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# Probabilistic methods for Jackson networks

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**Abstract**

We construct explicitly Lyapounov functions for Markovian Jackson networks. Two direct corollaries are obtained : first a proof of the necessary and sufficient conditions for ergodicity, without using the famous Jackson's product form ; secondly, an exponential convergence rate to the stationary distribution. We also consider small perturbations of the transition probabilities (yielding thus non Jackson networks) and prove that the corresponding stationary distribution is an analytic function of these perturbations.

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## INTRODUCTION

Jackson networks are now classical models for communication networks. Jackson [1] obtained the famous product form for their stationary probabilities. Sufficient ergodicity conditions follow from this product forme. The proof of the necessity of these conditions was obtained by many authors [2, 3, 4]. From the general theory of countable Markov chains, it follows that the n-step transition probabilities converge to the stationary probabilities when $n \to \infty$. But not much is known about the rate of this convergence. The only result is exponential convergence under some *smallness* assumption in [5].
Here we the consider the basic Markovian Jackson networks, i.e. with Poisson arrivals and exponential service times. In this case, they are equivalent to a class of random walks in $\mathbf{Z}_+^N = \{(z_1, ..., z_N) : z_i \geq 0 \textit{ are integers}\}$, where $N$ is the number of nodes in the network. In this paper, and this is the main result, we provide an explicit construction of Lyapounov functions, which appear to be piecewise linear.
Using the results of [7], we get a corollary showing exponential convergence for any ergodic Jackson network. Also we introduce a class of networks which are not of Jackson's type, but are weak perturbations of Jackson networks. Typical examples are e.g. weak dependence between nodes, simultaneous arrivals, etc... We prove that the stationary probabilities, which cannot be given by a product-form, nonetheless depend analytically on the parameters. In particular, *mutatis mutandis,* these networks have ergodicity conditions still obtained directly from the Jackson's system of linear equations for the mean number of visits to the various nodes.
Let us emphasize that we never use Jackson's product form in the proofs.

# 1 Ergodicity conditions for Jackson networks

Here we recall some well-known facts and prove a useful geometric lemma. We consider an open Jackson network with $N$ nodes restricting ourselves here to the simplest assumptions : independent Poisson inputs with parameter $\lambda_i > 0$ at any node $i$, exponential service times with parameters $\mu_i > 0$ and FIFO service discipline. After a customer completes service at the i-th node, he is immediately transferred with probability $p_{ij}$ to the end of the queue at



node $j$, $j = 1, ..., N$ and, with probability

$$p_{io} = 1 - \sum_{j=1}^{N} p_{ij} ,$$

he leaves the network. It will be convenient (and we shall do it, although it be not necessary) to assume that $p_{ii} = 0$, for all $i$.
In other words, we consider a continuous time random walk $\tilde{L}$ on $\mathbf{Z}_+^N$ with transition intensities $\lambda_{\alpha\beta}$, from the state $\alpha = (\alpha^1, ..., \alpha^N)$ to the state $\beta = (\beta^1, ..., \beta^N)$,

$$\lambda_{\alpha\beta} = \begin{cases} \mu_{oi} \doteq \lambda_i , & \text{if } \beta - \alpha = e_i, \\ \mu_{io} \doteq \mu_i \, p_{io} , & \text{if } \beta - \alpha = -e_i \\ \mu_{ij} \doteq \mu_i \, p_{ij} , & \text{if } \beta - \alpha = -e_i + e_j, \\ & 1 \leq i, j \leq N . \end{cases} \tag{1.1}$$

Here $e_i$ is the vector $(0, ..., 0, 1, 0, ..., 0)$, with i-th coordinate equal to 1. It is convenient to denote the zero vector by $e_0$. We recall now Jackson's equations. Assuming a stationary regime, we denote by $\nu_j$ "the mean number of customers" visiting node $j$ and coming from the outside world or from the other nodes during a unit time interval. Using the law of large numbers, Jackson wrote the following system of equations (we call it *Jackson's system*)

$$\nu_j = \lambda_j + \sum_{i=1}^{N} \nu_i \, p_{ij} \ , \ j = 1, ..., N . \tag{1.2}$$

Let us note that these equations can be solved by the following iteration scheme

$$\nu_j = \lambda_j + \sum_{k=1}^{\infty} \sum_{i=1}^{N} \lambda_i \, p_{ij}^{(k)} , \tag{1.3}$$

where

$$(p_{ij}^{(k)}) = \mathbf{P}^k \ , \ \mathbf{P} = \ (p_{ij})_{i,j=0,1,...,N},$$

and $p_{0i} \equiv 0$, $i \neq 0$, $p_{00} = 1$.
The series on the right-hand side of (1.3) converges if

$$p_{ij}^{(k)} \leq C(1 - \epsilon)^k, \tag{1.4}$$

for some $\epsilon > 0$, $C > 0$.
For this it is necessary and sufficient to assume the classical

*Condition (A): Starting from any state, we reach 0 with a positive probability (a.s.) in the Markov chain with $N+1$ states $0, 1, ..., N$, and defined by the stochastic matrix* $\mathbf{P}$.

Thus we can rewrite (1.3) as

$$\nu_j = \lambda_j + \sum_{i=1}^{N} \lambda_i \, m_{ij}^0 ,$$

where $m_{ij}^0$ is the mean number of hittings of $j$ starting from $i$ in this finite-state Markov chain. Thus it is immediate to see that the solution of (1.2) is unique. The following well-known theorem holds :

**Theorem 1.1** *(Jackson). The network is ergodic iff*

$$\nu_j < \mu_j \quad \textit{for all} \quad j = 1, ..., N .$$

Below we give a new proof of this theorem by means of a geometrical approach, which will be useful in the rest of the study. Later on, we will use several results for discrete time Markov chains, borrowed from [7]. We note that all of them could be easily rewritten for the continuous time case. To avoid this rewriting, we introduce the following discrete time random walk $L$ in $\mathbf{Z}_+^N$. Its transition probabilities are taken to be

$$p_{\alpha\beta} = w_\alpha \, \lambda_{\alpha\beta} , \tag{1.5}$$

for some constants $w_\alpha$ satisfying

$$0 < w_\alpha \leq (\sum_\beta \lambda_{\alpha\beta})^{-1} .$$

E.g., if we choose

$$w_\alpha = (\sum_\beta \lambda_{\alpha\beta})^{-1} ,$$



we get the natural and standard imbedded chain. In fact, it will be more convenient to choose

$$w_\alpha \equiv w \leq \min_\alpha (\sum_\beta \lambda_{\alpha\beta})^{-1} . \tag{1.6}$$

It is well known that $L$ is ergodic if and only if $\tilde{L}$ is, and that the corresponding stationary distributions $\pi$ and $\tilde{\pi}$ are the same :

$$\pi_\alpha = \tilde{\pi}_\alpha \tag{1.7}$$

We want to recall now some basic definitions from [7].
Consider the discrete time homogeneous Markov chain $L$, which is assumed to be irreducible and aperiodic unless otherwise stated. The set of states is $\mathbf{Z}_+^N$.
Let $p_{\alpha\beta}^{(k)}$ be the $k$-step transition probabilities on $L$, $M^k(\alpha) = (M_1^k(\alpha), ..., M_N^k(\alpha))$ be the vector of the mean jumps from the point $\alpha$ in $k$ steps,

$$M^k(\alpha) = \sum_\beta (\beta - \alpha) p_{\alpha\beta}^{(k)} .$$

For the sake of brevity, we shall write $p_{\alpha\beta}^{(1)} = p_{\alpha\beta}$ *and* $M^1(\alpha) = M(\alpha)$.
For any $\Lambda \subseteq \{1, 2, ..., N\}$, we define the *face* $B^\Lambda$ of $R_+^N = \{(r_1, ..., r_N) : r_i \geq 0 \textit{ real}\}$ by $B^\Lambda = \{(r_1, ..., r_N) : r_i > 0, \ i \in \Lambda; \ r_i = 0, \ i \notin \Lambda\}$.
It is sufficient for our purpose to consider random walks satisfying the following conditions, which are stronger than in [7] :

- *boundedness of the jumps* :

  $$p_{\alpha\beta} = 0 \ , \ \textit{for} \ \| \alpha - \beta \| > 1 ,$$

  where

  $$\| \alpha \| = \max_i |\alpha_i| \ , \quad \alpha = (\alpha_1, ..., \alpha_N) ;$$

- *homogeneity condition* : for any $\Lambda$ and for any $a \in B^\Lambda \cap \mathbf{Z}_+^N$,

  $$p_{\alpha\beta} = p_{\alpha+a, \beta+a} ,$$

for all $\alpha \in B^\Lambda \cap \mathbf{Z}_+^N$ , $\beta \in \mathbf{Z}_+^N$.
Obviously the random walk $L$, which is equivalent to the Jackson network under study, does meet these conditions.
We define now the *first* vector field on $\mathbf{R}_+^N$, which is taken to be constant in any $B^\wedge$ and equal to

$$M_\Lambda \equiv M(\alpha)\ ,\ \alpha \in B^\wedge\ .$$

For the Markov chain $L$, we have the following crucial property

$$M_\Lambda = f_0 + \sum_{i\in\Lambda} f_i\ , \tag{1.8}$$

where, for $i = 0, 1, ...N$,

$$f_i = w \sum_{j=0}^{N} \mu_{ij}(-e_i + e_j)\ . \tag{1.9}$$

So $f_i$ represents the contribution of the transition from the i-th node (including the virtual 0-node).
It is clear that the $2^N$ mean jump vectors $M_\Lambda$ are the vertices of a parallelepiped which we denote by $\prod$. Its initial point can be taken $f_0$ and the edges drawn from this point are $f_1, ..., f_N$. This parallelepiped can be degenerate if the vectors $f_1, ..., f_N$ are linearly dependent. We shall use below the following combinatorial criterion of ergodicity, equivalent to Jackson's one.

**Lemma 1.2** *Jackson's network is ergodic iff $\prod$ is not degenerate and the point $0 \in R^N$ is one of its interior points. Moreover, if the origin does not belong to $\prod$, then this chain is transient.*

**Proof** : Let us consider the following system of equations, with respect to $\epsilon_1, ..., \epsilon_N$,

$$f_0 + \epsilon_1\ f_1 + ... + \epsilon_N\ f_N = 0\ . \tag{1.10}$$

Note that $\prod$ is not degenerate iff this system is not degenerate. In this case the system (1.10) has a unique solution and 0 is an interior point of $\prod$ iff $0 < \epsilon_i < 1$, for $i = 1, ..., N$.
Inserting (1.1.), (1.9) into (1.10), we get



$$0 = f_0 + \sum_{j=1}^{N} \epsilon_j\ f_j = \sum_{j=1}^{N} \lambda_j\ e_j + \sum_{i=1}^{N} \epsilon_i \sum_{j=0}^{N} \mu_i\ p_{ij}(-e_i + e_j) =$$

$$= \sum_{j=1}^{N} \lambda_j\ e_j + \sum_{i=1}^{N} \epsilon_i\ \mu_i(-e_i + \sum_{j=0}^{N}\ p_{ij}\ e_j) =$$

$$= \sum_{j=1}^{N} \lambda_j\ e_j - \sum_{j=1}^{N} \epsilon_j\ \mu_j\ e_j + \sum_{i=1}^{N} \epsilon_i\ \mu_i \sum_{j=0}^{N} p_{ij}\ e_j$$

$$= \sum_{j=1}^{N} (\lambda_j - \epsilon_j\ \mu_j + \sum_{i=1}^{N} \epsilon_i\ \mu_i\ p_{ij}) e_j\ ,$$

which coincides with (1.2) for $\epsilon_i \equiv \rho_i = \frac{\nu_i}{\mu_i}$ .

Thus, when 0 is an interior point of $\prod$ and $\prod$ is not degenerate, ergodicity follows from Jackson's explicit formulae for the stationary probabilities *but in the next sections we prove it without using Jackson's results.*
Let now 0 lie on the boundary of $\prod$ (this includes the case of a degenerate $\prod$, when $\prod$ coincides with its boundary). Assume first that $\prod$ is not degenerate. Then there exists a hyperplane $\mathcal{L}$ of dimension $N - 1$ in $R^N$, such that $0 \in \mathcal{L}$ and $\prod$ belongs to the closure of one of the two half-spaces defined by $\mathcal{L}$. Denote this closure by $\mathcal{L}^+$ and consider the straight line $l$, passing through 0 and perpendicular to $\mathcal{L}$. Let $x$ be the coordinate on $l$ which is positive on $\mathcal{L}^+$. For any point $\alpha \in \mathbf{R}_+^N$, let $f(\alpha)$ be the value of the $x$-coordinate of the orthogonal projection of $\alpha$ onto $l$. But since all the $M_\lambda$ 's belong to $\mathcal{L}^+$, it follows that

$$\sum_\beta p_{\alpha\beta} f(\beta) - f(\alpha) \geq 0\ ,\ f(\alpha) > 0 \text{ for an infinite number of } \alpha \in \mathbf{Z}_+^N.$$

Consider the sequence of random variables $\xi_0, \xi_1, ...$ constituting the chain $L$, and the corresponding sequence $f(\xi_t)$. Let $\tau$ be the time of first visit of $\xi_t$ to the set $\{\alpha : f(\alpha) \leq 0\}$ and $\eta(t) = f(\xi_{t\wedge\tau})$. So we have

$$E(\eta_{t+1}/\xi_t, ..., \xi_0) - \eta_t \geq 0\ .$$

It is then well-known that $E\tau = \infty$, which proves the non-ergodicity. We refer the reader to lemma 1.4 and theorem 1.7 of [7] and to more general results in [10]. For a degenerate $\prod$, the proof is the same and, with regard to the transience, we have

$\sum_\beta p_{\alpha\beta} f(\beta) - f(\alpha) \geq \epsilon\,,\ f(\alpha) > 0$ for an infinite number of $\alpha \in \mathbf{Z}_+^N$,

for some $\epsilon > 0$. Then the proof follows by lemma 1.3 or theorem 1.6 of [7].

## 2 Main results

**Theorem 2.1** *Let $\tilde{L}$ be a Jackson network such that 0 lies inside $\prod$ and $\tilde{p}_{\alpha\beta}^{(t)}$ its time-t transition probabilities. Then there exist constants $C(\alpha) > 0$ and $\chi > 0$, such that, for any $\alpha, \beta, t$,*

$$|\tilde{\pi}_\alpha - \tilde{p}_{\alpha\beta}^{(t)}| < C(\alpha) e^{-\chi t}\ .$$

Let us now fix some Jackson network with transition intensities $\lambda_{\alpha\beta}$. In addition to the *Jacksonian* jumps of this network, we also permit any jump $\alpha \to \beta$ satisfying boundedness and homogeneity conditions, but the intensities $\nu_{\alpha\beta}$ of these additional jumps are *small*, in some sense to be made precise later and in particular in the following

**Theorem 2.2** *If a fixed Jackson network with intensities $\lambda_{\alpha\beta}$ is such that 0 lies inside $\prod$, then there exists $\nu_0 > 0$ such that, for*

$$\nu_{\alpha\beta} < \nu_0\ ,$$

*the resulting network (which is not necessarily Jacksonian) has stationary probabilities analytically depending on $\nu_{\alpha\beta}$. Moreover, this analytic family is in fact an analytic Lyapounov family (see section 4 for the definition).*

**Remark 1** *In particular, one can expand the stationary probabilities $\pi_\alpha$ as a convergent series in $\nu_{\alpha\beta}$. For such perturbed Markov chains, we also have an exponential convergence to the stationary state.*

**Remark 2** *If 0 lies inside $\prod$, ergodicity follows from Theorem 2.1. Thus, as it was stated, we proved ergodicity without using Jackson's product form.*



## 3 Geometric construction

Let us recall that $\prod$ is the convex hull of the *points* $M_\Lambda$ (the ends of the vectors $M_\Lambda$ having their initial point at 0).
Let $a$ be a fixed point of $R^N$. We define

$$\Gamma = \Gamma^a = \{a + \sum_{i=1}^{N} \beta_i f_i : \beta_i \geq 0\}\ ,$$

so that $\Gamma$ represents a multidimensional cone (with vertex $a$) generated by the vectors $f_i$. It will be convenient to put

$$\Gamma_\Lambda^a = \{a + \sum_{i\in\Lambda} \beta_i f_i : \beta_i \geq 0\}\ , \Lambda \subset \{1, ..., N\}\ .$$

Thus

$$\Gamma^a = \Gamma^a_{\{1,...,N\}}$$

and we define the *surface* $\tilde{\Gamma}$ of $\Gamma$ by

$$\tilde{\Gamma} = \tilde{\Gamma}^a = \bigcup_{\Lambda \neq \{1,...,N\}} \Gamma^a_\Lambda\ .$$

Whenever $a = f_0$, we shall simply write $\Gamma$, $\Gamma_\Lambda$, etc...

**Scaling** : Let us denote by $\alpha\Gamma$, $\alpha\tilde{\Gamma}$, $\alpha\Gamma_\Lambda$, $\alpha \geq 1$, the scaled geometrical objects respectively, with vertex $\alpha a$.

**Lemma 3.1** *When 0 lies inside $\prod$, the set*

$$R_+^N \cap (\alpha\Gamma)$$

*is compact for any $\alpha \geq 1$.*

**Proof** : Let us first note that, if it is compact for some $a$, then it is compact for any $a$. Hence we can choose $a$ in a convenient way, e.g. putting

$$a = f_0\ .$$

We now remark that the ray $f_0 + \beta_i f_i$, $0 \leq \beta_i < \infty$, intersects the face $x_i = 0$ of $R_+^N$, as $f_i$ has its $e_i$-component negative and the others are positive. From this, the announced compactness is readily seen. For instance, we can rewrite

$$f_0 + \sum_{i=1}^{N} \beta_i f_i = w \sum_{i=1}^{N} C_i e_i , \tag{3.1}$$

with

$$C_j = \lambda_j + \sum_{i=1}^{N} \alpha_i p_{ij} - \alpha_j ,$$

after having set

$$\alpha_j = \mu_j \beta_j .$$

Choosing a ray $\alpha_i = tr_i$, $r_i \geq 0$, $t \geq 0$, we see that its intersection with $R_+^N$ is an interval of finite length, since

$$\sum_{j=1}^{N} C_j = \sum \lambda_j + t[\sum r_i(1-p_{io}) - \sum r_i] = \sum \lambda_j - t \sum r_i p_{io} ,$$

and the coefficient of $t$ is negative.

Let us now consider some $\Gamma_\Lambda$, with $|\Lambda| = N-1$. This defines an affine hyperplane (of dimension $N-1$) in $R^N$, which subdivides $R^N$ into 2 half-spaces $\Gamma_\Lambda^+, \Gamma_\Lambda^-$. We denote by $\Gamma_\Lambda^+$ the half-space containing $\Gamma$.

**Lemma 3.2** *Under the conditions of lemma 3.1, let us consider an arbitrary $\Gamma_\Lambda$ with $|\Lambda| = N-1$. Then any vector $M_{\Lambda'}$, with*

$$\Lambda' \not\subset \Lambda , \tag{3.2}$$

*which has its initial point in $\Gamma_\Lambda$, lies in $\Gamma_\Lambda^+$.*

**Proof** : Let us first show that $M_{\{1,\ldots,N\}}$ has this property for all $\Gamma_\Lambda$, such that $|\Lambda| = N-1$. For this, choose $a = -M_{\{1,\ldots,N\}}$. Let us note that $0 \in \prod$ iff $0 \in (-\prod)$. Then the vector $M_{\{1,\ldots,N\}}$, with initial point $a$ (which belongs to all $\Gamma_\Lambda$ simultaneously), has 0 as its final point and is thus contained in $\Gamma_\Lambda^+$.

Let us take now e.g. $\Lambda = \{2,\ldots,N\}$ and any $\Lambda'$ such that $1 \in \Lambda'$. Choosing again $a = -M_{\{1,\ldots,N\}} = -f_0 - \sum_{i\in\Lambda'} f_i - \sum_{i\neq 0, i\notin\Lambda'} f_i$, we see that the point

$$b = a + \sum_{i\neq 0, i\notin\Lambda'} f_i$$

belongs to $\Gamma_\Lambda$ and $b + M_{\Lambda'} = 0 \in \Gamma_\Lambda^+$.



**Lemma 3.3** *If $a$ lies strictly inside $R_+^N$ then $\Gamma_\Lambda^a$ , $|\Lambda| = N-1$, has the property :*

$$\Gamma_\Lambda^a \cap \overline{B^{\Lambda'}} = \emptyset \ , \ \ for\ \Lambda' \subset \Lambda ,$$

*where $\overline{B}$ denotes the closure of $B$.*

**Proof** : Let again $\Lambda = \{2,\ldots,N\}$, then

$$\Gamma_\Lambda^a = \{a + \sum_{j=2}^{N} \beta_j f_j\} .$$

But the vector

$$a + \sum_{j=2}^{N} \beta_j f_j = \sum_{i=1}^{N} C_i e_i ,$$

since $C_1$ is strictly positive, cannot belong to the region $\overline{B^{\Lambda'}}$, where the first coordinate is zero.

Let us introduce the following function, with domain $R_+^N$,

$$f : x \to f_x = \alpha, \quad \text{for} \quad x \in \alpha\, \tilde{\Gamma}^a . \tag{3.3}$$

This "piecewise linear" function, obtained by scaling, is our main Lyapounov function, as will be shown in section 5.

# 4 Analytic Lyapounov families

Here we give a compact reformulation of some results in [7] which are necessary to prove the main results. Let us consider a family of Markov chains $\{L^\nu\}$, $\nu \in \mathcal{D}$, where $\mathcal{D}$ is an interval of the real axis containing 0, with the same state space $S$. The matrix $P_\nu = (p_{ij}(1,\nu))_{i,j\in S}$, the elements of which are transition probabilities, can be considered as a bounded linear operator in the Banach space $l_1(S)$. Let us assume that $P_\nu$ is analytic in $\nu$, as a function in $\mathcal{D}$ with values in the Banach algebra of bounded operators in $l_1(S)$. This means that $P_\nu$ can be expanded in Taylor series, as

$$P_\nu = \sum_{n=0}^{\infty} P_n \nu^n , \tag{4.1}$$

where the $P_n$ 's are bounded linear operators satisfying

$$\| P_n \| \leq Ca^n , \qquad (4.2)$$

for some $C$, $a > 0$, i.e. the series is convergent for $|\nu|$ sufficiently small. Under these conditions, we say that we have an *analytic family* of Markov chains.

**Definition 1** *We say that an analytic family is an analytic Lyapounov family if, in addition, the following conditions are fulfilled : there exist non-negative functions $f^\nu : i \to f_i^\nu$, and strictly positive integer valued functions $k^\nu : i \to k_i^\nu$, defined for $i \in S$, $\nu \in \mathcal{D}$, such that*

(i) $\sup_{i\in S,\nu\in\mathcal{D}} k_i^\nu = b < \infty$ ;

(ii) the series

$$\sum_{i\in S} exp\,(-b_1\, f_i^\nu) ,$$

defined for any $b_1 > 0$, converges uniformly in $\nu \in \mathcal{D}$ ;

(iii) there exist $d > 0$ such that

$$p_{ij}(1,\nu) = 0, \text{ for all } \nu \in \mathcal{D}, \text{ whenever } |f_i^\nu - f_j^\nu| > d ;$$

(iv) there exist $n > 0$ and $\delta > 0$ such that, for any $i \in S$ and any

$$j \in V_i \stackrel{\text{def}}{=} \{j : \sup_{\nu\in\mathcal{D}} p_{ji}(1,\nu) > 0\} ,$$

$$p_{ji}(n,0) > \delta ,$$

where $p_{ji}\ (n,\nu)$ are the n-step transition functions for the Markov chain $L^\nu$ ;

(v) for all $\nu \in \mathcal{D}$, $i \in S - B$, where $B$ is a finite subset of $S$, and for some $\epsilon > 0$,

$$\sum_{j\in S} p_{ij}(k_i^\nu,\nu) f_j^\nu - f_i^\nu < -\epsilon .$$

By Foster's criterion, $L_\nu$ is ergodic for any $\nu \in \mathcal{D}$.
We say that a Markov chain $L = L^0$ is an *analytic Lyapounov Markov chain*, if the family $L^\nu \equiv L^0$ is analytic Lyapounov.



**Remark 3** *In the rest of the paper, we shall essentially use $f_i^\nu \equiv f_i$.*

**Theorem 4.1** *If $L^\nu$ is an analytic Lyapounov family, then there exists $\nu_0 > 0$, such that the following results hold :*

1. *there exist $C_2$, $\delta_2 > 0$, such that*

$$\pi_i(\nu) < C_2\, exp\,(-\delta_2\, f_i^\nu) , \qquad (4.3)$$

*for all $i \in S$, $\nu \in \mathcal{D}$ ;*

2. *there exist constants $\sigma_2$, $C_3$, $\delta_3 > 0$, such that*

$$\sum_{j\in S} |p_{ij}(n,\nu) - \pi_j\,(\nu)| < C_3\, exp\,(-\delta_3 n) ,$$

*for all $\nu \in \mathcal{D}$, $i \in S$, $n > \sigma_2\, f_i^\nu$ ;*

3. *the stationary probabilities $\pi_i(\nu)$ are analytic in $\nu$, for $|\nu| < \nu_0$ and all $i \in S$.*

See theorem 4.2 and lemmas 4.3, 4.6 in [7].

## 5 Lyapounov functions

In this construction, we use the Lyapounov function (3.3), taking $k_x \equiv k$ sufficiently large. In the rest of this section, the main lemmas and theorems are quoted without proof.

**Theorem 5.1** *Let us consider a Jackson network such that $0 \in \prod$, and choose the function $f_x$ as in (3.3), with the point a lying inside $R_+^N$. Then, for this Lyapounov function, the Jackson network under consideration is an analytic Lyapounov Markov chain.*

**Lemma 5.2** *Let us fix $\epsilon > 0$ sufficiently small. There exists $\rho_0 > 0$ such that, for any $x \in \mathbf{Z}_+^N$ with*

$$\rho = \rho(x, \partial R_+^N) > \rho_0 ,$$

*and for any k such that*

$$\rho_0 < kd \equiv k \leq \rho ,$$

*we have*

$$\sum_y p_{xy}^{(k)} f_y - f_x < -\epsilon . \tag{5.1}$$

**Lemma 5.3** *Again choose $\epsilon > 0$ sufficiently small and $i = 1, ..., n$. Then there exist $\rho_i > 0$ such that, for any $x \in \mathbf{Z}_+^N$ with*

$$\rho = \max_{\Lambda: i \notin \Lambda} \rho(x, B^\Lambda) > \rho_i , \tag{5.2}$$

*and for any k such that*

$$\rho_i < kd \equiv k \leq \rho ,$$

*we have*

$$\sum_y p_{xy}^{(k)} f_y - f_x < -\epsilon . \tag{5.3}$$

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