

Convergence to equilibrium due to collisions with external particles

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Abstract. We consider a class of “most non ergodic” particle systems and prove that for most of them ergodicity appears if only one particle of N has contact with external world, that is this particle collides with external particles in random time moments.

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1. Introduction

In this paper we prove the main results of our previous paper [7] under more realistic assumptions. In [7] the external force was assumed to consist of 2 terms – continuous time random gaussian process (white noise) and simplest deterministic dissipative force $-\alpha v$. Now we consider the collisions of one internal particle with external particles having random velocities at random time moments. We show that the results appear to be quite similar to [7].

2. Model and results

2.1. Hamiltonian dynamics

We consider system of N point particles in \mathbf{R}^d with the linear phase space

$$L = \mathbf{R}^{2dN} = \left\{ \psi = \begin{pmatrix} q \\ p \end{pmatrix} : q = (q_1, \dots, q_N)^T, p = (p_1, \dots, p_N)^T, p_k, q_k \in \mathbf{R}^{d'} \right\},$$

where $()'$ means transposition. Coordinates and momenta $q_k = (q_{k1}, \dots, q_{kd})'$, $p_k = (p_{k1}, \dots, p_{kd})'$, are also considered as column vectors, so ψ is a column vector. L has obvious direct sum representation

$$L = l_2^{(dN)} \oplus l_2^{(dN)}, l_2^{(dN)} = \mathbf{R}^{dN}.$$

Introduce the Hamiltonian

$$H(\psi) = \sum_{k=1}^N \frac{|p_k|^2}{2M} + \frac{1}{2}(q, Vq), \tag{2.1}$$

where $M > 0$, and V is a positive definite $(dN \times dN)$ -matrix.

The evolution is defined by the following Hamiltonian system of equations on L :

$$\dot{q}_{kj} = \frac{\partial H}{\partial p_{kj}}, \tag{2.2}$$

$$\dot{p}_{kj} = - \frac{\partial H}{\partial q_{kj}}, \tag{2.3}$$

where $k = 1, \dots, N, j = 1, \dots, d$. The system (2.2), (2.3) can be rewritten as

$$\dot{\psi} = A\psi. \quad A = \begin{pmatrix} 0 & \frac{1}{M}E \\ -V & 0 \end{pmatrix} \tag{2.4}$$

The solution $\psi(t)$ of (2.4) with initial conditions $\psi(0)$ is

$$\psi(t) = e^{tA}\psi(0).$$

2.2. Collisions

Fix some particle, say with number $n = 1$ and assume that at random time moments

$$0 = t_0 < t_1 < \dots < t_k < \dots$$

the velocity of this particle changes instantaneously somehow, that is there are jumps $p_1(t_k-) \rightarrow p_1(t_k)$. On the intervals $[t_{k-1}, t_k)$ the dynamics is defined by the equations (2.2), (2.3) with the corresponding initial conditions $\psi(t_{k-1})$. We interpret the nature of these jumps as collisions with external particles on short time scale.

We use three exact models of “collisions”: concrete 1-1, 1- d , and abstract, 2.

Condition 1-1 Assume that masses of external particles are equal to m and internal masses are equal to $M > m$ (this condition is not necessary for our goals, but here it is technically convenient). We also assume that $d = 1$ and the collision conserves energy and momentum for this two particle system. Then it is known that (see for example [16])

$$v(t_k) = \alpha v(t_k-) + (1 - \alpha)u_k, \quad 0 < \alpha = \frac{M - m}{M + m} < 1 \quad (2.5)$$

or

$$p(t_k) = \alpha p(t_k-) + (1 - \alpha)Mu_k$$

where $v(t)$ is the velocity of the particle 1 at time t , and u_k is the velocity of external particle with which occurs the collision at time t_k .

Condition 1-d For dimension $d > 1$ we assume transformation

$$v(t_k) = Rv(t_k-) + w_k, \quad (2.6)$$

with some matrix (possibly random) R satisfying the following conditions

1. the distribution of the vector w_k has everywhere positive density on \mathbf{R}^d ;
2. the matrix R defines a contraction map of \mathbf{R}^d , i.e. there exists constant $0 < \alpha < 1$ such that

$$(Rp, Rp) \leq \alpha(p, p)$$

for all $p \in \mathbf{R}^d$. Or equivalently, the spectrum of the matrix RR^T lies in the open unit interval.

Condition 2 These are more general but seemingly more technical conditions:

$$p_1(t_k) = J(\xi_k, p_1(t_k-)) \quad (2.7)$$

where ξ_k is a random l -dimensional vector $l \geq 1$, and

$$J : \mathbf{R}^l \times \mathbf{R}^d \rightarrow \mathbf{R}^d = \{p_1\}$$

is some vector function. For example, in case 1-1 the vector $\xi_k = u_k$.

Transformation J is assumed to have the following properties:

1. J is everywhere differentiable and analytic almost (w.r.t. Lebesgue measure) for any point $(g, p) \in \mathcal{O}_\xi \times \mathbf{R}^d$, where \mathcal{O}_ξ is some open set in \mathbf{R}^l . For example, in case 1-1 $\mathcal{O}_\xi = \mathbf{R}^d$.
2. For any $p \in \mathbf{R}^d$ the image $J(\mathcal{O}_\xi, p) \subset \mathbf{R}^d$ contains sphere of radius $|p|$, and for any $h \geq 0$ there exists $g = g(p, h) \in \mathcal{O}_\xi$ such that

$$|J(g; p)| = h.$$

3. There exists compact subset $K \subset L$ and positive constant $\alpha < 1$ such that for any $p \notin K$ the following inequality holds:

$$E_\xi |J(\xi; p)|^2 \leq \alpha |p|^2,$$

Intuitively this means that kinetic energy of particle 1, being sufficiently large, should be decreased by collisions.

Assumptions on the nature of randomness We assume that:

1. $\tau_m = t_m - t_{m-1}, m = 1, 2, \dots$, are i.i.d. random variables with positive on R_+ density ρ_τ such that $E\tau_1 < \infty$;
2. random vectors $\xi_1, \xi_2, \dots, \xi_m, \dots$ are i.i.d. with distribution function F_ξ . Assume that F_ξ has density ρ_ξ with respect to Lebesgue measure on \mathbf{R}^l , which is positive on some open set $\mathcal{O}_\xi \subset \mathbf{R}^l$ and such that $P(\xi_1 \in \mathcal{O}_\xi) = 1$;
3. the arrays $(\xi_1, \xi_2, \dots, \xi_m, \dots)$ and $(\tau_1, \tau_2, \dots, \tau_m, \dots)$ are mutually independent.

Dynamics with collisions For $\psi = (q, p) \in L$, $q = (q_1, \dots, q_N)^T$, $p = (p_1, \dots, p_N)^T$ and $\xi \in \mathbf{R}^l$ we define the transformation:

$$J_L(\xi; \psi) = \psi' = (q', p'),$$

where

$$q' = q, \quad p'_1 = J(\xi, p_1), \quad p'_k = p_k, \quad k > 1.$$

For simplicity we will omit index L and denote $J_L(\cdot, \cdot)$ simply $J(\cdot, \cdot)$. For any $t \geq 0$ we will consider the following transformations of L ,

$$J(t, \xi; \psi) = J(\xi; e^{tA}\psi), \quad \psi \in L.$$

For any $\psi \in L$ and any integer $m \geq 1$ define the time lengths $\tau_1 = t_1, \tau_2 = t_2 - t_1, \dots, \tau_m = t_m - t_{m-1}$. For any $t \geq 0$ we define the dynamics with collisions as follows

$$\psi(t) = e^{A(t-t_m)}\psi_m, \quad \psi_m = J(\tau_m, \xi_m; \psi_{m-1}), \quad \psi_0 = \psi(0).$$

where m is the maximal integer such that $t_m < t$.

Note that if τ_k are exponentially distributed then $\psi(t)$ is a Markov process. Moreover, the defined random process $\psi(t)$ is piece wise deterministic continuous time Markov process with trajectories continuous from the right.

2.3. Dissipative subspace

Denote by \mathbf{H} the set of all positive definite $dN \times dN$ matrices V . Note that the set of all symmetric matrices is the linear space of dimension $\frac{dN(dN+1)}{2}$, and \mathbf{H} is its open subset with induced topology and induced Lebesgue measure λ from the space of symmetric matrices.

Define the dissipative subspace

$$L_-(V) = L_- = \left\{ \begin{pmatrix} q \\ p \end{pmatrix} \in L : q, p \in l_V \right\}$$

where l_V is the subspace of \mathbf{R}^{dN} generated by the vectors $V^k e_1, k = 0, 1, 2, \dots$, where $e_n, n = 1, \dots, dN$ is the standard basis of $l^{(dN)} = \mathbf{R}^{dN}$.

We say that V has **completeness** property if

$$L_-(V) = L. \tag{2.8}$$

Further we denote $\omega_1^2, \dots, \omega_{dN}^2$ all (positive) eigenvalues of V .

Denote by \mathbf{H}^+ the set of positive definite $dN \times dN$ matrices V having completeness property. Denote also $\mathbf{H}^{++} \subset \mathbf{H}^+$ the subset of matrices with completeness property and, moreover, having spectrum such that $\omega_1, \dots, \omega_{dN}$ are rationally independent. Further on we assume all ω_k to be positive.

Proposition 2.1. *The set \mathbf{H}^{++} is everywhere dense in the set \mathbf{H} of all positive definite $dN \times dN$ matrices, and moreover, the complement $\mathbf{H} \setminus \mathbf{H}^{++}$ has Lebesgue measure zero.*

2.4. Main results

In the theorems below the above mentioned assumptions are always assumed.

Ergodicity

Theorem 2.1. *There exists probability measure π on L such that:*

- 1) *it is absolutely continuous with respect to Lebesgue measure λ on L and has positive density;*
- 2) *for any measurable bounded function f on L and any initial condition $\psi(0)$*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\psi(t)) dt = \int_L f(\psi) \pi(d\psi), \quad \text{a.s.}$$

Stronger convergence of the embedded chain Denote by ψ_n the embedded chain:

$$\psi_n = \psi(t_n). \quad (2.9)$$

It is clear that ψ_n is a discrete time Markov chain. Let $P(\psi, A), \psi \in L, A \subset \mathcal{B}(L)$, be the transition probability of this chain, where $\mathcal{B}(L)$ is the Borel σ -algebra in L .

Theorem 2.2. *ψ_n has unique invariant measure π on L (up to multiplication on a constant). Moreover, it has the following properties:*

1. *π is the same in the previous theorem. Further on, we will suppose, that $\pi(X) = 1$.*
2. *For any $\psi \in L$ we have: $\sup_{A \in \mathcal{B}(L)} |P^k(\psi, A) - \pi(A)| \rightarrow 0$, as $k \rightarrow \infty$.*

Properties of the invariant measure Here we give some results concerning invariant measure π in case 1-1. Assume moreover that u_m have finite second moment $E u_m^2 < \infty$.

Assume also that the intervals τ_m between collisions have exponential distribution with parameter $\lambda > 0$. In this case $\psi(t)$ is time homogeneous Markov process. Denote

$$C(t) = E_\psi \{ (\psi(t) - E\psi(t)) (\psi(t) - E\psi(t))^T \}$$

its covariance matrix.

Theorem 2.3. *The following propositions holds:*

1. *The measure π is Gibbs measure, that is, it has density*

$$p_\beta(\psi) = \frac{1}{Z} \exp(-\beta H(\psi)),$$

$$Z = \int_L \exp(-\beta H(\psi)) d\lambda(\psi)$$

for some $\beta > 0$ iff u_m have gaussian distribution with mean zero and some variance σ^2 . In the gaussian case we have

$$\beta = \frac{(1 + \alpha)}{M(1 - \alpha)\sigma^2} = \frac{1}{m\sigma^2}, \quad (2.10)$$

where m is the mass of external particles.

2. However, if $Eu_m = 0$ but u_m is not gaussian, then for any $\psi(0)$ the limiting covariance matrix is the same as above, i.e.

$$\lim_{t \rightarrow \infty} C(t) = \beta^{-1} \begin{pmatrix} V^{-1} & 0 \\ 0 & ME \end{pmatrix},$$

where β is defined in (2.10).

Plan of the proof First of all we will prove Theorem 2.2 about convergence of the embedded chain. For this we will need some results from the theory of general Markov processes, which will be formulated just now. We will show in Appendix how they follow from the known results. After this, we will prove Theorems 2.1 and 2.3.

Stability of Markov processes Let X be complete locally compact separable metric space and $\mathcal{B}(X)$ – its Borel σ -algebra. Also we will consider measures on X which will be always assumed non-negative and countably additive. Consider Markov chain ξ_n , $n = 0, 1, 2, \dots$ on X with transition probabilities $P(x, A)$, $x \in X, A \in \mathcal{B}(X)$.

Theorem 2.4. Assume that the following conditions hold:

1. (strong irreducibility property) there exist integer $m \geq 1$ such that for any $x \in X$ the m -step transition probability $P^m(x, \cdot)$ is equivalent to some finite measure μ , having the property that $\mu(O) > 0$ for any open subset $O \subset X$;
2. (weak Feller property) for any open $O \subset X$ the function $P(x, O)$ is lower semi-continuous;
3. (drift condition) there exists compact subset $K \subset X$ and non-negative measurable function $f(x), x \in X$ (Lyapunov function), which tends to infinity with the distance from K , such that

$$\int f(y)dP(x, dy) - f(x) \leq -1, \text{ for all } x \in X \setminus K,$$

$$\int f(y)dP(x, dy) < \infty \text{ for all } x \in K.$$

Then there exists a unique (up to a multiplicative constant) invariant measure π for ξ_n and the following properties hold:

1. π is finite and absolutely continuous w.r.t. measure μ . We assume further that $\pi(X) = 1$.
2. For any $x \in X$

$$\lim_{n \rightarrow \infty} \sup_{A \subset B(X)} |P^n(x, A) - \pi(A)| = 0.$$

3. For any bounded measurable function h on X and any initial condition ξ_0 we have:

$$\frac{1}{n} \sum_{k=0}^n h(\xi_k) \rightarrow \pi(f) = \int_X h(x) \pi(dx), \text{ as } n \rightarrow \infty, \text{ a.s.}$$

3. Proof of Theorem 2.2

Consider now the embedded Markov chain $\psi_k = \psi(t_k)$, $\psi_0 = \psi(0)$. For $\psi \in L$ and Borel subset $A \subset L$ let $P(\psi, A)$ be the transition probability of the Markov chain ψ_k . Without loss of generality we assume here $M = 1$.

Theorem 2.2 immediately follows from theorem 2.4 and the following lemma:

Lemma 3.1. *The chain ψ_k satisfies all conditions of theorem 2.4 with any finite measure μ equivalent to the Lebesgue measure on L and $f(\psi) = H(\psi)$.*

Now we will prove this lemma.

3.1. Proof of weak Feller property

Let O be any fixed open subset of L . For any $\psi \in L$ denote $\mathbf{1}_\psi(\tau, y)$, $\tau \in \mathbf{R}_+$, $y \in \mathbf{R}^l$ the indicator function on $\mathbf{R}_+ \times \mathbf{R}^l$, that is $\mathbf{1}_\psi(\tau, y) = 1$ if $J(\tau, y; \psi) \in O$, and zero otherwise. Then we have

$$P(\psi, O) = \int_{\mathbf{R}_+ \times \mathbf{R}^l} \mathbf{1}_\psi(s, y) \rho_\tau(s) \rho_\xi(y) ds dy.$$

Let $\psi_n \rightarrow \psi$, $\psi_n \in L$ as $n \rightarrow \infty$. Fix $s \geq 0$ and $y \in \mathbf{R}^l$ and consider two cases:

1. $J(s, y; \psi) \in O$, then, as O is open and $J(s, y; \psi)$ is continuous in ψ , starting from some n the inclusion $J(s, y; \psi_n) \in O$ holds. That is why

$$\lim_{n \rightarrow \infty} \mathbf{1}_{\psi_n}(s, y) = \mathbf{1}_\psi(s, y) = 1.$$

2. $J(s, y; \psi) \notin O$. Then

$$\liminf_n \mathbf{1}_{\psi_n}(s, y) \geq \mathbf{1}_\psi(s, y) = 0.$$

Thus for any $s \geq 0$ and $y \in \mathbf{R}^l$

$$\liminf_n \mathbf{1}_{\psi_n}(s, y) \geq \mathbf{1}_\psi(s, y).$$

Then by Fatou's lemma

$$\liminf_n P(\psi_n, O) \geq P(\psi, O).$$

Thus the chain satisfies the weak Feller property.

3.2. Proof of the drift condition

The drift condition for the chain ψ_k of Theorem 2.4 immediately follows from the following lemma (stronger than Lemma 3.1).

Lemma 3.2. *There exists compact set $K' \subset L$, such that for all $\psi \notin K'$ the following inequality holds:*

$$E\{H(\psi_1)|\psi_0 = \psi\} - H(\psi) < -rH(\psi),$$

for some positive constant $r > 0$, not depending on ψ .

Proof. It will be convenient to use the following notation:

$$\begin{aligned} \psi &= (q, p)^T, \quad q = (q_1, \dots, q_N)^T, \quad p = (p_1, \dots, p_N)^T, \\ \psi^0(t) &= e^{tA}\psi = (q^0(t), p^0(t))^T, \\ q^0(t) &= (q_1^0(t), \dots, q_N^0(t))^T, \quad p^0(t) = (p_1^0(t), \dots, p_N^0(t))^T, \quad t \geq 0. \end{aligned}$$

Then the point ψ_1 can be written as

$$\psi_1 = J(\tau, \xi; \psi),$$

where $\xi = \xi_1$, $\tau = \tau_1$ are random variables defined above in the definition of $\psi(t)$. For the energy we have:

$$\begin{aligned} H(\psi_1) &= H(\psi^0(\tau)) + \frac{1}{2} (|J(\xi; p_1^0(\tau))|^2 - |p_1^0(\tau)|^2) = \\ &= H(\psi) + \frac{1}{2} (|J(\xi; p_1^0(\tau))|^2 - |p_1^0(\tau)|^2). \end{aligned}$$

And for the mean energy:

$$E\{H(\psi_1)|\psi_0 = \psi\} = H(\psi) + \frac{1}{2}(E\{|J(\xi; p_1^0(\tau))|^2\} - E\{|p_1^0(\tau)|^2\}).$$

By Condition 2 (item 3) on J we have the inequality:

$$E\{|J(\xi; p_1^0(\tau))|^2\} \leq M + \alpha E\{|p_1^0(\tau)|^2\},$$

where

$$M = \sup_{p \in K} E\{|J(\xi; p)|^2\}$$

and the compact set K was defined in item 3 of Condition 2 on J .

Again

$$E\{H(\psi_1)|\psi_0 = \psi\} \leq H(\psi) + \frac{M}{2} - c\Delta(\psi), \quad c = \frac{1-\alpha}{2} > 0,$$

where we used the notation

$$\Delta(\psi) = E\{|p_1^0(\tau)|^2\}.$$

From the definition of Δ we conclude that it is a non negative definite quadratic form of ψ . On the other hand as the completeness condition holds and due to the Proposition 6.1 (see below) we have $\Delta(\psi) = 0$ iff $\psi = 0$. So form $\Delta(\psi)$ is positive definite. As all positive definite forms define equivalent norms, there exists constant $r > 0$ such that for all $\psi \in L$ the following inequality holds:

$$\Delta(\psi) > rH(\psi).$$

It is clear that for some compact subset $K' \subset L$ and for all $\psi \notin K'$ we have

$$M < rcH(\psi).$$

That is why, for all $\psi \notin K'$:

$$E\{H(\psi_1)|\psi_0 = \psi\} - H(\psi) \leq -\frac{rc}{2}H(\psi).$$

Thus the lemma is proved. \square

3.3. Proof of the strong irreducibility property

For $\psi \in L$, any $m = 1, 2, \dots$ and arbitrary $t_1, \dots, t_m \geq 0$, $u_1, \dots, u_m \in \mathcal{O}_\xi$, $\psi \in L$ define the map J_m :

$$J_0 = J_0(\psi) = \psi, \quad J_1(t_1, u_1; \psi) = J(t_1, u_1; \psi),$$

$$J_2(t_1, u_1, t_2, u_2; \psi) = J(t_2, u_2; J_1(t_1, u_1; \psi)),$$

$$J_m(t_1, u_1, \dots, t_m, u_m; \psi) = J(t_m, u_m; J_{m-1}(t_1, u_1, \dots, t_{m-1}, u_{m-1}; \psi)).$$

Also for any point $\psi \in L$ and any integer $m \geq 1$ define the subset:

$$\mathcal{J}_m(\psi) = \{J_m(t_1, u_1, \dots, t_m, u_m; \psi) : t_1, \dots, t_m \geq 0, u_1, \dots, u_m \in \mathcal{O}_\xi\} \subset L.$$

Now we prove the following ‘‘Strong controllability Theorem’’.

Theorem 3.1. *There exists $m \geq 1$ such that for any $\psi \in L$*

$$\mathcal{J}_m(\psi) = L.$$

Proof. Due to condition 2 on the transformation J , for all $\psi \in L$ and all $t \geq 0$ there exists $u = u(t, \psi) \in \mathcal{O}_\xi$ such that

$$J(t, u; \psi) = J(u; e^{tA}\psi) = Ie^{tA}\psi,$$

where the linear transformation I is a velocity flip of the first coordinate of particle 1, i.e. if $\psi = (q, p) \in L$, $q = (q_1, \dots, q_N)^T$, $p = (p_1, \dots, p_N)^T$, then

$$I\psi = \psi' = (q', p'), \quad q' = q, \quad p'_{1,1} = -p_{1,1},$$

$$p'_{1,k} = p_{1,k}, \quad k = 2, \dots, d, \quad p'_j = p_j, \quad j = 2, \dots, N.$$

Thus, due to Theorem 1 of [9] there exists m such that for all ψ

$$\mathcal{M}_h \subset \mathcal{J}_m(\psi),$$

where $h = H(\psi)$ and

$$\mathcal{M}_h = \{\psi \in L : H(\psi) = h\}$$

is the constant energy manifold.

Consider now any point $\psi' \in L$, $h' = H(\psi')$. Let us denote $g = (0, p) \in L$, $p = (p_1, \dots, p_N)^T$, where

$$p_{1,1} = 1, \quad p_{1,k} = 0, \quad k = 2, \dots, d, \quad p_j = 0, \quad j = 2, \dots, N.$$

It is obvious that $\sqrt{2h}g \in \mathcal{M}_h$, and therefore there exist $t_1, u_1, \dots, t_m, u_m$ such that

$$J_m(t_1, u_1, \dots, t_m, u_m; \psi) = \sqrt{2h}g \in \mathcal{M}_h.$$

Due to the second part of condition 2 on the transformation J we can find $u_{m+1}, t_{m+1} = 0$ such that

$$J_{m+1}(t_1, u_1, \dots, t_m, u_m, t_{m+1}, u_{m+1}; \psi) = J(0, u_{m+1}; \sqrt{2h}g) = \tilde{\psi},$$

where $\tilde{\psi} = (0, \tilde{p})$, $\tilde{p} = (\tilde{p}_1, \dots, \tilde{p}_N)^T$ and

$$\tilde{p}_k = 0, \quad k = 2, \dots, N, \quad |\tilde{p}_1| = \sqrt{2h'},$$

and so $H(\tilde{\psi}) = h'$. Similar arguments show the existence of $t'_1, u'_1, \dots, t'_m, u'_m$ such that

$$J_m(t'_1, u'_1, \dots, t'_m, u'_m; \tilde{\psi}) = \psi'.$$

Finally we get

$$J_{2m+1}(t_1, u_1, \dots, t_m, u_m, t_{m+1}, u_{m+1}, t'_1, u'_1, \dots, t'_m, u'_m; \psi) = \psi'.$$

So, the assertion is proved. \square

Let us come back to the proof of the strong irreducibility property. Fix the number m from Theorem 3.1 and consider the following set:

$$\mathcal{U} = \{(u_1, t_1, \dots, u_m, t_m) : t_i \geq 0, u_i \in \mathcal{O}_\xi \text{ for all } i = 1, \dots, m\} = (\mathcal{O}_\xi \times \mathbb{R}_{\geq 0})^m.$$

When $\psi \in L$ is fixed, we can consider the map $J_m(u_1, t_1, \dots, u_m, t_m; \psi)$ as a map from \mathcal{U} to L . We will denote this map

$$G^\psi : \mathcal{U} \rightarrow L, \quad \mathbf{u} = (u_1, t_1, \dots, u_m, t_m) \mapsto J_m(u_1, t_1, \dots, u_m, t_m; \psi).$$

Theorem 3.1 states that the image of the map G^ψ coincides with L for all $\psi \in L$. Denote λ and μ Lebesgue measures on \mathcal{U} and L accordingly.

Lemma 3.3. *For any measurable $B \subset L$ its Lebesgue measure $\mu(B) = 0$ iff the Lebesgue measure λ of the set $(G^\psi)^{-1}(B)$ in \mathcal{U} is zero.*

The proof of this lemma is the same as the proof of lemma 7 in [9]. But for reader's convenience we will give the proof here in our notation.

Proof. 1) Assume that for some $B \subset L$ we have $\mu(B) = 0$. Let us show that $\lambda((G^\psi)^{-1}(B)) = 0$. Let A_{cr} be the set of critical points of the map G^ψ (that is points $\mathbf{u} = (u_1, t_1, \dots, u_m, t_m) \in \mathcal{U}$ where the rank of the Jacobin is not maximal) and let $E = G^\psi(A_{cr}) \subset L$ be the set of critical values of G^ψ . By Sard's theorem $\mu(E) = 0$. But as $G^\psi(\mathcal{U}) = L$, then there exists a non-critical point $\mathbf{u} \in \mathcal{U}$, that is such that the rank of dG^ψ at this point equals $2N$. As the map G^ψ is analytic in the variables $u_1, t_1, \dots, u_m, t_m$, the set of points A_{cr} , where the rank is less than $2N$, has Lebesgue measure zero. Then the equality $\lambda((G^\psi)^{-1}(B)) = 0$ follows from the Ponomarev's theorem (theorem 1 of [11]).

2) Assume that for some $B \subset L$ we have $\mu(B) > 0$, and let us show that $\lambda((G^\psi)^{-1}(B)) > 0$. By Lebesgue differentiation theorem there exists point $\psi' \in L \setminus E$ and its neighborhood $O(\psi')$ such that $\mu(O(\psi') \cap B) > 0$. Then there is point $\mathbf{u} = \mathbf{u}(\psi') \in (G^\psi)^{-1}(\psi')$ and some its neighborhood $O(\mathbf{u}) \subset \mathcal{U}$ such that the restriction of G^ψ on $O(\mathbf{u})$ is a submersion. Then $\mu(O(\psi') \cap B) > 0$ implies $\lambda((G^\psi)^{-1}(B) \cap O(\mathbf{u})) > 0$. So the lemma is proved. \square

Consider the probability density function of $\xi_1, \tau_1, \dots, \xi_m, \tau_m$ w.r.t. Lebesgue measure on \mathcal{U} :

$$p(\mathbf{u}) = \rho_\xi(u_1)\rho_\tau(t_1) \dots \rho_\xi(u_m)\rho_\tau(t_m), \quad \mathbf{u} = (u_1, t_1, \dots, u_m, t_m) \in \mathcal{U}.$$

We have an obvious equality:

$$P^m(\psi, B) = \int_{(G^\psi)^{-1}(B)} p(\mathbf{u}) d\mathbf{u}.$$

Due to Lemma 3.3 and this equality we conclude that the strong irreducibility property holds for the chain ψ_k .

4. Proof of Theorem 2.1

Theorem 2.1 follows from ergodicity of the embedded chain ψ_k (Theorem 2.2). Proof of this statement coincides almost verbatim with the proof of similar statement in [9] (theorem 2 from the section “Proof of theorem 2”).

5. Proof of Theorem 2.3

In this case the process $\psi(t)$ is a time homogeneous Markov process. Denote by \mathcal{A} the infinitesimal generator of $\psi(t)$, i.e.

$$(\mathcal{A}f)(\psi) = \lim_{t \rightarrow 0} \frac{E_\psi f(\psi(t)) - f(\psi)}{t}.$$

The well known formula is

$$(\mathcal{A}f)(\psi) = \{H, f\} + \lambda I(f), \quad (5.1)$$

where

$$I(f) = E_\xi f(J(\xi; \psi)) - f(\psi) = E_v f(q; \alpha p_1 + (1 - \alpha)Mv, p_2, \dots, p_N) - f(q; p)$$

and

$$\{H, f\} = \sum_{k=1}^N \frac{\partial H}{\partial p_k} \frac{\partial f}{\partial q_k} - \frac{\partial H}{\partial q_k} \frac{\partial f}{\partial p_k}$$

is a Poisson bracket.

Let us prove the first part of Theorem 2.3. Assume that $p_\beta(\psi)$ is the Gibbs probability density of the invariant measure for the process $\psi(t)$ for some $\beta > 0$. Thus $\mathcal{A}^* p_\beta(\psi) = 0$ for all $\psi \in L$ where \mathcal{A}^* is a formal adjoint operator to \mathcal{A} w.r.t. the standard inner product on $L_2(d\psi)$. For the bracket we have

$$\{H, \cdot\}^* = -\{H, \cdot\}.$$

Consequently, for all $\beta > 0$

$$\{H, p_\beta\}^* = 0.$$

Let us find I^* . For any functions $f, g \in L_2$ we have:

$$\begin{aligned} & \int E_v \{f(q; \alpha p_1 + (1 - \alpha)Mv, p_1, \dots, p_N)\} g(q, p) dqdp = \\ & = \frac{1}{\alpha} \int f(q, p) E_v \left\{ g\left(q; \frac{p_1 - (1 - \alpha)Mv}{\alpha}, p_2, \dots, p_N\right) \right\} dqdp. \end{aligned}$$

That is why

$$I^*(f) = \gamma E_v \{f(q; \gamma p_1 - (1 - \gamma)Mv, p_2, \dots, p_N)\} - f(q, p), \quad \gamma = \frac{1}{\alpha}.$$

In case $\alpha = 0$ one should take the formal limit in this formula for $I^*(f)$, then we will have $I^*(f) = 1$.

If $f = p_\beta$, then

$$\begin{aligned} & E_v \{p_\beta(q; \gamma p_1 - (1 - \gamma)Mv, p_2, \dots, p_N)\} = \\ & = \frac{1}{Z} e^{-\beta H(q, p)} \exp\left\{\frac{\beta}{2M} p_1^2\right\} E_v \left\{ \exp\left(-\frac{\beta}{2M} (\gamma p_1 - (1 - \gamma)Mv)^2\right) \right\}. \quad (5.2) \end{aligned}$$

Lemma 5.1. *If η is a gaussian random variable with mean a and variance σ^2 then*

$$E \exp(-\eta^2) = \sqrt{\frac{1}{1 + 2\sigma^2}} \exp\left(-\frac{a^2}{1 + 2\sigma^2}\right).$$

Proof. By definition we have

$$E \exp(-\eta^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} e^{-x^2} e^{-(x-a)^2/(2\sigma^2)} dx.$$

On the other hand

$$\begin{aligned} x^2 + \frac{(x-a)^2}{2\sigma^2} &= x^2 \left(1 + \frac{1}{2\sigma^2}\right) - x \frac{a}{\sigma^2} + \frac{a^2}{2\sigma^2} = \\ &= \left(1 + \frac{1}{2\sigma^2}\right) \left(x - \frac{a}{2\sigma^2} \frac{1}{1 + \frac{1}{2\sigma^2}}\right)^2 + \frac{a^2}{2\sigma^2} - \frac{a^2}{4\sigma^4} \frac{1}{1 + \frac{1}{2\sigma^2}} = \\ &= \left(1 + \frac{1}{2\sigma^2}\right) \left(x - \frac{a}{2\sigma^2} \frac{1}{1 + \frac{1}{2\sigma^2}}\right)^2 + \frac{a^2}{1 + 2\sigma^2} = \frac{\left(x - \frac{a}{2\sigma^2 + 1}\right)^2}{2 \frac{\sigma^2}{1 + 2\sigma^2}} + \frac{a^2}{1 + 2\sigma^2}. \end{aligned}$$

Thus,

$$\begin{aligned} E \exp(-\eta^2) &= \frac{1}{\sqrt{2\pi\sigma^2}} \sqrt{2\pi \frac{\sigma^2}{1+2\sigma^2}} \exp\left(-\frac{a^2}{1+2\sigma^2}\right) \\ &= \sqrt{\frac{1}{1+2\sigma^2}} \exp\left(-\frac{a^2}{1+2\sigma^2}\right). \end{aligned}$$

So the lemma is proved. \square

Now assume that the velocity of the external particle v is a zero mean gaussian random variable with the variance σ_v^2 . From the equation (5.2) and lemma 5.1 we have

$$\begin{aligned} &E_v \{p_\beta(q; \gamma p_1 - (1-\gamma)Mv, p_2, \dots, p_N)\} \\ &= \frac{1}{Z} e^{-\beta H(q,p)} \exp\left\{\frac{\beta}{2M} p_1^2\right\} \sqrt{\frac{1}{1+2\sigma^2}} \exp\left(-\frac{a^2}{1+2\sigma^2}\right), \end{aligned}$$

where

$$\begin{aligned} a &= E_v \sqrt{\frac{\beta}{2M}} (\gamma p_1 - (1-\gamma)Mv) = \sqrt{\frac{\beta}{2M}} \gamma p_1, \\ \sigma^2 &= D \sqrt{\frac{\beta}{2M}} (\gamma p_1 - (1-\gamma)Mv) = \frac{\beta M}{2} (1-\gamma)^2 \sigma_v^2. \end{aligned}$$

So,

$$\begin{aligned} &E_v \{p_\beta(q; \gamma p_1 - (1-\gamma)v, p_2, \dots, p_N)\} = \\ &= \frac{1}{Z} e^{-\beta H(q,p)} \sqrt{\frac{1}{1+2\sigma^2}} \exp\left(\frac{\beta}{2M} p_1^2 \left(1 - \frac{\gamma^2}{1 + \beta M (1-\gamma)^2 \sigma_v^2}\right)\right). \end{aligned}$$

Choose β such that

$$1 - \frac{\gamma^2}{1 + \beta M (1-\gamma)^2 \sigma_v^2} = 0,$$

i.e.

$$\beta = \frac{\gamma^2 - 1}{M(1-\gamma)^2 \sigma_v^2} = \frac{\gamma + 1}{M(\gamma - 1) \sigma_v^2} = \frac{1 + \alpha}{M(1 - \alpha) \sigma_v^2}.$$

For this β we have $\sigma^2 = (\gamma^2 - 1)/2$ and

$$E_v \{p_\beta(q; \gamma p_1 - (1-\gamma)Mv, p_2, \dots, p_N)\} = \frac{1}{Z} e^{-\beta H(q,p)} \frac{1}{\gamma}.$$

That is why

$$\mathcal{A}^* p_\beta = I^* p_\beta = 0$$

and so the Gibbs measure is an invariant for the process $\psi(t)$.

Conversely, let p_β satisfy the equation $\mathcal{A}^*p_\beta = 0$ for some $\beta > 0$. Now we prove that in this case the distribution of v is gaussian with zero mean. From the (5.2) we have that for all $p_1 \in \mathbf{R}$:

$$\gamma E_v \left\{ \exp \left(-\frac{\beta}{2M} (\gamma p_1 - (1-\gamma)Mv)^2 \right) \right\} = \exp \left\{ -\frac{\beta}{2M} p_1^2 \right\}. \quad (5.3)$$

Multiplying the both side of the last equality on $e^{i\lambda p_1}$ and integrating by p_1 over the \mathbf{R}^1 we obtain

$$\int_{\mathbf{R}^1} e^{i\lambda p_1} E_v \left\{ \exp \left(-\frac{\beta}{2M} (\gamma p_1 - (1-\gamma)Mv)^2 \right) \right\} dp_1 = \int_{\mathbf{R}^1} e^{i\lambda p_1} e^{-\frac{\beta}{2M} p_1^2} dp_1.$$

The right hand side is the characteristic function of the zero mean gaussian random variable with the variance $\sqrt{M/\beta}$ and so

$$\int_{\mathbf{R}^1} e^{i\lambda p_1} e^{-\frac{\beta}{2M} p_1^2} dp_1 = \sqrt{\frac{2\pi M}{\beta}} e^{-\frac{M\lambda^2}{2\beta}}. \quad (5.4)$$

If we denote by ρ the density of v then for the left hand side we have

$$\begin{aligned} l(\lambda) &= \int_{\mathbf{R}^1} e^{i\lambda p_1} E_v \left\{ \exp \left(-\frac{\beta}{2M} (\gamma p_1 - (1-\gamma)Mv)^2 \right) \right\} dp_1 = \\ &= \int_{\mathbf{R}^1} dv \rho(v) \int_{\mathbf{R}^1} dp_1 e^{i\lambda p_1} \exp \left(-\frac{\beta}{2M} (\gamma p_1 - (1-\gamma)Mv)^2 \right). \end{aligned}$$

The integral

$$\int_{\mathbf{R}^1} e^{i\lambda p_1} \exp \left(-\frac{\beta}{2M} (\gamma p_1 - (1-\gamma)Mv)^2 \right) dp_1$$

is the characteristic function of the gaussian random variable with mean $a = (1-\gamma)Mv/\gamma$ and variance $\sigma^2 = \sqrt{M/(\beta\gamma^2)}$. That is why

$$\begin{aligned} \int_{\mathbf{R}^1} e^{i\lambda p_1} \exp \left(-\frac{\beta}{2M} (\gamma p_1 - (1-\gamma)Mv)^2 \right) dp_1 &= \sqrt{\frac{2\pi M}{\beta\gamma^2}} \exp \left(i\lambda a - \frac{\sigma^2 \lambda^2}{2} \right) = \\ &= \sqrt{\frac{2\pi M}{\beta\gamma^2}} \exp \left(i\lambda \frac{(1-\gamma)Mv}{\gamma} - \frac{M\lambda^2}{2\beta\gamma^2} \right). \end{aligned}$$

Then

$$l(\lambda) = \sqrt{\frac{2\pi M}{\beta\gamma^2}} \exp \left(-\frac{M\lambda^2}{2\beta\gamma^2} \right) \int_{\mathbf{R}^1} \exp \left(i\lambda \frac{(1-\gamma)Mv}{\gamma} \right) \rho(v) dv =$$

$$= \sqrt{\frac{2\pi M}{\beta\gamma^2}} \exp\left(-\frac{M\lambda^2}{2\beta\gamma^2}\right) \varphi\left(\frac{(1-\gamma)M\lambda}{\gamma}\right),$$

where $\varphi(\lambda)$ is a characteristic function of v :

$$\varphi(\lambda) = \int_{\mathbf{R}} e^{i\lambda v} \rho(v) dv.$$

Consequently from the (5.3) and (5.4) for all $\lambda \in \mathbf{R}$ we have

$$\varphi\left(\frac{(1-\gamma)M\lambda}{\gamma}\right) = \exp\left(-\frac{M\lambda^2}{2\beta} + \frac{M\lambda^2}{2\beta\gamma^2}\right).$$

Immediately we conclude that the distribution of v is zero mean gaussian. So the first part of Theorem 2.1 is proved.

Now we prove the second part. Let us denote

$$Q_k(t) = E_\psi q_k(t) = E\{q_k(t)|\psi(0) = \psi\}, \quad P_k(t) = E_\psi p_k(t) = E\{p_k(t)|\psi(0) = \psi\}.$$

Then using formula (5.1) for the infinitesimal generator we get the following ordinary differential equations

$$\begin{aligned} \dot{Q}_k &= \frac{1}{M} P_k, \\ \dot{P}_k &= -\sum_{j=1}^N V_{k,j} Q_j - \lambda(1-\alpha) P_1(t) \delta_{k,1}, \end{aligned}$$

for all $k = 1, \dots, N$, where $\delta_{k,1}$ is a Kronecker symbol and the initial conditions are given by $(Q(0), P(0)) = \psi$. If we return to the velocity variables $V_k = P_k M$, then we get

$$\begin{aligned} \dot{Q}_k &= V_k, \\ \dot{V}_k &= -\frac{1}{M} \sum_{j=1}^N V_{k,j} Q_j - \lambda(1-\alpha) V_1(t) \delta_{k,1}, \end{aligned}$$

for all $k = 1, \dots, N$. Due to theorem 2.1 from [4] we obtain

$$\lim_{t \rightarrow \infty} Q_k(t) = \lim_{t \rightarrow \infty} V_k(t) = 0,$$

for all $k = 1, \dots, N$ and any initial condition $\psi \in L$, i.e.

$$\lim_{t \rightarrow \infty} E_\psi \psi(t) = 0, \tag{5.5}$$

for any initial $\psi(0) \in L$. Now let us consider the matrix

$$C(t) = E_\psi \psi(t) \psi^T(t).$$

Denote by Γ the $2N \times 2N$ matrix all elements of which are equal to 0 except $(2N+1, 2N+1)$ -element which is equal to 1.

Lemma 5.2. *The matrix $C(t)$ satisfies the following differential equation*

$$\dot{C}(t) = AC + CA^T - \lambda(1 - \alpha)(\Gamma C + C\Gamma - (1 - \alpha)\Gamma C\Gamma) + \lambda(1 - \alpha)^2 M^2 \sigma^2 \Gamma \quad (5.6)$$

where matrix A was defined in (2.4).

Proof. We will use formula (5.1) for the infinitesimal generator. To get the $\{H, C\}$ -term note that it can be received as a derivative of $\psi(t)\psi^T(t)$ by equation (2.4). In other words, if $\psi_0(t)$ is a solution of (2.4) then

$$\frac{d}{dt}\psi_0(t)\psi_0^T(t) = A\psi_0(t)\psi_0^T(t) + \psi_0(t)\psi_0^T(t)A^T = \{H, \psi_0(t)\psi_0^T(t)\}.$$

Now we want to find the term $I(f)$ in (5.1).

Denote by B the $(2N \times 2N)$ -diagonal matrix which has all diagonal elements equal to 1, except $(2N + 1, 2N + 1)$ -element which is equal to α and g is $2N$ -vector with all entries equal to zero except $(2N + 1)$ -element equal to 1. Then $\Gamma = gg^T$ and the transformation $p_1 \rightarrow \alpha p_1 + (1 - \alpha)Mv$ can be written as

$$\psi \rightarrow B\psi + (1 - \alpha)Mvg.$$

So we get

$$\begin{aligned} I(\psi\psi^T) &= E_v(B\psi + (1 - \alpha)Mvg)(B\psi + (1 - \alpha)Mvg)^T - \psi\psi^T = \\ &= B\psi\psi^T B + (1 - \alpha)^2 M^2 \sigma^2 gg^T - \psi\psi^T. \end{aligned}$$

Consequently, we have the equation:

$$\dot{C} = AC + CA^T + \lambda(BCB - C) + \lambda(1 - \alpha)^2 M^2 \sigma^2 \Gamma.$$

The matrix B can be written as $B = E - (1 - \alpha)gg^T$. So

$$BCB - C = -(1 - \alpha)(\Gamma C + C\Gamma) + (1 - \alpha)^2 \Gamma C\Gamma.$$

The lemma is proved. □

Let us introduce notation for the linear part of the right hand side of equation (5.6):

$$L(C) = AC + CA^T - \lambda(1 - \alpha)(\Gamma C + C\Gamma - (1 - \alpha)\Gamma C\Gamma).$$

Then the right hand side of the equation (5.6) can be rewritten as

$$\dot{C} = L(C) + \lambda(1 - \alpha)^2 M^2 \sigma^2 \Gamma.$$

Denote

$$C_G = \begin{pmatrix} V^{-1} & 0 \\ 0 & ME \end{pmatrix}$$

the Gibbs matrix. Then

$$A = \begin{pmatrix} 0 & \frac{1}{M}E \\ -V & 0 \end{pmatrix}$$

and we have

$$\begin{aligned} AC_G + C_G A^T &= AC_G + (AC_G)^T = 0, \\ \Gamma C_G + C_G \Gamma &= 2M\Gamma, \quad \Gamma C_G \Gamma = M\Gamma. \end{aligned}$$

Thus, for $\beta > 0$

$$L(\beta^{-1}C_G) + \lambda(1-\alpha)^2 M^2 \sigma^2 \Gamma = -\lambda(1-\alpha)M(2\beta^{-1} - (1-\alpha)\beta^{-1} - (1-\alpha)M\sigma^2)\Gamma.$$

If we choose $\beta > 0$ so that

$$2\beta^{-1} - (1-\alpha)\beta^{-1} - (1-\alpha)M\sigma^2 = 0,$$

i.e.

$$\beta^{-1} = \frac{1-\alpha}{1+\alpha} M\sigma^2,$$

then

$$L(\beta^{-1}C_G) + \lambda(1-\alpha)^2 M^2 \sigma^2 \Gamma = 0$$

and we conclude that

$$C_{G,\beta} = \beta^{-1}C_G$$

is a fixed point of the equation (5.6).

Lemma 5.3. *For any initial non-negative definite $(2N \times 2N)$ -matrix $C(0)$ the solution of the homogeneous equation*

$$\dot{C}(t) = L(C(t)) \tag{5.7}$$

converges to zero as $t \rightarrow \infty$.

Proof. For the proof we want to find a corresponding Lyapunov function. Consider the function

$$F(C) = \text{Tr}(C_G^{-1}C),$$

where Tr denotes the trace. Note that if C is non-negative definite then $F(C) \geq 0$ and $F(C) = 0$ iff $C = 0$. We will prove that $F(C)$ is non-increasing along the trajectories of the equation (5.7). If $C(t)$ a solution of the equation (5.7) we have

$$\begin{aligned} \frac{d}{dt}F(C(t)) &= \text{Tr}(C_G^{-1}L(C)) = \\ &= \text{Tr}(C_G^{-1}AC + C_G^{-1}CA^T) - \\ &\quad -\lambda(1-\alpha)\text{Tr}(C_G^{-1}\Gamma C + C_G^{-1}C\Gamma - (1-\alpha)C_G^{-1}\Gamma C\Gamma)\text{Tr}(C_G^{-1}AC + C_G^{-1}CA^T) = \end{aligned}$$

$$\begin{aligned}
&= \text{Tr}((C_G^{-1}A + A^T C_G^{-1})C) = 0. \\
&\text{Tr}(C_G^{-1}\Gamma C + C_G^{-1}C\Gamma - (1-\alpha)C_G^{-1}\Gamma C\Gamma) = \\
&= \text{Tr}((C_G^{-1}\Gamma + \Gamma C_G^{-1} - (1-\alpha)\Gamma C_G^{-1}\Gamma)C) = \\
&= \left(\frac{2}{M} - \frac{(1-\alpha)}{M}\right)\text{Tr}(\Gamma C) = \frac{(1+\alpha)}{M}C_{2N+1,2N+1}.
\end{aligned}$$

Thus

$$\frac{d}{dt}F(C(t)) = -\lambda \frac{(1-\alpha^2)}{M} C_{2N+1,2N+1}(t) \leq 0. \quad (5.8)$$

Further, we want to use Barbashin–Krasovskij’s theorem ([12], p. 19, Th. 3.2). For this we need to check that the set of the non-negative definite matrices C , whose $(2N+1, 2N+1)$ -element is zero, does not contain the solution of (5.7) except for the zero solution. Assume the contrary, i.e. that there exists solution $C(t)$ of (5.7) such that $C_{2N+1,2N+1}(t) = 0$ for all $t \geq 0$. For such solution we have $\Gamma C(t)\Gamma = 0$. Thus

$$\dot{C} = AC + CA^T - \lambda(1-\alpha)(\Gamma C + C\Gamma) = A_D C + CA_D^T,$$

where $A_D = A - \lambda(1-\alpha)\Gamma$. It is easy to see that the solution of the last equation is given by the formula:

$$C(t) = \int_0^t e^{sA_D} C(0) e^{sA_D^T} ds + C(0).$$

Denote by (\cdot, \cdot) the standard Euclidean inner product on L . Then we have for all $t \geq 0$

$$\begin{aligned}
0 = C_{2N+1,2N+1}(t) &= (g, C(t)g) = \int_0^t (g, e^{sA_D} C(0) e^{sA_D^T} g) ds = \\
&= \int_0^t (e^{sA_D^T} g, C(0) e^{sA_D^T} g) ds = 0.
\end{aligned}$$

Since $C(0)$ is non-negative definite, the last equality takes place iff $\exp\{tA_D^T\}g \in \text{Ker}(C(0))$ for all $t \geq 0$, where $\text{Ker}C(0)$ is the kernel of $C(0)$. Consequently, there exists non-zero vector $u \in L$ such that $(e^{tA_D^T} g, u) = 0$ for all $t \geq 0$. But

$$(e^{tA_D^T} g, u) = \sum_{k=0}^{\infty} \frac{t^k}{k!} ((A_D^T)^k g, u).$$

Then $((A_D^T)^k g, u) = 0$ for all $k = 0, 1, \dots$. Due to the completeness property of V and lemma 3.1 of [4] (see the proof of the latter lemma) we conclude that $u = 0$. This contradicts our assumptions. So the set of non-negative definite matrices C , whose $(2N+1, 2N+1)$ -element is zero, does not contain the whole trajectories of the solution of (5.7). Then using (5.8) and Barbashin – Krasovskij’s theorem we get the proof of the lemma. \square

Let us come back to the proof of the second part of theorem 2.3. Remind what we have already proved. If $\psi(t)$ is our process, then we have proved that

$$\lim_{t \rightarrow \infty} E_\psi \psi(t) = 0.$$

Then the matrix $C(t) = E_\psi \psi(t) \psi(t)^T$ satisfies the equation (5.6) which has fixed point $C_{G,\beta}$. Moreover, the solution of the homogeneous part of (5.6) converges to zero as $t \rightarrow \infty$ due to the last lemma. All these facts give us the second proposition of Theorem 2.3. So the Theorem is finally proved.

6. APPENDIX

6.1. Collisions

Here we prove that two simplest collision cases satisfy our general Condition 2.

Moreover, assume that their interaction radius is small, and collision time is short. That is during this time the potential between our particle of mass M and other part of the system does not change.

One-dimensional collisions Let $d = 1$. Assume that a particle of mass M of the N -particle system collides with external particles of mass m and velocity v . Then conservation of energy and momentum give the formula (see for example in this issue [16]):

$$p' = J(v; p) = \alpha p + M(1 - \alpha)v, \quad \alpha = \frac{M - m}{M + m},$$

where p, p' are the momenta of our particle of mass M before and after collision correspondingly.

Lemma 6.1. *Assume that the distribution of the velocity v has density on R and its support coincides with \mathbf{R} . Moreover, assume that $a_2 = Ev^2 < \infty$. Then Condition 2 holds.*

Proof. Points 1 and 2 evidently hold. Let us prove point 3:

$$E_v \{ J^2(v; p) \} = \alpha^2 p^2 + 2M\alpha(1 - \alpha)a_1 p + M^2(1 - \alpha)^2 a_2, \quad a_1 = Ev.$$

But it is clear that for any $\varepsilon > 0$ there exists P such that for any p such that $|p| > P$

$$E_v\{J^2(v; p)\} < (\alpha^2 + \varepsilon)p^2.$$

This gives the proof. \square

Two-dimensional non central collision Let $d = 2$. Consider elastic collision of two ideal two-dimensional balls with masses m_1 and m_2 on \mathbf{R}^2 . Assume that at the moment of collision the centers of the balls have coordinates

$$O_1 = (x_1, y_1), \quad O_2 = (x_2, y_2),$$

and velocities

$$\vec{v}_1 = (v_1^x, v_1^y)^T, \quad \vec{v}_2 = (v_2^x, v_2^y)^T,$$

(vector here are column vectors). Assume obviously that $O_1 \neq O_2$. Denote by

$$(\vec{v}_1)' = ((v_1^x)', (v_1^y)')^T, \quad (\vec{v}_2)' = ((v_2^x)', (v_2^y)')^T$$

the velocities of the balls after the collisions.

Define the normalized vector connecting the centers of the balls:

$$\vec{R} = \frac{1}{|O_1 O_2|} \overrightarrow{O_1 O_2},$$

where $|O_1 O_2|$ is the distance between points O_1 and O_2 .

Lemma 6.2. *There exists $\varphi \in [0, 2\pi)$ such that*

$$\vec{R} = R(\varphi) = (\cos \varphi, \sin \varphi)^T.$$

Then also

$$(\vec{v}_1)' = G_\alpha(\varphi)\vec{v}_1 + c_\alpha(\varphi, \vec{v}_2)R(\varphi),$$

where

$$G_\alpha(\varphi) = \begin{pmatrix} \alpha \cos^2 \varphi + \sin^2 \varphi & -\frac{1-\alpha}{2} \sin 2\varphi \\ -\frac{1-\alpha}{2} \sin 2\varphi & \alpha \sin^2 \varphi + \cos^2 \varphi \end{pmatrix}, \quad \alpha = \frac{m_1 - m_2}{m_1 + m_2},$$

$$c_\alpha(\varphi, \vec{v}_2) = (1 - \alpha)(v_2^x \cos \varphi + v_2^y \sin \varphi)$$

Proof. Denote by \vec{R}_\perp a unit vector (one of the two) orthogonal to \vec{R} . Then we can expand the velocity in terms of tangential and normal components with respect to the vector \vec{R} :

$$\vec{v}_1 = v_{1,n}\vec{R} + v_{1,t}\vec{R}_\perp, \quad \vec{v}_2 = v_{2,n}\vec{R} + v_{2,t}\vec{R}_\perp, \quad (6.1)$$

$$(\vec{v}_1)' = v'_{1,n}\vec{R} + v'_{1,t}\vec{R}_\perp, \quad (\vec{v}_2)' = v'_{2,n}\vec{R} + v'_{2,t}\vec{R}_\perp. \quad (6.2)$$

Then the conservation laws of energy and momentum are as follows:

$$\begin{aligned} m_1 v_{1,n} + m_2 v_{2,n} &= m_1 v'_{1,n} + m_2 v'_{2,n} \\ m_1 v_{1,t} + m_2 v_{2,t} &= m_1 v'_{1,t} + m_2 v'_{2,t} \\ m_1 (v_{1,n}^2 + v_{1,t}^2) + m_2 (v_{2,n}^2 + v_{2,t}^2) &= m_1 ((v'_{1,n})^2 + (v'_{1,t})^2) + m_2 ((v'_{2,n})^2 + (v'_{2,t})^2). \end{aligned}$$

Now we have 3 equations and four unknowns. But assumptions about elasticity condition and smoothness of the ball's boundary implies that the tangential components of the velocities rest unchanged

$$v_{1,t} = v'_{1,t}, \quad v_{2,t} = v'_{2,t}.$$

Then, for normal components we get the same equations as in the case of central elastic collision. Namely

$$v'_{1,n} = \alpha v_{1,n} + (1 - \alpha) v_{2,n}, \quad \alpha = \frac{m_1 - m_2}{m_1 + m_2}.$$

Now we want to write down velocity transformation after collision in the initial coordinate system. Note that

$$\vec{R}_\perp = (-\sin \varphi, \cos \varphi)^T.$$

which gives:

$$\begin{aligned} v_{1,n} &= v_1^x \cos \varphi + v_1^y \sin \varphi, & v_{1,t} &= -v_1^x \sin \varphi + v_1^y \cos \varphi, \\ v_{2,n} &= v_2^x \cos \varphi + v_2^y \sin \varphi, & v_{2,t} &= -v_2^x \sin \varphi + v_2^y \cos \varphi. \end{aligned}$$

and

$$\begin{aligned} (v_1^x)' &= v'_{1,n} \cos \varphi - v_{1,t} \sin \varphi = (\alpha v_{1,n} + (1 - \alpha) v_{2,n}) \cos \varphi - v_{1,t} \sin \varphi = \\ &= v_1^x (\alpha \cos^2 \varphi + \sin^2 \varphi) - v_1^y \frac{1 - \alpha}{2} \sin 2\varphi + (1 - \alpha) v_{2,n} \cos \varphi, \\ (v_1^y)' &= v'_{1,n} \sin \varphi + v_{1,t} \cos \varphi = (\alpha v_{1,n} + (1 - \alpha) v_{2,n}) \sin \varphi + v_{1,t} \cos \varphi = \\ &= -v_1^x \frac{1 - \alpha}{2} \sin 2\varphi + v_1^y (\alpha \sin^2 \varphi + \cos^2 \varphi) + (1 - \alpha) v_{2,n} \sin \varphi. \end{aligned}$$

□

For the next statement we assume that the interaction with the external media is defined by the elastic collision with external particle of mass m and random velocity $\vec{v} \in \mathbf{R}^2$. Introduce the transformation J as follows:

$$J(\xi; p) = G_\alpha(\varphi)p + M c_\alpha(\varphi, \vec{v}) \vec{R}(\varphi), \quad \xi = (\varphi, \vec{v}), \quad p \in \mathbf{R}^2$$

where $\alpha = (M - m)/(M + m)$ and the matrix G_α , vector $R(\varphi)$ and constant $c_\alpha(\varphi, \vec{v})$ are as in Lemma 6.2.

Lemma 6.3. *Assume that the support of the distribution density of velocity \vec{v} coincides with \mathbf{R}^2 , that the second moment $a_2 = E|\vec{v}|^2$ is finite and the density support of the angle φ coincides with $[0, 2\pi)$. Then Condition 2 holds for transformation J .*

Proof. The first item evidently holds. For the others we shall use notation and assertions from the proof of Lemma 6.2. Consider two arbitrary vectors $\vec{v}_1, \vec{v}^* \in \mathbf{R}^2$. To prove point 2 it is sufficient to show that there exists $\varphi \in [0, 2\pi)$ and $\vec{v}_2 \in \mathbf{R}^2$, such that

$$v_t^* = v_{1,t}, \quad (6.3)$$

$$v_n^* = \alpha v_{1,n} + (1 - \alpha)v_{2,n}, \quad (6.4)$$

where we used the expansion of our vectors in normal and tangential components of $R(\varphi)$:

$$\vec{v}^* = v_n^* \vec{R}(\varphi) + v_t^* \vec{R}_\perp(\varphi).$$

Similarly for the vectors \vec{v}_1, \vec{v}_2 accordingly to formula (6.1). It is clear that, for any given $\vec{R}(\varphi)$, vector \vec{v}_2 can be chosen so that the equality (6.4) holds. Define vector $\vec{R}(\varphi)$ as follows:

$$\vec{R}(\varphi) = \frac{1}{|\vec{v}_1 - \vec{v}^*|} (\vec{v}_1 - \vec{v}^*).$$

It is clear that

$$(\vec{v}_1 - \vec{v}^*, \vec{R}_\perp(\varphi)) = 0,$$

where $(,)$ is the standard euclidean scalar product on \mathbf{R}^2 . Then

$$v_t^* = v_{1,t}.$$

and we have proved that point 2 holds.

Now we shall prove that point 3 of Condition 2 on J holds. To do this we expand momentum $p = (p_1, p_2) \in \mathbf{R}^2$ and the velocity of external particle \vec{v} in normal and tangential components of the vector $\vec{R}(\varphi)$:

$$p = p_n \vec{R}(\varphi) + p_t \vec{R}_\perp(\varphi), \quad \vec{v} = v_n \vec{R}(\varphi) + v_t \vec{R}_\perp(\varphi).$$

Then as before we get the formula:

$$E_\xi |J(\xi; p)|^2 = p_t^2 + E_\xi (\alpha p_n + M(1 - \alpha)v_n)^2.$$

Simple calculation gives:

$$E_\xi |J(\xi; p)|^2 = |p|^2 - (1 - \alpha^2)E_\xi \{p_n^2\} + 2M\alpha(1 - \alpha)E_\xi \{p_n v_n\} + M^2(1 - \alpha)^2 E_\xi \{v_n^2\}.$$

We want now to obtain lower estimate of $E_\xi\{p_n^2\}$. We have:

$$E_\varphi p_n^2 = E_\varphi(p_1 \cos \varphi + p_2 \sin \varphi)^2 = f(p_1, p_2),$$

that defines the quadratic function f on \mathbf{R}^2 . It is clear that f is a non-negatively defined quadratic form with matrix

$$F = \begin{pmatrix} E\{\cos^2 \varphi\} & E\{\cos \varphi \sin \varphi\} \\ E\{\cos \varphi \sin \varphi\} & E\{\sin^2 \varphi\} \end{pmatrix}.$$

Let us prove that F is non-degenerate. In fact, the determinant of the matrix F is equal to:

$$\begin{aligned} \det(F) &= E\{\cos^2 \varphi\}E\{\sin^2 \varphi\} - (E\{\cos \varphi \sin \varphi\})^2 = \\ &= \frac{1}{4}(1 + E\{\cos 2\varphi\})(1 - E\{\cos 2\varphi\}) - \frac{1}{4}(E \sin 2\varphi)^2 = \\ &= \frac{1}{4}(1 - ((E \cos 2\varphi)^2 + (E \sin 2\varphi)^2)). \end{aligned}$$

The distribution φ was assumed to have density, that is why,

$$(E \cos 2\varphi)^2 + (E \sin 2\varphi)^2 < E(\cos^2 2\varphi + \sin^2 2\varphi) = 1.$$

Then

$$\det(F) \neq 0$$

and there exists a number $\lambda > 0$ such that for all $p = (p_1, p_2) \in \mathbb{R}^2$ the inequality

$$E_\xi\{p_n^2\} = E_\varphi(p_1 \cos \varphi + p_2 \sin \varphi)^2 = f(p_1, p_2) \geq \lambda|p|$$

holds. It follows that we have the following inequality for transformation J :

$$E_\xi|J(\xi; p)|^2 \leq \beta|p|^2 + 2M\alpha(1 - \alpha)E_\xi\{p_n v_n\} + M^2(1 - \alpha)^2 E_\xi\{v_n^2\},$$

where $\beta = 1 - \lambda(1 - \alpha^2) < 1$. It remains to note that $E_\xi\{p_n v_n\}$ is linear function of p , that proves point 3. \square

6.2. Proof of Theorem 2.4.

Note first that the third proposition follows from the first and second assertions. Note also that it is the law of large numbers for Markov chains (see [15], p. 140 and [13] p. 209).

We will use the definitions and the propositions from the book [2]. Strong irreducibility property and weak Feller property imply that the chain ψ_k is μ -irreducible, aperiodic (due to theorem 5.4.4, p. 113) and satisfies the T-property

(thanks to theorem 6.0.1, p. 124). Thus from the theorem 6.0.1 (p. 124) it follows that any compact subset of X is a ‘petite’ set.

Another short proof of the fact that any compact subset is a ‘small’ set (property ‘small’ is stronger than ‘petite’) in our case one can find in [9]. Lyapunov (drift) condition and theorem 9.1.8 (p. 206) imply Harris recurrence property of ψ_k . For the final proof it remains to use theorem 13.0.1 (p. 313) from [2]. In fact, we have proved Theorem 2.2.

Note that irreducibility, aperiodicity and T-property for the embedded chain (2.9) one could easily deduce from propositions for the Nonlinear State Space model (NSS) (see [2], p. 146) and our proofs of strong irreducibility and weak Feller property.

6.3. Dissipative subspace

Here we will prove Proposition 2.1 together with the following Proposition containing more detailed information concerning the dissipating space L_- even in more general setting. Let us remind the definition of L_- . For simplicity we write N instead of dN . Then fix any subset $\Lambda' \in \{1, \dots, N\}$ and define $L_- = L_-(\Lambda')$ as the subspace

$$L_-(\Lambda') = \{(q, p) : q, p \in l_V\}$$

where

$$l_V = l_V(\Lambda') = \langle \{V^k e_n : n \in \Lambda', k = 0, 1, 2, \dots\} \rangle,$$

compare this definition with that in [7].

$L_0 = L_0(\Lambda')$ is defined as the orthogonal complement $L_0 = L_-^\perp$ (with respect to scalar product $(\cdot, \cdot)_2$) to L_- .

Denote \mathbf{H}_{ind} the subset of \mathbf{H} with the spectrum of V such that $\omega_1, \dots, \omega_N$ are rationally independent.

Proposition 6.1.

1. The set \mathbf{H}^+ is open, everywhere dense in \mathbf{H} and $\lambda(\mathbf{H} \setminus \mathbf{H}^+) = 0$.
2. The complement of $\mathbf{H}^{++} = \mathbf{H}^+ \cap \mathbf{H}_{ind}$ to \mathbf{H} has Lebesgue measure zero.
3. For any $n = 1, 2, \dots, N$ denote $g_n = (0, e_n)^T$, where $\{e_n\}$ is the standard basis in \mathbf{R}^N . Then L_- is invariant w.r.t. A . Moreover, L_- can be presented as follows:

$$L_- = \langle \{A^k g_n : k = 0, 1, \dots; n \in \Lambda'\} \rangle, \quad (6.5)$$

$$L_- = \langle \{(A^*)^k g_n : k = 0, 1, \dots; n \in \Lambda'\} \rangle \quad (6.6)$$

where A^* is the adjoint operator to A in the scalar product $(\cdot, \cdot)_2$, and angle brackets $\langle \cdot \rangle$ is the linear span of the corresponding set of vectors.

4. Let $\psi(t) = (q(t), p(t))^T$, $q(t) = (q_1(t), \dots, q_N(t))^T$, $p(t) = (p_1(t), \dots, p_N(t))^T$ denote the solution of the equation (2.4) with initial condition $\psi(0) = \psi$. Then $\psi \in L_0$ if and only if for any $n \in \Lambda'$ and all $t \geq 0$ the equality $p_n(t) = 0$ holds. Moreover, the subspace L_0 is invariant with respect to A .

More physically: the initial conditions from L_0 are the conditions where momentum $p_k(t) = 0$ for all $n \in \Lambda'$ and $t \geq 0$.

Proof of 1 and 2 The fact that \mathbf{H}^+ is open is evident. It is sufficient to prove that $\lambda(\mathbf{H} \setminus \mathbf{H}^+) = 0$ and $\lambda(\mathbf{H} \setminus \mathbf{H}_{ind}) = 0$. The first equality $\lambda(\mathbf{H} \setminus \mathbf{H}^+) = 0$ follows from the fact that $\mathbf{H} \setminus \mathbf{H}^+$ is an algebraic manifold ([7], lemma 2.3). Let us prove the second equality. Without loss of generality we can assume that $d = 1$. For the array of numbers $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$ we define the subset $D_\alpha \subset \mathbf{H}$ of the diagonal matrices with positive diagonal elements x_1, \dots, x_N such that

$$\sum_{k=1}^N \alpha_k \sqrt{x_k} = 0.$$

Consider the subset $\mathbf{H}_\alpha \subset \mathbf{H}$ of matrices with eigenvalues μ_1, \dots, μ_N such that

$$\sum_{k=1}^N \alpha_k \sqrt{\mu_k} = 0.$$

It is easy to see that

$$\mathbf{H}_\alpha = \{C^{-1}DC : D \in D_\alpha, C \in O(N)\},$$

where $O(N)$ is a set of the orthogonal matrices. In other words, \mathbf{H}_α is an orbit of the D_α under the action of $O(N)$ by the conjugation. So the dimension of \mathbf{H}_α equals $N - 1 + N(N - 1)/2 < N + N(N - 1)/2 = N(N + 1)/2 = \dim(\mathbf{H})$ and hence $\lambda(\mathbf{H}_\alpha) = 0$. We have

$$\mathbf{H}_{ind} = \cup_{\alpha \in \mathbb{Q}^N} \mathbf{H}_\alpha.$$

And consequently $\lambda(\mathbf{H} \setminus \mathbf{H}_{ind}) = 0$. Then Proposition 2.1 is also proved.

Proof of 3. Invariance follows from (6.5). Since

$$A^2 = - \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix},$$

we get for any $k = 0, 1, \dots$ and any $n = 1, \dots, N$

$$A^{2k} g_n = (-1)^k \begin{pmatrix} 0 \\ V^k e_n \end{pmatrix} \quad A^{2k+1} g_n = (-1)^k \begin{pmatrix} V^k e_n \\ 0 \end{pmatrix}.$$

The (6.5) follows from this. Moreover we have:

$$A^* = \begin{pmatrix} 0 & -V \\ E & 0 \end{pmatrix}, \quad (A^*)^2 = - \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix} = -A^2.$$

Hence for any $k = 0, 1, \dots$ and any $n = 1, \dots, N$ we have:

$$(A^*)^{2k} g_n = A^{2k} g_n \quad (A^*)^{2k+1} g_n = (-1)^k \begin{pmatrix} V^{k+1} e_n \\ 0 \end{pmatrix}.$$

Thus we proved that

$$\{(A^*)^k g_n : k = 0, 1, \dots; n \in \Lambda'\} = \text{Bigl}\left\{ \begin{pmatrix} q \\ p \end{pmatrix} : q \in V(l_V) p \in l_V \right\}.$$

Then by Hamilton–Cayley theorem for any $n = 1, \dots, N$ we have

$$e_n \in \langle V e_n, \dots, V^N e_n \rangle.$$

It follows that $V(l_V) = l_V$, that proves (6.6).

Proof of 4. For any $\psi \in L$ we have

$$e^{tA} \psi = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \psi.$$

If $n \in \Lambda'$ then for the corresponding momenta:

$$p_n(t) = (e^{tA} \psi, g_n)_2 = \sum_{k=0}^{\infty} \frac{t^k}{k!} (A^k \psi, g_n)_2 = \sum_{k=0}^{\infty} \frac{t^k}{k!} (\psi, (A^*)^k g_n)_2. \quad (6.7)$$

If $\psi \in L_0$, then due to Lemma 3.3 all coefficients in the latter expansion equal zero, and then $p_n(t) = 0$ for any $t \geq 0$ and any $n \in \Lambda'$. Vice-versa, assume that $p_n(t) = 0$ for all $t \geq 0$ and all $n \in \Lambda'$. From expansion (6.7) it follows that $(\psi, (A^*)^k g_n)_2 = 0$ for all $k = 0, 1, \dots$ and all $n \in \Lambda'$. Using Lemma 3.3 we get that $\psi \in L_0$. Thus, we have proven the first assertion of the Lemma 5.1. Invariance of L_0 follows because L_- is invariant with respect to A^* (this is direct consequence of the formula (6.6)).

Now we want to formulate some results concerning the dimension of the dissipative subspace in terms of the spectrum of V .

The dimension of the dissipative subspace Denote $l = \mathbf{R}^N$. Let λ be an eigenvalue of V . Then the subspace

$$l(\lambda) = \{u \in l : Vu = \lambda u\}$$

we call the corresponding eigensubspace. Let $\sigma(V)$ be the spectrum of V . As V is symmetric then $l(\lambda) \perp l(\lambda')$ for $\lambda \neq \lambda'$. The space l can be presented as the direct sum of such eigensubspaces:

$$l = \bigoplus_{\lambda \in \sigma(V)} l(\lambda). \quad (6.8)$$

Consider the subspace

$$l_{\Lambda'} = \langle \{e_i\}_{i \in \Lambda'} \rangle \subset l_V.$$

For $\lambda \in \sigma(V)$ denote $l_{\Lambda'}(\lambda)$ the orthogonal projection of $l_{\Lambda'}$ onto $l(\lambda)$.

Theorem 6.1. *The following expansion holds:*

$$l_V = \bigoplus_{\lambda \in \sigma(V)} l_{\Lambda'}(\lambda).$$

Proof. As l_V is invariant w.r.t. to V , there exists the presentation of l_V as the direct sum of eigensubspaces for the restriction of V onto l_V :

$$l_V = \bigoplus_{\lambda \in \sigma(V)} l_V(\lambda),$$

where $l_V(\lambda) = \{u \in l_V : Vu = \lambda u\}$. Let us prove that

$$l_V(\lambda) = l_{\Lambda'}(\lambda)$$

for all $\lambda \in \sigma(V)$. From the orthogonality of eigensubspaces it follows that $l_V(\lambda)$ is the orthogonal projection of l_V on $l(\lambda)$. Thus, $l_{\Lambda'}(\lambda) \subset l_V(\lambda)$. Let us show that the reverse assertion also holds. Following (6.8) we can represent the vector e_i , $i \in \Lambda'$ as:

$$e_i = \sum_{\lambda \in \sigma(V)} u_\lambda,$$

for some $u_\lambda \in l(\lambda)$, $\lambda \in \sigma(V)$. From the definition of $l_{\Lambda'}(\lambda)$ it follows that $u_\lambda \in l_{\Lambda'}(\lambda)$ for all $\lambda \in \sigma(V)$. Then for any $j = 0, 1, \dots$ we get:

$$V^j e_i = \sum_{\lambda \in \sigma(V)} \lambda^j u_\lambda. \quad (6.9)$$

From (6.9) it follows that the projection of $V^j e_i$ on $l(\lambda)$ equals $\lambda^j u_\lambda$ and thus it belongs to $l_{\Lambda'}(\lambda)$. Then the projection of l_V on $l(\lambda)$, equals $l_V(\lambda)$ and thus belongs to $l_{\Lambda'}(\lambda)$. The theorem is thus proved. \square

From this theorem we have the following corollary.

Lemma 6.4. *If $L_- = L$, then the multiplicity of any eigenvalue of V cannot exceed $|\Lambda'|$, where $|\Lambda'|$ is the number of boundary indices Λ' .*

Proof. From the definition of $l_{\Lambda'}(\lambda)$ it follows that for all $\lambda \in \sigma(V)$ the following inequality holds

$$\dim(l_{\Lambda'}(\lambda)) \leq \dim(l_{\Lambda'}) \leq |\Lambda'|.$$

On the other hand, the condition $L_- = L$ is equivalent to that $l_V = l$. This gives the equality $l_{\Lambda'}(\lambda) = l(\lambda)$ for all $\lambda \in \sigma(V)$. \square

In particular for $|\Lambda'| = 1$ the necessary (but, as we will see below, not sufficient) condition of completeness of the dissipative subspace is that the spectrum of V is simple.

Lemma 6.5. *Let the spectrum of V be simple and let $\{v_1, \dots, v_N\}$ be the basis of the space l , where v_k are the eigenvectors of V . Then*

$$\dim(L_0) = 2\#\{k \in \{1, \dots, N\} : v_k \in l_{\Lambda'}^\perp\}.$$

Otherwise speaking Lemma 6.5 says that in case of simple spectrum of V the dimension of L_0 is equal to the double number of the eigenvectors of V , having all “boundary” coordinates $i \in \Lambda'$ zero.

Proof. By Lemma 6.1 l_V is spanned by those vectors v_k , for which $(v_k, e_i)_1 \neq 0$ at least for one index $i \in \Lambda'$. It follows that the complement to l_V coincides with the linear span of the vectors v_k for which $(v_k, e_i)_1 = 0$ for all $i \in \Lambda'$. The lemma is proven. \square

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