Critical states of strongly interacting many-particle systems on a circle

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1 Introduction

In multicomponent systems with strong local interaction one can encounter some phenomena absent in the standard systems of statistical physics and other multicomponent systems. Namely, a system with N components in the bounded volume of order 1 (macroscale) has the natural microscale of the order $\frac{1}{N}$. Applying the macroscopic force (of order 1) on the system, and thus on any of its components, one normally gets changes on the macroscale itself and simultaneously small, of the order $\frac{1}{N}$, changes of the microcomponents, see for example [7]. In the systems, considered below, with the strong Coulomb repulsion between the particles, however, one can observe the influence of such force on the equilibrium state only on a scale, much smaller that the standard microscale. Otherwise speaking, the information about the macroforce is not available neither on the macrocale nor on the standard microscale, but only on a finer scale. If this phenomenon does not depend on the continuity properties of the applied force, then the mere existence of the equilibrium depends essentially on the continuity properties of the external force.

The model Consider the system

$$0 \leq x_1(t) < ... < x_N(t) < L \tag{1}$$

of identical classical point particles on the interval $[0, L]$ with periodic boundary conditions (that is on the circle S of length L). The dynamics of this system of points is defined by the system of N equations

$$m\frac{d^2x_i}{dt^2} = -\frac{\partial U}{\partial x_i} + F(x_i) - A\frac{dx_i}{dt} \tag{2}$$

where $A \geq 0$, F is the external force, and the interaction is given by

$$U(x_1, ..., x_N) = V(x_2 - x_1) + V(x_3 - x_2) + ... + V(x_1 - x_N)$$

where $x_1 - x_N$ should be understood as $x_1 + (L - x_N)$, otherwise speaking, here and further on the differences are taken clockwise. It is assumed that the potential V is symmetric and repulsive, and moreover

$$V(x) = V(-x) > 0, V(r) = \alpha r^{-a+1} > 0, r = |x|$$

$$f(r) = -\frac{dV(r)}{dr} = \alpha(a-1)r^{-a} > 0 \tag{3}$$

We assume further that $a > 1$ and put $\alpha(a-1) = 1$, that is $f(r) = r^{-a}$. Fixed (critical) configurations $X = (x_1, ..., x_N)$ of the system are defined by the equations

$$f(x_k - x_{k-1}) + F(x_k) - f(x_{k+1} - x_k) = 0, k = 1, ..., N \quad (4)$$

assuming that positive forces are directed clockwise, and that $x_0 = x_N, x_{N+1} = x_1$. Our goal will be to study these equations.

The results If $F \equiv 0$, then it is evident that the fixed configuration is unique up to translation, and for any $i = 1, ..., N$

$$|x_{i+1} - x_i| = \frac{L}{N}$$

If F is not identically zero, the situation is essentially more complicated. However, we have the following general result.

Теорема 1 *Let the external force $F(x)$ be a bounded function. Assume that there exists a sequence $(x_1^{(p)}, ..., x_{N_p}^{(p)}), p = 1, 2, ...,$ of fixed configurations with $N_p \to \infty$. Then for $p \to \infty$ uniformly in $i = 1, ..., N_p$*

$$|x_{i+1}^{(p)} - x_i^{(p)}| \sim \frac{L}{N_p}$$

Existence and uniqueness of the fixed configuration do not have such general general results, but have interesting effects, related to $N \to \infty$. Let the force be potential on the circle, that is

$$\int_S F(x)dx = 0$$

then the potential can be defined as

$$W(x) = \int_0^x F(x)dx$$

If $W(x)$ is smooth, then the minimum of the potential

$$U(x_1, ..., x_N) + \sum_{i=1}^{N} W(x_i)$$

satisfies the equation (4), this minimum thus being the solution of these equations. At the same time we have

Лемма 1 *Let $F(x)$ be left-continuous function with finite number of gaps. If there exists a sequence $N_p \to \infty$ such that for any p there exists at least one fixed configuration $(x_1^{(p)}, ..., x_{N_p}^{(p)})$, then*

$$\int_S F dx = 0 \quad (5)$$

that is the force $F(x)$ is potential on the circle.

The proof of this lemma will be obtained during the proof of lemma 2.

If the potential $W(x)$ is not smooth, even if the force is potential, the equations (4) are far from being always solvable, that is critical points cannot exist at all. This is not surprising, similar phenomenon exists for example in the following simplest model with one particle, where there is no critical point on the interval $[0, L]$: the particle moves in the field of the external force $F(x) = F_1 > 0, x \in [0, M], 0 < M < L$, and $F(x) = F_2 < 0, x \in (M, L]$. However we concentrate only on phenomena related to the number of particles $N \to \infty$

Even the simplest case of the piecewise constant function shows that the existence of the equilibrium is not generic. Let $F(x) = F_1$ on the interval $(0, M]$ of the circle S, $F(x) = F_2$ on its complement $(M, L]$. We shall often denote $M_1 = M, M_2 = L - M$, N_1, N_2 - the number of particles on the intervals $(0, M]$ and $(M, L]$ correspondingly, $N = N_1 + N_2$.

Our goal is to prove the following results.

Теорема 2 *Fix $L, M = \frac{L}{2}$ and $F(x) = F > 0, x \in (0, M], F(x) = -F, x \in (M, L]$. Then for sufficiently large even N there is a continuum of fixed configurations. More exactly, for any point $x \in S$ there is a fixed configuration*

$$0 < x_1 < ... < x_{\frac{N}{2}} \leq \frac{M}{2} < x_{\frac{N}{2}+1}, ..., x_N \leq L,$$

containing x (that is x coinsides with some x_k). Moreover, the number of points on the intervals $(0, M]$ and $(M, L]$ is the same, and $\Delta_k = x_{k+1} - x_k = \Delta_{N-k}$ for all $k = 1, ..., \frac{N}{2}$.

If N is odd, there is no fixed configurations.

During the proof we shall explicitely construct the existing fixed configurations.

Теорема 3 *For any $0 < C_1 < C_2 < \infty$ and any sequence $N_1^{(p)}, N_2^{(p)} \to \infty$ so that $\frac{N_1^{(p)}}{N_2^{(p)}} \to \gamma \neq 1$, starting from some p there is no fixed configurations for any L, M, F_i such that*

$$C_1 < L, M, F_i < C_2$$

At the same time one can always change the value of the function F at one or two gap points (that is at the points M or L) so, that there exists a fixed configuration with one or two particles at the gap points.

2 Proofs

2.1 Uniform asymptotics

Let us prove theorem 1. For any p there exists at least one $1 \leq k(p) \leq N$ with

$$\Delta_{k(p)}^{(p)} = x_{k(p)+1}^{(p)} - x_{k(p)}^{(p)} \leq \frac{L}{N_p}$$

Thus there can be two cases. Either for $p \to \infty$

$$\Delta_{k(p)}^{(p)} \sim \frac{L}{N_p}$$

either there is a subsequence p_n such that for any $\epsilon > 0$ and all p_n

$$\Delta_{k(p_n)}^{(p_n)} \leq \frac{L}{N_{p_n}}(1-\epsilon)$$

Let us prove that in the first case the theorem holds. In fact, let us sum up the equations (4) with $k = k(p)+1, ..., m$. Here m can be any number of

$$m = k(p)+1, ..., N, 1, ..., k(p)-1$$

(we consider the indices modulo N, that is we identify x_k and x_{k+N} for any integer k). Then

$$f(\Delta_m^{(p)}) = f(\Delta_{k(p)}^{(p)}) + F(x_{k(p)+1}^{(p)}) + ... + F(x_m^{(p)})$$

and thus for some $C = \sup |F(x)| > 0$

$$f(\Delta_m^{(p)}) = f(\Delta_{k(p)}^{(p)}) + r_m^{(p)}, |r_m^{(p)}| \leq CN_p$$

or

$$\Delta_m^{(p)} = ((\Delta_{k(p)}^{(p)})^{-a} + r_m^{(p)})^{-\frac{1}{a}} \sim \frac{L}{N_p}(1 + r_m^{(p)}\frac{L^a}{N_p^a})^{-\frac{1}{a}}$$

The result follows from this. Let us prove now that the second case is impossible. Quite similarly, for any sufficiently large p_n and all m

$$\Delta_m^{(p_n)} \leq (1-\epsilon)\frac{L}{N_{p_n}}(1 + r_m^{(p_n)}\frac{L^a(1-\epsilon)}{N_{p_n}^a})^{-\frac{1}{a}} \leq (1-\frac{\epsilon}{2})\frac{L}{N_{p_n}}$$

From this, summing over по $m = 1, ..., N$, we get $L \leq (1-\frac{\epsilon}{2})L$, which is impossible.

2.2 Existence conditions

The following lemma (together with lemma 1) gives a list of obstructions for the existence of the fixed points.

Лемма 2

- *If the quotient $\frac{F_2}{F_1}$ is irrational, then there are no fixed points.*
- *For fixed M_i, F_i, N the fixed configuration can exist not more than for one partition of the number $N = N_1 + N_2$, where N_i is the number of particles on the interval of length M_i correspondingly.*
- *Let M_i, F_i be fixed. Then a necessary condition of existence of at least one fixed configuration for any p in a sequence $N^{(p)}, p = 1, 2...,$ is the existence of the partition $N_1^{(p)} + N_2^{(p)} = N$ such that*

$$\frac{M_1}{M_2} = -\frac{F_2}{F_1} = \frac{N_1^{(p)}}{N_2^{(p)}} \tag{6}$$

Proof. Note that for fixed N a necessary condition for the configuration to be fixed is the condition

$$\sum F(x_i) = 0 \tag{7}$$

which is obtained by summing the equations (4). In our case (7) becomes

$$F_1 N_1 + F_2 N_2 = 0 \tag{8}$$

From this the two first assertions of the lemma follow. Let us note that there appears the necessary condition of the arithmetic character: F_1 and F_2 should be commensurable.

Note now that by theorem 1 for $p \to \infty$

$$|\frac{L}{N_p}\sum_{i=1}^{N_p} F(x_i^{(p)}) - \sum_{i=1}^{N_p} F(x_i^{(p)})(x_{i+1}^{(p)} - x_i^{(p)})| \to 0$$

and thus

$$\frac{L}{N_p}\sum_{i=1}^{N_p} F(x_i^{(p)}) \to_{p\to\infty} \int_S F dx = F_1 M_1 + F_2 M_2$$

which gives the proof of lemma 1, as by (7) the left part is identically zero. Besides this, we get the first equality in (6), and the second follows from (8).

2.3 Auxiliary problem on the segment

Consider the system of identical classical point particles on the interval $[0, L] \in R$

$$0 \leq x_1 < ... < x_N \leq L$$

with the same interaction (3) but with the completely inelastic boundary conditions. This means that if one of the extreme particles reaches one of the end points of the interval, it stops and can leave this point only if the resulting force becomes directed to inside the interval. Then the equilibrium condition is the following system

$$F(0) - f(x_2) \leq 0, F(L) + f(L - x_{N-1}) \geq 0$$

$$f(x_k - x_{k-1}) + F(x_k) - f(x_{k+1} - x_k) = 0, k = 2, ..., N-1 \tag{9}$$

We will consider the fixed configurations such that $x_1 = 0, x_N = L$. One can show that there are no others but we will not need this.

Лемма 3 *Assume that the external force $F > 0$ is constant. Then for sufficiently large N the fixed point $(x_1, x_2, ..., x_N)$ such that $x_1 = 0, x_N = L$, exists and is unique, moreover $\Delta_k = x_{k+1} - x_k, k = 2, ..., N-1$, analytically depends on L, F and Δ_1.*

Proof. It is convenient to introduce δ_k by

$$\Delta_k = x_{k+1} - x_k = \frac{L}{N-1}(1 + \delta_k)$$

We have from (9)

$$f(x_{k+1} - x_k) = f(x_2) + (k-1)F \tag{10}$$

or

$$f(\frac{L}{N-1}(1+\delta_k)) - f(\frac{L}{N-1}(1+\delta_1)) = (k-1)F$$

Rewrite

$$(1+\delta_k)^{-a} - (1+\delta_1)^{-a} = Q_k = (k-1)q, q = (\frac{L}{N-1})^a F \tag{11}$$

or

$$1 + \delta_k = [(1+\delta_1)^{-a} + Q_k]^{-\frac{1}{a}} \tag{12}$$

that defines δ_k as the real analytic function of L, F, δ_1 for $L > 0, F > 0, |\delta_1| < 1$. One can write

$$\delta_k = [1 - a\delta_1 + \frac{a(a+1)}{2}\delta_1^2 + O(\delta_1^3) + Q_k]^{-\frac{1}{a}} - 1 =$$

$$= \delta_1 - a^{-1}Q_k + \frac{a^{-1}(a^{-1}+1)}{2}Q_k^2 - (a^{-1}+1)\delta_1 Q_k + g_3(\delta_1, N, k) \tag{13}$$

where

$$g_3(\delta_1, N, k) = O((|\delta_1| + Q_k)^3)$$

is the real analytic function of L, F, δ_1 for $L > 0, F > 0, |\delta_1| < 1$.

Summing over k and using the condition

$$\sum_{k=1}^{N-1} \delta_k = 0$$

we get

$$\sum_{k=1}^{N-1} \delta_k = 0 = (N-1)\delta_1 - (a^{-1} + (a^{-1}+1)\delta_1) \sum_{k=2}^{N-1} Q_k + \frac{a^{-1}(a^{-1}+1)}{2} \sum_{k=2}^{N-1} Q_k^2$$

$$+ \sum_{k=2}^{N-1} g_3(\delta_1, N, k)$$

It follows

$$\delta_1 = (N-1)^{-1}((a^{-1} + (a^{-1}+1)\delta_1) \sum_{k=2}^{N-1} Q_k - \frac{a^{-1}(a^{-1}+1)}{2} \sum_{k=2}^{N-1} Q_k^2) + G(\delta_1, N) =$$

$$= (a^{-1} + (a^{-1}+1)\delta_1) q(\frac{N}{2} - 1) - \frac{a^{-1}(a^{-1}+1)}{2} q^2 \frac{(N-2)(2N-3)}{6} + G(\delta_1, N)$$

$$G(\delta_1, N) = (N-1)^{-1} \sum_{k=2}^{N-1} g_3(\delta_1, N, k) = O((|\delta_1| + Q_{N-1})^3) = O((|\delta_1| + N^{-a+1})^3)$$

As the latter equation can be written as $\delta_1 = \sum_{n=0}^{\infty} c_n \delta_1^n$, where in the righthand part there is an analytic function with small coefficients c_n, then by subsequent iterations we get the unique solution for δ_1. Moreover

$$\delta_1 = a^{-1}q(\frac{N}{2} - 1) + \frac{a^{-1}(a^{-1}+1)}{12} q^2 (N-2)(N-3) + J_1(N), J_1(N) = o(N^{-2a+2}) \tag{14}$$

$$\delta_{N-1} = \delta_1 - a^{-1}Q_{N-1} + \frac{a^{-1}(a^{-1}+1)}{2}Q_{N-1}^2 - (a^{-1}+1)\delta_1 Q_{N-1} + g_3(\delta_1, N, N-1) =$$

$$= -a^{-1}q(\frac{N}{2} - 1) + \frac{a^{-1}(a^{-1}+1)}{12} q^2 (N-2)(N-3) + J_{N-1}(N), J_{N-1}(N) = o(N^{-2a+2}) \tag{15}$$

2.4 Construction of fixed configurations

Note that if $X = (x_1, ..., x_N)$ is a fixed configuration on the circle with given $F(x)$, then any its subsequence (without gaps) $X_{kl} = (x_k, ..., x_l)$ is a fixed configuration on the segment $[x_k, x_l]$ (with the same force $F(x)$) in the sense of the section 2.3. In fact, restricting on the segment we neglect the part of the force at the end points, directed to inside the interval. It means that at the end points the force becomes directed to outside the segment, which gives first two inequalities in (9). Vice-versa, if for a given configuration $X = (x_1, ..., x_N)$ $X_{kl} = (x_k, ..., x_l)$ will be the fixed configuration (with the same force $F(x)$) on the segment of the circle in-between the points x_k and x_l (in the clockwise order), and the configuration $X_{kl} = (x_{l-1}, ..., x_{k+1})$ will be the fixed configuration on the segment in-between the points x_{l-1} and x_{k+1}, then X is the fixed configuration on the circle. Such situation is called glueing. In the symmetric case, that is under the conditions of theorem 2, in glueing we use the mirror symmetry.

Symmetric case Let us prove theorem 2. Let N be even and $x \in S$. We will construct the fixed configuration, containing x, for the case of equal number $\frac{N}{2}$ of points on the intervals $(0, M]$ and $(M, L]$.

Let us consider the fixed configuration with $\frac{N}{2} + 2$ points

$$0 = y_1 < ... < y_{\frac{N}{2}+2} = M + m$$

on the interval $[0, M + m]$ ($m > 0$ being a small real number) with constant force $F > 0$, as in the section 2.3. By lemma 3 it is unique and has the differences

$$\Delta_k = y_{k+1} - y_k = \frac{M + m}{\frac{N}{2} + 1}(1 + \delta_k), k = 1, ..., \frac{N}{2} + 1$$

$$\sum_{k=1}^{\frac{N}{2}+1} \Delta_k = M + m$$

which were calculated in the section 2.3. With these differences we will construct a fixed configuration $X = (x_1, ..., x_N)$ on the circle. Define by clockwise induction, for some $b > 0$,

$$x_N = L - b, x_1 = -b + \Delta_1, x_2 = x_1 + \Delta_2, ..., x_{\frac{N}{2}+1} = x_{\frac{N}{2}} + \Delta_{\frac{N}{2}+1} = -b + \sum_{k=1}^{\frac{N}{2}+1} \Delta_k = -b + M + m$$

and similarly by counter-clockwise induction

$$x_N = L + x_1 - \Delta_1, x_{N-1} = x_N - \Delta_2, ..., x_{N-k} = x_{N-k+1} - \Delta_{k+1}, ..., x_{\frac{N}{2}} = x_{\frac{N}{2}+1} - \Delta_{\frac{N}{2}+1}$$

We see that these definitions are compatible and it follows from them

$$x_{\frac{N}{2}+1} - x_1 = x_N - x_{\frac{N}{2}} = M + m - \Delta_1$$

For the constructed configuration to be a configuration on the circle of length L, it is necessary the additional condition $2(M + m) - \Delta_1 - \Delta_{\frac{N}{2}+1} = L$ (as two intervals of length $M + m$ cover the circle, but the intervals $\Delta_1, \Delta_{\frac{N}{2}+1}$ are taken into account twice) or

$$2m = \Delta_1 + \Delta_{\frac{N}{2}+1} \qquad (16)$$

This gives the equation for m

$$2m = \frac{M+m}{\frac{N}{2}+1}(1+\delta_1) + \frac{M+m}{\frac{N}{2}+1}(1+\delta_{\frac{N}{2}+1}) = \frac{M}{N} + h(m,N)$$

where the function h is analytic in m and by following the expansions (14,15)

$$h(m,N) = o(\frac{|m|}{N} + \frac{1}{N^2})$$

That is why this equation has a unique solution $m = O(\frac{1}{N})$.

From the definitions above it follows that $\Delta_k = \Delta_k(m)$ depends on m and $x_k = x_k(m,b)$ depends on m and b, and moreover, as we know fron section 2.3, for all k $\Delta_k(m) > \Delta_{k+1}(m)$. In particular, for given m, $\Delta_1(m)$ is the maximal of the intervals $\Delta_k(m)$. Besides that,

$$\Delta_k(m') > \Delta_k(m), m' > m$$

$$x_k(m, b+c) = x_k(m,b) + c$$

To get a fixed point on the circle from this glueing (that is to satisfy equilibrium conditions), one should demand that the points $x_1, ..., x_{\frac{N}{2}}$ belonged to the interval $(0, M]$, and the rest belonged to the interval $(M, L]$. Necessary and sufficient conditions for this will be the inequalities

$$x_1 > 0 \tag{17}$$

$$x_{\frac{N}{2}} = -b + M + m - \Delta_{\frac{N}{2}+1} < M \tag{18}$$

$$x_{\frac{N}{2}+1} = -b + M + m > M \tag{19}$$

The first one can be reduced to

$$0 < b < \Delta_1(m) \tag{20}$$

The second and the third ones can be reduced to

$$b < m < \Delta_{\frac{N}{2}+1}(m) + b \tag{21}$$

Put

$$b(m) = \frac{\Delta_1(m)}{2}$$

It is easy to see that all inequalities (20,21) are fullfilled.

Assume that $x_k(m, b(m))$ is the point of the fixed configuration, with the parameters $m, b(m)$, which is the nearest to x . Let for example $x_k < x$. Then

$$x - x_k \leq \frac{\Delta_{k-1}(m)}{2}$$

Choosing now $b = b(m) + x - x_k$, we get the point x as the k-th point of the new configuration

$$x = x_k(m,b)$$

Nonexistence for odd N follows from the third assertion of lemma 2.

Remark 1 *For given x and k, the fixed configuration with $x = x_k$ is unique, which follows from the monotonicity of the function $x_k(b)$ in b. The question whether there can be, for given x, two fixed configurations such that for one of them $x = x_k$, and for the other one $x = x_{k+1}$, acquires more exact calculations and is not considered here.*

Asymmetric case Let us prove theorem 3. Let for some p there exist a fixed configuration, further on we omit the index p. Then it looks like

$$0 < x_1 < x_2 < ... < x_{N_1} \leq M < x_{N_1+1} < ... < x_N \leq L$$

that is the points $x_1, ..., x_{N_1+1}$ belong to the interval where the force $F_1 > 0$ is applied, and the points $x_{N_1+1}, ..., x_N$ belong to the interval where the force $F_2 < 0$ is applied. This configuration defines two auxiliary fixed configurations on the intervals $[0, M_i + m_i], i = 1, 2$, with $N_i + 2$ points

$$0 = y_1^i < ... < y_{N_1+2}^i = M_i + m_i$$

and the forces F_i correspondingly, which are defined by their differences

$$u_{i,k} = y_{k+1}^i - y_k^i = \frac{M_i + m_i}{N_i + 1}(1 + \delta_{i,k})$$

Moreover, $u_{1,k}$ are defined by the coordinates $x_N, x_1, ..., x_{N_1+1}$

$$x_1 = L - x_N + u_{1,1}, x_2 = x_1 + u_{1,2}, ..., x_{N_1+1} = x_{N_1} + u_{1,N_1+1}$$

and $u_{2,k}$ is defined by the coordinates $x_1, x_N, x_{N-1}, ..., x_{N_1}$ (in reverse order)

$$x_N = x_1 + L - u_{2,1}, x_{N-1} = x_N - u_{2,2}, ..., x_{N_1} = x_{N_1+1} - u_{2,N_2+1}$$

For compatibility the following two conditions should be fullfilled

$$u_{1,1} = u_{2,1}, u_{1,N_1+1} = u_{2,N_2+1} \tag{22}$$

and the length L will be defined by

$$L = M_1 + m_1 + M_2 + m_2 - u_{1,1} - u_{1,N_1+1} \tag{23}$$

that is simalarly to (16) as the circle of length L is covered by two segments, where $u_{1,1}$ and u_{N_1+1} in the union of two segments are counted twice. Also it should be $F_2N_2 + F_1N_1 = 0$. It is convenient to denote $\hat{M}_i = M_i + m_i$.

As (by lemma 2 or by theorem 1)

$$\frac{\hat{M}_1}{N_1 + 1} \sim \frac{\hat{M}_2}{N_2 + 1} \tag{24}$$

then one can write

$$\frac{\hat{M}_1 + m}{N_1 + 1} = \frac{\hat{M}_2}{N_2 + 1}, m = o(1) \tag{25}$$

and find m from equation (22), that will give two equations for m

$$\frac{\hat{M}_1}{N_1 + 1}(1 + \delta_{1,1}) = \frac{\hat{M}_1 + m}{N_1 + 1}(1 + \delta_{2,1})$$

$$\frac{\hat{M}_1}{N_1 + 1}(1 + \delta_{1,N_1+1}) = \frac{\hat{M}_1 + m}{N_1 + 1}(1 + \delta_{2,N_2+1})$$

Rewrite the latter equations as

$$m = \hat{M}_1(\delta_{1,1} - \delta_{2,1})(1 + \delta_{2,1})^{-1} \tag{26}$$

$$m = \hat{M}_1(\delta_{1,N_1+1} - \delta_{2,N_2+1})(1 + \delta_{2,N_2+1})^{-1} \tag{27}$$

We obtained two equations with one unknown m. Let us show that they are incompatible for small m. To do this let us first compare the main terms of the two series: for $\delta_{1,1} - \delta_{2,1}$ and $\delta_{1,N_1+1} - \delta_{2,N_2+1}$. More exactly, using the expansion (14), calculate the difference of the two first terms in the series for $\delta_{1,1}$ and $\delta_{2,1}$. Note that, doing this, we will use $|F_2|N_2 = F_1N_1$, and in the formulae for $\delta_{2,1}$ we should take $|F_2|$, as y_k^2 corresponds to the inverse order of the coordinates x_i. Direct calculation gives

$$\delta_{1,1} - \delta_{2,1} = R_1 + R_2$$

where

$$R_1 = \frac{1}{2}a^{-1}F_1N_1(\frac{\hat{M}_1}{N_1+1})^a[1 - (1 + \frac{m}{\hat{M}_1})^a] = \frac{1}{2}a^{-1}F_1N_1(\frac{\hat{M}_1}{N_1+1})^a[-a\frac{m}{\hat{M}_1} + O(m^2)]$$

$$R_2 = \frac{a^{-1}(a^{-1}+1)}{12}F_1N_1(\frac{\hat{M}_1}{N_1+1})^{2a}[N_1 - (1 + \frac{m}{\hat{M}_1})^{2a}(N_2 - 1)] =$$

$$= \frac{a^{-1}(a^{-1}+1)}{12}F_1N_1(\frac{\hat{M}_1}{N_1+1})^{2a}[(N_1 - N_2) - (2a\frac{m}{\hat{M}_1} + O(m^2))(N_2 - 1)]$$

Thus the equation (26) will take the form

$$m = [c_1N_1^{-a+1}m + c_2N_1^{-2a+2} + c_3N_1^{-2a+1}m + O(m^2)N_1^{-a+1}](1 + c_4N_1^{-a+1} + o(N_1^{-a+1})) \tag{28}$$

where $c_i = c_i(N)$ tend, as $N \to \infty$, to nonzero constants d_i, from which we will need only

$$c_1 = -\frac{F_1}{2}\frac{N_1}{N_1+1}(\frac{\hat{M}_1}{N_1+1})^{a-1} \to d_1 = -\frac{F_1}{2}M_1^{a-1}$$

$$c_4 = -\frac{a^{-1}F_1}{2}N_1(\frac{\hat{M}_1}{N_1+1})^a \to d_4 = -\frac{a^{-1}F_1}{2}M_1^a$$

as $\hat{M}_1 \to M_1$ (note that c_1, c_4 are obtained only from the first terms of the expansion). It is evident that the equation (28) has the unique solution $m = o(1)$, which is asymptotically equal to

$$m \sim d_2N_1^{-2a+2} \tag{29}$$

At the same time, as it can be seen from the comparison of the expansions (14) and (15), $\delta_{1,N_1+1} - \delta_{2,N_2+1}$ look similarly, but have minus sign in front of c_1 and of c_4. Subtracting the second equation from the first we get

$$0 = 2c_1N_1^{-a+1}m + 2c_4c_2N_1^{-3a+3} + 2c_4c_3N_1^{-3a+2}m + O(m^2)N_1^{-a+1}$$

from where

$$m \sim -\frac{2c_4c_2}{c_1}N_1^{-2a+2} \sim -\frac{2d_4d_2}{d_1}N_1^{-2a+2} \tag{30}$$

Comparing (29) and (30) we get the necessary compatibility condition of the two equations

$$-\frac{2d_4}{d_1} = 2a^{-1}M_1 = 1$$

But as M_1 and M_2 are completely symmetric, we could perform the same calculations for M_2. At the end of these calculations we get the similar condition

$$2a^{-1}M_2 = 1$$

However, these conditions cannot hold simultaneously as $\gamma \neq 1$ and thus $M_1 \neq M_2$. Remind that the case $M_1 = M_2$ was considered separately above. This proves the first assertion of theorem 3.

The proof of the last assertion fo theorem 3 is sufficiently simple. Let us construct two fixed configurations on the intervals $[0, M]$ and $[M, L]$, as the section 2.3. To get a fixed configuration on the circle, the particles at the points 0 и M should be in equilibrium. For this to happen it is sufficient to adjust the value of the external force at these points so that it compensated the difference of the forces from the neighbor particles.

3 Remarks

Earlier the ground states (fixed configurations) of classical particles were studied in connection with the problem of existence and the structure of the lattice for the condensed matter state (see [1], [2],[3]-[6]). There are also other continuum one-dimensional models, most known are the Toda chains and the Frenkel-Kontorova model. In all these models it is assumed that the potential V has a minimum. Thus in the Frenkel-Kontorova model mainly the quadratic hamiltonian is considered [8], but there are papers where the latter model is understood in a wider sense, see for example [10, 9]. However the ground states are considered on the whole real line. In our paper we pursue completely different goals and study different phenomena in a finite volume, related to the appearance of a finer scale, that is in fact related to the second term of the asymptotics of distances between particles. We use direct approach, which is rather straightforward ideologically, but demands cumbersome calculations.

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