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Moment-closed Processes with Local Interaction*

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I. General definitions and the simplest model

§1. *Introduction*

We consider Markov processes $\vec{\xi}_t = (\vec{\xi}_t(x), x \in \mathbf{Z}^\nu)$ where the time t is discrete, $t \in \mathbf{Z}_+$, or continuous, $t \in \mathbf{R}_+$. The process $\vec{\xi}_t$ assumes values in $S^{\mathbf{Z}^\nu}$, where S is the set of values of each random variable $\vec{\xi}_t(x)$. We take a new look at the theory of such processes, selecting and studying a class of processes for which, roughly speaking, the compound moments of order n at a given moment of time are expressed in terms of the compound moments of orders $m \leq n$ at the previous moment. Such processes are quite diverse and they have been studied in particular cases from different points of view by various authors. Among them are the voter model, the exclusion process and other concrete models **[6]**. However, a general point of view on such "explicitly-solvable" processes makes sense and is worked out here, apparently for the first time.

Let us enumerate the basic phenomena which we observe in this paper:

1. The class of processes that we introduce is in some sense "completely integrable": the first and second moments are always explicitly analyzed. We introduce different versions of moment closure and obtain a more intrinsic description of such processes.

2. Marginally-closed processes are well analyzed in a number of cases with the help of the elementary technique of cluster expansions or with the help of Holley–Stroock duality. However, such a situation is uncommon. We propose a new simple method, giving, apparently, the capability of analyzing an arbitrary

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marginally-closed process. For this we use the direct process rather than the dual.

3. Despite what was said in Section 2, the main ideology is based on the concept of the cluster operator in the sense of [7]: the equation for the n-particle function is an n-particle cluster operator.

4. The fundamental problem is to describe all the types of phase transitions which are possible for our class of processes. Another basic problem is to find moment-closed systems that are natural from a physical point of view.

We have attempted to make this article elementary and to formulate additional new problems.

The structure of the paper is as follows. We introduce (§2.1) two classes of processes with local interaction: the moment-closed and marginally-closed processes $\vec{\xi}_t$ for discrete time. We obtain their intrinsic characterization in §2.1, §1.11 and §5.11. The case of continuous time is considered in Sections 1, 2 and 3 of Part III. The case $|S| = 2$, in general, is well-known (the voter model). We investigate it from a somewhat different point of view in §3.1. Here, the elementary technique of cluster expansions gives a complete answer in some cases.

The latter case is considered in Part II. Central here is §3, where an explicit representation is given of the two-particle correlation functions in terms of a direct process. In §2 the possibility of analyzing the dual operator is studied.

The Gaussian case considered in §4 is characterized by its simplicity. In Part III §4 the diffusion processes are nonergodic and moment closure is used to solve quite another type of problem.

§2. *Marginal and moment closure*

The general definition of a process with local interaction is usually given in Gibbsian language [2]. We present a simpler definition, embracing numerous applications.

Let us introduce the probability spaces: $(\Omega_0, \Sigma_0, \mu_0)$ on which the random variables $\xi_0(x)$ are given at the moment $t = 0$, "the initial data"; $(\Omega_{t,x}, \Sigma_{tx}, \mu_{tx})$, $t \geq 1$, $x \in \mathbf{Z}^\nu$ describe the "stochastic mechanism" at the point (t, x).

All the subsequent events will turn around the product of these spaces

$$(\Omega, \Sigma, \mu) = (\Omega_0, \Sigma_0, \mu_0) \times \left[\prod_{(t,x)} (\Omega_{tx}, \Sigma_{tx}, \mu_{tx})\right].$$

In other words, we assume that the initial data and the "coin-tossing" at the points (t, x) are mutually independent.

Definition 1. The process $\xi_t(x)$ is called a (Markovian) process with local interaction if for all t, x there exist functions F_{tx} such that

$$\xi_{t+1}(x) = F_{tx}(\xi_t(x+y), \omega_{t,x+y}, y \in Q), \tag{1}$$

where Q is some fixed finite subset of $\mathbf{Z}^\nu$, generating all $\mathbf{Z}^\nu$. Let $d = \operatorname{diam} Q$. This means that for fixed values of $\vec{\xi}_t$, $\xi_{t+1}(x)$ is a d-dependent sequence of random variables. If the dependence in (1) is only on $\omega_{t,x}$, then the process is called conditionally-independent.

From now on, complete translational (relative to the shifts on $\mathbf{Z}^\nu$) invariance is assumed. Since in (1), F and Q are now identical for all t and x, it is enough to assume the translational invariance of the field $\xi_0(x)$.

Remark 1. Usually in processes with local interaction, there are also included cases of infinite Q, but with weak dependence on distant $\xi_t(x+y)$, $\omega_{t,x+y}$. It is more natural to call this a process with quasi-local interaction.

Now let S be at most countable. Let us consider the finite-dimensional distribution

$$P(S_X; t) = P(\xi_t(x) = s_x, x \in X), \qquad |X| < \infty, \qquad S_X = (s_x, x \in X).$$

Using (1), we obtain

$$\begin{aligned} P(\xi_{t+1}(x) = s_x, x \in X \mid \vec{\xi}_t) &= \int \prod_{z \in X+Q} d\mu_{t,z} \\ &= A(S_X; \xi_t(z), z \in X+Q), \end{aligned} \tag{2}$$

where the integration is conducted over the set ω such that for all $x \in X$

$$F(\xi_t(x+y), \omega_{t,x+y}, y \in Q) = s_x.$$

Therefore,

$$\begin{aligned} P(S_X, t+1) &= \sum_{S_{X+Q}} A(S_X, S_{X+Q}) \times \left\langle \prod_{z \in X+Q} \delta_{s_z}(\xi_t(z)) \right\rangle \\ &= \sum_{S_{X+Q}} A(S_X, S_{X+Q}) P(S_{X+Q}; t). \end{aligned} \tag{3}$$

The chain of equations (3) is similar to the *BBGKY* chain in statistical physics.

Definition 2. The process $\xi_t(x)$ is called marginally closed if, using the consistency conditions for $P(S_Y; t)$, it is possible to reduce (3) to the form

$$P(S_X; t+1) = \sum_Y \sum_{S_Y} \hat{A}_{X,Y}(S_X, S_Y) P(S_Y; t), \tag{4}$$

where the summation is over all $Y \subset X + Q$ such that

$$|Y| \leq |X|. \tag{5}$$

Similar definitions with obvious changes can be presented for arbitrary S.
Now let S be an arbitrary subset of $\mathbf{R}$.

Definition 3. The process $\xi_t(x)$ is called moment-closed if its compound moments

$$m(K_X; t) = \left\langle \prod_{x \in X} \xi_t^{k_x}(x) \right\rangle, \qquad K_X = (k_x, x \in X),$$

and k_x are nonnegative integers, satisfying

$$m(K_X; t+1) = \sum_Y \sum_{K_Y} B_{X,Y}(K_X, K_Y) m(K_Y; t) \tag{6}$$

where the sum is over all $Y \in X + Q$ and all K_Y satisfying

$$|K_Y| = \sum_{y \in Y} K_y \leq |K_X|. \tag{7}$$

For $|S| = 2$, one may suppose that $S = \{0, 1\}$ and the concepts of marginal and moment closure coincide. In this case, it is possible to consider only

$$P(\xi_t(x_1) = 1, \ldots, \xi_t(x_n) = 1) = \langle \xi_t(x_1) \cdots \xi_t(x_n) \rangle$$

for pairwise distinct x_i.

Let us now consider the conditionally independent process $\xi^t(x)$ with $S = \{0, 1\}$. Such a process is called conditionally-linear if

$$P(\xi_{t+1}(x) - 1 \mid \vec{\xi}_t) = \sum_{y \in Q} a_y \xi_t(x+y) + c \tag{8}$$

for some constants a_y and c.

Assertion 1. *The process $\xi_t(x)$ is marginally closed if and only if it is conditionally linear.*

The proof follows from the fact that the most general function has, in the case $S = \{0, 1\}$, the form of a polynomial of degree at most $|Q|$ and degree at most 1 in each variable.

For (1) to be a probability, it is necessary and sufficient that we have in force the following consistency conditions

$$0 \leq \sum_{x \in Q'} a_x + c \leq 1 \qquad \text{for all } Q' \subset Q. \tag{9}$$

It follows, in particular, from these that

$$\sum_{x \in Q} |a_x| \leq 1, \tag{10}$$

always, and we consider two cases, either

(i)
$$\sum |a_x| < 1, \tag{11}$$

or

(ii)
$$\sum a_x = 1, \qquad a_x \geq 0, \qquad c = 0. \tag{12}$$

In the latter case we obtain in this way what is known as the voting process.

Using (8), we obtain

$$\langle \xi_{t+1}(x_1) \cdots \xi_{t+1}(x_n) \rangle = \sum_{k=0}^{n} C^{n-k} \sum_{z_1 \in Q} \cdots \sum_{z_k \in Q} a_{z_1} \cdots a_{z_k} \times \sum_{i \leq i_1 < \cdots < i_k \leq n} \langle \xi_t(x_{i_1} + z_1) \cdots \xi_t(x_{i_k} + z_k) \rangle. \tag{13}$$

§3. *Expansion by paths*

It turns out that the limiting behavior of $\vec{\xi}_t$ as $t \to \infty$ with $S = \{0, 1\}$ can be analyzed by the simplest technique of cluster expansions. This capability is tied to the fact that inequality (10) §2 is always satisfied. The basis of the expansion is the convenient graph interpretation of relation (13) §1.

Let us fix the vertices $(x_1, t+1), \ldots, (x_n, t+1)$ in $\mathbf{Z}^\nu \times \mathbf{Z}_+$. They will designate the left-hand side of (3). To each term of the sum in the right-hand side of (3), that is, to fixed $k, z_1, \ldots, z_k, x_{i_1}, \ldots, x_{i_k}$, we relate the vertices $(z_1 + x_{i_1}, t), \ldots, (z_k + x_{i_k}, t)$ and edges ζ_1 between $(x_{i_1}, t+1), (z_1 + x_{i_1}, t)$, ζ_2 between $(x_{i_2}, t+1), (x_{i_2}, z_2, t)$ etc. To each edge ζ_i we assign the number a_{z_i}, and to the vertex $(x_j, t+1)$, where j is different from all $i_1, \ldots, i_k$, we assign the number c. We call the product of all such c and Q_{z_i}, the contribution of the diagram so obtained (the cluster of edges).

It is possible to iterate formula (13) §1, $t+1$ times, until the moments on the left-hand side of (13) are expressed in terms of the moments of the initial distribution $\vec{\xi}_0$. It is also convenient to put the expression obtained in this case in the form of the sum of contributions of the diagrams (clusters of paths). It is possible to carry out the construction of the cluster of paths by induction on the moments of time t'. Now let there be constructed the vertices $(y_1, t'), \dots, (y_m, t')$ (in the time band $\mathbf{Z}^\nu + \{t'\} \subset \mathbf{Z}^\nu \times \mathbf{Z}_+$). Then, we either declare the vertex (y_i, t'), $i = 1, \dots, m$ to be a terminal one and compare the factor c with it, or take an edge from (y_i, t') to the vertex $(y_i + z, t'-1)$ for some $z \in Q$ and compare the factor a_z with this edge. The process concludes at the moment t' if all the vertices in the time band t' are terminal ones.

In this way, it is possible to write

$$\langle \xi_{t+1}(x_1) \cdots \xi_{t+1}(x_n) \rangle = \sum_G I_G = \sum_G c^{K(G)} \prod_{\zeta \in G} a_{z(\zeta)} \left\langle \sum_{(y,0) \in G} \xi_0(y) \right\rangle, \tag{1}$$

where the sum is over all possible diagrams (clusters of paths) G, the first product runs over all edges ζ of G, and the second over all vertices of G in the zero time band (slice) (if there are any), and $K(G)$ is the number of terminal vertices for G.

We call $t+1-t'$ the length $l(G)$ of G, where t' is the least time band in which there are vertices in G. The diagram G, which has no vertices in the zero band, is assigned to type 1; and the remaining ones, to type 2.

It is possible to write

$$\langle \xi_{t+1}(x_1) \cdots \xi_{t+1}(x_n) \rangle = I_1(t) + I_2(t), \tag{2}$$

where $I_1(t)$ is the sum over all diagrams of type 1 of length not exceeding t, and $I_2(t)$ is over all diagrams of length $t+1$ (the points $x_1, \dots, x_n$ are always fixed). Let us define $I_1(\infty)$ as the sum of the contributions of all diagrams of type 1 of an arbitrary finite length.

Theorem 1. *If $\alpha = \sum |a_y| < 1$, then for any initial distribution $\vec{\xi}_0$, the finite-dimensional distributions or the moments $\langle \xi_t(x_1) \cdots \xi_t(x_n) \rangle$ converge as $t \to \infty$ to the unique distribution of the limiting random field $\vec{\xi}_\infty$. Moreover,*

$$\langle \xi_\infty(x_1) \cdots \xi_\infty(x_n) \rangle = I_1(\infty) \tag{3}$$

and the convergence is exponential, that is,

$$\begin{aligned} |I_1(\infty) - I_1(t)| &\le \left(\frac{2}{1-\alpha}\right)^n \alpha^t, \\ |I_2(t)| &\le \left(\frac{2}{1-\alpha}\right)^n \alpha^t. \end{aligned} \tag{4}$$

The semi-invariants of $\vec{\xi}_\infty$ admit a uniform strong exponential bound in the sense of [**8**], *in particular*

$$\langle \xi_\infty(x), \xi_\infty(y) \rangle \le C\, e^{+\beta|x-y|}, \qquad C, \beta > 0. \tag{5}$$

Proof. Let us observe that the sum of the contributions of all diagrams for $\langle \xi_{t+1}(x_1) \rangle$, that is, with one upper vertex, of type 1 of length l equals

$$ca^l \le c\alpha^l, \qquad \alpha = \sum |a_y|, \qquad a = \sum a_y.$$

In this case each diagram is a single path. Therefore, the sum of all diagrams for $\langle \xi_\infty(x_1) \cdots \xi_\infty(x_n) \rangle$ of type 1 of length l does not exceed

$$\sum_{\varnothing \ne T \subset \{x_1 \cdots x_n\}} \alpha^{|T|l} c^{|T|} \prod_{x \in T} (\alpha^{r(x)} + \alpha^{r(x)+1} + \cdots), \tag{6}$$

where

$$r(x) = \frac{\rho(x, T_x)}{\max\limits_{y,z \in Q} \rho(y, z)},$$

and $\rho(x, y)$ is the distance between x and y.

The bound (6) is obtained as follows: from each point of the chosen nonempty set $T \subset \{x_1, \ldots, x_n\}$ we draw an arbitrary path of length l with a terminal point. Let us enumerate the points $x_{j_1}, \ldots$ from T and, sequentially from each, draw a path (of length less than l) up to the intersection with one of the paths constructed earlier. Expression (6) is the majorant of this procedure. T_x is the union of T and of all the points $\bar{T}$ preceding x.

Relations (3) and (4) clearly follow from (6). The exponential decrease of the correlations of the limiting field is proved by the standard technique of the theory of cluster expansions. Quite another situation occurs in case (ii).

Let $\sum a_x = 1$, $a_x > 0$, $x \in Q$, and let the initial distribution $\vec{\xi}_0$ satisfy the following conditions:

(a) it is translationally invariant, in particular

$$\langle \xi_0(x) \rangle \equiv p,$$

(b) $$\langle \xi_0(x_1), \ldots, \xi_0(x_n) \rangle \to 0, \qquad \text{if } \operatorname{diam}\{x_1, \ldots, x_n\} \to \infty.$$

Let us first observe that then we also have

$$\langle \xi_t(x) \rangle \equiv p.$$

Therefore, the stationary distribution here is not unique. Let us now consider the second moment $\langle \xi_t(x_1)\xi_t(x_2) \rangle$. In this case, the projections on $\mathbf{Z}^\nu$ of the two paths from (x_1, t) and (x_2, t) can be considered as a random walk on $\mathbf{Z}^\nu$ of two independent particles, emanating from x_1 and x_2, respectively. Let $x_1(t)$ and $x_2(t)$ be the random position of these particles at the moment τ (τ is the "reverse" time of the dual process; see §3). The difference $x_1(t) - x_2(t)$ can be considered as a random walk of one particle, exiting from $x_1 - x_2$ with transition probabilities

$$p(z_1, z_2) = \sum_{x, y: x - y = z_2 - z_1} a_x a_y.$$

It is easy to see that this walk is symmetric. Let us now observe that

$$I_1(t) = pP(\hat{\tau} < t), \tag{7}$$

where $\hat{\tau}$ is the stochastic moment at which this walk reaches zero.

In dimensions $\nu = 1, 2$, we have $P(\hat{\tau} < t) \to 1$ as $t \to \infty$, and this means that $I_1(t) \to p$ and $I_2(t) \to 0$. Therefore, $\langle \xi_\infty(x_1)\xi_\infty(x_1) \rangle \equiv p$. This means that the limit field is concentrated on two configurations, "all l" with probability p and "all 0" with probability $1 - p$. There is quite a different situation in dimensions $\nu \geq 3$. Here $p(\hat{\tau} < t)$ approaches some probability $q(x_1 - x_2)$ and this means that

$$I_1(\infty) = pq(x_1 - x_2). \tag{8}$$

Let us denote by $p^{(0)}(x; \tau)$ the probability that $x_1(\tau) - x_2(\tau)$ did not hit 0 for $\tau' < \tau$ if $x_1 - x_2 = x$. It is known that for all $d < \infty$

$$\sum_{x: |x| < d} p^{(0)}(x; \tau) \to 0 \qquad \text{as } \tau \to \infty. \tag{9}$$

Therefore if $\langle \xi_0(x_1), \xi_0(x_2) \rangle \to 0$ as $|x_1 - x_2| \to \infty$ then

$$\langle \xi_0(x_1)\xi_0(x_2) \rangle \to p^2. \tag{10}$$

A comparison of (9) and (10) gives a definitive answer in dimensions $\nu \geq 3$:

$$\langle \xi_\infty(x_1)\xi_\infty(x_2) \rangle = pq(x_1 - x_2) + p^2(1 - q(x_1 - x_2)). \tag{11}$$

Similar reasoning also holds for the higher moments.

Theorem 2. *Let* $\sum a_x = 1$, $a_x > 0$, $x \in Q$ *and let* ξ_0 *satisfy* (1) *and* (2). *Then for* $\nu = 1, 2$, *the limit distribution is concentrated on the two configurations, "all l" with probability p and "all 0" with probability* $1 - p$. *If* $\nu \geq 3$, *then for each p its* *own limit distribution exists, satisfying* (11) *and property* (b) *of the decay of the correlations.*

An obvious corollary of Theorem 2 for $\nu = 1$ is the observable growth of long series of zeros and long series of units, which alternate. A precise statement of this fact yields $\langle \xi_t(x_1)\xi_t(x_2)\rangle \to p$. Griffeath [3] observed this effect by means of computer modeling.

Griffeath's effect is important in many questions about the division of the phases in a multiphase system (crystallization from a multiphase melt, etc.). The transitional situation between Theorems 1 and 2 is of interest. Let $\sum a_x = 1 - \varepsilon$, $\varepsilon \ll 1$, $\nu = 1, 2$. Direct computations show that the ergodic state, considered in Theorem 1, is characterized by a correlational scale $l_\varepsilon \simeq \varepsilon^{-1/2}$. If "spontaneous voting mistakes" are improbable (this is the meaning of the constant $\varepsilon = 1 - \sum a_x$), then long $(O(\varepsilon^{-1/2}))$ series of l's and 0's arise.

§4. *Duality and clustering*

Let $\mathscr{F}$ be the set of all finite subsets of $\mathbf{Z}^\nu$ (including the empty subset). Let us denote by A the linear operator in $l_\infty(\mathscr{F})$ which maps the function $f(Y)$, $Y = \{y_1, \ldots, y_m\}$ into

$$(Af)(X) = \sum A_Y^X f(Y), \qquad f(\varnothing) = 1, \tag{1}$$

where A_Y^X are the coefficients in the moment chain

$$\begin{gathered}\langle \xi_{t+1}(x_1) \cdots \xi_{t+1}(x_n)\rangle = \sum_{m=0}^{\infty} \sum_{\{y_1, \ldots, y_m\}} A_Y^X \langle \xi_t(y_1) \cdots \xi_t(y_m)\rangle \\ X = \{x_1, \ldots, x_m\}, \qquad Y = \{y_1, \ldots, y_m\}.\end{gathered} \tag{2}$$

Let us observe that the dual (adjoint) operator A^* takes the function δ_X, equal to 1 at X and 0 elsewhere, into

$$A^*\delta_X = \sum_Y A_Y^X \delta_Y. \tag{3}$$

We observe that A^* leaves the spaces $l_2(\mathscr{F})$ and $l_1(\mathscr{F})$ invariant.

If $A_Y^X \geq 0$ and $\sum_Y A_Y^X = 1$, then A^* determines a countable Markov chain with discrete time on $\mathscr{F}$ with a transition matrix A_Y^X (from X to Y). This Markov chain is the process dual to $\vec{\xi}_t$ in the sense of Holley and Stroock (see [6]). We used this process in the preceding section. A wide class of linear operators in $l_2(\mathscr{F})$, to which many mathematical physics operators belong, was introduced in

[7]. It is not hard to prove that the operators A and A^* (considered in $l_2(\mathcal{F})$) also belong to this class.

Proposition 1. *The operators A and A^* are multiplicative cluster operators.*

In this way, it is possible to say that the process $\vec{\xi}_t$ is moment-closed if and only if A or A^* is a finite-particle cluster operator.

Let us remark that in contrast to the transfer matrix which acts in L_2, according to the invariant measure (that is, near to the equilibrium state), the operators considered here encompass also the "far from equilibrium" situation, in physics terminology.

II. Processes with discrete time and an arbitrary set of values

§1. *Marginally closed processes*

Let S be a finite or countable set and let us consider the conditionally-independent process $\xi_t(x)$ with values in S, while maintaining all the other assumptions of Part I. In the set of complex functions on S, let us select the linear basis $\{\delta_s\}$, where $\delta_s(s') = 1$ if $s = \bar{s}$, and 0 in the opposite case.

Let us consider the finite-dimensional distributions

$$p_{x_1 \cdots x_n}(s_1, \ldots, s_n; t) = P\{\xi_t(x_1) = s_1, \ldots, \xi_t(x_n) = s_n\}$$
$$= \langle \delta_{s_1}(\xi_t(x)) \cdots \delta_{s_n}(\xi_t(x_n)) \rangle \quad (1)$$

where all x_i are pairwise distinct.

We call the process $\vec{\xi}_t$ conditionally-linear if

$$P(\delta_s(\xi_{t+1}(x)) = 1 \mid \vec{\xi}_t) = \sum_{s' \in S} \sum_{y \in Q} a_y(s, s')\delta_{s'}(\xi_t(x+y)). \quad (2)$$

Let us observe that here in (2) there are no constants on the right-hand side, since it is possible to make the substitution $1 = \sum \delta(\xi_t(x))$.

It is clear that here conditional linearity is equivalent to marginal closure. Actually, the sufficiency of condition (2) is verified by a simple computation, but at the same time condition (2) is already necessary for the one-particle functions.

The correctness conditions of relations (2) have the following form: for any s and any function $s'(y)$, $y \in Q$,

$$\text{(i)} \qquad \sum_{y \in Q} a_y(s, s'(y)) \geq 0,$$
$$\text{(ii)} \qquad \sum_s \sum_y a_y(s, s'(y)) = 1. \quad (3)$$

From here it easily follows that, for all y and s',

$$\sum_{s} a_y(s, s') = q_y; \tag{4}$$

that is, the sum, depends only on y, where $\sum_y q_y = 1$.

Let us further note that the right-hand side in (2) does not change under the replacement of $a_y(s, s')$ by $a_y(s, s') + c_y(s)$, where $\sum_y c_y(s) = 0$ for any $s \in S$. Consequently, by (i) and (ii), $a_y(s, s')$, $y \in Q$, $s' \in S$, can be chosen in such a way that

$$a_y(s, s') \geq 0$$

for all $y \in Q$ and all $s, s' \in S$.

Now we move on to analyzing the properties of marginally-closed processes.

One-particle correlation functions

From (2) we have

$$p_x(s; t+1) = \sum_{s', y} a_y(s, s') p_{x+y}(s'; t). \tag{5}$$

Since the initial conditions are translation-invariant, then $p_x(s, t) = p(s, t)$ does not depend on x and

$$p(s, t+1) = \sum_{s'} b(s, s') p(s'; t), \qquad b(s, s') = \sum_{y} a_y(s, s'). \tag{6}$$

The function $b(s, s')$ is, by (3), a matrix of transition probabilities (from s' to s) of some countable Markov chain with a set S of states, which we denote by $\mathscr{L}_1$. The ergodic property of $\vec{\xi}_t$ depends in an obvious way on the properties of $\mathscr{L}_1$.

Two-particle correlation functions

For $x_1 \neq x_2$ we have

$$\begin{aligned} p_{x_1 x_2}(s_1, s_2; t+1) = & \sum_{s_1', s_2' \in S'} \sum_{y_1 y_2} a_{y_1}(s_1, s_1') a_{y_2}(s_2, s_2') \\ & \times \langle \delta_{s_1'}(\xi_t(x_1 + y_1)) \delta_{s_2'}(\xi_t(x_2 + y_2)) \rangle \\ = & \sum{}' a_{y_1}(s_1, s_1') a_{y_1}(s_2, s_2') p_{y_1 + x_1, y_2 + x_2}(s_1', s_2'; t) \\ & + \sum{}'' a_{y_1}(s_1, s') a_{y_2}(s_2, s') p(s', t) \end{aligned} \tag{7}$$

where $\sum'$ ($\sum''$) are summations over all y_1 and y_2 such that $x_1 + y_1 + x_2 + y_2$ ($x_1 + y_1 = x_2 + y_2$) and over all $s_1', s_2'(s')$.

In distinction from the "direct" process $\mathscr{L}_1$, we consider the dual process $\mathscr{L}_2^*$ (if it exists) with a set of states

$$E \times E = \{(x_1, s_1), (x_2, s_2)\}, \qquad E = \mathbf{Z}^\nu \times S.$$

In view of translational invariance, it is possible to reduce it to a process on

$$\mathbf{Z}^\nu \times S \times S = \{(x = x_1 - x_2, s_1, s_2)\}.$$

The transition probability from (x, s_1, s_2) to (x', s_1', s_2') equals

$$\sum_{y_1 - y_2 = x' - x} a_{y_1}(s_1, s_1')a_{y_2}(s_2, s_2') \tag{8}$$

if $x \neq 0$. But if $x = 0$ and $s_1 \neq s_2$, then the point "perishes"; and if $s_1 = s_2$, then it assumes the value $p(s_1, \infty)$. Finally, (8) is the actual probability quite rarely, but if this is so, then it is possible to calculate the limiting behavior of the two-particle correlation functions.

The following cases are possible. First let $\mathscr{L}_1$ contain one essential class and be nonperiodic.

(A) If $\mathscr{L}_1$ is nonergodic, then $p(x, t) \to 0$ and this means in general that all finite-dimensional probabilities

$$p_{x_1 \cdots x_n}(s_1, \ldots, s_n; t) \to 0.$$

(B) If $\mathscr{L}_1$ is ergodic, then $p(s, t)$ converges to a unique distribution. From considerations of compactness, it is easy to prove that there is at least one invariant (translationally invariant) distribution. The following questions arise.
  (B1) When is this distribution unique?
  (B2) If it is unique, then for which initial distributions is there convergence to it?
  (B3) The decay of the correlations and the dependence on the parameters (phase transitions of the second kind).

(C) If there are several essential classes and $|S| < \infty$, then does nonuniqueness hold?

The voting model is related to this case.

It would be very good if the method of expansions into paths allowed one to conduct an exhaustive analysis here. Apparently, this is not possible. Here we show that the dual process exists very rarely.

Proposition 1. *For the existence of $\mathscr{L}_2^*$ it is necessary and sufficient that there exists a dual process $\mathscr{L}_1^*$, that is, that $b(s, s')$ is a doubly stochastic matrix and that $a_y(s, s') \geq 0$.*

The process $\mathscr{L}_2^*$ is a countable Markov chain. Let it have one essential class and let $\mathscr{L}_2$ be ergodic. Then two cases are possible.

(1) The probability of hitting the set $x = 0$ equals unity. Let us then denote by $q^*(x, s_1, s_2; s)$ the probability that the dual process $\mathscr{L}_2^*$, exiting from (x, s_1, s_2), at the first hitting of $x = 0$ hits at $(0, s, s)$. This case corresponds, roughly speaking, to dimensions $\nu = 1, 2$. However, we do not know whether this is always so in dimensions $D = 1, 2$. The answer apparently depends on the speed of convergence of the chain $\mathscr{L}_1^*$ to the stationary distribution. This is always so if $|S| < \infty$, $\nu = 1, 2$, when the convergence is exponential.

Theorem 1. *In case* (1) *there exists a unique limiting two-particle correlation function*

$$p_{x_1 x_2}(s_1, s_2; \infty) = \sum_s p(s; \infty) q^*(x, s_1, s_2; s). \tag{9}$$

(2) The probability of hitting $x = 0$ is less than 1. Then the limiting conditional probability (under the condition of not hitting at $x = 0$) is such, as $t \to \infty$, that the dual process at $x \neq 0$ approaches zero for all x. This case corresponds to dimensions $\nu \geq 3$. The result is similar to the situation in §3.1 and we will not consider this further.

If the dual process does not exist, in general, another approach is preferable (see the next section).

In the case of continuous S (we restrict ourselves to the case $S = \mathbf{R}$) we will understand by the correlation functions the joint densities

$$p_{x_1 \cdots x_n}(s_1, \ldots, s_n; t) = \langle \delta(s_1 - \xi_t(x_1)) \cdots \delta(s_n - \xi_t(x_n)) \rangle;$$

and in the definition of conditional linearity

$$M(\delta(s - \xi_{t+1}(x)) \mid \xi_t) - \sum_y \int a_y(s, s')\delta(s' - \xi_t(x + y))\, ds' = \sum_y a_y(s, \xi_t(x + y)), \tag{10}$$

we restrict ourselves to the case of sufficiently smooth $a_y(s, s')$.

All the rest is fully analogous to the case of discrete S with the exception that $q^*(x, s_1, s_2; s) = 0$ always. We will not discuss this further.

It is of interest to compare the classes of marginally-closed and moment-closed processes $\vec{\xi}_t$ (if $S \subseteq \mathbf{R}$). They have a nonempty intersection (for example, if $S = \{0, 1\}$ they coincide). There are marginally-closed processes that are not moment-closed. This follows because the process $\xi_t(x)$ at a given point x is always marginally-closed, but very rarely moment-closed. At the same time, on the other hand, Gaussian processes (see below) are moment-closed, but not marginally closed.

§2. *The dual operators* A_2^*

Instead of $l_2(\mathscr{F})$ as in §4.1, we consider a subset $\mathscr{F}_0(\mathbf{Z}^\nu, S)$ of a Fock symmetric space $\mathscr{F}_{\mathrm{sym}}(l_2(\mathbf{Z}^\nu \times S))$ of vectors

$$(f_0, f_1(x_1, s_1), f_2(x_1, s_1, x_2, s_2), \dots)$$

such that $f_n(x_1, s_1, \dots, x_n, s_n) = 0$ if at least two of the $x_1, \dots, x_n$ coincide.

The restriction of A^* to the direct sum of the two-particle and one-particle subspaces $\mathscr{F}_0(\mathbf{Z}^\nu; S)$ is denoted by A_2^*. Because of translational invariance, we may suppose that A_2^* is an operator in $l_2(\mathbf{Z}^\nu\backslash\{0\}, S^2) \oplus l_2(S)$. Let $\delta_{x,s_1,s_2}(\cdot)$, $x \neq 0$, be a function from $l_2(\mathbf{Z}^\nu\backslash\{0\}, S^2)$ equal to 1 at the point (x, s_1, s_2) and 0 in the remaining cases. Let us define the operator A_2^* by the equalities (as in §1)

$$A_2^*\delta_{x,s_1,s_2} = \sum{}' a_{y_1}(s_1, s_1')a_2(s_2, s_2')\delta_{x+y_2-y_1,s_1',s_2'} + \sum{}'' a_{y_1}(x_1, s)a_{y_2}(s_2, s)\delta_s \qquad (1)$$

for $x \neq 0$, where $\delta_s = \delta_{0,s,s}$ and

$$A_2^*\delta_s = A_1^*\delta_s. \qquad (2)$$

In equations (1) and (2) and below it is convenient to identify $l_2(\mathbf{Z}^\nu\backslash\{0\}, S^2) \oplus l_2(S)$ with the subspace $l_2(\mathbf{Z}^\nu, S^2)$ of functions equal to 0 at the points $(0, s_1, s_2)$, $s_1 \neq s_2$.

The basic problem is the analysis of the limiting behavior

$$((A_2^*)^t\delta_{x,s_1,s_2}, f) = (\delta_{x,s_1,s_2}, A_2^t f) \qquad (3)$$

where $f = \{p_{0,x}(s_1, s_2; 0)\}$ is the vector of the initial distribution belonging to l_∞.

It is convenient to pass to the Fourier transform

$$l_2(\mathbf{Z}^\nu \times S^2) \to L_2([-\pi, \pi]^\nu) \otimes l_2(S^2).$$

If, for the initial distribution, the correlations decrease sufficiently quickly, then the Fourier transform of the initial distribution has the form

$$\begin{aligned} \tilde{f}(\lambda) &= \sum_x e^{i(\lambda,x)} p_{0,x}(s_1, s_2; 0) \\ &= \delta(\lambda)p(s_1; 0)p(s_2; 0) + \tilde{f}_1(\lambda, s_1, s_2) \qquad (4) \end{aligned}$$

with sufficiently smooth functions $\tilde{f}_1(\lambda, s_1, s_2)$. It is convenient to rewrite (1) in the form

$$A_2^*\delta_{x,s_1,s_2} = \sum_{y_1,y_2,s_1',s_2'} a_{y_1}(s_1, s_1')a_{y_2}(s_2, s_2')\delta_{x+y_2-y_1,s_1',s_2'}$$

$$-(1-\delta_{x,0}) \sum_{\substack{s_1' \neq s_2', y_1, y_2 \\ x+y_2-y_1=0}} a_{y_1}(s_1, s_1')a_{y_2}(s_2, s_2')\delta_{0,s_1',s_2'}$$

$$-\delta_{x,0} \sum_{y_1,y_2,s_1',s_2'} a_{y_1}(s_1, s_1')a_{y_2}(s_2, s_2')\delta_{x+y_2-y_1,s_1',s_2'} + \delta_{x,0}\delta_{s_1,s}\delta_{s_2,s}A_1^*\delta_s, \tag{5}$$

where $\delta_{x,0}$ and $\delta_{s,s'}$ are Kronecker symbols.

Now let $|S| < \infty$. Then, in the right-hand side of (5), all besides the first sum is a finite-dimensional operator.

Let us introduce an $S \times S$-matrix $a(\lambda)$ with elements

$$a_{s',s}(\lambda) = \sum_y e^{i(\lambda,y)}a_y(s, s'), \tag{6}$$

and the operator $\tilde{A}_0$ in $L_2([-\pi, \pi]^\nu) \otimes l_2(s^2)$ of multiplication by the matrix $a(\lambda) \otimes a(-\lambda)$. Then the operator A_2^* acts in $L_2([-\pi, \pi]^\nu) \otimes l_2(S^2)$ and acts as

$$\tilde{A}_2^* = \tilde{A}_0 + \tilde{A}_1 \tag{7}$$

where $\tilde{A}_1$ is a finite-dimensional operator. In this manner, we have reduced the problem to the analysis of Friedrich's model (7).

The operator A_2^* has an eigenvector

$$e_0 = \sum_s \delta_s.$$

If $\mathscr{L}_1$ is reducible then e_0 is not a unique eigenvector. The representation (7), and the fact that iterations of $\tilde{A}_2^*\tilde{\delta}_{x,s_1,s_2}$ do not lead out of the class of smooth functions, suggest that the asymptotics (3) are equal to

$$((\tilde{A}_2^*)^t\tilde{\delta}_{x,s_1,s_2}, \tilde{f})$$

where $\tilde{f}$ has the form (4), if $\mathscr{L}_1$ is ergodic, and is determined by the eigenvector e_0 with the eigenvalue 1 and the behavior of $(\tilde{A}_2^*)^t\tilde{\delta}$ at $\lambda = 0$. In the case of the existence of a dual process, there corresponds to this, respectively, the hitting of the "dual particle" at $x = 0$ and its departure to infinity.

§3. *The operators A_2*

Let there be given at time 0 translationally-invariant one-particle $p(s; 0)$ and two-particle $p_{0,x}(s_1, s_2; 0)$ correlation functions. We may consider the set of all these functions as a vector

$$f \in l_1(S) \oplus l_\infty((\mathbf{Z}^\nu \backslash \{0\}) \times S^2). \tag{1}$$

It will always be assumed in addition that at time 0 the process $\vec{\xi}_t$ possesses the property of decomposition of the correlations in the sense that (if $\langle \cdot, \cdot \rangle$ denotes a semi-invariant)

$$\langle \delta_{s_1}(\xi_0(0)), \delta_{s_2}(\xi_0(x)) \rangle \equiv p_{0,x}(s_1, s_2; 0) - p(s_1; 0)p(s_2, 0) \in l_1((\mathbf{Z}^\nu \backslash \{0\}) + S^2). \tag{2}$$

Let us then expand the vector

$$\vec{f} = \vec{f}_1 \oplus \vec{f}_2 \oplus \vec{f}_3 \in l_1(s) \oplus l_1((\mathbf{Z}^\nu \backslash \{0\}) + S^2) \oplus l_1(S^2) = \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \mathscr{H}_3 \equiv \mathscr{H}, \tag{3}$$

where $\vec{f}_1$ has the components $f_1(s) = p(s; 0)$, $\vec{f}_2$ has the components $f_2(x, s_1, s_2) = \langle \delta_{s_1}(\xi_0(0)), \delta_{s_2}(\xi_0(x)) \rangle$ and $\vec{f}_3$ the components $f_3(s_1, s_2) = p(s_1; 0)p(s_2; 0)$.

Let us define the operator $A_2 \colon \mathscr{H} \to \mathscr{H}$,

$$A_2 \vec{f} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} \vec{f}_1 \\ \vec{f}_2 \\ \vec{f}_3 \end{pmatrix} \tag{4}$$

where $A_{ij} \colon \mathscr{H} \to \mathscr{H}_i$,

(i) $A_{12} = A_{13} = 0$,

$$(A_{11} f_1)(s) = \sum_{s'} b(s, s') f_1(s');$$

that is, A_{11} is the transition matrix of the chain $\mathscr{L}_1$;

(ii) $A_{31} = A_{32} = 0$

$$(A_{33} f_3)(s_1, s_2) = \sum_{s_1', s_2'} b(s_1, s_1') b(s_2, s_2') f_3(s_1', s_2'),$$

that is, A_{33} is the transition matrix of the chain $\mathscr{L}_1 \times \mathscr{L}_1$;

(iii) $(A_{21} f_1)(x, s_1, s_2) = \sum_{y_1, y_2 \colon y_1 = y_2 = x} \sum_s a_{y_1}(s_1, s) a_{y_2}(s_2, s) f_1(s)$;

that is, A_{21} acts as $\sum''$ in formula (7) §1;

$$(A_{22} f_2)(x, s_1 . s_2) = \sum_{y_1, y_2} \sum_{s_1', s_2'} a_{y_1}(s_1, s_1') a_{y_2}(s_2, s_2') f_2(x + y_2 - y_1, s_1', s_2'),$$

that is, A_{22} acts as $\sum'$ in formula (7) §1,

$$(A_{23}f_3)(x, s_1, s_2) = -\sum_{y_1, y_2: x = y_1 - y_2} \sum_{s'_1, s'_2} a_{y_1}(s_1, s'_1) a_{y_2}(s_1, s'_2) f_3(s'_1, s'_2).$$

Proposition 1. *A_2 is a bounded operator in $\mathscr{H}$.*

Proposition 2. *Let $\vec{f}$ be the vector of initial data. Let us write*

$$A_2^t \vec{f} = \vec{f}_1^t \oplus \vec{f}_2^t \oplus \vec{f}_3^t.$$

Then

$$p_{0,x}(s_1, s_2; t) = f_2^t(x, s_1, s_2) + f_3^t(s_1, s_2). \tag{5}$$

Proof. Induction on t. It is enough to compare formula (7) §1 with the definition of A_2. Let us remark that A_{23} is defined for the compensation "linearly" of the action of A_{33} "from the point $p_{0,0}s_1, s_2)$".

Now use the definition of A_2 to obtain an explicit form of the limiting two-particle correlation functions.

Theorem 1. *Let the chain $\mathscr{L}_1$ be ergodic and all $a_y(s, s') \geq 0$. Let $\mathscr{L}_1$ be such that*

$$\sum_{t,s} |p(s; t) - p(s; \infty)| < \infty. \tag{6}$$

Then the limit correlation function exists, is unique and equals

$$p_{0,x}(s_1, s_2; \infty) = \sum_{t=0}^{\infty} A_{22}^t (A_{21}\vec{f}_1 + A_{23}\vec{f}_3)(x; s_1, s_2) + \vec{f}_3(s_1, s_2), \tag{7}$$

where

$$f_1(s) = p(s, \infty), \qquad f_3(s_1, s_2) = p(s_1; \infty) p(s_2; \infty).$$

Proof. If, at the initial moment of time, $\xi_0(x)$ are independent and have stationary one-dimensional distributions $p(s; \infty)$, then formula (7) follows from the definition of the operator A_2. In the general case we write

$$\alpha(s; t) = p(s; t) - (p(s; \infty) \tag{8}$$

$$\alpha(s_1, s_2; t) = p(s_1; t) p(s_2; t) - p(s_1; \infty) p(s_2; \infty), \tag{9}$$

and introduce the vectors $\vec{\psi}_1(t) \in \mathscr{H}_1$ and $\vec{\psi}_3(t) \in \mathscr{H}_3$ with components (8) and

(9), respectively. Then

$$p_{0,x}(s_1, s_2; t) = \sum_{t' < t} A_{22}^{t'}(A_{21}(\vec{f}_1 + \psi_1(t - t' - 1)) + A_{23}(\vec{f}_3 + \psi_3(t - t' - 1))$$
$$+ A_{22}\vec{f}_2(x; s_1, s_2) + f_3^t(s_1, s_2). \tag{10}$$

For the proof of the theorem it is sufficient to show that, for all $x, y \neq 0, s_1, s_2, s_1', s_2'$,

$$(\delta_{y,s_1',s_2'}, A_{22}^t \delta_{x,s_1,s_2}) \to 0. \tag{11}$$

For the proof of (11), let us consider the random walk on $\mathbf{Z}^\nu$ with transition probabilities

$$p(x \to x') = \sum_{y_1, y_2 : x' = x + y_2 - y_1} q_{y_1} q_{y_2}.$$

It is symmetric, and with probability 1 either is absorbed at 0, or departs to ∞. But the operator A_{22} admits an interpretation as a random walk on $\mathbf{Z}^\nu$ with internal degrees of freedom $(s_1, s_2) \in S^2$. The theorem is proved.

From this it is possible to obtain more precise representations for the limiting two-particle correlation function.

Theorem 1 can be carried over to the case of n-particle correlation functions.

§4. *Gaussian processes*

We call the process $\xi_t(x) \in \mathbf{R}$ linear if

$$\xi_{t+1}(x) = \sum_{y \in Q} a_{t,x}(y)\xi_t(x + y) + a_{t,x}, \tag{1}$$

where the vectors $\vec{a}_{t,x} = (a_{t,x}(y), a_{t,x})$ are independent for different values of t or x.

We say that such a process is "homogeneous Gaussian" if

(1) $a_{t,x}(y) = a_y$ are constants,
(2) $a_{t,x}$ are independent Gaussian variables with mean m and variance σ^2, and
(3) $\vec{\xi}_0$ is a stationary Gaussian process.

It is easy to see that Gaussian processes are moment-closed (but not marginally-closed).

The convergence of the means $m_t \equiv \langle \xi_t(x) \rangle$ is trivial: they converge if $|a| < 1$, where $a = \sum a_y$. Later, we shall consider the case when $m = 0$ and $\langle \xi_0(x) \rangle \equiv 0$. Here everything is determined by the two-particle correlation functions.

Let us write

$$b_t(x) = \langle \xi_t(x')\xi_t(x'+x) \rangle.$$

Then

$$b_{t+1}(x) = \sum_{y,y' \in Q} a_y a_{y'} b_t(x+y-y') + \sigma^2 \delta(x) \tag{2}$$

or

$$\tilde{b}_{t+1}(\lambda) = \alpha(\lambda)\alpha(-\lambda)\tilde{b}_t(\lambda) = \sigma^2, \tag{3}$$

where

$$\alpha(\lambda) = \sum_y a_y \, e^{i(\lambda, y)},$$

$$\tilde{b}_t(\lambda) = \sum_x b_t(x) \, e^{i(\lambda, x)}.$$

We can regard (2) as a linear operator in $l_2(\mathbf{Z}^\nu) \oplus \mathbf{C}$ (if one remembers that it takes the vector $(0, 1)$ to $(\sigma^2, 1)$).

Its restriction to $l_2(\mathbf{Z}^\nu)$ is a cluster operator such that, in the terminology of [7], its cluster functions are different from zero only for one-point sets. Therefore, under the Fourier transform, it becomes the multiplication operator on the smooth functions on the torus $L_2(S^\nu)$ which arises from a much wider space ($l_\infty(\mathbf{Z}^\nu)$ or even the tempered distributions on $\mathbf{Z}^\nu$).

We now present a complete classification of such processes. We begin with definitions [4], [5].

Definition 1. Let us denote by $p_t^\varepsilon(x)$ the probability that $|\xi_t(x)| \leq \varepsilon$. We say that the process $\vec{\xi}_t$ is recurrent at x if $\sum_t p_t^\varepsilon(x) = \infty$, whereas zero recurrence holds if $p_t^\varepsilon(x) \to 0$ as $t \to \infty$. But if the series $\sum p_t^\varepsilon(x)$ converges, we say that the process $\vec{\xi}_t$ is nonrecurrent at x.

Definition 2. We say that the Gaussian process $\vec{\xi}_t$ converges as $t \to \infty$ for a given $\vec{\xi}_0$ if, for any $x_1, \ldots, x_n \in \mathbf{Z}^\nu$, the distribution $\xi_t(x_1), \ldots, \xi_t(x_n)$ has a nonzero Gaussian limit.

It is clear that, for a convergent process, nonzero recurrence holds at each point $x \in \mathbf{Z}^\nu$. For homogeneous Gaussian processes, we have the following classification.

(1) A homogeneous Gaussian process converges as $t \to \infty$ if $|\alpha(\lambda)| < 1$ for all $\lambda \in [-\pi, \pi]^\nu$.

(2) If

$$\max_{\lambda \in [-\pi, \pi]^\nu} |\alpha(\lambda)| = 1,$$

then the process converges if and only if $\nu \geq 3$. In the contrary case, nonzero recurrence holds at every point $x \in \mathbf{Z}^\nu$.

(3) However, if

$$\max_{\lambda \in [-\pi, \pi]^\nu} |\alpha(\lambda)| > 1,$$

then $\vec{\xi}_t$ is nonrecurrent at every point $x \in \mathbf{Z}^\nu$. (See the proof in [**5**].)

§5. *Moment-closed processes*

Definition 1. Let there be given a probability space

$$(\Omega, \Sigma, \mu) = (\Omega_1 \times \Omega_2, \Sigma_1 \times \Sigma_2, \mu_1 \times \mu_2).$$

Let us consider the function

$$F(\omega, \vec{\xi}) = F(\omega, \xi_1, \dots, \xi_n),$$

where $\omega \in \Omega_1$ and ξ_j are random variables on Ω_2. The function $F(\omega, \vec{\xi})$ is called linear in the mean if

$$\int F(\omega, \vec{\xi})\, d\mu_1(\omega) = \sum_y a_y \xi_y + c \tag{1}$$

for real a_y and c.

The class of functions that are linear in the mean contains not only the linear functions of the form (1) §3, but many others also. Typical examples for applications are the following.

(1) Let there be given on $\Omega_{x,t}$ the processes (stationary in s)

$$\vec{\eta}_{t,x}(s, \omega_{t,x}) = (\eta_{t,x,y}(s, \omega_{t,x}), y \in Q).$$

Then the function

$$F(\omega_{t,x}, \xi_t(x+y), y \in Q) = \sum_{y \in Q} \int_0^{\xi_t(x+y)} \eta_{t,s,y}(s)\, ds \tag{2}$$

is linear in the mean.

(2) Let $\omega \in \mathbf{R}$. Then the function

$$F(\omega, \xi) = \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}(\omega - \xi)^2) \frac{1}{f(\omega)}, \qquad f(\omega) > 0,$$

is linear in the mean if $d\mu_1(\omega) = f(\omega)\, d\omega$.

In contrast to the case of marginal closure, where the closure of the equations for the one-particle correlation functions implies their closure for all, this is not so in the case of moment closure.

Here, linearity in the mean of the function F from (1) §2 is equivalent to the closure of the equations for the first-order moments.

It is easy, moreover, to construct examples for which there is not closure for the second-order moments, and so on.

Remark 1. Another definition of moment closure is of interest when, not $\sum K_x$, but $|X|$ is expressed by the order of the compound moment $m(K_x, t)$ (see No. 2.1). This definition in many cases (but not always, due to the problem of moments) is equivalent to marginal closure.

Good sufficient conditions for moment closure are given by relationship (2) for certain restrictions on $\vec{\eta}_{t,x}(s, \omega_t, x)$. We will not state them, but limit ourselves to an important example.

A branching process with diffusion [**1**]

Let $\xi_t(x) \in \mathbf{Z}_+$ be interpreted as the number of particles at the point x at time t. Each particle at x, independently of the others, moves with probability q_y to $x + y$, $y \neq 0$, $y \in Q$; perishes with probability p, and with probability q_0 duplicates itself, remaining at x, $p + \sum q_y = 1$.

Let us define for each point (t, x) an infinite set of independent identically distributed vectors

$$\vec{\eta}^{(\alpha)}_{t,x} = (\eta^{(\alpha)}_{t,x+y}(\omega_{t,x}), \ y \in Q), \qquad \alpha = 1, 2, \ldots.$$

Each component $\eta^{(\alpha)}_{t,x,y} = 1$ for $y \neq 0$ and 2 for $y = 0$ with probability q_y or 0, where not more than one of the components can be different from zero.

Then

$$\xi_{t+1}(x) = \sum_{y \in Q} \sum_{\alpha=1}^{\xi_t(x-y)} \eta^{(\alpha)}_{t,x-y,y}. \tag{3}$$

Moment closure is obtained automatically from this.

It is possible to consider similarly the more complicated models of diffusion with chemical reactions.

Let us now consider a more general case.

The Markov process $\xi_t(x)$ with values in $\mathbf{Z}^1_+$ is naturally interpreted in terms of particles. Equation (1) can be rewritten in the form

$$P\{\xi_{t+dt}(x) = \xi_t(x) + K \mid \vec{\xi}_t\} = f_K(\xi_t(x+y), y \in Q)\, dt, \qquad K \geq -\xi_t(x),$$

$$P\{\xi_{t+dt}(x) = \xi_t(x) \mid \vec{\xi}_t\} = 1 - \sum_{K=-\xi_t(x)} f_K(\xi_t(x+y))\, dt.$$

For the first moment $m_1(t, x) = \langle \xi_t(x) \rangle$ (even without an assumption of homogeneity) we obtain in the standard manner

$$\frac{\partial m_1}{\partial t} = \sum_K K \langle f_K(\xi_t(x+y), y \in Q) \rangle.$$

From this it is apparent that for the closure of the equation for m_1 it is necessary and sufficient that

$$\sum_K K f_K(x_y, y \in Q) = \sum_{y \in Q} \alpha_y x_y. \tag{4}$$

This equation assumes the form

$$\frac{\partial m_1}{\partial t}(t, x) = \sum_y \alpha_y m_1(t, x+y)$$

and can be solved by the Fourier method.

To ensure nonnegativity, it is necessary to require additionally that $f_K(\xi_t(x+y)) = 0$ for $K \leq -\xi_t(x)$. Let us note that the conditional independence of $\xi_{t+dt}(x)$ with the condition $\vec{\xi}_t$ has so far not been used. If one requires it, then it is easy to obtain closed equations even for the higher moments under (4). For example, if

$$m_2(t, s_1, x_2) = \langle \xi_t(x_1) \xi_t(x_2) \rangle$$

then

$$\frac{\partial m_1}{\partial t} = \sum_{y \in Q} a_y[m_2(t; x_1 + y, x_2) + m_2(t; x_1, x_2 + y)], \qquad x_1 \neq x^2,$$

$$\frac{\partial m_2}{\partial t} = \sum_{y, y' \in Q} a_y a_{y'} m_2(t, x+y, x+y'), \qquad x_1 = x_2 = x,$$

etc.

The independence condition is restrictive in a number of cases, since it does not embrace diffusion processes with reproduction, when one particle, moving after a time dt from one point to another, changes $\xi_t(x)$ immediately at both points (see above, where the case of discrete time was discussed). If one supposes that the diffusion occurs independently for different particles, and, let us say, each completes the transition $x \to x', |x - x'| = 1$ with probability $\kappa\, dt$ ($\kappa = \text{const}$), and the branching intensity is a linear function of the form $\sum a_y \xi_t(x + y)$ ($a_y \geq 0$), where the distribution of the number of streams at each branching action is fixed and conditional independence holds for the branching process, then we obtain an explicitly solvable (in the sense of moments) branching process scheme with diffusion and local interaction.

§6. *Other problems*

Many new problems are tied to the nontranslational invariance of the initial distribution. Besides the natural question of convergence, we present two more.

1. *Processes with local interaction in a random medium*

Let us limit ourselves to the case $|S| = 2$ and a conditionally independent process $\vec{\xi}_t$,

$$P\{\xi_{t+1}(x) = 1 \mid \vec{\xi}_t\} = \sum_{y \in Q} a_y^x \xi_t(x + y). \tag{1}$$

In this case, we suppose that the realization a_y^x is determined on some auxiliary probability space (Ω, Σ, μ), and form a "random mean". The vectors $a^x = (a_y^x, y \in Q)$ are mutually independent and identically distributed so that with probability 1

$$\sum_y a_y^x = 1.$$

We are interested in the typical (in the sense of μ) behavior of $\xi_t(x)$ as $t \to \infty$.

Let us remark that for a given $\omega \in \Omega$, it is easy to see that $\xi_t(x)$ is moment-closed; but if it is averaged over $d\mu$, then the moment closure is lost, since a_y^x and $\xi_t(x + y)$ are dependent in the sense of $d\mu$.

The methods of §3.1 can in many cases be transferred to the generalized voting model. In this case, instead of the typical symmetric random walks, there arise random walks in a random medium.

2. *The hydrodynamic limit*

Papers on the construction of the hydrodynamic picture for an exclusion process and the voting model (see **[9]** and the references there) can apparently be generalized to arbitrary moment or marginally closed processes. In this case, it is supposed that the initial data are slowly changing functions of some auxiliary parameter ε, which subsequently approaches zero; for example,

$$\langle \xi_0(x) \rangle = m(\varepsilon x)$$

for some smooth function $m(r)$, $r \in \mathbf{R}$.

Two versions of the hydrodynamic picture are known, differing among themselves: the case of the local law of preservation in an exclusion process and the preservation of mean "magnetization" in the voting model. Consideration of the general class can yield new effects.

A similar problem occurs for kinetic equations. Let us remark that marginal closure corresponds to the linear kinetic equations, and moment closure to the hydrodynamic equations.

3. *Quantum processes with local interaction*

Such processes (which it is natural also to call open systems) are usually introduced for C^*-algebras. We will indicate the Grassmann version, which has a number of advantages; in particular, there is a complete analog for it of the conditional mathematical expectation.

Let us consider a Grassmann algebra $\mathfrak{A}$ with generators $\xi_0^\alpha(x)$, $\bar{\xi}_0^\alpha(x)$, $x \in \mathbf{Z}^\nu$, $\alpha = 1, \dots, N$, and an extended Grassmann algebra $\bar{\mathfrak{A}}$ with generators $\omega_{t,x}^\beta$, $\bar{\omega}_{t,x}^\beta$, $\beta = 1, \dots, N$, supplementary to $\mathfrak{A}$.

Let there be given on $\bar{\mathfrak{A}}$ a quasi-state $\langle\cdot\rangle_0$ which is white noise on the generators ω (see [8], §I.VII):

$$\langle \omega_{t,x}^\beta \bar{\omega}_{t',x'}^{\beta'} \rangle = \delta_{tt'} \delta_{xx'} \delta_{\beta\beta'},$$

and with a restriction of this quasi-state on $\mathfrak{A}$ independent of the white noise. Then the process with local interaction determines the temporal evolution of the quasi-state $\langle\cdot\rangle_0$ on $\mathfrak{A}$. It can be given in the Schrödinger picture by the homomorphism $\varphi\colon \bar{\mathfrak{A}} \to \mathfrak{A}$:

$$\varphi(\xi_0^\alpha(x)) \equiv \xi_1^\alpha(x) = F_\alpha(\xi_0^{\alpha'}(x+y), \bar{\xi}_0^{\alpha'}(x+y), \omega, \bar{\omega}),$$

$$\varphi(\bar{\xi}_0^\alpha(x)) \equiv \bar{\xi}_1^\alpha(x) = \bar{F}_\alpha(\xi_0^{\alpha'}(x+y), \bar{\xi}_0^{\alpha'}(x+y), \omega, \bar{\omega}),$$

where, as earlier, F and $\bar{F}$ depend only on the generators ξ, $\bar{\xi}$, ω and $\bar{\omega}$ at the points $x+y$, $y \in Q$.

Inductively, let

$$\varphi(\xi_t^\alpha(x)) = \xi_{t+1}^\alpha(x) = \varphi^{t+1}(\xi_0^\alpha(x))$$

and

$$\langle A \rangle_t = \langle \varphi^{t+1}(A) \rangle_0.$$

The latter formula determines a process with local interaction and, as above, it is of interest to distinguish the class of moment-closed processes. Among them, of course, are the Gaussian processes, where $\langle\rangle_0$ is Gaussian and φ is linear in ξ and $\vec{\xi}$.

III. Processes with continuous time

§1. *The analog of conditional independence*

If the set S of values is finite or countable, then to define a process $\vec{\xi}_t$ with continuous time, it is possible to act as follows: first introduce the process $\vec{\xi}_t^\Lambda = (\xi_t^\Lambda(x), x \in \Lambda)$ on the cube $\Lambda \subset \mathbf{Z}^\nu$ by means of the matrix $H_\Lambda = (h(\sigma_\Lambda, \sigma'_\Lambda))$, so that

$$p(s_\Lambda, t + dt) = p(s_\Lambda, t) + \sum_{s'_\Lambda} h(s_\Lambda, s'_\Lambda) p(s'_\Lambda, t)\, dt, \tag{1}$$

where s_Λ is a configuration on Λ and

$$h(s_\Lambda, s'_\Lambda) \geq 0, \qquad s_\Lambda \neq s'_\Lambda,$$
$$h(s_\Lambda, s_\Lambda) = -\sum_{s'_\Lambda : s'_\Lambda \neq s_\Lambda} h(s_\Lambda, s'_\Lambda).$$

In the case of continuous time, the matrices H_Λ, the nondiagonal elements $h(s_\Lambda, s'_\Lambda)$ of which are different from zero only if s_Λ and s'_Λ differ at a single point, correspond to conditionally independent processes with discrete time. We will denote by $s_\Lambda(x, s)$ the configuration $s_\Lambda = (s_x, x \in \Lambda)$ in which s_x is replaced by s.

Conditional linearity means, by definition, that

$$h(s_\Lambda(x, s), s_\Lambda) = f(s_{x+y}, y \in Q) = \sum_{y \in Q} a_y(s, s_{x+y}) \tag{2}$$

for some real functions $a_y(\cdot, \cdot)$ on S^2, where, without loss of generality, we may suppose that $a_0(s, s) = 0$. From the general theory **[6]**, **[8]**, the existence of a limit process for $\Lambda \nearrow \mathbf{Z}^\nu$ follows.

Here conditional linearity does not guarantee marginal closure. Before proving this, let us observe that it is easy to show from (1) that the conditional probability

$$P\{\xi_{t+dt}(x) = s \mid \vec{\xi}_t\} = (1 - \delta_s(\xi_t(x))) \sum_y a_y(s, \xi_t(x + y))\, dt$$
$$+ \delta_s(\xi_t(x))\left(1 - dt \sum_{y, s' : s' \neq s} a_y(s', \xi_t(x + y))\right). \tag{3}$$

Hence

$$\frac{\partial p_x(s,t)}{\partial t} = \sum_y \sum_{s'} a_y(s,s')p_{x+y}(s',t) - \sum_y \sum_{s'} a_y(s,s')p_{x,x+y}(s,s';t)$$

$$- \sum_{y\neq 0} \sum_{s'':s''\neq s} \sum_{s'} a_y(s'',s')p_{x,x+y}(s,s';t) - \sum_{s'':s''\neq s} a_0(s'',s)p_x(s,t)$$

$$= \sum_y \sum_{s'} a_y(s,s')p_{x+y}(\bar{s},t) - \sum_y \sum_{s'} b_y(s')p_{x,x+y}(s,s';t); \qquad (4)$$

and if

$$\sum_{s''} a_y(s'',s') = b_y(s') = b_y, \qquad (5)$$

that is, it does not depend on s', then

$$\frac{\partial p_x(s,t)}{\partial t} = \sum_y \sum_{s'} a_y(s,s')p_{x+y}(s',t) - bp_x(s;t), \qquad b = \sum b_y. \qquad (6)$$

Theorem 1. *Condition* (5) *is necessary and sufficient for a conditionally linear process to be marginally-closed.*

Proof. Necessity was proved above. For the proof of sufficiency, we need to deduce an equation for the higher correlation functions.

We have, as in (3),

$$P\{\xi_{t+dt}(x_i) = s_i, i = 1, \ldots, n \mid \vec{\xi}_t\}$$

$$= \sum_{i=1}^n (1 - \delta_{s_i}(\xi_t(x_i))) \prod_{j:j\neq 8} \delta_{s_j}(\xi_t(x_j)) \sum_y a_y(s_i, \xi_t(x_i+y))\, dt$$

$$+ \prod_{i=1}^n \delta_{s_i}(\xi_t(x_i))\left(1 - dt \sum_{j=1}^n \sum_y \sum_{s'\neq s_j} a_y(s', \xi_t(x_j+y))\right).$$

Hence

$$\frac{\partial p_{x_1,\ldots,x_n}(s_1,\ldots,s_n;t)}{\partial t}$$

$$= \sum_{i=1}^n \sum_{(y,s')} p_{x_1,\ldots,x_i+y,\ldots,x_n}(s_1,\ldots,s',\ldots,s_n)a_y(s_i,s')$$

$$- \sum_{i=1}^n \sum_{(y,s')} p_{x_1,\ldots,x_i,x_i+y,x_{i+1},\ldots,x_n}(s_1,\ldots,s_i,\bar{s},\ldots,s_n;t)a_y(s_i,s')$$

$$- \sum_{i=1}^{n} \sum_{(y,s')} \sum_{s'' \neq s_i} p_{x_1,\ldots,x_i,x_i+y,x_{i+1},\ldots,x_n}(s_1,\ldots,s_i,s',\ldots,s_n;t)a_y(s'',s')$$

$$= \sum_{i=1}^{n} \sum_{(y,s')} p_{x_1,\ldots,x_i+y,\ldots,x_n}(s_1,\ldots,s',\ldots,s_n;t)a_y(s_i,s')$$

$$- b_n p_{x_1,\ldots,x_i,\ldots,x_n}(s_1,\ldots,x_n;t),$$

where $\sum_{(y,s')}$ is over all (y, s') such that $x_i + y$ does not coincide with any $x_j, j \neq i$; or if $x_i + y = x_j$, then $s' = s_j$.

§2. *Marginal closure without conditional linearity*

The processes of the previous section admit a generalization to the case when, after a time dt, the configuration can simultaneously change at several points. Such processes can be marginally closed, but they are never conditionally linear in the earlier sense.

We also define a class of processes that we will consider here. Let us consider a system of subsets $\mathscr{B} = \{B\}$, $B \subset \mathbf{Z}^\nu$, with the following properties:

(1) translational invariance: if $B \in \mathscr{B}$, then all shifts B belong to $\mathscr{B}$;
(2) diam $B < d_0 < \infty$ for all $B \in \mathscr{B}$;
(3) if $B' \subset B \in \mathscr{B}$, then $B' \in \mathscr{B}$.

Let $S_\Lambda(B, \tilde{S}_B)$ mean the configuration S_Λ in which the values at the points B are replaced by $\tilde{S}_B$. Let us consider the matrix H_Λ with

$$h(S_\Lambda(B, \tilde{S}_B), S_\Lambda) = f_B(\tilde{S}_B, S_{O(B)}), \tag{1}$$

where $O(B)$ is some set containing B. We will subsequently consider only the particular case when $O(B) = B$. Then, as in §1, we have

$$P\{\xi_{t+dt}(x) = \tilde{S}_x, x \in C \mid \vec{\xi}_t\}$$

$$= \sum_{B':\, B' \subset C, B' \in \mathscr{B}} \left(1 - \prod_{x \in B'} \delta_{\tilde{S}_x}(\xi_t(x))\right)$$

$$\times \prod_{y \in C \setminus B'} \delta_{\tilde{S}_y}(\xi_t(x)) \sum_{B:\, B \cap C = B'} \sum_{S_B:\, S_{B \setminus B'} = \tilde{S}_{B'}} f_B(S_B, \xi_{t,B})\, dt$$

$$+ \prod_{y \in C} \delta_{\tilde{S}_y}(\xi_t(y))\left(\left(1 - dt \sum_{B \in \mathscr{B}:\, B \cap C \neq \varnothing} \sum_{S_B:\, S_{B \cap C} \neq \tilde{S}_{B \cap C}} f_B(S_B, \xi_{t,B})\right)\right). \tag{2}$$

Then, in obvious notation,

$$\frac{\partial p(S_C : t)}{\partial t}$$

$$= \sum_{B:\, B\cap C = B \neq \varnothing} \sum_{S_{B\setminus B'}} \sum_{S'_B} f_B(S_{B\setminus B'} \cup \tilde{S}_{B'}, S_{B'}) p(\tilde{S}_{C\setminus B'}, S'_B; t)$$

$$- \sum_{B':\, B' \subset C, B' \in \mathscr{B}} \sum_{B:\, B\cap C = B'} \sum_{S_{B\setminus B'}} \sum_{S'_{B\setminus B'}} f_B(\tilde{S}_{B'} \cup S_{B\setminus B'}, \tilde{S}_{B'} \cup S'_{B\setminus B'}) p(\tilde{S}_C, S'_{B\setminus B'}; t)$$

$$- \sum_{B:\, B\cap C \neq \varnothing} \sum_{S_B:\, S_{B\cap C} \neq \tilde{S}_{B\cap C}} \sum_{S'_{B\setminus C}} f_B(S_B, S'_{B\setminus C} \cup \tilde{S}_{B\cap C}) p(\tilde{S}_C, S'_{B\setminus C}; t).$$

The second and third terms on the right of (3) give, under union,

$$- \sum_{B:\, B\cap C = B' \neq \varnothing} \sum_{S_B} \sum_{'B\setminus C}^{S} f_B(S_B, S'_{B\setminus C} \cup \tilde{S}_{B\cap C}) p(\tilde{S}_C, S'_{B\setminus C}; t). \tag{4}$$

If, for all B, $S'_{B\setminus C}$ and $\tilde{S}_{B\cap C}$,

$$\sum_{S_B} f_B(S_B, S'_{B\setminus C} \cup \tilde{S}_{B\cap C}) = \psi_B(\tilde{S}_{B\cap C}), \tag{5}$$

that is, it does not depend on $\sigma'_{B\setminus C}$, then (4) equals

$$- \sum_{B:\, B\cap C \neq \varnothing} \psi_B(\tilde{S}_{B\cap C}) p(\tilde{S}_C; t) \stackrel{\text{def}}{=} p(\tilde{S}_C, t) b(c, \tilde{S}_C). \tag{6}$$

Condition (5) is similar to the condition (5) of the preceding section. Here, however, there is a second (more restricted) condition for the analysis of the first term. We will require that, for all B and for all $\tilde{S}_{B'}$,

$$\sum_{S_{B\setminus B'}} \sum_{S_{B'}} f_B(S_{B\setminus B'} \cup \tilde{S}_{B'}, S'_B) p(S'_B) = \sum_{A \subset B:\, |A| \leq |B'|} \sum_{S'_A} \psi_A(S'_A, \tilde{S}_{B'}) p(S'_A). \tag{7}$$

Theorem 1. *For the process ξ_t, defined above, to be marginally closed, it is necessary and sufficient that conditions* (5) *and* (7) *are satisfied.*

An example of the slight generalization of the exclusion process [**1**], *where* $S = \{0, 1\}$ *on* $\mathbf{Z}^\nu$, *where B is any pair of nearest neighbors x, x', $|x - x'| = 1$ and where* $f_B = (S_B, S'_B) = \lambda$ *in any of the following cases*:

$$S_B = B_{B'} = (1, 1),$$

$$S_B = (0, 1), \qquad S_{B'} = (1, 0),$$

and $f_B = \lambda$ *if* $S_B = S_{B'} = (0, 0)$ *or* $S_B = (1, 0)$, $S_{B'} = (0, 1)$.

§3. *Diffusion systems with local interaction and their moment closure*

Let us consider the system of stochastic equations

$$d\xi_t(x) = \sigma(\xi_t(x+y), y \in Q)\, dW_t(x) + b(\xi_t(x+y), y \in Q)\, dt,$$

which can be rewritten in the form

$$\xi_t(x) = \xi_0(x) + \int_0^t \sigma\, dW_{t'}(x) + \int_0^t b\, dt', \tag{1}$$

$x \in \mathbf{Z}^\nu$, $W_t(x)$ are independent Wiener processes, and the stochastic integrals are understood in the Itô sense. If the function σ is smooth, and b grows not faster than linearly, then the system (1) is solvable by the method of sequential approximations in an appropriate Hilbert space (say, with the norm $\|f\|^2 = \sum_x 2^{-|x|} f^2(x)$). For the general theory of such equations, including also their continuous analogs when $\mathbf{Z}^\nu$ is replaced by $\mathbf{R}^\nu$, see Rozovskiĭ's book **[10]**. We limit ourselves to formal computations, which in our case are easily substantiated.

An equation of the first order has the form

$$dm_1(t, x) = \langle b(\xi_t(x+y), y \in Q)\rangle\, dt,$$

from which it follows that

$$b(\xi_t(x+y), y \in Q) = \sum_{y \in Q} A_y \xi_t(x+y) \tag{2}$$

is necessary and sufficient for the closure of the equation of the first moment. In case (2), as earlier, we obtain

$$\frac{\partial m_1}{\partial t}(t; x) = \sum_{y \in Q} a_y m_1(t; x+y).$$

In order to compute $m_2(t; x_1, x_2) = \langle \xi_t(x_1)\xi_t(x_2)\rangle$, let us apply the Itô formula to $\xi_t(x_1)$ and $\xi_t(x_2)$. Because of the independence of $w_t(x_1)$ and $w_t(x_2)$ for $x_1 \neq x_2$, this gives

$$\frac{\partial m_2}{\partial t}(t, x_1, x_2) = \sum_{y \in Q} a_y[m_2(t; x_1+y, x_2) + m_2(t; x_1, x_2+y)].$$

For $x_1 = x_2$, there are the supplemental terms

$$\frac{\partial m_2}{\partial t}(t; x, x) = \sigma^2(\xi_t(x+y), y \in Q) + \sum_{y \in Q} a_y m_2(t; x+y, x) \qquad (y \in Q).$$

The closure of m_2 will be ensured in the three cases: (a) $\sigma^2 = \text{const}$; (b) $\sigma^2(\xi_t(x+y), y \in Q) = \sum_{y \in Q} b_y \xi_t(x+y)$—in this case it is necessary to require additionally nonnegativity of ξ_t, which is not hard to ensure; and (c) $\sigma^2(\xi_t(x+y), y \in Q) = \sum_{y,y' \in Q} C_{yy'} \xi_t(x+y)\xi_t(x+y')$. Cases (b) and (c) require additional analysis of the solvability of the original stochastic equation.

The diffusion processes $\xi_t(x), x \in \mathbf{Z}^\nu$ with local interaction appear, in a natural way, in the solution of linear evolution problems in random rapidly-changing media, for approximately δ-form correlations. See, in this connection, the recent surveys [**11**], [**12**]. We will limit ourselves to one example.

§4. *Intermittency in chemical kinetics problems*

For diffusion processes, instead of general considerations, as above, we consider the example of a linearized chemical kinetics scheme, where we will discuss quite different statements of the problem.

Let $W_t(x)$ be standard Wiener processes, exiting from the origin, and independent for different $x \in \mathbf{Z}^\nu$.

Let us consider the equation

$$d\xi_t(x) = (\kappa\, \Delta\xi_t(x))\, dt + \xi_t(x)\, dW_t(x) \tag{1}$$

for the concentration $\xi_t(x)$ at the point x at time t.

From general theorems [**11**], it is not hard to deduce that the solution of (1) exists and is unique.

This process is nonergodic, but displays very curious asymptotic behavior. Here intermittency also occurs, but somewhat differently than in §3.1. As regards the physics interpretation, see [**12**] for more detail.

In order to obtain intuition about the behavior of $\xi_t(x)$, let us observe that for $\kappa = 0$ we have, at any point x,

$$\xi_t = \exp(W_t - t/2). \tag{2}$$

Thus, as $t \to \infty$,

$$\frac{\ln \xi_t}{t} \to -\tfrac{1}{2}, \tag{3}$$

$$\begin{aligned} m_p(t) \equiv \langle \xi_t^p \rangle &= \left\langle \exp\left(pW_t - \frac{pt}{2}\right)\right\rangle \\ &= \exp\left\{\frac{p^2 t}{2} - \frac{pt}{2}\right\}, \end{aligned} \tag{4}$$

and therefore

$$\frac{\ln m_p(t)}{t} \to \frac{p(p-1)}{2} \equiv \gamma_p. \tag{5}$$

It follows from (2)–(5) that the concentration at each point decreases exponentially, and the moments $m_p(t)$ grow exponentially. This means that high (but all the more rare at $t \to \infty$) peaks of concentration last a time proportional to the logarithm of their height (for $W_t - t/2 = \ln \xi_t$).

Now let us consider how the diffusion κ influences the growth of the moments and the asymptotics of $\xi_t(x)$.

Let us first observe that the process $\xi_t(x)$ is moment-closed. In fact, setting

$$m_p(x_1, \ldots, x_p; t) = \langle \xi_t(x_1) \cdots \xi_t(x_p) \rangle$$

and having observed that

$$\xi_{t+\Delta t}(x) = \xi_t(x) + \Delta t\Bigg(\kappa \sum_{x': |x-x'|=1} \xi_t(x') - 2\nu\kappa\xi_t(x)\Bigg) + \xi_t(x)\,\Delta W_t(x) + O(\Delta t)$$

and

$$\Delta W_t(x)\,\Delta W_t(y) = \Delta t \delta(x-y),$$

we obtain the moment chain

$$\frac{\partial m_p}{\partial t} = H_p m_p = \kappa(\Delta_{x_1} + \cdots + \Delta_{x_p}) m_p(x_1, \ldots, x_p; t) + \Big(\sum_{i<j} \delta(x_i - x_j)\Big) m_p. \tag{6}$$

We have obtained the Schrödinger equation for p particles on a lattice with the positive δ-form interaction

$$V(x_1, \ldots, x_p) = \sum_{i<j} \delta(x_i - x_j).$$

For the first moment we have

$$\frac{\partial m_1}{\partial t}(x, t) = \kappa\,\Delta m_1(x, t), \qquad m_1(x, 0) \equiv 1,$$

and this means that for all $\kappa \geq 0$,

$$\gamma_1 = \lim_{t \to \infty} \frac{m_1(x, t)}{t} = 0.$$

For the second moment, using translational invariance and setting

$$m_2(y, t) = m_2(x_1, x_2; t), \qquad y = x_2 - x_1,$$

we have

$$\frac{\partial m_2}{\partial t} = 2\kappa\, \Delta_y m_2 + \delta(y) m_2. \tag{7}$$

Let us denote by $\lambda^+(\kappa)$ the greatest eigenvalue of the operator in the right-hand side of (7). It turns out always to exist and gives the limit of the spectrum of this operator. For $\nu = 1, 2$,

$$\lambda^+(\kappa) \to 1,\ \kappa \to 0, \qquad \lambda^+(\kappa) \to 0,\ \kappa \to \infty.$$

For $\nu \geq 3$, $\lambda^+(\kappa)$ decreases from 1 to 0 as κ varies from 0 to some $\kappa_2 > 0$. For $\kappa > \kappa_2$, $\lambda^+(\kappa) = 0$.

For arbitrary p it is also easy to show that

$$\lim_{t\to\infty} \frac{\ln m_p}{t} = \lambda_p^+(\kappa)$$

exists, and is an upper boundary of the spectrum of H_p. In **[12]** more detailed information has been obtained about the behavior of $\lambda_p^+(\kappa)$.

A general picture of the behavior of $\lambda_p^+(\kappa)/p$ is given by

$$0 \leq \lambda_1^+(\kappa) \leq \lambda_2^+(\kappa)/p \leq \cdots \leq \lambda_p^+(\kappa)/p \leq \cdots. \tag{8}$$

Physical intermittency signifies an extreme nonuniformity of the field $\xi_t(x)$ over the space as $t \to \infty$. A hierarchy emerges of more and more uncommon and higher and higher peaks of the field, where the peaks on "one scale" correspond to the formation of the corresponding statistical moment. This leads to the fact that, for example, the mass (or energy) of a physical field is concentrated as $t \to \infty$ "almost entirely" in a very small part of the space. We consider that this point of view yields a qualitative explanation of many physical effects (activity centers in catalysis, nonuniformity of the magnetic field of the sun and stars, etc.). See, in this connection, the cited surveys. The effects of intermittency relative to random media that are nonstationary (quickly changing) in time, are related to the effects of localization in the physics of solid disordered bodies, which are connected to the appearance of exponentially decreasing eigenfunctions for random Hamiltonians of the form $H = -\Delta + \xi(x)$ with a potential, now independent of t.

The strict inequalities in (8), as explained in **[11]** signify physical intermittency. The strict inequalities can be rigorously proved for small κ and for $p = 2$.

We understood the initial equation $\partial u/\partial t = \kappa\, \Delta u + \dot{w}_t(x) u$ in the Itô sense. However, this standard point of view is not the only one possible, and what is more, the construction of the Stratonovich stochastic integral is physically more natural. Indeed, the δ-form time correlations are the limiting case of the correlations with the time scale τ as $\tau \to 0$. It is possible, for example, as a physical model of "white noise" $\dot{w}_t$ to consider the "renewal" process across time τ, $\dot{w}_\tau^{(\tau)} = \xi[t/\tau]/\sqrt{\tau}$, where ξ_i, $i = 0, \pm 1, \ldots$, are standard Gaussian variables. For $\kappa = 0$ the equation

$$\frac{\partial u(\tau)}{\partial t} = \dot{w}_t^{(\tau)} u, \qquad u(0) = 1,$$

has the solution

$$u^{(\tau)}(t) = \exp\left\{\int_0^t \dot{w}_s^{(\tau)}\, ds\right\},$$

and as $\tau \to 0$ in any natural sense it yields the limit

$$u^{(0)}(t) = \exp\{w_t\}$$

and not $\exp(w_t - t/2)$, which is obtained using the Itô integral. In the concrete case of the equation of (1), linearized chemical kinetics, this leads to a shift of all the moment increments $\gamma_p(\kappa)$ by 1/2. However, in other situations (say, in the theory of a hydromagnetic turbulent dynamo) the differences are more noticeable. In the physics literature, the symmetric Stratonovich stochastic integral is always used in such problems.

Conclusion

The monograph of Liggett [6] consists largely of models which in one sense or another admit an explicit solution. In this paper, it is noted that the majority of these examples belong to the wide class of processes with local interaction: marginally-closed and moment-closed processes. Marginally-closed processes admit an explicit solution in purely probabilistic terms. The moment-closed processes reduce to "finite-particle problems". In this paper, the foundations are laid for the theory of this class of processes and many directions and problems are presented for subsequent analysis.

Afterword

After twenty years from the introduction into the mathematical literature of the concept of a Markov process with interaction, it has become clear that this class of stochastic processes is a natural one for modeling a very wide circle of real phenomena: from the interaction of microscopic particles in the framework of statistical physics to the interaction of living individuals and the interaction of computer processors, connected to the system of information exchange. However, until now there had not existed sufficiently general methods of studying such processes, and hundreds of papers are dedicated to the investigation of the properties of particular, but in exchange in some sense explicitly solvable (integrable), models. Against this background, the value of the present paper is clear: in it there is introduced a new broad class of explicitly solvable models, including those known earlier; moreover, at first a general idea is disclosed, one that lies at the basis of many previous analyses. The results of the analysis of the particular models can be extrapolated to the general case, giving useful hypotheses and predictions, important both for the theory of Markov processes with interaction, and for the applications of this theory. I am convinced that this paper will find a wide class of interested readers.

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