1)\cdots a^*(f_m)a(g_1)\cdots a(g_n)\,,\ m\,,\ n \geqslant 0\,,\ \widehat{f}_i\,,\ \widehat{g}_j \in C_0^\infty(\mathbb{R}^\nu \setminus \{S_\ell\})\}.$$

Using q-fold integration by parts, for $|t| \geqslant 1$, we obtain the following estimate for the function $\widehat{f} \in C_0^\infty(\mathbb{R}^\nu \setminus \{S_\ell\})$:

$$|(f\,,\ e^{ith}f_{i,j}^{(k)})| \leqslant \frac{C_q(f\,,\ f_{i,j}^{(k)})}{|t|^q}\,, \tag{2.2.5}$$

where the constant $C_q(f\,,\ f_{i,j}^{(k)}) > 0$ also depends on q. If we take $q \geqslant 2$, then in (2.2.4) we obtain a function belonging to $L_1(\mathbb{R} \setminus [-1\,,\ 1])$.

Theorem 2.4 has been proved for interactions belonging to $\mathscr{A}_e^0$. To prove Theorem 2.4 for $V \in \mathscr{A}_e$, note that we have only to verify the estimate (2.2.4).

For this, it is sufficient to represent $V \in \mathscr{A}_e$ in the following form

$$V = \sum_{i=1}^{d}\sum_{N_1}\cdots\sum_{N_{m_i+n_i}} C_i(N_1\,,\ \ldots\,,\ N_{m_i+n_i})a^*(e_{N_1})\cdots a^*(e_{N_{m_i}})$$
$$\times\, a^*(e_{N_{m_i+1}})\cdots a^*(e_{N_{m_i+n_i}})\,, \tag{2.2.6}$$

where $\|e_N\| = 1$, $e_N \in C_0^\infty(\mathbb{R}^\nu)$, N_j take on values in a countable set $\mathscr{N}$, with

$$\sum_{i=1}^{d}\sum_{N_1}\cdots\sum_{N_{m_i+n+i}} |C_i(N_1\,,\ \ldots\,,\ N_{m_i+n_i})| < \infty\,, \tag{2.2.7}$$

and, for all $N\,,\ N' \in \mathscr{N}$ and $f \in C_0^\infty(\mathbb{R}^\nu)$, the following estimates hold

$$|(e_N\,,\ e^{-ith}f)| \leqslant \frac{C(f)}{(1+|t|)^\delta}\,, \tag{2.2.8}$$

$$|(e_N\,,\ e^{-ith}e_{N'})| \leqslant \frac{C}{(1+|t|)^\delta}\,, \tag{2.2.9}$$

where $C(f)$ is a constant depending only on f, and C, δ are absolute constants, with $\delta > 1$.

It is easily seen that the estimate (2.2.4) follows from (2.2.6)–(2.2.9).

We shall prove that $V \in \mathscr{A}_e$ may be represented in the form (2.2.6). Suppose first that the Fourier transform of the kernel $\mathscr{V}_i$ satisfies

$$\widehat{\mathscr{V}_i} \in C_0^\infty(\mathbb{R}^{\nu(m_i+n_i)})\,;$$

then we may choose an N so that

$$\operatorname{supp}\widehat{\mathscr{V}_i} \subseteq [-N\,,\ N]^{\nu(m_i+n_i)}$$

for all i. Suppose that

$$e_n^{(k)} = \frac{\varkappa(k)\exp(2\pi i n k/A)}{d_n}, \quad k \in \mathbb{R}, \tag{2.2.10}$$

where the function $\varkappa(k)$ belongs to $C_0^\infty(\mathbb{R})$, $0 \leqslant \varkappa(k) \leqslant 1$, $\varkappa(k) = 1$ for $|k| \leqslant B$, and $\varkappa(k) = 0$ for all $|k| \geqslant B+1$. The constants d_n are chosen so that $\|e_n\| = 1$.

We select A and B so that $N < B < 2N < A$ and

$$\widehat{\mathscr{V}_i} = \sum_{\overline{N}} C_i(\overline{N}) \prod_{j=1}^{D_i} e_{n_j}(k_j), \tag{2.2.11}$$

where $\overline{N} = (n_1, \dots, n_{D_i})$, $D_i = \nu(m_i + n_i)$. It is readily seen that for any q there exists a constant $C(q)$ such that

$$|C_N| \leqslant \frac{C(q)}{|\overline{N}|}, \qquad |\overline{N}| = \sum_i |n_i|. \tag{2.2.12}$$

It is clear that for $q > \max_i D_i$ there exists an estimate

$$\sum_{\overline{N}} |C_{\overline{N}}| < \infty.$$

For the functions $e_N = \prod_{j=1}^{\nu} e_{n_j}(k_j)$, $e_{N'} = \prod_{j=1}^{\nu} e_{n_j'}(k_j)$, the following estimates hold uniformly in $N, N' \in \mathbb{Z}^\nu$, $t, t' \in \mathbb{R}$:

(a)

$$|(e_N, e^{-ith} e_{N'})| \leqslant \frac{C}{(|t|+1)^{\nu/2-\delta'}}, \tag{2.2.13}$$

(b)

$$|(e_N, e^{-ith} f)| \leqslant \frac{C(f)}{(|t|+1)^{\nu/2-\delta'}}, \tag{2.2.14}$$

where the constant $\delta' > 0$ may be made arbitrarily small, $C = C(\nu, \delta')$, and $C(F) = C(f, \nu, \delta')$. By choosing in (2.2.13) and (2.2.14) $\delta = \nu/2 - \delta' > 1$ for $\nu \geqslant 3$ and $\delta' < \frac{1}{2}$, we obtain inequalities (2.2.8) and (2.2.9).

But if $\widehat{V}_i \in S(\mathbb{R}^{m_i+n_i})$, then we shall use the partition of unity

$$\sum_{\overline{N}} \alpha_{\overline{N}}(k) = 1, \tag{2.2.15}$$

where $\operatorname{diam}(\operatorname{supp}\alpha_{\overline{N}}) \leqslant \text{const}$ uniformly in $\overline{N}$.

Further, we shall represent the kernel $\widehat{\mathscr{V}_i}$ in the form of a sum of finite kernels $\widehat{\mathscr{V}_i}\alpha_{\overline{N}}$, and using an expansion for $\widehat{\mathscr{V}_i}\alpha_{\overline{N}}$ analogous to (2.2.6) with an appropriate shift of the functions $e_{\overline{N}}$, we can repeat the proof.

The proof of Theorem 2.4 is complete. ■

§2.3. The invertibility of Møller morphisms for small bounded perturbations of the free dynamics

To prove the invertibility of direct Møller morphisms, it is sufficient to prove the existence of inverse Møller morphisms. As in the case of wave operators, it is more difficult to prove the existence of inverse Møller morphisms than that of direct Møller morphisms.

THEOREM 2.5 [BM1] (The invertibility of direct Møller morphisms). *Let* $\nu \geqslant 3$; *then for* $V = V^* \in \mathscr{A}_e$ *there exists an* $\varepsilon_0 > 0$ *such that for* $|\varepsilon| \leqslant \varepsilon_0$, $\varepsilon \in \mathbb{R}$, *the direct Møller morphisms*

$$\gamma_\pm(A) = \lim_{t\to\pm\infty} \tau_{-t}^{\varepsilon V}\tau_t^0(A), \quad \forall A \in \mathfrak{A},$$

exist and are invertible, with

$$\tau_t^{\varepsilon V} = \gamma_\pm \circ \tau_t^0 \circ \gamma_\pm^{-1}, \quad \forall t \in \mathbb{R}. \tag{2.3.1}$$

COROLLARY. *Let* $\nu \geqslant 3$; *then for* $V = V^* \in \mathscr{A}_e$ *there exists an* $\varepsilon_0 > 0$ *such that for* $|\varepsilon| \leqslant \varepsilon_0$, $\varepsilon \in \mathbb{R}$, *the* C^**-dynamical systems* $(\mathfrak{A}(\mathscr{H}), \tau_t^0)$ *and* $(\mathfrak{A}(\mathscr{H}), \tau_t^{\varepsilon V})$ *are equivalent.*

Before proving Theorem 2.5, let us make two remarks.

REMARK 1. In contrast to direct Møller morphisms, inverse ones may exist or not for $\nu = 1, 2$ [Ma] and for arbitrarily small values of the coupling constant ε. The corresponding example may be constructed as follows.

EXAMPLE. Let $V = -a^*(f_0)a(f_0)$; then $\tau_t^{\varepsilon V}$ is also the free dynamics generated by the operator $h_e = h + \varepsilon P_0$, where P_0 is the projection onto the vector $-f_0$. For $A = a^\#(f)$ the inverse Møller morphisms are of the form

$$\begin{aligned}\widehat{\gamma}_\pm(A) &= \lim_{t\to\pm\infty} \tau_{-t}^0\tau_t^{\varepsilon V}(a^\#(f)) \\ &= \lim_{t\to\pm\infty} a^\#(e^{-ith}e^{ith_\varepsilon}f) = a^\#(\widehat{W}_\pm f),\end{aligned} \tag{2.3.2}$$

where $\widehat{W}_\pm$ are the usual inverse wave operators. As is well known, if $h = -\Delta$ and the function f_0 is such that $\widehat{f}_0 \in C_0^\infty(\mathbb{R}^\nu)$ and

$$\int_{\mathbb{R}^\nu} \widehat{f}_0(k)\,dk < 0, \tag{2.3.3}$$

then the operator h_ε has an eigenvalue $\lambda_\varepsilon < 0$ for arbitrarily small ε. Therefore $\widehat{W}_\pm(e_\varepsilon)$ do not exist, and consequently $\widehat{\gamma}_\pm(a^\#(e_\varepsilon))$ also do not exist.

REMARK 2. In Theorem 2.3, the coupling parameter for $\nu \geqslant 3$ must be small, and this is essential, since for large values of the coupling constant ε bound states may occur. As is easily seen, for $\nu \geqslant 3$ and large $|\varepsilon|$, the previous example demonstrates this phenomenon.

PROOF OF THEOREM 2.5. To prove the existence of $\widehat{\gamma}_+$, by the Cook criterion and by the observation in the proof of Theorem 2.4, it is sufficient to prove that

$$\|[\tau_t^{\varepsilon V}(V), a^\#(f)]\| \in L_1(\mathbb{R}_+),$$

i.e., $\|[\tau_t^{\varepsilon V}(V), a^{\#}(f)]\| < \infty$ for all $f \in C_0^\infty(\mathbb{R}^\nu)$. We have

$$\int_0^\infty \|[\tau_t^{\varepsilon V}(V), a^{\#}(f)]\| \, dt \leqslant \int_0^\infty \|[\tau_t^0(V), a^{\#}(f)]\| \, dt + \sum_{n=1}^\infty |\varepsilon|^n \int_0^\infty \int_{\Delta_n^t} \|[a^{\#}(f), [\tau_{s_1}^0(V), [\dots[\tau_{s_n}^0(V), \tau_t^0(V)]\dots]]]\| \times ds_1 \cdots ds_n \, dt. \quad (2.3.4)$$

LEMMA 2.6. *In the conditions of Theorem* 2.5, *there exist a constant* $C = C(V, \nu) > 0$ *independent of* f *and a constant* $C(f, V) > 0$ *such that the following estimates hold*:

(a)

$$\int_0^\infty \|[\tau_s^0(V), a^{\#}(f)]\| \, ds \leqslant C C(f, V), \quad (2.3.5)$$

(b)

$$\int_0^\infty \int_{\Delta_n^s} \|[a^{\#}(f), [\tau_{s_1}^0(V), [\tau_{s_2}^0(V), [\dots[\tau_{s_n}^0(V), \tau_s^0(V)]\dots]]]]\| \times ds_1 \cdots ds_n \, ds \leqslant C^{n+1} C(f, V). \quad (2.3.6)$$

REMARK. It is easy to see that Theorem 2.5 follows from Lemma 2.6 for $\varepsilon_0 = 1/C$.

PROOF OF LEMMA 2.6. Consider the nth term of the series. First, we shall estimate the integrand in (2.3.6) by means of a sum

$$\|[a^{\#}(f), [\tau_{s_1}^0(V), [\dots[\tau_{s_n}^0(V), \tau_s^0(V)]\dots]]]\| \leqslant \sum_G W_G(s_1, \dots, s_n, s),$$

where $\sum_G$ is taken over all admissible diagrams with weights W_G. The admissible diagrams G and their weights are described below. After this, we shall estimate

$$\int_0^\infty \int_{\Delta_n^s} \sum_G W_G(s_1, \dots, s_n, s) \, ds_1 \cdots ds_n \, ds \leqslant C^{n+1} C(f, V)$$

by means of a new technique for evaluating sums of diagrams. We shall put each s_i, $i = 0, 1, \dots, n+1$ (where $s_{n+1} \equiv s$ and $s_0 \equiv 0$) into correspondence with the vertex with number $n + 1 - i$.

We have

$$\tau_{s_v}^0(V) = \sum_{i=1}^d \sum_{k=1}^M c_k a^*(e^{is_v h} f_{i,1}^{(k)}) \cdots a^*(e^{is_v h} f_{i,m_i}^{(k_v)}) \times a(e^{is_v h} f_{i,m_i+1}^{(k)}) \cdots a(e^{is_v h} f_{i,m_i+n_i}^{(k_v)}). \quad (2.3.7)$$

Consider the integrand. This is an $(n+1)$-fold commutant. We shall treat it step by step in $n + 1$ steps.

At the first step, in the innermost commutant $[\tau^0_{s_n}(V), \tau^0_s(V)]$ using canonical anticommutation relations, we carry the creation-annihilation operators belonging to $\tau^0_{s_n}(V)$ to the right, i.e., we carry operators like

$$a^\#(e^{is_n h} f^{(k_1)}_{i,j})$$

through the operator $\tau_s(V)$. In other words, we use the relations

$$a(f)a(g) = -a(g)a(f),$$
$$a(f)a^*(g) = -a^*(g)a(f) + (f, g)I,$$

or similar ones for $a^*(f)$. In this procedure, multipliers like

$$(e^{is_n h} f^{(k_1)}_{i,j}, e^{ish} f^{(k_0)}_{i',j'}), \tag{2.3.8}$$

or their conjugates will appear. Note that the procedure described above is performed by turn: first, $\tau^0_{s_n}(V)$ is carried through the leftmost creation-annihilation operator in $\tau^0_s(V)$. If a pairing occurs, then we stop carrying $\tau^0_{s_n}(V)$ through $\tau^0_s(V)$ and say that a "line" with contribution (2.3.8) has appeared. But if no pairing took place, then $\tau^0_{s_n}(V)$ is carried through another creation-annihilation operator in $\tau^0_s(V)$, etc. Note that due to parity of V the terms with no pairings cancel each other out since they enter into the expression for the commutant

$$W_1 = [\tau^0_{s_n}(V), \tau^0_s(V)]$$

with opposite signs. Therefore only one edge appears at the first step.

Further, we shall represent the next commutant $[\tau^0_{s_{n-1}}(V), W_1]$ in the same way, that is by means of a similar procedure of carrying creation-annihilation operators like

$$a^\#(e^{is_{n-1}h} f^{(k_2)}_{i,j})$$

through W_1. At the second step, multipliers like

$$(e^{is_{n-1}h} f^{(k_2)}_{i,j}, e^{ish} f^{(k_0)}_{i',j'}) \quad \text{or} \quad (e^{is_{n-1}h} f^{(k_{n-1})}_{i,j}, e^{is_n h} f^{(k_n)}_{i',j'})$$

appear and again only one edge. And so forth, until step $n+1$ when the operator $a^\#(f)$ will be carried.

At step v, an edge appears, with a corresponding multiplier like

$$r_v = (e^{is_v h} f^{(k_v)}_{i,j}, e^{is_{v'} h} f^{(k_{v'})}_{i',j'}), \tag{2.3.9}$$

where $v' = v'(v) > v$.

It is readily seen that for all $1 \leqslant i, j, i', j' \leqslant d$, $0 \leqslant v, v' \leqslant n+1$ the following estimate holds:

$$|r_v| \leqslant \frac{C}{(|s_v - s_{v'(v)}| + 1)^\delta}, \tag{2.3.10}$$

where $\delta = \nu/2 - \delta' > 1$, $\delta' > 0$ is an arbitrary small constant, and $C > 0$ does not depend on $i, j, i', j', s_v, s_{v'}, k_v, k_{v'}$.

A diagram is a graph with vertices $n+1, n, \dots, 1, 0$. The edge of the graph between the vertices v and v' appears in our "carrying" procedure together with the multiplier

$$\frac{C}{(|s_v - s_{v'(v)}| + 1)^{\delta}}.$$

All the resulting diagrams will be called the *admissible* diagrams.

Exactly one edge goes out to the right from each vertex v, and this edge is paired with the vertex $v'(v) < v$. Note that at each vertex there are no more than $2m_{\max}$ edges going out to the left. By construction, each admissible diagram is connected. We let $\mathbb{G}^{n+1}$ denote the set of all admissible diagrams.

Each diagram is correspondingly weighted in (3.3.6).

The weight of a diagram. Let the set of edges $\{v, v'(v)\}$ correspond to an admissible diagram G; then its weight is determined by

$$W_G = C^n \prod_v \frac{1}{(|s_v - s_{v'(v)}| + 1)^{\delta}}.$$

From the estimate (2.3.10), after a change of variables in each summand, it follows that the nth term of the series may be estimated by

$$\int_{\Delta_{n+1}^{\infty}} C^n \Bigg(\sum_{\{v'(v)\} \in \mathbb{G}^{n+1}} \prod_v \frac{1}{(|s_v - s_{v'(v)}| + 1)^{\delta}} \Bigg) ds_1 \cdots ds_{n+1}, \qquad (2.3.11)$$

where a unique $v'(v) > v$ corresponds to each v, and the sum is taken over all collections $\{v'(v),\ v = 1, \dots, n+1\}$ such that among the numbers $v'(1), \dots, v'(n+1)$ no more than $2m_{\max}$ coincide with l, where $l = 0, 1, \dots, n$.

To estimate the overall contribution of all the diagrams, we will make use of the lemma that follows.

LEMMA 2.7. *Suppose that* $g \in L_1(\mathbb{R})$, $g(t) \geqslant 0$, *for all* $t \in \mathbb{R}$. *Then for all* n *the following estimate holds*:

$$\int_{\Delta_n^{\infty}} \Bigg(\sum_{\{v'(v)\} \in \mathbb{G}^n} \prod_v g(t_v - t_{v'(v)}) \Bigg) ds_1 \cdots ds_n \leqslant C^n \left(\int_{\mathbb{R}} g(t)\, dt \right)^n, \qquad (2.3.12)$$

where the sum is taken over all collections of admissible diagrams, and the constant $D > 0$ *does not depend on* n.

It is clear that the estimates (2.3.11) and (2.3.12) yield Theorem 2.5.

PROOF OF THE LEMMA. The proof of (2.3.12) is similar to that of (4.1) in [BM1], where it was proved for the case $m_{\min} = m_{\max} = 2$ and

$$g(t) = \frac{1}{(1 + |t|)^{\nu/2}}.$$

We consider the Riemann sums of the two sides of inequality (2.3.12) and prove that this inequality holds for the Riemann sums for any d, where d is the approximation step:

$$d^n\Bigl(\sum_{0<t_1<\cdots<t_n}\sum_{\{v'(v)\}}\prod_v g(t_v - t_{v'(v)})\Bigr) \leqslant d^n C^n\Bigl(\sum_{s\neq 0} g(s)\Bigr)^n$$

$$\leqslant d^n C^n\Bigl(\sum_{s_1\neq 0}\cdots\sum_{s_{n+1}\neq 0}\prod_{i=1}^{n} g(s_i)\Bigr). \quad (2.3.13)$$

The sums in (2.3.13) are taken over all $t_i, s_i \in \mathbb{Z}_d$, where $\mathbb{Z}_d$ is the one-dimensional lattice with step $d > 0$.

By means of an algorithm, we show that to any collection $(s_0, \dots, s_n)$, $s_0 = 0$, from the sum on the right-hand side of (2.3.13) no more than C^n diagrams from the left-hand side will be assigned, with contribution

$$g(s_1)\cdots g(s_n).$$

The algorithm will consist of no more than $2m_{\max}n$ steps. We shall enumerate the steps

$$(1, 1), \dots, (1, 2m_{\max}), \dots, (n, 1), \dots, (n, 2m_{\max}).$$

At step $(1, 1)$, we take s_1 and construct an edge from the first vertex to s_1. So we have constructed the vertices $1, s_1$ and the edge between them. Further, we proceed by induction. Suppose that edges with lengths $s_1, \dots, s_q$ have been constructed, and the next step is (i, j).

The rules of the algorithm are the following.

(1) At each step, we construct either one edge or no edge and, correspondingly, either one vertex or no vertex.

(2) If at step (i, j) we decide not to construct an edge, then at the succeeding steps (i, j'), $j' > j$, we do not construct edges, either.

(3) At step $(i, 1)$, we select one of the already constructed vertices v_i, and at the succeeding steps $(i, 1), \dots, (i, 2m_{\max})$ we may construct edges only from the vertex v. The vertex v_i will be called *used* at step $(i, 1)$.

(4) The selection of a vertex v_i is done uniquely by means of the following rule: v_i is the first (having the least number) of the already constructed vertices that has not been "used" at previous steps, excluding the zero vertex; but if all constructed vertices have already been "used", then an unconstructed vertex having the least number is selected, excluding the zero vertex.

(5) The algorithm stops either at step $(n, 2m_{\max})$ or in the case when there are no unused vertices or else all n edges have been constructed, i.e., all the $s_1, \dots, s_n$ have been used up. It is easy to see that each diagram will be constructed my means of the algorithm, and each collection $s_1, \dots, s_n$ will have been used no more than C^n times, where C depends on $m_{\max}$, since at steps $(i, 1), \dots, (i, 2m_{\max})$, for each i, $1 \leqslant i \leqslant n$, the algorithm has the following branches:

(a) the edge is drawn either to the right or to the left;

(b) at the vertex v_i, during these steps no more than $2m_{\max}$ lines are constructed;

(c) if the edge is drawn to the right, we can either reach the zero vertex or not.

For $d \to 0$, inequality (2.3.13) yields inequality (2.3.12). Lemma 2.7 is proved, and so is Theorem 2.5 for interactions belonging to $\mathscr{A}_e^0$. ■

To generalize the proof of Lemma 2.6 to the class $\mathscr{A}_e$, it is easy to see that it is sufficient to represent $V \in \mathscr{A}_e$ in the form (2.2.6) so that the estimates (2.2.7)–(2.2.9) hold, but this has been proved in §2.2. The proof of Theorem 2.5 is complete. ■

§2.4. Unitary equivalence of Hamiltonians for free and locally perturbed Fermi gas in the ground state

We shall consider free and locally perturbed Fermi gas in the ground state, i.e., for $\beta \to \infty$. The ground state $\langle \cdot \rangle$ is specified by

$$\langle A \rangle = (A\Omega, \Omega). \tag{2.4.1}$$

We shall prove the theorem that follows.

THEOREM 2.8. *Let* $\mathscr{H} = L_2(\mathbb{R}^\nu, dx)$, $H_0 = d\Gamma(h)$ *and* $H_\varepsilon = H_0 + \varepsilon V$, *where* $h \in \mathbb{H}$ *and* $V \in \mathscr{A}$. *Then for* $\nu \geqslant 3$ *there exists* $\varepsilon_0 > 0$ *such that for* $|\varepsilon| \leqslant \varepsilon_0$ *the direct wave operators*

$$W_\pm = \operatorname*{s-lim}_{t\to\pm\infty} e^{-it(H_0+\varepsilon V)} e^{itH_0}, \tag{2.4.2}$$

exist and are invertible, with

$$H_0 + \varepsilon V = W_\pm H_0 W_\pm^{-1}. \tag{2.4.3}$$

REMARK. The statement in Theorem 2.8 concerning the interaction $V \in \mathscr{A}_e$ follows from Theorem 2.5. Indeed, since in this case there is no polarization of vacuum, i.e., $V\Omega = 0$, we have

$$e^{-it(H_0+\varepsilon V)}\Omega = e^{itH_0}\Omega \equiv 0 \tag{2.4.4}$$

for all $t \in \mathbb{R}$. Therefore, for

$$A = a^*(f_1)\cdots a^*(f_n), \quad \hat{f}_i \in C_0^\infty(\mathbb{R}^\nu), \tag{2.4.5}$$

the limit

$$\lim_{t\to\pm\infty} e^{-it(H_0+\varepsilon V)} e^{ith_0} A\Omega$$
$$= \lim_{t\to\pm\infty} e^{-it(H_0+\varepsilon V)} e^{itH_0} A e^{-itH_0} e^{it(H_0+\varepsilon V)}\Omega = \hat{\gamma}(A)\Omega \tag{2.4.6}$$

exists for sufficiently small ε by Theorem 2.5. But since linear combinations of vectors like (2.4.5) are dense in $\mathscr{F}_a$, it follows that inverse wave operators exist. Obviously, the same also applies to direct wave operators.

PROOF OF THEOREM 2.8. In the case $V \in \mathscr{A}$, there is also no polarization of vacuum, but we cannot use Theorem 2.5, since there may be odd monomials in V. However, we shall use the idea behind the proof of Theorem 2.5. Put

$$\Phi_N = (\varphi_0, \varphi_1, \dots, \varphi_N, 0, \dots), \quad \Phi_N \in \mathscr{F}_{a,0}, \quad \widehat{\varphi}_i \in C_0^\infty(\mathbb{R}^{\nu i}).$$

The inverse wave operator $\widehat{W}_\pm \Phi_N$ may be represented as a perturbation theory expansion

$$\begin{gathered} \widehat{W}_\pm(t)\Phi_N = \Phi_N + \sum_{i=1}^{\infty}(-i\varepsilon)^n \int_{\Delta_n^{0,t}} dt_1 \cdots dt_n\, V(t_n)\cdots V(t_1)\Phi_N, \\ \widehat{W}_\pm \Phi_N = \lim_{t\to\pm\infty} \widehat{W}_\pm(t)\Phi_N, \end{gathered} \tag{2.4.6}$$

where

$$\Delta_n^{0,t} = \{(t_1, \dots, t_n), \quad 0 < t_1 < \cdots < t_n < t\},$$
$$V(t) = e^{itH_0} V e^{-itH_0}.$$

The integrand is the product of Wick monomials and may be represented as the sum

$$V(t_1)\cdots V(t_n) = \sum_G V_G(t_1, \dots, t_n) \tag{2.4.7}$$

of Wick monomials indexed by Friedrichs diagrams G (see §3.1). But the number of such diagrams is very large, and so we shall use another expansion. In this new expansion, we shall practically perform a partial resumming of Friedrichs diagrams.

To $V(t_v)$ we assign a vertex v of a certain graph and select in it the rightmost annihilation operator of the form $a(f^{(v)}(t_v))$, where $f^{(v)}(t_v) = e^{it_v h} f^{(v)}$; by means of the anticommutation relations

$$\begin{aligned} &a(f^{(v)}(t_v))a^*(f^{(v')}(t_{v'})) \\ &\qquad = -a^*(f^{(v')}(t_{v'}))a(f^{(v)}(t_{v'}) + (f^{(v')}(t_{v'}), f^{(v)}(t_v))), \end{aligned} \tag{2.4.8}$$

let us carry it to the right up to the vacuum vector Ω.

At each such permutation, one of the two terms on the right-hand side of (2.4.6) arises. If the first term appears, we go on with the translation, if the second one does, i.e., a pairing takes place, then we say that the edge (v, v') of the diagram has appeared. If no pairing has resulted from permutations, such a term makes a zero contribution, since $a(f)\Omega \equiv 0$ for all $f \in \mathscr{H}$. Thus we obtain exactly n edges $(v, v'(v))$, $v = 1, \dots, n$. Besides, if the pairing with the vector Φ_N took place, then we shall say that $v'(v) = 0$.

Obviously,

$$|(f^{(v')}t_{v'}), f^{(v)}(t_v))| \leqslant C\frac{1}{(1 + |t_v - t_{v'}|)^{\nu/2}}. \tag{2.4.9}$$

We consider the resulting graph with vertices n, $n-1, \dots, 1, 0$ and edges (lines) $(v, v'(v))$; by construction this graph is connected. We let $\mathbb{G}_N^n$ denote the class of all such graphs.

Note that the graphs from $\mathbb{G}_N^n$ differ from those belonging to $\mathbb{G}^n$ only in that there are not one but several (no more than N) lines incident to the zero vertex.

It is not difficult to see that in this case the following estimate is true:

$$\|v(t_n)\cdots v(t_1)\Phi_N\| \leqslant C(\Phi_N)C^n\left(\sum_{\{v'(v)\}\in\mathbb{G}_N^n}\prod_v \frac{1}{(|t_v - t_{v'(v)}|+1)^\delta}\right), \quad \delta > 1. \quad (2.4.10)$$

LEMMA 2.7′. *Let* $g \in L_1(\mathbb{R})$, $g(t) \geqslant 0$, *for all* $t \in \mathbb{R}$. *Then the following estimate holds for all* n:

$$\int_{\Delta_n^\infty}\left(\sum_{\{v'(v)\}\in\mathbb{G}_N^n}\prod_v g(t_v - t_{v'(v)})\right) ds_1 \cdots ds_n \leqslant (N!)C^n\left(\int_{\mathbb{R}} g(t)\,dt\right)^n, \quad (2.4.11)$$

where the sum is taken over all admissible diagrams, and the constant C *does not depend on* n.

PROOF. The proof of Lemma 2.7′ is completely similar to that of Lemma 2.7.

Lemma 2.7′ implies Theorem 2.8. ■

§2.5. Unitary equivalence of Hamiltonians for free and locally perturbed Fermi gas in KMS-state

We shall consider free and locally perturbed Fermi gas in the temperature state, i.e., for $\beta \in \mathbb{R}$, $\beta \neq 0$. Let $\mathscr{H} = L_2(\mathbb{R}^\nu, dx)$ and

$$h = -\Delta + \mu, \quad (2.5.1)$$

where $\mu \in \mathbb{R}$ is the chemical potential, and Δ is the Laplace operator in $\mathscr{H}$.

Define a free Fermi gas and a Fermi gas with local interaction $V = V^* \in \mathfrak{A}(\mathscr{H})$ for temperature $1/\beta$.

DEFINITION 2.4 [BR2]. $(\mathfrak{A}(\mathscr{H}), \tau_t^0, \langle\cdot\rangle_0, \beta)$ is called a *free Fermi gas at temperature* $1/\beta$, where the free dynamics τ_t^0 is generated by the one-particle Hamiltonian of the form (2.5.1), and $\langle\cdot\rangle_0$ is a single (τ_t^0, β)-KMS-state.

DEFINITION 2.5 [BR2]. $(\mathfrak{A}(\mathscr{H}), \tau_t^V, \langle\cdot\rangle_V, \beta)$ is called a *Fermi gas with local interaction* $V = V^* \in \mathfrak{A}(\mathscr{H})$ *at temperature* $1/\beta$, where τ_t^V is a locally perturbed dynamics, and $\langle\cdot\rangle_V$ is a single (τ_t^V, β)-KMS-state, which is defined in terms of $\langle\cdot\rangle_0$ as follows

$$\langle A\rangle_V = \frac{\langle F^*AF\rangle_0}{\langle F^*F\rangle_0}, \qquad \forall A \in \mathfrak{A}(\mathscr{H}), \quad (2.5.2)$$

where $F \in \mathfrak{A}(\mathscr{H})$ (see [BR2]), and

$$F \stackrel{\text{def}}{=} e^{-\beta/2(H_0+V)} e^{\beta/2H_0}$$
$$= 1 + \sum_{n=1}^{\infty} (-1^n \int_0^{\beta/2} \int_0^{s_1} \cdots \int_0^{s_{n-1}} \tau_{is_1}(V) \cdots \tau_{is_n}(V)\, ds_1 \cdots ds_n, \tag{2.5.3}$$

where the series (2.5.3) converges in the norm.

DEFINITION 2.6. A C^*-dynamical system $(\mathfrak{A}, \tau)$ is called $L_1(\mathfrak{A}^0)$*-asymptotically abelian* if

$$\int_{-\infty}^{\infty} \|[A, \tau_t(B)]\|\, dt < \infty \tag{2.5.5}$$

for all A, B from a dense (in the norm) $*$-subalgebra $\mathfrak{A}^0$ in $\mathfrak{A}$.

There exists the following simple connection between KMS-states and Møller morphisms.

PROPOSITION 2.9 [BR2]. *Let* $(\mathfrak{A}, \tau)$ *be* $L_1(\mathfrak{A}^0)$*-asymptotically abelian, then for each* $V = V^* \in \mathfrak{A}^0$ *there exist direct Møller morphisms*

$$\gamma_\pm(A) = \lim_{t\to\pm\infty} \tau_{-t}^V \tau_t^0(A), \qquad A \in \mathfrak{A}(\mathscr{H}),$$

and if $\langle \cdot \rangle_V$ *is* (τ_t^V, β)*-KMS-state for* $\beta \in \mathbb{R} \setminus \{0\}$, *then the state* $\langle \gamma_\pm(\cdot)\rangle$ *is also a* (τ_t, β)*-KMS-state.*

Note that $(\mathfrak{A}_e(\mathscr{H}), \tau_t^0)$ is $L_1(\mathfrak{A}_e^0(\mathscr{H}))$-asymptotically abelian, where τ_t^0 is generated by the one-particle Hamiltonian h.

THEOREM 2.10 [BM1]. *Suppose that* $\nu \geqslant 3$, $\beta \in \mathbb{R} \setminus \{0\}$, *and* $V = V^* \in \mathscr{A}_e$. *Then there exists* $\varepsilon_0 = \varepsilon_0(V, \nu) > 0$ *such that the operators* H_{GNS}^0 *and* $H_{\text{GNS}}^{\varepsilon V}$ *are unitarily equivalent for* $|\varepsilon| \leqslant \varepsilon_0$.

PROOF. By Theorem 2.5, we can find an $\varepsilon_0 = \varepsilon_0(V, \nu) > 0$ such that for $|\varepsilon| \leqslant \varepsilon_0$ Møller morphisms $\gamma_\pm$ exist and are invertible. Proposition 2.9 implies that the state $\langle \gamma_\pm(\cdot)\rangle_{\varepsilon V}$ is a (τ_t, β)-KMS-state on $\mathfrak{A}_e(\mathscr{H})$, but such a state is unique (see [BR2]). However, due to the gauge-invariance of $\langle \cdot \rangle_0$ and $\langle \cdot \rangle_{\varepsilon V}$ the following equality holds

$$\langle \gamma_\pm(A)\rangle_{\varepsilon V} \equiv \langle A \rangle_0 \tag{2.5.6}$$

for all $A \in \mathfrak{A}(\mathscr{H})$.

Define the operator $U_\pm : \mathscr{H}_{\text{GNS}}^0 \to \mathscr{H}_{\text{GNS}}^{\varepsilon V}$ as

$$U_\pm(\pi_0(A)\Omega_0) = \pi_{\varepsilon V}(\gamma_\pm(A))\Omega_{\varepsilon V}, \tag{2.5.7}$$

where for conciseness we set $\pi_0 \equiv \pi_{\langle \cdot \rangle_0}$, $\pi_{\varepsilon V} \equiv \pi_{\langle \cdot \rangle_{\varepsilon V}}$, $\Omega_0 \equiv \Omega_{\langle \cdot \rangle_0}$, and $\Omega_{\varepsilon V} \equiv \Omega_{\langle \cdot \rangle_{\varepsilon V}}$. The operators $U_\pm$ are unitary. In fact, we have

$$\begin{aligned}(U_\pm\pi_0(A)\Omega_0, U_\pm\pi_0(B)\Omega_0) &= (\pi_{\varepsilon V}(\gamma_\pm(A))\Omega_{\varepsilon V}, \pi_{\varepsilon V}(\gamma_\pm(B))\Omega_{\varepsilon V}) \\ &= \langle (\gamma_\pm(B))^*\gamma_\pm(A)\rangle_{\varepsilon V} = \langle \gamma_\pm(B^*A)\rangle_{\varepsilon V} \\ &= \langle B^*A\rangle_0 = (\pi_0(A)\Omega_0, \pi_0(B)\Omega_0).\end{aligned}$$

Obviously $\operatorname{Ran} U_{\pm} = \mathscr{H}_{\text{GNS}}^{\varepsilon V}$, and the inverse operators $U_{\pm}^{-1}$ can be defined as follows

$$U_{\pm}^{-1}\left(\pi_{\varepsilon V}(A)\Omega_{\varepsilon V}\right) = \pi_0\left(\gamma_{\pm}^{-1}(A)\right)\Omega_0 \tag{2.5.8}$$

for all $A \in \mathfrak{A}(\mathscr{H})$. By the definition of the operators H_{GNS}^0 and $H_{\text{GNS}}^{\varepsilon V}$, we have

$$e^{itH_{\text{GNS}}^0}\pi_0(A)\Omega_0 = \pi_0\left(\tau_t^0(A)\right)\Omega_0,$$
$$e^{itH_{\text{GNS}}^{\varepsilon V}}\pi_{\varepsilon V}(A)\Omega_{\varepsilon V} = \pi_{\varepsilon V}\left(\tau_t^{\varepsilon V}(A)\right)\Omega_{\varepsilon V}.$$

Hence by the relation $\tau_t^{\varepsilon V}(A) = \gamma_{\pm}\tau_t^0\gamma_{\pm}^{-1}$ we get

$$\begin{aligned} U_{\pm}^{-1}e^{itH_{\text{GNS}}^{\varepsilon V}}U_{\pm}\pi_0(A)\Omega_0 &= U_{\pm}^{-1}e^{itH_{\text{GNS}}^{\varepsilon V}}\pi_{\varepsilon V}(\gamma_{\pm}(A))\Omega_{\varepsilon V} \\ &= U_{\pm}^{-1}\pi_{\varepsilon V}\left(\tau_t^{\varepsilon V}(\gamma_{\pm}(A))\right)\Omega_{\varepsilon V} = \pi_0\left(\gamma_{\pm}^{-1}\tau_t^{\varepsilon V}\gamma_{\pm}(A)\right)\Omega_0 \\ &= \pi_0\left(\tau_t^0(A)\right)\Omega_0 = e^{itH_{\text{GNS}}^0}\pi_0(A)\Omega_0. \end{aligned}$$

Thus

$$e^{itH_{\text{GNS}}^0} \equiv U_{\pm}^{-1}e^{itH_{\text{GNS}}^{\varepsilon V}}U_{\pm}, \quad \forall t \in \mathbb{R}, \tag{2.5.9}$$

and consequently

$$U_{\pm}^{-1}H_{\text{GNS}}^{\varepsilon V}U = H_{\text{GNS}}^0. \tag{2.5.10}$$

The theorem is proved. ■

§2.6. The operator H_{GNS}^V in Hilbert space $\mathscr{H}_{\text{GNS}}^0$

Suppose that $(\mathscr{H}_{\text{GNS}}^0, \pi_0, \Omega_0)$ and $(\mathscr{H}_{\text{GNS}}^V, \pi_V, \Omega_V)$ are cyclic representations of $\mathfrak{A}(\mathscr{H})$ for the states $\langle\cdot\rangle_0$ and $\langle\cdot\rangle_V$, respectively. We let H_{GNS}^0 and H_{GNS}^V denote the generators of unitary groups acting in $\mathscr{H}_{\text{GNS}}^0$ and $\mathscr{H}_{\text{GNS}}^V$ and generated by the dynamics τ_t^0 and τ_t^V in $\mathfrak{A}(\mathscr{H})$.

We shall define the operator $U: \mathscr{H}_{\text{GNS}}^V \to \mathscr{H}_{\text{GNS}}^0$ by

$$U\pi_V(A)\Omega_V = C_V\pi_0(AF)\Omega_0, \tag{2.6.1}$$

where F is defined in (2.5.3) and

$$C_V = \langle F^*F\rangle_0. \tag{2.6.2}$$

Note that the operator U is unitary. In fact, for all $A, B \in \mathfrak{A}(\mathscr{H})$ we have

$$\begin{aligned} \left(U\pi_V(A)\Omega_V, U\pi_V(B)\Omega_V\right) &= C_V^2\langle(BF)^*AF\rangle_0 = C_V^2\langle F^*B^*AF\rangle_0 \\ &= \langle B^*A\rangle_V = \left(\pi_V(A)\Omega_V, \pi_V(B)\Omega_V\right). \end{aligned}$$

We shall assume that the perturbation V is defined by polynomials in analytic vectors of the operator h. The element F of the C^*-algebra $\mathfrak{A}(\mathscr{H})$ is invertible for small ε, and (see [BR2])

$$\begin{aligned} F^{-1} &= e^{-\beta/2H_0}e^{\beta/2(H_0+V)} \\ &= 1 + \sum_{n=1}^{\infty}(-\varepsilon)^n\int_0^{\beta/2}\int_0^{s_n}\cdots\int_0^{s_2}\tau_{is_1}(V)\cdots\tau_{is_n}(V)\,ds_1\cdots ds_n. \end{aligned} \tag{2.6.3}$$

Hence the inverse operator $U^{-1}: \mathscr{H}^0_{\mathrm{GNS}} \to \mathscr{H}^V_{\mathrm{GNS}}$ acts as follows

$$U^{-1}\pi_0(A)\Omega_0 = C_V^{-1}\pi_V(AF^{-1})\Omega_V, \quad \forall A \in \mathfrak{A}(\mathscr{H}). \tag{2.6.4}$$

PROPOSITION 2.11. *The operator H^V_{GNS} defined in the space $\mathscr{H}^V_{\mathrm{GNS}}$ is unitarily equivalent to the operator H' acting in $\mathscr{H}^0_{\mathrm{GNS}}$ for all $A \in \mathfrak{A}(\mathscr{H})$ such that $\pi_0(A)\Omega_0 \in \mathscr{D}(H^0_{\mathrm{GNS}})$ according to the formula*

$$H'\pi_0(A)\Omega_0 = H^0_{\mathrm{GNS}}\pi_0(A)\Omega_0 + i(\pi_0(VA)\Omega_0 - \pi_0(AV_{\beta/2}))\Omega_0, \tag{2.6.5}$$

where

$$V_{\beta/2} = \tau_{i\beta/2}(V) \in \mathfrak{A}(\mathscr{H})$$

and $\mathscr{D}(H^0_{\mathrm{GNS}})$ is the domain of the operator H^0_{GNS}.

PROOF. We shall prove that the operator in (2.6.5) coincides with the operator $UH^V_{\mathrm{GNS}}U^{-1}$. We have

$$\begin{aligned} UH^V_{\mathrm{GNS}}U^{-1}\pi_0(A)\Omega_0 &= UH^V_{\mathrm{GNS}}(C_V^{-1}\pi_V(AF^{-1})\Omega_V) \\ &= C_V^{-1}U\pi_V(i[H_0 + V, AF^{-1}])\Omega_V, \end{aligned} \tag{2.6.6}$$

where the latter equality follows from the definition of H^V_{GNS}. Further, by transforming (2.6.6), we obtain

$$\begin{aligned} UH^V_{\mathrm{GNS}}U^{-1}\pi_0(A)\Omega_0 &= \pi_0(i[H_0 + V, AF^{-1}]F)\Omega_0 \\ &= \pi_0(i(H_0 + V)A)\Omega_0 - \pi_0(iAF^{-1}(H_0 + V)F)\Omega_0 \\ &= \pi_0(i(H_0 + V)A)\Omega_0 - \pi_0(iA\tau_{i\beta/2}(H_0 + V))\Omega_0, \end{aligned} \tag{2.6.7}$$

since

$$\begin{aligned} F_{-1}(H + V)F &= e^{-\beta/2H_0}e^{\beta/2(H_0+V)}(H_0 + V)e^{-\beta/2(H_0+V)}e^{\beta/2H_0} \\ &= e^{-\beta/2H_0}(H_0 + V)e^{\beta/2H_0} \\ &= e^{-\beta/2H_0}Ve^{\beta/2H} + H_0, \end{aligned} \tag{2.6.8}$$

where we have first used the commutativity of the operators $H_0 + V$ and $e^{\pm\beta/2(H_0+V)}$ and the property that the equalities hold

$$e^{\beta/2(H_0+V)}e^{-\beta/2(H_0+V)} = I_{\mathscr{H}}, \tag{2.6.9}$$

for the set of analytic vectors of the operator H_0 in $\mathscr{F}_a$, and then the commutativity of the operators H_0 and $e^{\pm\beta H_0}$ and the relation

$$e^{-\beta/2H_0}e^{\beta/2H_0} = I_{\mathscr{H}} \tag{2.6.10}$$

which holds for the set of analytic vectors of the operator H_0. It is obvious that (2.6.5) follows from (2.6.7) and (2.6.8). ■

We have defined the operator $H' = UH^V_{\mathrm{GNS}}U^{-1}$ in the Hilbert space $\mathscr{H}^0_{\mathrm{GNS}}$, and now we shall consider the action of the operator H' in $\widehat{\mathscr{H}}_{\mathrm{GNS}}$ constructed in Theorem 1.2. Suppose that

$$V = a^*(f)a^*(g)a(g)a(f), \quad f, g \in \mathscr{H}, \quad (f, g) = 0, \tag{2.6.11}$$

where f and g belong to the set of analytic vectors h. Then

$$V_{\beta/2} = a^*(e^{-\beta/2h}f)a^*(e^{-\beta/2h}g)a(e^{\beta/2h}g)a(e^{\beta/2h}f).$$

Consider the operator H' in $\widehat{\mathscr{H}}_{\mathrm{GNS}}$. By Theorem 1.4, the subspaces $\mathscr{H}_{m,n}$ are invariant with respect to the operator H^0_{GNS}, but this is not so for the operator H':

$$H'\colon \mathscr{H}_{m,n} \to \mathscr{H}_{m-2,n-2} \oplus \mathscr{H}_{m-1,n-1} \oplus \mathscr{H}_{m,n} \oplus \mathscr{H}_{m+1,n+1} \oplus \mathscr{H}_{m+2,n+2}.$$

PROPOSITION 2.12. *Denote by $b^*(f)$, $b(f)$ ($c^*(f)$, $c(f)$) the creation-annihilation operators in F_b (F_c). The right multiplication by the operator $a^*(f)$ corresponds to the operator*

$$b^*(B_1f) \otimes 1 + (-1)^m 1 \otimes c(B_2f), \tag{2.6.12}$$

and the multiplication by the operator $a(f)$ corresponds to the operator

$$b(B_1f) \otimes 1 + (-1)^m 1 \otimes c^*(B_2f). \tag{2.6.13}$$

The left multiplication by the operator $a^*(f)$ corresponds to the operator on the subspace $\mathscr{H}_{mn}$

$$(-1)^{m+n} b^*(B_1f) \otimes 1 + (-1)^n 1 \otimes c(B_2f), \tag{2.6.14}$$

and the multiplication by the operator $a(f)$ corresponds to the operator

$$(-1)^{m+n} b(B_1f) \otimes 1 + (-1)^n 1 \otimes c^*(B_2f). \tag{2.6.15}$$

PROOF. We shall prove that the right multiplication by the operator $a^*(f)$ is given by (2.6.12); the rest of the proof is completely analogous.

For $A =: a^*(f_1)\cdots a^*(f_m)a(f_{m+1})\cdots a(f_{m+n}):$, we have

$$\pi_0(A)\Omega_0 = b^*(B_1f_1)\cdots b^*(B_1f_m)\Omega_b \otimes c^*(B_2f_{m+1})\cdots c^*(B_2f_{m+n})\Omega_c,$$

$$\begin{aligned}\pi_0(a^*(f)A)\Omega_0 &= \pi_0(: a^*(f)a^*(f_1)\cdots a^*(f_m)a(f_{m+1})\cdots a(f_{m+n}) :)\Omega_0 \\ &\quad + \pi_0\Bigg(\sum_{i=m+1}^{m+n} (-1)^{i-1}(f, Bf_i) : a^*(f_1)\cdots a^*(f_m) \\ &\qquad\qquad \times a(f_{m+1})\cdots \overset{i}{\check{a(f_i)}}\cdots a(f_{m+n}) : \Omega_0\Bigg) \\ &= (b^*(B_1f) \otimes 1 + 1 \otimes c(B_2f))b^*(B_1f_1)\cdots b^*(B_1f_m)\Omega_b \\ &\qquad \otimes c^*(B_2f_{m+1})\cdots c^*(B_2f_{m+n})\Omega_c. \quad \blacksquare\end{aligned}$$

The right multiplication by V corresponds to the operator

$$V_1' = (b^*(B_1 f) \otimes 1 + 1 \otimes c(B_2 f))(b^*(B_1 g) \otimes 1 + 1 \otimes c(B_2 g)) \times (b(B_1 g) \otimes 1 + 1 \otimes c^*(B_2 g))(b(B_1 f) \otimes 1 + 1 \otimes c^*(B_2 f)), \quad (2.6.16)$$

and the left multiplication by $V_{\beta/2}$ corresponds to the operator

$$V_2' = (b^*(B_3 f) \otimes 1 + 1 \otimes c(B_4 f))(b^*(B_3 g) \otimes 1 + 1 \otimes c(B_4 g)) \times (b(B_3 g) \otimes 1 + 1 \otimes c^*(B_4 g))(b(B_3 f) \otimes 1 + 1 \otimes c^*(B_4 f)). \quad (2.6.17)$$

Observe that $B_3 = B_1 e^{\beta/2h}$, $B_4 = B_2 e^{-\beta/2h}$.

It follows from (2.6.16) and (2.6.17) that

$$\begin{aligned} V_1' - V_2' = &\sum_{i,j,k,l} \alpha_{i,j,k,l} b^\#(u_i) b^\#(u_j) b^\#(u_k) b^\#(u_l) \otimes 1 \\ &+ \sum_{i,j,k,l} \beta_{i,j,k,l} b^\#(u_i) b^\#(u_j) b^\#(u_k) \otimes c^\#(u_l) \\ &+ \sum_{i,j,k,l} \gamma_{i,j,k,l} b^\#(u_i) b^\#(u_j) \otimes c^\#(u_k) c^\#(u_l) \\ &+ \sum_{i,j,k,l} \delta_{i,j,k,l} b^\#(u_i) \otimes c^\#(u_j) c^\#(u_k) c^\#(u_l) \\ &+ \sum_{i,j,k,l} \varepsilon_{i,j,k,l} 1 \otimes c^\#(u_i) c^\#(u_j) c^\#(u_k) c^\#(u_l), \end{aligned}$$

where $\alpha_{i,j,k,l}$, $\beta_{i,j,k,l}$, $\gamma_{i,j,k,l}$, $\delta_{i,j,k,l}$, $\varepsilon_{i,j,k,l}$ are some constants that may be equal to zero, $u_1 = B_1 f$, $u_2 = B_1 g$, $u_3 = B_2 f$, $u_4 = B_2 g$, $u_5 = B_3 f$, $u_6 = B_3 g$, $u_7 = B_4 f$, $u_8 = B_4 g$.

Observe that there is a term of the form $b^* b^* \otimes c^* c^*$ in the expression for $V_1' - V_2'$, which yields the relation

$$(V_1' - V_2')\Omega_b \otimes \Omega_c \neq 0.$$

This means polarization of vacuum, as was to be expected, since

$$\langle (V - V_{\beta/2})^*(V - V_{\beta/2}) \rangle_0 > 0 \qquad \text{for } \beta > 0.$$

Note that $\langle (V - V_{\beta/2})^*(V - V_{\beta/2}) \rangle_0 = 0$ for $\beta = 0$, and there is no polarization of vacuum.

CHAPTER 3. THE LINKED CLUSTER THEOREM. ASYMPTOTIC COMPLETENESS FOR INTERACTIONS POLARIZING VACUUM

§3.0. Introduction

We shall now remove the constraint on the interaction $V\Omega = 0$. We shall prove that, generally speaking, the spectrum of the Hamiltonian for the perturbed dynamics $H_0 + \varepsilon V$ is shifted or, to be more precise, for small ε there exists a real number λ_ε such that $H_0 + \varepsilon V$ is unitarily equivalent to the operator $H_0+\lambda_\varepsilon E$. Formally speaking, this fact is well known, e.g. from the famous linked cluster theorem, which represents the main computation instrument for perturbation theory in the quantum theory of many particles. The proof of the convergence of the series in this theorem is the main result of this chapter. This permits us to rigorously prove all its formal corollaries that can be found in the classical books by Friedrichs [Fr] and Hepp [H].

We shall consider the following two situations:

(1) $\mathcal{F}_a = \mathcal{F}_{\text{as}}(L_2(\mathbb{R}^\nu))$ is an antisymmetric Fock space over the space $L_2(\mathbb{R}^\nu)$, $\nu \geqslant 3$, $H = H_0+\varepsilon V$, where $H_0 = d\Gamma(h)$ or in the k-representation

$$H_0 = \int_{\mathbb{R}^\nu} \mathfrak{h}(k) a^*(k) a(k)\, dk, \tag{3.0.1}$$

where $V \in \mathscr{A}_e$ and

$$\mathfrak{h}(k) = \sqrt{m^2+k^2}, \quad m > 0,$$

(in the relativistic case) or

$$\mathfrak{h}(k) = Kk^2$$

(in the massless case).

Since $V = V^* \in \mathfrak{A}(\mathscr{H})$, it follows that $H = H_0 + \varepsilon V$ is a selfadjoint operator in $\mathcal{F}_a$ with the same dense domain as for H_0.

(2) $\mathcal{F}_a = \mathcal{F}_{\text{as}}(l_2(\mathbb{Z}^\nu))$, $H_0 = d\Gamma(h)$, $h = -\Delta + \mu$, is a lattice Laplacian plus a constant μ. Consider the operator H_0 in the k-representation. As a result of the Fourier transformation, $l_2(\mathbb{Z}^\nu)$ is mapped into $L_2(\mathbb{T}^\nu)$, where $\mathbb{T}^\nu = [0, 2\pi]^\nu$ is the ν-dimensional torus, $\mathcal{F}_a = \mathcal{F}_{\text{as}}(L_2(\mathbb{T}^\nu))$, and

$$H_0 = \int_{\mathbb{T}^\nu} \mathfrak{h}(k) a^*(k) a(k)\, dk, \tag{3.0.2}$$

where $V = V^* \in \mathscr{A}_e$ and

$$\mathfrak{h}(k) = \sum_{i=1}^{\nu} 2(1-\cos(k_i)) + \mu. \tag{3.0.3}$$

We shall assume in this case that $\mu \geqslant 0$. This implies that the spectrum of the operator H_0 is nonnegative. Note that in this case we shall restrict ourselves to the interaction V of the form

$$V = \sum_{i=1}^{d} a^*(f_{i,1}) \cdots a^*(f_{i,m_i}) a(f_{i,m_i+1}) \cdots a(f_{i,m_i+l_i}), \tag{3.0.4}$$

where $m_i + l_i$ is even and $f_{i,j} \in C_0^\infty(\mathbb{T}^\nu)$ for all i, j.

All the results proved below hold in both cases. For the sake of readability, however, we shall carry out the proofs only for case (2).

Below (§3.6) we shall no longer require that the one-particle Hamiltonian be nonnegative ($\mu \geqslant 0$).

§3.1. Friedrichs diagrams. The algebra of Wick exponentials. The operations $\Gamma_\pm$ and Γ

We shall consider the vacuum (ground) state on the C^*-algebra CAR $\mathfrak{A}(L_2(\mathbb{R}^\nu))$

$$\langle A\rangle = (A\Omega, \Omega), \tag{3.1.1}$$

which is a quasifree gauge-invariant state with $B \equiv 0$.

The Wick brackets with respect to this state (see Chapter 1) simply mean that annihilation operators must be moved to the right of creation operators with due regard for the sign rule. In other words, the Wick brackets act identically on monomials of the form

$$W = a^*(f_1)\cdots a^*(f_m)a(f_{m+1})\cdots a(f_{m+p}).$$

The product of several Wick monomials may be expanded into the sum of Wick monomials. The relevant rule can be easily formulated in terms of diagrams.

With every monomial

$$W_i = \int w_i(k_{i1}, \dots, k_{im_i}, k_{i,m_i+1}, \dots, k_{i,m_i+l_i})a^*(k_{i1})\cdots a^*(k_{i,m_i}) \times a(k_{i,m_i+1})\cdots a(k_{i,m_i+l_i})\, dk_{i1}\cdots dk_{i,m_i+l_i}, \tag{3.1.2}$$

we may associate a diagram G_i with m_i (l_i) numbered left (right) legs, with $a^*(k_{i,j})$ corresponding to the jth left leg and $a(k_{i,m_i+j})$ to the jth right leg. Then

$$W_1\cdots W_n = \sum_G W_G, \tag{3.1.3}$$

where the sum is taken over all possible pairings leading to the diagram G in the disjoint union of the diagrams G_i, and

$$W_G = \int \prod_i w_i \prod_{ij} dk_{ij} \prod{}'\delta(k_{ij} - k_{i'j'}) : \prod{}'' a^\#(k_{ij}) : (-1)^{\pi(G)}, \tag{3.1.4}$$

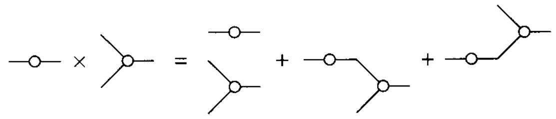
where the product $\prod'$ is taken over all edges (pairs of connected legs i, j and i', j') of the diagram G and $\prod''$ over all unpaired legs. The relations (3.1.3) and (3.1.4) are easy to prove if we transpose each annihilation operator to the right of creation operators using the anticommutation relations, with $\pi(G)$ representing the total number of such transpositions.

The following example will illustrate the notation:

$$W_1 = a^*(f_1)a(f_2), \qquad W_2 = a^*(f_3)a^*(f_4)a(f_5),$$

$$\begin{aligned} W_1W_2 &= \,:W_1W_2: + W_1 -\!\circ\!- W_2 \\ &= a^*(f_1)a^*(f_3)a^*(f_4)a(f_2)a(f_5) \\ &\quad + (f_3, f_2)a^*(f_1)a^*(f_4)a(f_5) - (f_4, f_2)a^*(f_1)a^*(f_3)a(f_5) \end{aligned}$$

or schematically



Note that : $W_1 W_2$: represents the term in the expansion of $W_1 W_2$ in which not a single pairing has occured during the transpositions of annihilation operators.

We shall introduce the following notation (see [H]):

$(W_1 \cdots W_n)_C$ is the sum over all connected diagrams G in (3.1.3);

$(\cdots)_{00}$ is the sum over all connected diagrams without external edges in (3.1.3);

$(\cdots)_L = (\cdots)_C - (\cdots)_{00}$ is the sum over all connected diagrams with at least one external edge in (3.1.3);

$(\cdots)_{CR}$ is the sum over all diagrams from the class $(\cdots)_L$ with external edges belonging only to creation operators.

Henceforth, we shall only need some algebraic properties of series involving Wick monomials.

For a formal power series

$$A = \sum_{m,l=0}^{\infty} x^m y^l A_{ml},$$

where A_{ml} are Wick monomials, we set by definition

$$:A: = \sum_{m,l=0}^{\infty} x^m y^l :A_{ml}:.$$

Proposition 3.1 [H]. *Let*

$$A = \sum_{\substack{m,l=0\\ m+l>0}}^{\infty} x^m y^l A_{ml}, \qquad B = \sum_{\substack{m,l=0\\ m+l>0}}^{\infty} x^m y^l B_{ml}$$

be formal power series in even Wick monomials, where m (l) *is the degree with respect to creation (annihilation) operators.*

Then the following identities hold:

$$A : \exp B : \; = : (A : \exp B :)_C \exp B :, \tag{3.1.5}$$

$$: \exp B : A = : (: \exp B : A)_C \exp B :, \tag{3.1.6}$$

where $(\cdot)_C$ *refers to all connected diagrams.*

Proof. This can be reduced to a simple combinatorial argument with reference to the same powers $x^m y^l$ (see [Fr], [H]). ■

DEFINITION 3.1 [Fr], [H]. A left connected product $W_1 \angle : W_2 \cdots W_n :$ (right connected product $: W_2 \cdots W_n : \llcorner\, W_1$) is the sum of all Wick monomials belonging to $W_1 : W_2 \cdots W_n :$ $(: W_2 \cdots W_n : W_1)$ whose graphs are connected, each W_i corresponding to a particular vertex.

REMARK. Usually, identities (3.1.5) and (3.1.6) are written in the form (see [Fr], [H])

$$A : \exp B : \; = : (A \angle : \exp B :) \exp B : ,$$
$$: \exp B : A = : (: \exp B : \llcorner A) \exp B : .$$

DEFINITION 3.2 [H]. We define Friedrichs operations $\Gamma_{\pm\varkappa}$ on the monomials

$$U_{mp} = \int u_{mp}(k_1, \dots, k_{m+p}) a^*(k_1) \cdots a^*(k_m) \times a(k_{m+1}) \cdots a(k_{m+p})\, dk_1 \cdots dk_{m+p},$$

for $\varkappa > 0$, by

$$\begin{aligned} \Gamma_{\pm\varkappa}(U_{mp}) &\stackrel{\text{def}}{=} i \int_{\pm\infty}^{0} e^{-\varkappa|t|} \tau_t^0(U_{mp})\, dt \\ &= \int u_{mp}(k_1, \dots, k_{m+p})(E_C - E_A \pm i\varkappa)^{-1} a^*(k_1) \cdots a^*(k_m) \\ &\qquad \times a(k_{m+1}) \cdots a(k_{m+p})\, dk_1 \cdots dk_{m+p}, \end{aligned} \tag{3.1.7}$$

and $\Gamma_{\pm}(U_{mp})$ as the strong limit of $\Gamma_{\pm\varkappa}(U_{mp})$ as $\varkappa \to 0$, where

$$E_C = \sum_{j=1}^{m} \mathfrak{h}(k_j), \qquad E_A = \sum_{j=m+1}^{m+p} \mathfrak{h}(k_j),$$

and the Glimm operation Γ by

$$\Gamma(U_{mp}) = \int u_{mp}(k_1, \dots, k_{m+p}) E_C^{-1} a_1^*(k) \cdots a_m^*(k) \times a(k_{m+1}) \cdots a(k_{m+p})\, dk_1 \cdots dk_{m+p}. \tag{3.1.8}$$

Also we define the operation

$$\left[H_0^{(\pm\varkappa)}, U_{mp}\right] \stackrel{\text{def}}{=} \frac{d}{dt} \left[e^{-\varkappa|t|} \tau_t^0(U_{mp})\right]\Big|_{t=\pm 0}. \tag{3.1.9}$$

It can be reduced to replacing the kernel U_{mp} by the kernel

$$u_{mp}(k_1, \dots, k_{m+p})(E_A - E_C \pm i\varkappa).$$

PROPOSITION 3.2. *For $\varkappa > 0$, the following relations are valid*:

$$\left[H_0^{(\pm\varkappa)}, \Gamma_{\pm\varkappa}(U_{mp})\right] = U_{mp}, \tag{3.1.10}$$

$$\left[H_0^{(\pm\varkappa)}, : \exp(\Gamma_{\pm\varkappa}(U_{mp})) :\right] = : U_{mp} \exp \Gamma_{\pm\varkappa}(U_{mp}) : . \tag{3.1.11}$$

PROOF. The first equality follows from the definition of $\Gamma_{\pm\varkappa}$. The first equality of Proposition 3.2 yields

$$[H_0^{(\pm\varkappa)}, : \exp(\Gamma_{\pm\varkappa}(U_{mp})) :]$$
$$= [H_0^{(\pm\varkappa)} \underset{1}{-\circ-} \Gamma_{\pm\varkappa}(U_{mp}) - \Gamma_{\pm\varkappa}(U_{mp}) \underset{1}{-\circ-} H_0^{(\pm\varkappa)}] : \exp(\Gamma_{\pm\varkappa}(U_{mp})) :,$$

where $\underset{1}{-\circ-}$ means that Wick monomials with one pairing are taken. But the expression in brackets is equal to (3.1.11). ■

§3.2. Adiabatic wave operators. Linked cluster theorem

We introduce the following additional notation (see [H]).

Define for $-\infty < t, s < \infty$ and $\varkappa > 0$ the evolution operator without adiabatic cut-off

$$U(t, s) = e^{itH_0} e^{-i(t-s)H} e^{isH_0} \tag{3.2.1}$$

and with adiabatic cut-off

$$U^{(\varkappa)}(t, s) = 1 - \int_s^t dr\, V^{(\varkappa)}(r) U^{(\varkappa)}(t, s), \tag{3.2.2}$$

where

$$V^{(\varkappa)}(r) \overset{\text{def}}{=} e^{-i\varkappa|r|} e^{irH_0} V e^{-irH_0}.$$

It is well known that for finite t, s and $\varkappa \geqslant 0$ (see [H]) the following relation holds:

$$U^{(\varkappa)}(t, s) = 1 + \sum_{i=1}^{\infty} (-i\varepsilon)^n \int_{\Delta_n^{s,t}} dt_1 \cdots dt_n\, V^{(\varkappa)}(t_1) \cdots V^{(\varkappa)}(t_n), \tag{3.2.3}$$

where $\Delta_n^{s,t} = \{(t_1, \dots, t_n), s < t_1 < \cdots < t_n < t\}$, and the series (3.2.3) converges in the norm.

The integrand is the product of Wick monomials and may be represented as the sum

$$V^{(\varkappa)}(t_1) \cdots V^{(\varkappa)}(t_n) = \sum_G W_G(t_1, \dots, t_n) \tag{3.2.4}$$

of Wick monomials indexed by Friedrichs diagrams G (see [Fr]).

Using the equality for $\varkappa > 0$

$$(\Gamma_{\pm\varkappa}(U))(t) = i \int_{\pm\infty}^t U^{(\varkappa)}(s)\, ds,$$

and integrating each term in the series (3.2.3) with respect to $\Delta_n^{0,\pm\infty}$ we get

$$U^{(\varkappa)}(0, \pm\infty) = 1 + \sum_{n=1}^{\infty} (-i\varepsilon)^n \Gamma_{\pm n\varkappa}(V \cdots \Gamma_{\pm 2\varkappa}(V \Gamma_{\pm\varkappa}(V)) \cdots). \tag{3.2.5}$$

Integration with respect to $\Delta_n^{\pm\infty, 0}$ yields

$$U^{(\varkappa)}(\pm\infty, 0) = 1 + \sum_{n=1}^{\infty} (-i\varepsilon)^n \Gamma_{\pm n\varkappa}(\cdots \Gamma_{\pm 2\varkappa}(\Gamma_{\pm\varkappa}(V) V) \cdots V). \tag{3.2.6}$$

THEOREM 3.3. *There exists* $\varepsilon_0 > 0$ *such that for* $|\varepsilon| \leqslant \varepsilon_0$ *and either of the two cases*:

(1) $-\infty < t, s < \infty$ *and* $\varkappa \geqslant 0$;

(2) $-\infty \leqslant t, s \leqslant \infty$ *and* $\varkappa > 0$

the series

$$\sum_{i=1}^{\infty}(-i\varepsilon)^n \int_{\Delta_n^{t,s}} dt_1 \cdots dt_n \, (V^{(\varkappa)}(t_1)\cdots V^{(\varkappa)}(t_n))_C \stackrel{\text{def}}{=} U^{(\varkappa)}(t, s)_C \qquad (3.2.7)$$

converges in the norm, and ε_0 *does not depend on* $t, s, \varkappa$. *The same holds if we substitute* $(\cdots)_{00}$, $(\cdots)_L$, *and* $(\cdots)_{CR}$ *instead of* $(\cdots)_C$.

The next theorem can be deduced from this statement in a nonformal way.

THEOREM 3.4 (Linked Cluster Theorem) (see Theorem 2.7 in [H]). *In the hypotheses of Theorem* 3.3, *the following equalities hold*:

$$U^{(\varkappa)}(t, s) = :\exp\left(U^{(\varkappa)}(t, s)_C\right):, \qquad (3.2.8)$$

$$\frac{U^{(\varkappa)}(t, s)}{(\Omega, U^{(\varkappa)}(t, s)\Omega)} = :\exp\left(U^{(\varkappa)}(t, s)_L\right):, \qquad (3.2.9)$$

where the series on the right-hand sides of (3.2.8) *and* (3.2.9) *converge in the norm.*

Let

$$T_{t,s}^{(\varkappa)} \stackrel{\text{def}}{=} \frac{U^{(\varkappa)}(t, s)}{(\Omega, U^{(\varkappa)}(t, s)\Omega)}. \qquad (3.2.10)$$

THEOREM 3.5. *If* $\nu \geqslant 3$, *then there is an* $\varepsilon_0 > 0$ *such that for* $|\varepsilon| < \varepsilon_0$ *the following limits*

$$\operatorname*{s-lim}_{\varkappa \to 0} T_{0,\pm\infty}^{(\varkappa)} \stackrel{\text{def}}{=} T^{\pm} \qquad (3.2.11)$$

(*direct adiabatic wave operators*) *and*

$$\operatorname*{s-lim}_{\varkappa \to 0} T_{\pm\infty,0}^{(\varkappa)} \stackrel{\text{def}}{=} \widehat{T}^{\pm} \qquad (3.2.12)$$

(*inverse adiabatic wave operators*) *exist.*

THEOREM 3.6. *In the hypotheses of Theorem* 3.5, *the renormalization constant*

$$Z^{-1} = \left\| \exp\left(\sum_{n=1}^{\infty} (-\varepsilon)^n (\Gamma(V \cdots \Gamma(V)))_{CR} \Omega \right) \right\|^2, \qquad (3.2.13)$$

where n *is the number of Friedrichs operations* (*see* [H]), *is finite. The operator* $\sqrt{Z}T^{\pm}$ *is unitary and defines unitary equivalence*

$$HT^{\pm} = T^{\pm}(H_0 + \lambda_\varepsilon), \qquad (3.2.14)$$

where λ_ε *is defined by a converging Goldstone series* (*see* [H])

$$\lambda_\varepsilon = (\Omega, \varepsilon V \angle T^{\pm}\Omega), \qquad (3.2.15)$$

where $V \angle T^{\pm}$ *is the left connected product.*

Some formal analogs of Theorems 3.3–3.6 are given in [H], in which the identity (3.2.10) is proved formally, i.e., without proving the convergence of the relevant series. A nonformal proof of these theorems was first given in [M2].

Before proving Theorems 3.3–3.6, we shall give a formal proof of Theorem 3.6 (see [H]).

A FORMAL PROOF OF THEOREM 3.6. It follows from (3.2.5) and (3.2.9) that

$$T^{(\varkappa)}(0, \pm\infty) =: \exp\left(\sum_{n=1}^{\infty}(-i\varepsilon)^n \Gamma_{\pm n\varkappa}(V \cdots \Gamma_{\pm 2\varkappa}(V\Gamma_{\pm\varkappa}(V))\cdots)_L\right) :.$$

This yields

$$T^{\pm} =: \exp(\Gamma_{\pm}(Q_{\pm})) : \tag{3.2.16}$$

as $\varkappa \to 0$, where

$$Q_{\pm} = \sum_{n=1}^{\infty}(-i\varepsilon)^n (V\Gamma_{\pm}(V \cdots \Gamma_{\pm}(V\Gamma_{\pm}(V))\cdots))_L.$$

Using Propositions 3.1 and 3.2, we obtain

$$H_0 T^{\pm} = T^{\pm} H_0 + : Q_{\pm} T^{\pm} : \tag{3.2.17}$$

and

$$\begin{aligned} \varepsilon V T^{\pm} &= \varepsilon : (V \angle T^{\pm}) T^{\pm} : = \varepsilon : (V T^{\pm})_C T^{\pm} : \\ &= \varepsilon : (V T^{\pm})_L T^{\pm} : + \varepsilon : (V T^{\pm})_{00} T^{\pm} : \\ &= - : \sum_{n=1}^{\infty}(-\varepsilon)^n (V\Gamma_{\pm}(V \cdots \Gamma_{\pm}(V\Gamma_{\pm}(V))\cdots))_L T^{\pm} : \\ &\qquad + (\Omega, \varepsilon(V T^{\pm})_{00}\Omega) T^{\pm}, \end{aligned} \tag{3.2.18}$$

where the last equality follows from the relation ([Fr])

$$Q_{\pm} = \varepsilon V \measuredangle : \exp(-\Gamma_{\pm}(Q_{\pm})) : -\varepsilon(\Omega, V \measuredangle : \exp(-\Gamma_{\pm}(Q_{\pm})) : \Omega).$$

The relations (3.2.17) and (3.2.18) imply Theorem 3.6 with

$$\varepsilon_{\lambda} = (\Omega, \lambda(V T^{\pm})_{00}\Omega) = \sum_{n=1}^{\infty}(-\varepsilon)^n (\Omega, (V\Gamma(V \cdots \Gamma(V\Gamma(V))\cdots))_{00}\Omega), \tag{3.2.19}$$

where n is the number of Γ, and we have replaced $\Gamma_{\pm}$ by Γ because a nonzero contribution in (3.2.19) is possible only in the absence of annihilation operators since $a(f) \equiv 0$ for all $f \in \mathscr{H}$. ■

REMARK. We can prove all these assertions without using the operations $\Gamma_{\pm}$ but using the following identity instead:

$$\begin{aligned} &\frac{d}{dt} U^{(\varkappa)}(t, s)_C = -i\varepsilon(V^{(\varkappa)}(t) : \exp U^{(\varkappa)}(t, s)_C :), \\ &U^{(\varkappa)}(t, t)_C = 1. \end{aligned} \tag{3.2.20}$$

DEFINITION 3.3. Define the *S-matrix* (or *scattering matrix*) by

$$S \stackrel{\text{def}}{=} ZT_+^* T_- = : \exp\left(2\pi i \sum_{n=1}^{\infty} (-\varepsilon)^n \Delta(V\Gamma_-(V \cdots \Gamma_-(V\Gamma_-(V))\cdots))\right)_L :, \tag{3.2.21}$$

where the operation Δ applied to the Wick monomial U_{mp} means replacing its kernel u_{mp} by the kernel

$$u_{mp}(k_1, \dots, k_{m+p})\delta(E_C - E_A).$$

It follows from Theorem 3.6 that the S-matrix is unitary.

§3.3. Proof of Theorem 3.3. Partitioning into clusters and decomposition into modes

We shall first treat the case $\varkappa = 0$, $-\infty < s, t < \infty$, $\nu \geqslant 3$.

We shall further consider the estimate for the expression

$$\int_{\Delta_n^{s,t}} (V(t_1) \cdots V(t_n))_C \, dt_1 \cdots dt_n \tag{3.3.1}$$

and show that its modulus is bounded by $|t - s|C^n$, where C is a constant independent of s, t, n. We shall derive Theorem 3.3 using this estimate.

The main difficulty in the proof of this estimate is related to great number of diagrams involved in it. We shall cancel the diagrams, wherever possible, in "time clusters", or "sectors".

Partitions. The indices $1, \dots, n$ are the vertices of diagrams. Each subset $\alpha = (\alpha_1, \dots, \alpha_k)$, $\alpha_1 < \alpha_2 < \cdots < \alpha_k$, of the set $(1, \dots, n)$ defines a *partition* of $(1, \dots, n)$ into intervals

$$I_1 = [1, \alpha_1) = \{i : 1 \leqslant i < \alpha_1\}, \dots, I_k = [\alpha_{k-1}, \alpha_k), I_{k+1} = [\alpha_k, n].$$

Sectors. Each partition defines a subset Δ_α of the region $\Delta_n^{s,t}$ which we shall call a *sector*. It is uniquely determined by the following conditions:

(1) if i, j belong to the same interval I_l of the partition α, then there exists an integer M such that

$$t_i, t_j \in [M, M+1) \stackrel{\text{def}}{=} \widehat{I}_l;$$

(2) if i, j belong to the partitions I_m and I_l, respectively, then t_i and t_j belong to different intervals $[M, M+1) = \widehat{I}_m$, $[L, L+1) = \widehat{I}_l$, i.e. $M \neq L$.

It is obvious that $\bigcup_\alpha \Delta_\alpha = \Delta_n^{s,t}$. Henceforth, I_l will be called the lth *group* of the sector Δ_α.

Subsectors. A *subsector* $(M_1, \dots, M_{k+1})$ of the sector Δ_α is defined by the partition α and integers $M_1 < M_2 < \cdots < M_{k+1}$. This is a set of all $(t_1, \dots, t_n)$, $s < t_1 < \cdots < t_n < t$, such that $t_i \in [M_l, M_l + 1)$ for $i \in I_l$.

Modes. In the space $L_2(\mathbb{T}^\nu)$, we select the orthonormal basis $\{e_N\}_{N\in\mathbb{Z}^\nu}$, where $e_N = C\exp[i(N, k)]$, $k \in \mathbb{T}^\nu$. The elements of this basis will be called *modes.*

Let

$$S \stackrel{\text{def}}{=} \bigcup_{i=1}^{d} \{f_{i,1}, \ldots, f_{i,m_i+l_i}\}.$$

We fix a subsector $\Delta_\alpha(M_1, \ldots, M_{k+1})$. If $t \in [M_l, M_l+1]$, then $t = M_l + \delta t$, $0 \leqslant \delta t \leqslant 1$. This yields

$$V(t) = \sum_{i=1}^{d} a^*(e^{iM_l h} e^{i\delta t h} f_{i,1}) \cdots a(e^{iM_l h} e^{i\delta t h} f_{i,m_i+l_i}). \tag{3.3.2}$$

Decomposition into modes. We decompose vectors of the form $e^{i\delta t h} f$, $f \in S$, in the basis $\{e_N\}$:

$$e^{i\delta t h} f = \sum_{N\in\mathbb{Z}^\nu} C_{N,f}(\delta t) e_N. \tag{3.3.3}$$

Observe that for any $\gamma > 0$ there exists a constant $C(\gamma)$ such that the coefficients $C_{N,f}(\delta t)$ of the series (3.3.3) satisfy the following inequality uniformly in $f \in S$, $|\delta t| \leqslant 1$:

$$|C_{N,f}(\delta t)| \leqslant \frac{C(\gamma)}{|N|^\gamma}, \quad |N| = \sum_{i=1}^{\nu} |N^i|. \tag{3.3.4}$$

We can prove this inequality using integration by parts. We choose $\gamma > \nu + 1$ so that

$$\sum_{N\in\mathbb{Z}^\nu} |C_{N,f}(\delta t)| < C < \infty. \tag{3.3.5}$$

For a partition $\alpha = (\alpha_1, \ldots, \alpha_k)$, we let $\mathfrak{B}_\alpha$ denote the following subset of $[0, 1)^\nu$:

$$\mathfrak{B}_\alpha = \{(\delta t_1, \ldots, \delta t_n) : 0 \leqslant \delta t_1 < \cdots < \delta t_{\alpha_1 - 1}, \ 0 \leqslant \delta t_{\alpha_1} < \cdots < \delta t_{\alpha_2 - 1}, \ \ldots, \ 0 < \delta t_{\alpha_k} < \cdots < \delta t_n < 1\}.$$

Using the notation introduced above, we may write expression (3.3.1) in the form

$$\sum_{\alpha} \sum_{M_1, \ldots, M_{k+1}} \int_{\mathfrak{B}_\alpha} \prod_{i=1}^{n} d(\delta t_i) \sum_{\{N_i, f_j\}} \sum_{G} W_G, \tag{3.3.6}$$

where, from left to right, we have the following:

(1) sums over all partitions;
(2) sums over all subsectors;
(3) integrating within all subsectors;
(4) sums over all modes;
(5) sums over all admissible diagrams.

Diagrams. A *diagram* is a graph with vertices $1, 2, \dots, n$. In it, m_i right-hand legs (corresponding to annihilation operators) and l_i left-hand legs (corresponding to creation operators) are incident to each vertex. The selection of modes assigns an element e_N of the basis $\{e_N\}_{N\in\mathbb{Z}^\nu}$ to each leg. Each leg is numbered by the index (v, p), where v is the vertex number and p is the number of the leg at the vertex v. A connected graph formed by a pairing of some of the legs (v_1, p_1) and (v_2, p_2), $v_1 < v_2$, is an admissible diagram if the former leg is right-hand and the latter leg is left-hand. Paired legs are edges of the admissible diagram and will be called *internal* (or "int", for short). Unpaired legs will be called *external* (or "ext", for short).

Each diagram appears in expression (3.3.6) with its weight.

Weight of a diagram. Suppose that the partition $\alpha = (\alpha_1, \dots, \alpha_k)$, the subsector $(M_1, \dots, M_{k+1})$, the vector $(\delta t_1, \dots, \delta t_n)$, and the mode N_{vp} are fixed. We introduce the function $M(v)$, $v = 1, \dots, n$, with $M(v) = M_i$ if $v \in \delta_i$, $i = 1, \dots, k+1$. The following expression will be called the *weight* of a diagram G:

$$W_G = (-1)^{\pi(G)} \prod_{\text{int}} \left(e^{ih(M(v)-M(v'))}, e_{N_{vp}}, e_{N_{v'q}}\right) \times C_{N_{vp}}(\delta t_v) C_{N_{v'q}}(\delta t_{v'}) \prod_{\text{ext}} a^{\#}\left(e^{iM(v)h} e_{N_{vp}}\right), \quad (3.3.7)$$

where $a^{\#} = a^*$ if the leg (v, p) is right-hand, and $a^{\#} = a$ if the leg (v, p) is left-hand.

REMARK. We shall consider a diagram G whose internal edge $(vp, v'q)$ lies entirely in the interval I_l for some l. Expression (3.3.7) shows that if $N_{vp} \neq N_{v'q}$, then $W_G = 0$, since $M(v) = M(v')$ and $(e_{N_{vp}}, e_{N_{v'q}}) = 0$. This will help us to cancel a large number of diagrams.

A subset of vertices I_l of the diagram G will be called the lth *group of vertices*. Suppose that A_l (B_l) is a set of diagram edges (both internal and external) going out to the right (left) from the lth group of vertices without pairing the two vertices of this group.

We fix a sector $\Delta_\alpha(M_1, \dots, M_{k+1})$, $\alpha = (\alpha_1, \dots, \alpha_k)$, of the mode $\{N_{vp}\}$. For each $l = 1, \dots, k+1$, we let N_l denote some fixed set of modes, $N_l = \{e_1, \dots, e_i\}$. Let $G(N_1, \dots, N_{k+1})$ be the set of diagrams for which the modes of legs A_l are exactly N_l.

LEMMA 3.7. (1) *There exists no more than one diagram* $G \in G(N_1, \dots, N_{k+1})$ *with nonzero weight* W_G.

(2) *For each such diagram, the sets* N_l *contain different modes.*

PROOF OF LEMMA 3.7. It is obvious that the assertions of this lemma follow from

$$\{a^{\#}(e^{iMh} e_{N_1}), a^{\#}(e^{iMh} e_{N_2})\} = 0, \quad (3.3.8)$$

where $\{\cdot, \cdot\}$ is the anticommutator, if $N_1 \neq N_2$, or else from

$$(a^{\#}(e^{iMh}e_N))^2 = 0. \tag{3.3.9}$$

If $I_l = \{i_1, \dots, i_q\}$, then we use the notation $V_{I_l} = V(t_{i_1}) \cdots V(t_{i_q})$. Further, we fix some l, $1 \leqslant l \leqslant k+1$.

We represent V_{I_l} as a sum over Wick monomials (Friedrichs diagrams). It follows from (3.3.8) and (3.3.9) that there exists only one nonzero diagram for which the modes of right-hand free legs coincide with N_l, the modes being different. Assertions (1) and (2) of Lemma 3.7 follow from these remarks. ■

We now return to the estimate for expression (3.3.6). Since all the sums in it, except the sums over modes, are finite, we may transpose the summation over modes and the integration over $\mathfrak{B}_\alpha$ to the left. It follows from (3.3.4) and $\mathfrak{B}_\alpha \subseteq [0, 1)^n$ that to prove Theorem 3.3 it is sufficient to establish, uniformly in modes, the following inequality

$$\left|\sum_{\alpha} \sum_{M_1, \dots, M_{k+1}} \sum_{G} W_G\right| \leqslant C^n |t - s|, \tag{3.3.10}$$

where C is independent of n, s, t, modes, and W_G is defined by an expression of the form (3.3.7) in which all $C_{N_{vp}}(\delta t_v)$ have been removed.

Further, suppose that the collection of modes is fixed. We can select sets $N_1, \dots, N_{k+1}$ in no more than $(2^{m_{\max}})^n$ ways, where $m_{\max}$ is the maximum number of creation operators in Wick monomials that enter into V. Hence, we may assume that $N_1, \dots, N_{k+1}$ are also fixed. It follows from Lemma 3.7 that each admissible diagram G can be brought into correspondence with a connected diagram $\widetilde{G}$ having $k+1$ vertices $M_1, \dots, M_{k+1}$ and no more than $m_{\max} n$ edges.

The weight $\widetilde{W}_{\widetilde{G}}$ of the diagram $\widetilde{G}$ is specified by

$$\widetilde{W}_{\widetilde{G}} = \prod_{\text{int}} |(e^{ih(M(v) - M(v'))} e_{N_{vp}}, e_{N_{v'q}})|, \tag{3.3.11}$$

with

$$\|W_G\| \leqslant C_1^n \widetilde{W}_{\widetilde{G}}, \tag{3.3.12}$$

where C_1 is a constant.

For all $M, N \in \mathbb{Z}^\nu$, the following estimate holds

$$\left|\int_{\mathbb{R}^\nu} dk\, e_N(k) e_M(k) e^{it\mathfrak{h}(k)}\right| \leqslant \frac{C}{(1 + |t|)^{\nu/2}}, \tag{3.3.13}$$

where C is independent of N, M, t. In this case, taking account of (3.3.11) and (3.3.13), we obtain

$$\|W_G\| \leqslant (\text{const})^n \prod_{\text{int}} \frac{1}{|M_{l(v,p)} - M_{l(v',q)}|^{\nu/2}}, \tag{3.3.14}$$

where $l(v, p)$ means that the leg (v, p) belongs to Δ_l.

Taking account of the above, we have

$$\left|\sum_{\alpha} \sum_{M_1, \dots, M_{k+1}} \sum_{G} W_G\right| \leqslant (\text{const})^n \sum_{M_1, \dots, M_{k+1}} \prod_{\text{int}} \frac{1}{|M_{l(v,p)} - M_{l(v',q)}|^{\nu/2}}. \tag{3.3.15}$$

We fix M_1 as an integer belonging to the segment $[s, t]$.

Lemma 3.8. *The following estimate holds*

$$\left|\sum_{k=1}^{\infty} \sum_{M_2, \dots, M_{k+1}} \prod_{\text{int}} \frac{1}{|M_{l(v,p)} - M_{l(v',q)}|^{\nu/2}}\right| \leqslant \left(\sum_{M=-\infty}^{\infty} \frac{C}{(1+|M|)^{\nu/2}}\right)^{nm_{\max}}. \tag{3.3.16}$$

This result can be obtained by applying the standard cluster expansion techniques (see [MM3]) to the left-hand side of (3.3.16).

Summing over M_1 yields the multiplier $|t-s|$ in the estimate (3.3.10). The proof of Theorem 3.3 is complete. ■

§3.4. Asymptotic completeness

We can now give informal proofs of Theorems 3.5 and 3.6, obtaining the asymptotic completeness of the Hamiltonian $H_0 + \varepsilon V$ for small ε in the case when the interaction V polarizes vacuum but is of even parity.

Proof of Theorem 3.5. We shall prove that for $\psi = a^*(f_1) \cdots a^*(f_m)\Omega$, $f_i \in C_0^\infty(\mathbb{T}^\nu)$, the limit

$$\lim_{\varkappa \to 0} \sum_n (-i\varepsilon)^n \int_{\pm\infty}^0 \cdots \int_{\pm\infty}^0 dt_1 \cdots dt_n \, (V^{(\varkappa)}(t_1) \cdots V^{(\varkappa)}(t_n))_L \psi \tag{3.4.1}$$

exists.

Let us consider the nth term of the series (3.4.1). We first prove this assertion for diagrams belonging to L', i.e., such that they have at least one external annihilation edge. Repeating the proof of Theorem 3.3 for these diagrams, we shall obtain the upper bound $C^n \varepsilon^n$, uniformly in $\varkappa$.

Further, we shall treat the sum over diagrams having no external annihilation edges. They possess external creation edges that give the contribution

$$\prod_{v,p} a^*(e^{it_v h - \varkappa|t_v|} e_{N_{vp}}) = \int \prod_{v,p} a^*(k_{vp}) e^{it_v \mathfrak{h}(k_{vp}) - \varkappa|t_v|} e_{N_{vp}}(k_{vp}) \, dk_{vp}. \tag{3.4.2}$$

We change the variables

$$t_1' = t_1, \; t_2' = t_2 - t_1, \; \dots, \; t_n' = t_n - t_{n-1}$$

and integrate with respect to t_1'. Observe that the limit

$$\lim_{\varkappa \to 0} i \int_{\pm\infty}^0 dt_1' \exp\left(it_1' \sum_{v,p} \mathfrak{h}(k_{vp}) - \varkappa|t_1'|\right) = \frac{1}{\sum_{v,p} \mathfrak{h}(k_{vp})} \tag{3.4.3}$$

belongs locally to L_2 if $\mu \geqslant 0$. Therefore we may expand $\left(\sum_{v,p} \mathfrak{h}(k_{v.p})\right)^{-1}$ in modes. Integration with respect to t_1' is equivalent to moving the entire diagram to the point $t_1 = 0$. Then we shall use the techniques elaborated in the proof of Theorem 3.3. ■

REMARK (ON THE THE ADIABATIC CUT-OFF). In the above, we considered wave operators depending on two parameters: the the adiabatic cut-off parameter $\varkappa$ and time t. In that case, repeated limits $\lim_{\varkappa\to 0} \lim_{t\to\infty}$ were investigated. It follows from Theorems 3.3–3.6 that these limits exist in the norm or in the strong sense. The question arises of the role of adiabatic cut-off and whether we can do without it. Note that the use of the adiabatic cut-off is characteristic for stationary scattering theory.

Note also that if the limit $\text{s-lim}_{t\to\infty} U(0, t)$ exists without cut-off, then the limit $\text{s-lim}_{\varkappa\to 0} U^{(\varkappa)}(0, \infty)$ also exists with adiabatic cut-off, and they are equal. For example, this situation arises when $V \in \mathscr{A}$ and $V\Omega = 0$.

If we analyze the proofs of Theorems 3.3–3.6, we see that these theorems correspond to similar assertions concerning the pertinent wave operators without the adiabatic cut-off. In that case, however, we must replace all strong limits with weak limits. But it is not clear now how this can be used to prove asymptotic completeness.

§3.5. The existence of a perturbed vacuum vector within a continuous spectrum

In this and the next section, we shall show that the dynamics of the Fermi system under consideration does not depend significantly on the chemical potential μ. The results of §§3.2–3.4 show that a perturbed system is unitarily equivalent to a "shifted" free system if $\mu \geqslant 0$. This condition is essential in proving Theorems 3.3–3.6.

Note that if $\mu \geqslant 0$, the point $\lambda_0 = 0$ of the discrete spectrum of the operator $H_0 = d\Gamma(h)$ lies outside or on the boundary of the continuous part of the spectrum of this operator. When the interaction εV is "turned on", it is shifted to the point λ_ε.

If $\mu < 0$, then the discrete spectrum of H_0 is contained within its continuous spectrum. It can be verified that also in this case the discrete spectrum does not vanish, as could be expected. For example, a similar situation arises for the model of an interacting Fermi gas with spin when the eigenvalue lying within the continuous spectrum vanishes as soon as the interaction ($\varepsilon \neq 0$) is "turned on".

In this section, we prove the above assertion. We use the procedure developed in §§3.2–3.4 for obtaining the estimates for diagrams. The notation of those sections will be used.

THEOREM 3.9. *Suppose that the one-particle Hamiltonian* h *has the form* (3.0.3), $\mu \in \mathbb{R}$, *and the operator* V *is of the form* (3.0.4). *Then for sufficiently small* ε, *the operator* $H_\varepsilon = H_0 + \varepsilon V$ *has an eigenvector* Ω_ε.

PROOF. According to Theorem 3.6, for sufficiently small ε the quantity

$$Z^{-1} = \left\| \exp\left\{\sum_{n=1}^{\infty}(-\varepsilon)^n(\Gamma(V\cdots\Gamma(V)))_{CR}\right\}\Omega\right\|^2$$

is finite. On the other hand, it follows from the results of §3.3 that

$$Z^{-1} = \lim_{t\to\infty}\frac{1}{|(\Omega, U(0, t)\Omega)|^2} = \lim_{t\to\infty}\frac{1}{|(e^{itH_\varepsilon}\Omega, \Omega)|^2}. \tag{3.5.1}$$

Hence the expression $e^{itH_\varepsilon}/(e^{itH_\varepsilon}\Omega, \Omega)$ is uniformly bounded in t. The lemma that follows can be proved in the same way as Theorem 3.3.

LEMMA 3.10. *For a set $\mathscr{D}$ dense in $\mathscr{F}_a$, there exists a finite limit*

$$\langle F\rangle = \lim_{t\to\infty}\frac{(e^{itH_\varepsilon}\Omega, F)}{(e^{itH_\varepsilon}\Omega, \Omega)}, \quad F\in\mathscr{D}. \tag{3.5.2}$$

As is seen from the following general lemma, this limit exists and is finite for any $F\in\mathscr{F}$.

LEMMA 3.11. *Let $\mathscr{H}$ be a separable Hilbert space, α_t a uniformly bounded function, $\alpha_t\colon \mathbb{R}\to\mathscr{H}$, i.e.,*

$$\|\alpha_t\| < M < \infty. \tag{3.5.3}$$

For some vector set $\{F\}$ dense in $\mathscr{H}$, let the finite limit

$$\lim_{t\to\infty}(\alpha_t, F) \tag{3.5.4}$$

exist. Then this limit exists and is finite for every $F\in\mathscr{H}$.

Hilbert space is weakly complete. Hence there exists a vector Ω_ε such that for all $F\in\mathscr{F}_a$

$$\lim_{t\to\infty}\frac{(e^{itH_\varepsilon}\Omega, F)}{(e^{itH_\varepsilon}\Omega, \Omega)} = (\Omega_\varepsilon, F). \tag{3.5.5}$$

We show that Ω_ε is an eigenvector of the perturbed operator H_ε.

First, assuming $F=\Omega$ in (3.5.5), we have $(\Omega_\varepsilon, \Omega) = 1$, i.e., $\Omega_\varepsilon\neq 0$.

Further, for any $s\in\mathbb{R}$

$$\begin{aligned}(\Omega_\varepsilon, F) &= \lim_{t\to\infty}\frac{(e^{i(t+s)H_\varepsilon}\Omega, F)}{(e^{i(t+s)H_\varepsilon}\Omega, \Omega)} = \lim_{t\to\infty}\frac{(e^{itH_\varepsilon}\Omega, e^{-isH_\varepsilon}F)}{(e^{itH_\varepsilon}\Omega, e^{-isH_\varepsilon}\Omega)}\\ &= \lim_{t\to\infty}\frac{(e^{itH_\varepsilon}\Omega, e^{-isH_\varepsilon}F)}{(e^{itH_\varepsilon}\Omega, e^{-isH_\varepsilon}\Omega)}\frac{(e^{itH_\varepsilon}\Omega, \Omega)}{(e^{itH_\varepsilon}\Omega, \Omega)}\\ &= \frac{(\Omega_\varepsilon, e^{-isH_\varepsilon}F)}{(\Omega_\varepsilon, e^{-isH_\varepsilon}\Omega)} = \frac{(e^{isH_\varepsilon}\Omega_\varepsilon, F)}{(e^{isH_\varepsilon}\Omega_\varepsilon, \Omega)}.\end{aligned}$$

In other words, for any $s \in \mathbb{R}$, $F \in \mathscr{F}$ the following identity holds

$$(\Omega_\varepsilon, F) = \frac{(e^{isH_\varepsilon}\Omega_\varepsilon, F)}{(e^{isH_\varepsilon}\Omega_\varepsilon, \Omega)}, \qquad (3.5.6)$$

or

$$(\Omega_\varepsilon, F)(e^{isH_\varepsilon}\Omega_\varepsilon, \Omega) = (e^{isH_\varepsilon}\Omega_\varepsilon, F). \qquad (3.5.7)$$

Differentiating both sides of (3.5.7) with respect to s and assuming $s = 0$, we get

$$(\Omega_\varepsilon, F)(H_\varepsilon\Omega_\varepsilon, \Omega) = (H_\varepsilon\Omega_\varepsilon, F). \qquad (3.5.8)$$

Since (3.5.8) holds for all $F \in \mathscr{F}_a$, it follows that

$$H_\varepsilon\Omega_\varepsilon = (H_\varepsilon\Omega_\varepsilon, \Omega)\Omega_\varepsilon, \qquad (3.5.9)$$

i.e., Ω_ε is the eigenvector of H_ε. The proof of Theorem 3.8 is complete. ■

§3.6. Unitary equivalence. The general case

In this section, we shall consider a further generalization of the results of §3.1, which consists in removing the restriction on chemical potential.

THEOREM 3.12. *Let the one-particle Hamiltonian* h *be of the form* (3.0.3), $\mu \in \mathbb{R}$, *and the operator* V *have the form* (3.0.4). *Then we can find an* $\varepsilon_0 > 0$ *such that for* $|\varepsilon| \leqslant \varepsilon_0$ *there exists a* $\lambda_\varepsilon \in \mathbb{R}$ *for which the operators* H_ε *and* $H_0 + \lambda_\varepsilon E$ *are unitarily equivalent.*

REMARK. The results of the previous section concerning the existence of a perturbed vacuum vector Ω_ε will form the basis for the proof of Theorem 3.12.

PROOF. We consider the following two dynamics in the C^*-algebra $\mathfrak{A}(\mathscr{H})$:

$$\tau_t^0(A) = e^{itH_0}Ae^{-itH_0},$$
$$\tau_t^{\varepsilon V}(A) = e^{it(H_0+\varepsilon V)}Ae^{-it(H_0+\varepsilon V)}.$$

By Theorem 2.5, for sufficiently small ε, there exist invertible Møller morphisms in the C^*-algebra $\mathfrak{A}(\mathscr{H})$

$$\gamma_\pm(A) = \operatorname*{s-lim}_{t\to\pm\infty} \tau_{-t}^{\varepsilon V}\tau_t^0(A), \qquad A \in \mathfrak{A}(\mathscr{H}).$$

Let Ω_ε be a perturbed vacuum vector of the operator $H_0 + \varepsilon V$, which exists by Theorem 3.9. We may assume this vector to be normalized, i.e., $\|\Omega_\varepsilon\| = 1$. We set $\gamma \equiv \gamma_+$.

LEMMA 3.13. *For all* $f \in \mathscr{H}$, *the following equality holds:*

$$(\gamma a(f))\Omega_\varepsilon = 0. \qquad (3.6.1)$$

PROOF. We denote $\tilde{a}(f) = \gamma a(f)$. On one hand, we have

$$\lim_{t\to+\infty} \tau_{-t}^{\varepsilon V}\tau_t^0(a(f))\Omega_\varepsilon = \tilde{a}(f)\Omega_\varepsilon. \qquad (3.6.2)$$

On the other hand, due to the fact that Ω_ε is the eigenvector of the operator $H_\varepsilon = H_0 + \varepsilon V$, we obtain

$$\|\tau_{-t}^{\varepsilon V}\tau_t^0(a(f))\Omega_\varepsilon\| = \|e^{-itH_\varepsilon}a(e^{ith}f)e^{itH_\varepsilon}\Omega_\varepsilon\| \\ = \|a(e^{ith}f)\Omega_\varepsilon\| \to 0 \tag{3.6.3}$$

for $t \to +\infty$. Indeed, let $\Omega_\varepsilon^{(n)} \to \Omega_\varepsilon$ in the norm as $n \to \infty$, where $\Omega_\varepsilon^{(n)}$ is a finite linear combination of the vectors of the form

$$a^*(f_1)\cdots a^*(f_m)\Omega, \quad f_i \in C_0^\infty(\mathbb{R}^\nu).$$

Let $\|\Omega_\varepsilon^{(n)} - \Omega_\varepsilon\| < \delta$ for $n > N_\delta$. Then

$$\|a(e^{-ith}f)\Omega_\varepsilon\| \leqslant \|a(e^{-ith}f)\Omega_\varepsilon^{(n)}\| + \delta\|f\|. \tag{3.6.4}$$

Using the anticommutation relations, we may carry the operator $a(e^{ith}f)$ from left to right through the operators $a^*(f_1), \dots, a^*(f_m)$. After this, the operator $a(e^{ith}f)$ will contain a finite number of summands, each having a multiplier of the form

$$(e^{ith}f, f_j). \tag{3.6.5}$$

It follows from spectral theory and the Lebesgue theorem (using the absolute continuity of the spectrum of h) that the expressions of the form (3.6.5) vanish for $t \to \infty$. Due to the arbitrary choice of $\delta > 0$, this proves the assertion of Lemma 3.13. ■

Lemma 3.14 follows immediately from Lemma 3.12.

LEMMA 3.14. *For any $A \in \mathfrak{A}(\mathscr{H})$, we have*

$$(A\Omega, \Omega) = (\gamma(A)\Omega_\varepsilon, \Omega_\varepsilon). \tag{3.6.6}$$

Define the operator $U: \mathscr{F}_a \to \mathscr{F}_a$ by

$$U(A\Omega) = \gamma(A)\Omega_\varepsilon. \tag{3.6.7}$$

The operator U preserves the norm, since

$$\|A\Omega\|^2 = (A^*A\Omega, \Omega) = (\gamma(A^*A)\Omega_\varepsilon, \Omega_\varepsilon) = \|\gamma(A)\Omega_\varepsilon\|^2.$$

Therefore, it is well defined and isometric. Since $\mathfrak{A}(\mathscr{H})$ is irreducible in $\mathscr{F}_a$, it follows that $\mathfrak{A}\Omega_\varepsilon = \mathscr{F}_a$, and consequently, the image of the operator U coincides with the entire space $\mathscr{F}_a$. Thus, the operator U is unitary.

For any $A \in \mathfrak{A}(\mathscr{H})$, we have

$$e^{it(H_0+\varepsilon V)}UA\Omega = e^{it(H_0+\varepsilon V)}\gamma(A)\Omega_\varepsilon = \tau_t^{\varepsilon V}(\gamma(A))e^{it(H_0+\varepsilon V)}\Omega_\varepsilon \\ = e^{it\lambda_\varepsilon}\gamma(\tau_t^0(A))\Omega_\varepsilon = e^{it\lambda_\varepsilon}U\tau_t^0(A)\Omega = e^{it\lambda_\varepsilon}Ue^{ith}A\Omega. \tag{3.6.8}$$

Here we have used the intertwining property of Møller morphisms:

$$\tau_t^{\varepsilon V} \circ \gamma = \gamma \circ \tau_t^0$$

and the fact that Ω_ε is the eigenvector of the operator $H_0 + \varepsilon V$ with the eigenvalue λ_ε. It follows from (3.6.8) that

$$e^{it(H_0+\varepsilon V)} = e^{ith\lambda_\varepsilon} U e^{itH_0} U^*, \tag{3.6.9}$$

or

$$H_0 + \varepsilon V = U(H_0 + \lambda_\varepsilon)U^*. \tag{3.6.10}$$

The proof of Theorem 3.12 is complete. ■

THEOREM 3.15. *The Møller morphisms* $\gamma_\pm$ *defined by* (3.6.2) *are unitarily representable.*

PROOF. For any $B \in \mathfrak{A}$ and the operator U in (3.6.7), the following relation holds

$$\gamma(A)B\Omega_\varepsilon = \gamma(A\gamma^{-1}(B))\Omega_\varepsilon = UA\gamma^{-1}B\Omega = UAU^*B\Omega_\varepsilon,$$

and since B is arbitrary, it follows that

$$\gamma(A) = UAU^*. \tag{3.6.11}$$

CHAPTER 4. THE WEAK COUPLING LIMIT FOR A QUANTUM SCHRÖDINGER PARTICLE INTERACTING WITH A FERMI GAS

§4.1. The weak coupling limit

Suppose that $\mathscr{H}_S = L_2(\mathbb{R}^\nu)$, $\nu \geqslant 3$, $\mathscr{H}_R$ is an antisymmetric Fock space over $\mathscr{H}_S$, Com($\mathscr{H}_S$) is a C^*-algebra of compact operators in $\mathscr{H}_S$, $\mathfrak{A}_S$ is a C^*-algebra of compact operators in $\mathscr{H}_S$ with unit, and $\mathfrak{A}_R$ is a C^*-algebra generated by the creation-annihilation operators $\{a(f), a^*(f), f \in \mathscr{H}_S\}$ in $\mathscr{H}_R$. The selfadjoint operator

$$H_0 = H_S \otimes 1 + 1 \otimes H_R, \tag{4.1.1}$$

where $H_S = -\Delta$ is a Laplace operator in $\mathscr{H}_S$ and $H_R = d\Gamma(H_S)$ is a second quantized Laplace operator in $\mathscr{H}_R$, defines the free dynamics

$$\tau_t^0(A) = \exp(itH_0)A\exp(-itH_0), \quad A \in \mathfrak{A}, \quad t \in \mathbb{R},$$

on the C^*-algebra $\mathfrak{A}$.

Suppose that ω_β is a KMS-state on the algebra $\mathfrak{A}_R$ at the inverse temperature β. Consider the interaction

$$\mathbb{V} = \sum_{j=1}^{\nu} p_j \otimes V_j, \tag{4.1.2}$$

where $p = (p_1, \dots, p_\nu) = i\nabla$ is the momentum operator in $\mathscr{H}_S$, and

$$V_j = \sum_{k=1}^{M_j} :a^*(f_{j,l}^{(k)}) \cdots a^*(f_{j,m_j}^{(k)}) a(f_{j,m_j+1}^{(k)}) \cdots a(f_{j,m_j+n_j}^{(k)}):,$$

$$\widehat{f}_{ji}^{(k)} \in C_0^\infty(\mathbb{R}^\nu), \quad 1 \leqslant j \leqslant \nu, \ 1 \leqslant i \leqslant m_j + n_j, \ 1 \leqslant k \leqslant M_j, \ M_j < \infty, \tag{4.1.3}$$

where $:\cdot:$ are Wick brackets with respect to a quasifree gauge-invariant state ω_β (see Chapter 1). Denote by

$$\mathscr{S} = \{f_{ji}^{(k)} \in C_0^\infty(\mathbb{R}^\nu),\ 1 \leqslant j \leqslant \nu,\ 1 \leqslant i \leqslant N_j,\ 1 \leqslant k \leqslant M_j\}$$

the set of all functions appearing in the definition of $\mathbb{V}$.

Remark. The choice of V_j in the special form (4.1.3) is deliberate. It corresponds to the renormalization of the interaction V_j. If Wick brackets in (4.1.3) are omitted, then the weak coupling limit may not exist.

We set

$$m_{\max} = \max_j (m_j + n_j), \qquad m_{\min} = \min_j (m_j + n_j).$$

Let $\mathscr{D}_0$ be the linear hull of all functions $f \in \mathscr{H}_S$ such that $\widehat{f} \in C_0^\infty(\mathbb{R}^\nu)$, and $\mathscr{F}_0$ be the subspace in $\mathscr{H}_R$ generated by vectors of the form

$$a^*(f_1)\cdots a^*(f_k)\Omega, \quad f_j \in \mathscr{D}_0,\ j = 1, \dots, k,$$

where Ω is the vacuum vector, and $\widehat{f}$ is the Fourier transform of the function f. The operator $H_\varepsilon = H_0 + \varepsilon\mathbb{V}$, $\varepsilon \in \mathbb{R}$, is essentially selfadjoint on $\mathscr{D}_0 \otimes \mathscr{F}_0$ and defines a perturbed dynamics

$$\tau_t^\varepsilon(A) = \exp(itH_\varepsilon)A\exp(-itH_\varepsilon), \quad A \in \mathfrak{A},\ t \in \mathbb{R}, \tag{4.1.4}$$

on $\mathfrak{A}$. Let $A \otimes B \in \mathfrak{A}$. Assuming that $\omega(A \otimes B) = A\omega_\beta(B)$ and extending the mapping ω by linearity and continuity to the entire algebra $\mathfrak{A}$, we define $\gamma_t^\varepsilon: \mathfrak{A}_S \to \mathfrak{A}_S$ as follows

$$\gamma_t^\varepsilon(A) = \omega(\tau_t^{\varepsilon\mathbb{V}}\tau_{-t}^0(A \otimes 1)), \quad A \in \mathfrak{A}_S, \quad t \in \mathbb{R}. \tag{4.1.5}$$

In this chapter, we shall be interested in proving the existence of the weak coupling limit

$$\lim_{\substack{\varepsilon \to 0 \\ \varepsilon^2 t = s}} \gamma_t^\varepsilon(A) = T_s(A), \quad A \in \mathfrak{A}_S,\ s \geqslant 0, \tag{4.1.6}$$

where T_S is a quantum dynamical semigroup on $\mathfrak{A}_S$. With this aim in mind, it is sufficient to prove the existence of the limit on the right-hand side of (4.1.6) for all $A \in \mathfrak{A}_S^0$, $0 \leqslant s \leqslant s_A$, where $\mathfrak{A}_S^0 \subseteq \mathfrak{A}_S$ is a dense subalgebra and $s_A > 0$.

§4.2. The existence of the weak coupling limit

Suppose that $\mathfrak{A}_{S,r}^0$ is the subalgebra in $\mathfrak{A}_S$ generated by the unit operator and projections of the form $(g, \cdot)f$, $\widehat{f}, \widehat{g} \in C_0^\infty(\mathbb{R}^\nu)$, $\operatorname{supp}\widehat{f} \in S_r$, $\operatorname{supp}\widehat{g} \in S_r$, where S_r is the sphere of radius r in $\mathbb{R}^\nu$. Set

$$\mathfrak{A}^0 = \bigcup_{r \geqslant 0} \mathfrak{A}_{S,r}^0.$$

Define the correlation functions $\{g_{ij},\ 1 \leqslant i, j \leqslant \nu\}$ by

$$g_{ij}(s) = \omega_\beta(V_{i,s}V_j), \tag{4.2.1}$$

$$V_{k,s} = \exp(isH_R)V_k\exp(-isH_R). \tag{4.2.2}$$

It follows from the definition of the operator V_k and the properties of the KMS-state ω_β that

(1)

$$|g_{ij}(s)| < \text{const}(1+|s|)^{-\nu m_{\min}}; \tag{4.2.3}$$

(2) if $\alpha_{ij} = \int_0^\infty g_{ij}(s)\,ds$, then

$$\Im\alpha_{ij} = \Im\alpha_{ji}, \quad 1 \leqslant i, j \leqslant \nu; \tag{4.2.4}$$

(3) the matrix $\{\Re\alpha_{ij},\ 1 \leqslant i, j \leqslant \nu\}$ is nonnegative definite over $\mathbb{C}^\nu$.

Suppose that $Q = \{Q_{ij},\ 1 \leqslant i, j \leqslant \nu\}$ is the positive square root of the matrix $\{\Re(\alpha_{ij} + \alpha_{ji})\}$. For $u = (u_1, \dots, u_\nu) \in \mathbb{R}^\nu$ and $p = (p_1, \dots, p_\nu)$, we set

$$P = \sum_{i,j=1}^{\nu} \Im\alpha_{ij}p_ip_j, \qquad (u, Qp) = \sum_{i,j=1}^{\nu} Q_{ij}u_ip_j, \qquad u^2 = \sum_{i=1}^{\nu} u_i^2. \tag{4.2.5}$$

THEOREM 4.1. *There is a quantum dynamical semigroup $\{T_s,\ s \geqslant 0\}$ on $\mathfrak{A}_S$ such that for all $A \in \mathfrak{A}_S^0$ there exists $s_A \geqslant 0$ and*

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon^2 t = s}} \gamma_t^\varepsilon(A) = T_s(A),$$

uniformly in $s \in [0, s_A]$. If $A \in \mathfrak{A}_{S,r}^0$, then s_A depends only on r. The generator L of the semigroup T_s is of the form (for $A \in \mathfrak{A}_S^0$):

$$L(A) = -\sum_{k,l=1}^{\nu}\left\{\frac{1}{2}\Re(\alpha_{kl} + \alpha_{lk})[p_k, [p_k, A]] + i\Im\alpha_{kl}[p_kp_l, A]\right\}, \tag{4.2.6}$$

and the semigroup itself acts on $\mathfrak{A}_S$ as follows

$$T_s(A) = (2Pt)^{-\nu/2}\int_{\mathbb{R}^\nu} \exp(-u^2/(2t)^\nu)\exp\{itP + i(u, Qp)\} \times A\exp\{-itP - i(u, Qp)\}\,du. \tag{4.2.7}$$

PROOF. We shall only outline the main steps in the proof of the theorem. For $A \in \mathfrak{A}_{S,r}^0$, we may consider the following converging series for $\gamma_t^\varepsilon(A)$

$$\gamma_t^\varepsilon(A) = A + \sum_{n=1}^{\infty}(i\varepsilon)^n\int_{\Delta_n^t} \omega_\beta([\mathbb{V}_{t_n}, [\mathbb{V}_{t_{n-1}}, \dots [\mathbb{V}_{t_q}, A\otimes 1]\dots]])dt_1\cdots dt_n, \tag{4.2.8}$$

where

$$\mathbb{V}_s = \exp(isH_0)\mathbb{V}\exp(-isH_0) = \sum_{j=1}^{\nu} p_j \otimes V_{j,s}.$$

The framework for the proof of Theorem 4.1 is provided by the following estimate for the nth term $I_n^t(A)$ of the series (4.2.8)

$$\|I_n^t(A)\| \leqslant C^n |\varepsilon|^n |t|^{[n/2]} C(A), \tag{4.2.9}$$

where $[\cdot]$ denotes the integer part of a number, the constant $C > 0$ is independent of n, and $C(A) > 0$ depends on A.

The main difficulty in obtaining (4.2.9) consists in estimating, after the removal of commutants, expressions of the form

$$\int_{\Delta_n^t} \omega_\beta(V_{i_1, t_{i_1}} \cdots V_{i_n, t_{i_n}})\, dt_1 \cdots dt_n, \tag{4.2.10}$$

where $\pi\colon (1, \dots, n) \to (i_1, \dots, i_n)$ is a permutation. Assigning each V_{i, t_i} to the vertex numbered i, we may represent the integrand as a sum of diagrams, which will be referred to as *admissible.* Each edge r_{ij} of an admissible diagram will make a multiplicative contribution to the weight of the diagram:

$$r_{ij}(t_i - t_j) = \omega_\beta(a^\#(f_{t_i})a^\#(g_{t_j})),$$

where $f, g \in \mathcal{S}$. Note that several edges may connect two vertices. It is obvious that all $r_{ij} \in L_1(\mathbb{R})$. Admissible diagrams do not possess loops. This can be seen directly from the properties of Wick brackets. There are many unconnected diagrams among admissible diagrams.

The proof of estimate (4.2.9) is simple for the case in which $m_{\max} = m_{\min} = 2$ if we add edges to make each admissible diagram connected and then use Lemma 2.7 (see [Do2]).

The case $m_{\max} > 2$ is much more difficult and requires a combination of procedures for obtaining the estimates described in Chapters 2 and 3. In this case, it is necessary to make fermion cancellations. We shall treat that case in a separate paper. ■

On the algebra $\mathfrak{A}_S$, there exists a state ρ_0 invariant with respect to arbitrary quantum dynamical semigroups T_s (since $T_s(I) = I$):

$$\rho_0(a1 + B) = a, \quad a \in \mathbb{C}, \quad B \in \mathrm{Com}(\mathscr{H}_S). \tag{4.2.11}$$

THEOREM 4.2. *If the operator* $\mathbb{V}$ *is nonzero, then* ρ_0 *is a unique invariant state for the semigroup* T_s.

REMARK. Theorem 4.1 holds for interactions $\mathbb{V}$ of the form

$$\mathbb{V} = \sum_{j=1}^{\nu} f_j(p_j) \otimes V_j, \tag{4.2.12}$$

where V_j is of the form (4.1.3) and $f_j \in C_0^\infty(\mathbb{R}^\nu)$ are bounded from above and from below by a polynomial.

§4.3. Discussion

One of the first examples of deduction of the kinetic equation by using the weak coupling limit is due to Davies [D3]. In the paper referred to, the

Hilbert space of the system was taken to be finite-dimensional:

$$\mathscr{H}_S = \mathbb{C}^N,$$

and the interaction $\mathbb{V}$ was of the form

$$\mathbb{V} = Q \otimes \Phi, \tag{4.3.1}$$

where Q is a symmetric matrix of size $N \times N$ and

$$\Phi = i\varphi(f_1)\varphi(f_2), \qquad \varphi(f) = \frac{a^*(f) + a(f)}{\sqrt{2}}, \tag{4.3.2}$$

in which case the functions f_1 and f_2 satisfy the supplementary condition

$$(e^{ith} f_1, f_2) = 0 \tag{4.3.3}$$

for all $t \in \mathbb{R}$. Condition (4.3.3) holds when the supports of the Fourier transforms of the functions f_1 and f_2 do not intersect. It follows that

$$\omega_\beta(\Phi) = 0,$$

i.e., Φ is of the form (4.1.2):

$$\Phi = i : \varphi(f_1)\varphi(f_2) : .$$

In the case under consideration, when interaction with respect to the reservoir Φ is quadratic and condition (4.3.3) is fulfilled, we can prove the existence of a quantum dynamical semigroup without using a special procedure for obtaining the estimates of the sums of a large number of diagrams (see [D3]). The same remark can also be made about interactions of the first degree.

REFERENCES

[Ai] V. V. Aizenshtadt, *Unitary equivalence of Hamiltonians in Fock space*, Uspekhi Mat. Nauk **39** (1988), no. 2, 220–221. (Russian)

[AiM] V. V. Aizenshtadt and V. A. Malyshev, *Spin interaction with an ideal Fermi gas*, J. Statist. Phys. **48** (1987), 51–68.

[Ar1] H. Araki, *On the dynamics and ergodic properties of the XY-model*, J. Statist. Phys. **31** (1983), no. 2, 327–346.

[Ar2] ———, *On the XY-model on two-sided infinite chain*, Publ. Res. Inst. Math. Sci. **20** (1984), 277–296.

[ArW] H. Araki and W. Wyss, *Representations of the canonical anticommutation relations*, Helv. Phys. Acta **37** (1964), no. 2, 136–159.

[ArB] H. Araki and E. Barouch, *On the dynamics and ergodic properties of the XY-model*, J. Statist. Phys. **31** (1983), no. 2, 327–346.

[BDS] C. Boldrighini, R. L. Dobrushin, and Yu. M. Sukhov, *One-dimensional caricature of hydrodynamics*, J. Statist. Phys. **31** (1983), no. 3, 577–615.

[BPT] C. Boldrighini, A. Pellegrinotti, and L. Triolo, *Convergence to stationary states for infinite harmonic systems*, J. Statist. Phys. **30** (1983), 123–155.

[B] D. D. Botvich, *Spectral properties of GNS-Hamiltonian in quasifree state*, Lecture Notes in Math., vol. 1021, Springer, 1983, pp. 65–71.

[BDM] D. D. Botvich, A. Sh. Domnenkov, and V. A. Malyshev, *Examples of asymptotic completeness in a translation-invariant system with an unbounded number of particles*, Acta Appl. Math. (to appear).

[BM1] D. D. Botvich and V. A. Malyshev, *Unitary equivalence of temperature dynamics of ideal and locally perturbed Fermi-gas*, Comm. Math. Phys. **91** (1983), no. 4, 301–312.

[BM2] ———, *A proof of asymptotic completeness, uniformly in the number of particles*, Izv. Akad. Nauk SSSR Ser. Mat. **54** (1990), no. 1, 132–145; English transl. in Math. USSR Izv. **36** (1991), no. 1.

[BR1] O. Bratteli and D. W. Robinson, *Operator algebras and quantum statistical mechanics*. I, Springer-Verlag, Berlin, 1979.

[BR2] ———, *Operator algebras and quantum statistical mechanics*. II, Springer-Verlag, Berlin, 1981.

[BF] D. Buchholz and K. Fredenhagen, *Clustering, charge-screening and mass-spectrum in local quantum field theory*, Mathematical Problems in Theoretical Physics (K. Osterwalder, ed.), Springer-Verlag, Berlin, 1980.

[C] J. M. Cook, *The mathematics of second quantization*, Trans. Amer. Math. Soc. **74** (1953), 222–245.

[Ch] A. L. Chistyakov, *On the scattering operator in the space of second quantization*, Dokl. Akad. Nauk SSSR **158** (1964), 66–70; English transl. in Soviet Math. Dokl. **5** (1965).

[D1] E. B. Davies, *Diffusion for weakly coupled quantum oscillators*, Comm. Math. Phys. **27** (1972), 309–325.

[D2] ———, *The harmonic oscillator in a heat bath*, Comm. Math. Phys. **33** (1973), 171–182.

[D3] ———, *Markovian master equation*, Comm. Math. Phys. **39** (1974), 91–110.

[D4] ———, *Dynamics of a multilevel Wigner-Weisskopf atom*, J. Math. Phys. **15** (1974), 2036–2039.

[D5] ———, *Particle-boson interactions and the weak coupled limit*, J. Math. Phys. **20** (1979), 345–351.

[D6] ———, *Markovian master equation*. III, Ann. Inst. H. Poincaré B **11** (1976), 265–273.

[D7] ———, *Markovian master equation*. II, Math. Ann. **219** (1976), 147–158.

[D8] E. B. Davies, *Resonances, spectral concentration and exponential decay*, Lett. Math. Phys. **1** (1975), 31–35.

[D9] ———, *Quantum theory of open systems*, Academic Press, London, 1976.

[D10] ———, *One-parameter semigroups*, Academic Press, London, 1982.

[DF] R. L. Dobrushin and J. Fritz, *Nonequilibrium dynamics of two-dimensional infinite particles systems with a singular interaction*, Comm. Math. Phys. **57** (1977), 67–75.

[Do1] A. Sh. Domnenkov, *Asymptotic completeness for the particle Fermi gas system*, Teoret. Mat. Fiz. **71** (1987), no. 3, 120–127; English transl. in Theoret. and Math. Phys. **71** (1987).

[Do2] ———, *The Markov limit for a particle interacting with gas*, Teoret. Mat. Fiz. **79** (1989), no. 2, 263–271; English transl. in Theoret. and Math. Phys. **79** (1989).

[DM] A. Sh. Domnenkov and V. A. Malyshev, *Translation-invariant interaction of the quantum particle with Fermi gas*, Dokl. Akad. Nauk SSSR **304** (1989), 326–329; English transl. in Soviet Phys. Dokl. **25** (1989).

[Du1] R. Dümcke, *Convergence of multi-time correlation functions in the weak and singular coupling limit*, J. Math. Phys. **24** (1983), 311–315.

[Du2] ———, *The low density limit for an N-level system interacting with free Bose or Fermi gas*, Comm. Math. Phys. **97** (1985), 331–357.

[E1] D. E. Evans, *Scattering in CAR-algebra*, Comm. Math. Phys. **48** (1976), 23–30.

[E2] ———, *Positive linear maps on operator algebra*, Comm. Math. Phys. **48** (1976), 15–22.

[E3] ———, *Complete positive quasi-free maps on the CAR algebra*, Comm. Math. Phys. **70** (1979), 53–68.

[F1] L. D. Faddeev, *Mathematical questions in the theory of quantum scattering for a system of three particles*, Trudy Mat. Inst. Steklov. **69** (1963), 1–122; English transl., *Mathematical aspects of the three-body problem in the quantum scattering theory*, Israel Program for Scientific Translations, Jerusalem; Davey, New York, 1965.

[F2] ———, *On the separation of the effects of self-action and scattering*, Dokl. Akad. Nauk SSSR **152** (1963), 573–576; English transl. in Soviet Phys. Dokl. **8** (1963).

[F3] ———, *On the Friedrichs model in the theory of perturbations of a continuous spectrum*, Trudy Mat. Inst. Steklov. **73** (1964), 292–313; English transl., Amer. Math. Soc. Transl. (2) **62** (1967), 177–203.

[Fr] K. O. Friedrichs, *Perturbations of spectra in Hilbert space*, Amer. Math. Soc., Providence, RI, 1965.

[FG1] A. Frigerio and V. Gorini, *N-level systems in contact with singular reservoir*, J. Math. Phys. **12** (1976), 2123–2127.

[FG2] ——, *On stationary Markov dilations of quantum dynamical semigroups*, Lecture Notes in Math., vol. 1055, Springer, Berlin and New York, 1984, pp. 119–125.

[GJ] J. Glimm and A. Jaffe, *Quantum Physics. A functional integral point of view*, 2nd ed., Springer-Verlag, New York, 1988, p. 535.

[GK] V. Gorini and A. Kossakowski, *N-level system in contact with a singular reservoir*, J. Math. Phys. **17** (1976), 1298–1305.

[GKS] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, *Completely positive dynamical semigroups on N-level systems*, J. Math. Phys. **17** (1976), 821–825.

[GuJ] K. Gustafson and G. Jonson, *On the absolutely continuous subspace of a self-adjoint operator*, Helv. Phys. Acta. **47** (1974), 163–166.

[H] K. Hepp, *Théorie de la renormalisation*, Springer-Verlag, Berlin, 1969.

[HL] K. Hepp and E. Lieb, *Phase transition in reservoir-driven open systems with applications to laser and superconductors*, Helv. Phys. Acta. **46** (1973), 573-603.

[H1] J. R. Hoegh-Krohn, *Partly gentle perturbations with application to perturbations by annihilation-creation operators*, Comm. Pure Appl. Math. **21** (1968), 313–342.

[H2] ——, *Gentle perturbations by annihilation-creation operators*, Comm. Pure Appl. Math. **21** (1968), 343–357.

[H3] ——, *Boson fields under a general class of cut-off interactions*, Comm. Math. Phys. **12**:3 (1969), 216–223.

[H4] ——, *Boson field with bounded interaction densities*, Comm. Math. Phys. **17** (1970), 179–187.

[H5] ——, *On the scattering operator for quantum fields*, Comm. Math. Phys. **18** (1970), 109–126.

[H6] *A general class of quantum fields without cut-offs in two space-time dimensions*, Comm. Math. Phys. **21** (1971), 244–255.

[IO] R. Iorio and M. O'Carroll, *Asymptotic completeness for multi-particle Schrödinger Hamiltonians with weak potentials*, Comm. Math. Phys. **27** (1972), 137–145.

[LR1] O. E. Lanford and D. W. Robinson, *Statistical mechanics of quantum spin systems*, Comm. Math. Phys. **9** (1968), 327–338.

[LR2] ——, *Approach to equilibrium of free quantum systems*, Comm. Math. Phys. **24** (1972), 193–210.

[L] G. Lindblad, *On the generators of quantum dynamical semi-groups*, Comm. Math. Phys. **48** (1976), 119–130.

[Ma] H. Maassen, *On the invertibility of Møller morphisms*, J. Math. Phys. **23** (1982), 1848–1851.

[M1] V. A. Malyshev, *Uniform expansion for the lattice models*, Comm. Math. Phys. **64** (1979), 131–157.

[M2] ——, *Convergence in the linked cluster theorem for many body fermion systems*, Comm. Math. Phys. **119** (1988), 501–508.

[M3] ——, *Cluster expansions in lattice models of statistical physics and the quantum theory of fields*, Uspekhi Mat. Nauk **35** (1980), no. 2, 3–53; English transl. in Russian Math. Surveys **35** (1980), no. 2, 1–62.

[MM1] V. A. Malyshev and R. A. Minlos, *Invariant subspaces of clustering operators*. I, J. Statist. Phys. **21** (1979), 231–242.

[MM2] ——, *Invariant subspaces of clustering operators*. II, Comm. Math. Phys. **82** (1981), 211–226.

[MM3] ——, *Clustering operators*, Trudy Sem. Petrovsk. **9** (1983), 63–80; English transl. in J. Soviet Math. **5** (1976).

[MM4] ——, *Gibbs random fields*, "Nauka", Moscow, 1985; English transl., Riedel, Dordrecht (to appear).

[MNT] V. A. Malyshev, I. Nikolaev, and Yu. A. Terletskii, *Temperature dynamics of the locally perturbed classical ideal gas*, J. Statist. Phys. **40** (1985), 133–146.

[M8] N. I. Muskhelishvili, *Singular integral equations*, 3rd ed., "Nauka", Moscow, 1968; English transl. of 1st ed., Noordhoff, Groningen, 1953; reprinted, 1972.

[P] P. F. Palmer, *The singular coupling and weak coupling limits*, J. Math. Phys. **18** (1977), 527–529.

[PS] R. Powers and E. Störmer, *Free states of the canonical anti-commutation relations*, Comm. Math. Phys. **16** (1970), 1–33.

[PSS] E. Presutti, Ya. G. Sinaĭ, and M. R. Soloveĭchik, *Hyperbolicity and Møller-morphism for a model of classical statistical mechanics*, Progr. Phys. **10** (1985), 253–284.

[Pu] G. V. Pule, *The Bloch equations*, Comm. Math. Phys. **38** (1974), 241–256.

[RS1] M. Reed and B. Simon, *Methods of modern mathematical physics*, vol. II, Academic Press, New York, 1975.

[RS2] ——, *Methods of modern mathematical physics*, vol. III, Academic Press, New York, 1979.

[RS3] ——, *Methods of modern mathematical physics*, vol. IV, Academic Press, New York, 1978.

[R1] P. A. Rejto, *On gentle perturbations*. I, J. Pure Appl. Math. **16** (1963), 297–303.

[R2] ——, *On gentle perturbations*. II, J. Pure Appl. Math. **17** (1964), 253–292.

[Ro1] D. W. Robinson, *Statistical mechanics of quantum spin systems*, Comm. Math. Phys. **6** (1967), 151–160.

[Ro2] ——, *Statistical mechanics of quantum spin system*. II, Comm. Math. Phys. 7 (1968), 337–348.

[Ro3] ——, *Return to equilibrium*, Comm. Math. Phys. **31** (1973), 171–189.

[RST1] F. Rocca, M. Sirague, and D. Testard, *On a class of equilibrium states under Kubo-Martin-Schwinger boundary condition* I. *Fermions*, Comm. Math. Phys. **13** (1969), 317–334.

[RST2] ——, *On a class of equilibrium states under Kubo-Martin-Schwinger boundary condition* II. *Bosons*, Comm. Math. Phys. **19** (1970), 119–141.

[R] D. Ruelle, *Statistical mechanics, rigorous results*, Benjamin, New York, 1969.

[SS] D. Shale and W. F. Stinespring, *States on the Clifford algebra*, Ann. Math. **80** (1964), 365–381.

[Sp] H. Spohn, *Kinetic equations from Hamiltonian dynamics: Markovian limits*, Rev. Modern Phys. **52** (1980), 569–615.

[S1] E. Störmer, *Positive linear maps of operator algebra*, Acta Math. **110** (1963), 233–278.

[S2] ——, *Spectra of states, and asymptotically abelian C^*-algebras*, Comm. Math. Phys. **28** (1972), 279–294.

[Su1] Yu. M. Sukhov, *Convergence to an equilibrium state for a one-dimensional system of hard rods*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), 1274–1315; English transl. in Math. USSR Izv. **21** (1983).

[Su2] ——, *On the convergence to the equilibrium state for free Fermi gas*, Teoret. Mat. Fiz. **55** (1983), no. 2, 282–290; English transl. in Theoret. and Math. Phys. **55** (1983).

[SuS1] Yu. M. Sukhov and A. G. Shukhov, *On the convergence to the stationary state for the one-dimensional lattice quantum models of solid rods*, Teoret. Mat. Fiz. **73** (1987), no. 1, 125–140; English transl. in Theoret. and Math. Phys. **73** (1987).

[SuS2] ——, *Hydrodynamical approximations for Bogolyubov transformation groups in quantum statistical mechanics*, Trudy Moskov. Mat. Obshch. **50** (1987), 156–208; English transl. in Trans. Moscow Math. Soc. **1988**.

[T] M. Takesaki, *Tomita's theory of modular Hilbert algebras and its applications*, Lecture Notes in Math., vol 128, Springer, Berlin, New York, 1970.

[Te] Yu. A. Terletskiĭ, *Metric isomorphism of a classic ideal gas with respect to its local perturbation*, Teoret. Mat. Fiz. **81** (1989), 323–335; English transl. in Theoret. and Math. Phys. **81** (1989).

Translated by N. K. KULMAN