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One particle subspaces for two particle quantum walks with ultralocal interaction

V. Malyshev and A. Zamyatin

Mechanics and Mathematics Faculty, Lomonosov Moscow State University, Leninskie Gory 1, Moscow, 119991, Russia.

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Abstract. We study one particle subspaces for two particles of different masses with ultra local interaction on a lattice of arbitrary dimension.

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1. Introduction

We consider here continuous time quantum walks on the integer lattice $\mathbf{Z}^d \subset \mathbf{R}^d$. It is well known that for Hamiltonians invariant w.r.t. some translation group there cannot be discrete spectrum. The convenient and at the same time absolutely rigorous language (among other approaches) which allows to formulate exactly what means bound state in this case of N particle problem is the language of 1-, 2-, $\ldots$ particle subspaces.

Here we consider two particle random walk in any dimension with ultralocal interaction. Main results are Theorems 4.1 and 4.2.

2. One particle quantum walk

In this section we will work in the complex Hilbert space $l_2(\mathbf{Z}^d)$. Elements of this space will be denoted as $f = (f(x), x \in \mathbf{Z}^d)$, the scalar product is

$$(f_1, f_2) = \sum_{x \in \mathbf{Z}^d} f_1(x) f_2^*(x).$$

The standard orthonormal basis consists of the vectors δ_y, $y \in \mathbf{Z}^d$ such that $\delta_y(x) = \delta_{x,y}$, where $\delta_{x,y} = 1$ if $x = y$ and zero otherwise.

Define the following linear bounded operators

$$H = H_0 + H_1 : \ l_2(\mathbf{Z}^d) \to l_2(\mathbf{Z}^d), \tag{2.1}$$

$$(H_0 f)(x) = -\lambda \sum_{k=1}^{d} (f(x+e_k) - 2f(x) + f(x-e_k)), \tag{2.2}$$

$$(H_1 f)(x) = \mu \delta_{x,0} f(x), \tag{2.3}$$

where $e_k = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbf{Z}^d$ is the unit vector in the kth direction, that is with all zero coordinates except the kth coordinate, λ, μ are real parameters. Moreover, we assume that $\lambda > 0$ as it is standardly accepted, although it is not really a restriction. It is clear that all these Hamiltonians are selfadjoint.

Put

$$\gamma(\varphi) = \cos\varphi_1 + \ldots + \cos\varphi_d$$

where $\varphi = (\varphi_1, \ldots, \varphi_d)$ belongs to d-dimensional torus

$$\mathbf{T}^d = (-\pi, \pi] \times \cdots \times (-\pi, \pi].$$

Below the following integrals are important

$$c(d) = \frac{1}{(2\pi)^d} \int_{\mathrm{T}^d} \frac{d\varphi}{\gamma(\varphi) + d} > 0$$

$$c_1(d) = \frac{1}{(2\pi)^d} \int_{\mathrm{T}^d} \frac{d\varphi}{\gamma(\varphi) - d} < 0$$

where $d\varphi = d\varphi_1 \ldots d\varphi_d$.

Note that $c(d) = -c_1(d)$. In fact,

$$(2\pi)^{-d} \int_{\mathrm{T}^d} \frac{d\varphi}{\gamma(\varphi) + d} = (2\pi)^{-d} \int_{\mathrm{T}^d} \frac{d\varphi}{\gamma(\varphi + \pi) + d} =$$
$$= (2\pi)^{-d} \int_{\mathrm{T}^d} \frac{d\varphi}{d - \gamma(\varphi)} = -c_1(d)$$

where $\varphi + \pi = (\varphi_1 + \pi, \ldots, \varphi_d + \pi)$.

These integrals coincide with the classical Watson integral [1]

$$w(d) = \pi^{-d} \int_0^\pi \ldots \int_0^\pi \frac{d\varphi}{d - \gamma(\varphi)}.$$

Namely,

$$w(d) = -c_1(d) = c(d).$$

Note that for $d = 1, 2$ the integral $c(d) = +\infty$, and for $d \geq 3$ it is convergent. In the paper [1] integral $c(3)$ was calculated explicitly. For $d \to \infty$ the asymptotic series is known, see [2]:

$$c(d) \sim \frac{1}{d} + \frac{1}{2d^2} + \frac{3}{4d^3} + \dots$$

Let $\sigma_{ess}(H)$, $\sigma_p(H)$ be essential and point spectra of H [3].

The following theorem gives complete description of the spectrum.

Theorem 2.1. *Let* $\lambda > 0$.

- *For all* μ *and all dimensions* d *we have* $\sigma_{ess}(H) = [0, 4\lambda d]$;
- *For* $\mu = 0$ $\sigma_p(H) = \emptyset$;
- *For* $d = 1, 2$ *and* $\mu \neq 0$ *the point spectrum* $\sigma_p(H)$ *consists of exactly one eigenvalue* $\nu = \nu(\mu, \lambda)$, *moreover* $\nu \notin \sigma_{ess}(H)$;
- *For* $d = 3, 4$

  if $|\frac{2\lambda}{\mu}| < c(d)$, *then the point spectrum* $\sigma_p(H)$ *consists of exactly one eigenvalue* $\nu = \nu(\mu, \lambda)$, *moreover* $\nu \notin \sigma_{ess}(H)$;

  if $|\frac{2\lambda}{\mu}| \geq c(d)$, *then* $\sigma_p(H) = \emptyset$.
- *For* $d \geq 5$

  if $|\frac{2\lambda}{\mu}| < c(d)$, *then the point spectrum* $\sigma_p(H)$ *consists of exactly one eigenvalue* $\nu = \nu(\mu, \lambda)$, *moreover* $\nu \notin \sigma_{ess}(H)$;

  if $\frac{2\lambda}{\mu} = c(d)$, *then the point spectrum* $\sigma_p(H)$ *consists of exactly one eigenvalue* $\nu = 4\lambda d$, *moreover* $\nu \in \sigma_{ess}(H)$;

  if $\frac{2\lambda}{\mu} = -c(d)$, *then the point spectrum consists of exactly one eigenvalue* $\nu = 0$, *moreover* $\nu \in \sigma_{ess}(H)$;

  if $|\frac{2\lambda}{\mu}| > c(d)$, *then* $\sigma_p(H) = \emptyset$.
- *In all cases, if* $\mu > 0$, *then the eigenvalue* $\nu \geq 4\lambda d$; *if* $\mu < 0$, *then* $\nu \leq 0$; *equality in these inequalities is achieved only in the case when* $d \geq 5$ *and* $|\frac{2\lambda}{\mu}| = c(d)$.

Thus, in dimension $d \geq 5$ and if $|\frac{2\lambda}{\mu}| = c(d)$, one of the boundary points of the essential spectrum belongs to the point spectrum. In all other cases the (unique) eigenvalue belongs to the discrete spectrum (that is, does not belong to the essential spectrum).

Theorem 2.1 is not new – similar result was proved in [4] for $\mu > 0$ and $\lambda = 1$, but this not a restriction for their method. We included this theorem for our paper were self-contained.

3. Direct integral

The following definitions are from [3] and [5].

Let X_0 be a separable Hilbert space, and (Ω, τ) measurable space with $\sigma-$finite measure τ. Vector function $f(\omega) : \Omega \to X_0$ is called measurable, if for any $\xi \in X_0$ the function $(\xi, f(\omega))_{X_0}$ is measurable. Consider the Hilbert space $X = L_2(\Omega, d\tau;\, X_0)$ of measurable square integrate functions with values in X_0. The scalar product

$$(f_1, f_2)_X = \int\limits_{\Omega} (f_1(\omega), f_2(\omega))_{X_0} d\tau, \quad f_1, f_2 \in X,$$

is finite as

$$\int\limits_{\Omega} \|f(\omega)\|^2 \, d\tau < \infty$$

Then we will call X the direct integral with the layers isomorphic to X_0, and will write

$$X = \int\limits_{\Omega}^{\oplus} X_0 \, d\tau.$$

Let $\mathcal{L}(X_0)$ be the space of linear bounded operators in X_0. Operator function $A(\omega) : \Omega \to \mathcal{L}(X_0)$ is called measurable, if the functions $(\chi, A(\omega)\chi')$ are measurable for all $\chi, \chi' \in X_0$. Let $L_\infty(\Omega, d\tau;\, \mathcal{L}(X_0))$ be the space of measurable functions from Ω to $\mathcal{L}(X_0)$ such that $ess\,sup\, \|A(\omega)\| < \infty$.

Consider the class of decomposable operators in $X = L_2(\Omega, d\tau;\, X_0)$. We shall say that linear bounded operator $A : X \to X$ is decomposed in the direct integral if there exists measurable operator function $A(\omega) \in L_\infty(\Omega, d\tau;\, \mathcal{L}(X_0))$ such that for any $F \in X$, $(AF)(\omega) = A(\omega)F(\omega)$. It is commonly written as

$$A = \int\limits_{\Omega}^{\oplus} A(\omega) \, d\tau(\omega).$$

For any ω the operator $A(\omega)$ will be called the restriction of the operator A on the corresponding layer.

4. One particle subspaces for two particle Hamiltonian

Free one particle quantum walk is defined by the Hamiltonian

$$(h_i f)(x) = -\lambda_i \sum_{k=1}^{d} \left(f(x + e_k) - 2f(x) + f(x - e_k) \right), \; x \in \mathbf{Z}^d, \lambda_i \in \mathbf{R}, i = 1, 2$$

in the Hilbert space $l_2(\mathbf{Z}^d)$. Then the Hamiltonian for two non-interacting particles is defined as

$$
\begin{aligned}
H_0 = h_1 \otimes 1 + 1 \otimes h_2 = \\
& - \lambda_1 \sum_{k=1}^{d} (f(x_1 + e_k, x_2) + f(x_1 - e_k, x_2) - 2f(x_1, x_2)) - \\
& - \lambda_2 \sum_{k=1}^{d} (f(x_1, x_2 + e_k) + f(x_1, x_2 - e_k) - 2f(x_1, x_2))
\end{aligned}
$$

in the space $L = l_2(\mathbf{Z}^d) \otimes l_2(\mathbf{Z}^d) = l_2(\mathbf{Z}^{2d})$ of functions $f(x_1, x_2), (x_1, x_2) \in \mathbf{Z}^{2d}$. We put for concreteness $\lambda_1, \lambda_2 > 0$.

We will consider Hamiltonian $H = H_0 + H_1$ for two particles with the δ-interaction term $(H_1 f)(x_1, x_2) = \mu \delta_{x_1, x_2} f(x_1, x_2)$ where $\mu \in \mathbf{R}$, and δ_{x_1, x_2} is the Kronecker symbol. Let U_y, $y \in \mathbf{Z}^d$, be the translation group in L, $(U_y f)(x_1, x_2) = f(x_1 + y, x_2 + y)$. Note that H commutes with U_y.

The change of variables $x_1 = x_1$, $x = x_2 - x_1$ defines one-to-one transformation $\mathbf{Z}^{2d} = \{(x_1, x_2)\} \to \mathbf{Z}^{2d} = (x_1, x)\}$ and unitary transformation $W_1 : L = \{f(x_1, x_2)\} \to L = \{g(x_1, x) = f(x_1, x_1 + x)\}$.

The translation group now acts only on the first argument: $(U_y g)(x_1, x) = g(x_1 + y, x)$.

In these coordinates H (in fact $W_1 H W_1^{-1}$) can be written as follows

$$
\begin{aligned}
(Hg)(x_1, x) = & - \lambda_1 \sum_{k=1}^{d} (g(x_1 + e_k, x - e_k) + g(x_1 - e_k, x + e_k) - 2g(x_1, x)) - \\
& - \lambda_2 \sum_{k=1}^{d} (g(x_1, x - e_k) + g(x_1, x + e_k)) - 2g(x_1, x)) \\
& + \mu \delta_{x,0} g(x_1, x)
\end{aligned}
$$

Consider the Hilbert space

$$\hat{L} = L_2(\mathbf{T}^d) \otimes l_2(\mathbf{Z}^d)$$

of square integrable functions $F = F(\varphi, x)$, where $\varphi \in \mathbf{T}^d, x \in \mathbf{Z}^d$. The scalar product is defined as

$$\langle F_1, F_2 \rangle = \sum_{x \in \mathbf{Z}^d} \int_{\mathrm{T}^d} F_1(\varphi, x) \overline{F_2(\varphi, x)} d\varphi.$$

It is known [3], that the Hilbert space $\hat{L} = L_2(\mathbf{T}^d) \otimes l_2(\mathbf{Z}^d)$ is isomorphic to the space of square integrable functions $l_2(\mathbf{Z}^d)$-valued functions

$$L_2(\mathbf{T}^d, d\varphi; l_2(\mathbf{Z}^2)),$$

where $d\varphi$ is the Lebesgue measure on $\mathbf{T}^d$.

Thus, according to the above definition, the space $\hat{L}$ can be represented as the direct integral

$$\hat{L} = \int_{\mathrm{T}^d}^{\oplus} M \, d\varphi$$

with identical layers $M = l_2(\mathbf{Z}^d)$.

The elements of $\hat{L}$ can also be considered either as complex functions of two variables φ, x (then we can denote them as $F(\varphi, x)$) or as functions of one variable φ with values in the Hilbert space $l_2(\mathbf{Z}^2)$ (in this case we can use notation $\hat{F}(\varphi)$).

Define the linear transformation $\mathcal{F} : L \to \hat{L} = L_2(\mathbf{T}^d) \otimes l_2(\mathbf{Z}^d)$

$$(\mathcal{F}g)(\varphi, x) = F(\varphi, x) = \frac{1}{(2\pi)^{d/2}} \sum_{x_1 \in \mathbf{Z}^d} g(x_1, x) e^{i(x_1, \varphi)} \tag{4.1}$$

where $\varphi = (\varphi_1, \ldots, \varphi_d) \in \mathbf{T}^d$. Note that $\mathcal{F}$ is a unitary operator. In fact, for any $x \in \mathbf{Z}^d$, by Parseval equality,

$$\sum_{x_1 \in \mathbf{Z}^d} |g(x_1, x)|^2 = \int_{\mathrm{T}^d} |F(\varphi, x)|^2 d\varphi,$$

whence

$$\sum_{x \in \mathbf{Z}^d} \sum_{x_1 \in \mathbf{Z}^d} |g(x_1, x)|^2 = \sum_{x \in \mathbf{Z}^d} \int_{\mathrm{T}^d} |F(\varphi, x)|^2 d\varphi.$$

The adjoint operator $\mathcal{F}^*$ is inverse to $\mathcal{F}$ and acts as

$$(\mathcal{F}^* F)(\varphi, x) = g(x_1, x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathrm{T}^d} F(\varphi, x) e^{-i(x_1, \varphi)} d\varphi. \tag{4.2}$$

Consider the operator $\hat{H} = \mathcal{F} H \mathcal{F}^* : \hat{L} \to \hat{L}$ which is unitarily equivalent to H:

$$\begin{aligned}(\hat{H}F)(\varphi, x) &= \\ &= -\sum_{k=1}^{d} \big((\lambda_1 e^{-i\varphi_k} + \lambda_2) F(\varphi, x - e_k) + (\lambda_1 e^{i\varphi_k} + \lambda_2) F(\varphi, x + e_k) \big) + \\ &\qquad + 2d(\lambda_1 + \lambda_2) F(\varphi, x) + \mu \delta_{x,0} F(\varphi, x) \end{aligned} \tag{4.3}$$

where $F(\varphi, x) \in \hat{L}$.

Define the operator function $\varphi \longrightarrow \hat{H}(\varphi)$, $\varphi \in \mathbf{T}^d$, where $\hat{H}(\varphi) : l_2(\mathbf{Z}^d) \longrightarrow l_2(\mathbf{Z}^d)$, such that

$$(\hat{H}(\varphi)u)(x) = -\sum_{k=1}^{d} \big((\lambda_1 e^{-i\varphi_k} + \lambda_2)u(x - e_k) + (\lambda_1 e^{i\varphi_k} + \lambda_2)u(x + e_k)\big) \\ + 2d(\lambda_1 + \lambda_2)u(x) + \mu\delta_{x,0}u(x) \tag{4.4}$$

for the vector $u = \{u(x),\, x \in \mathbf{Z}^d\} \in l_2(\mathbf{Z}^d)$. Then $\hat{H}(\varphi)$ is measurable in the sense that the scalar product $(u_1, \hat{H}(\varphi)u_2)$ is measurable for any $u_1, u_2 \in l_2(\mathbf{Z}^d)$.

By (4.3) and (4.4) the operator $\hat{H}$ can be represented as

$$(\hat{H}\hat{F})(\varphi) = \hat{H}(\varphi)\hat{F}(\varphi) \tag{4.5}$$

where $\hat{F}(\varphi) \in \hat{L}$.

It follows from this representation that $\hat{H}$ can be decomposed in the direct integral

$$\hat{H} = \int_{\mathrm{T}^d}^{\oplus} \hat{H}(\varphi)\, d\varphi.$$

Consider now the spectrum of the operator $\hat{H}(\varphi) : l_2(\mathbf{Z}^d) \to l_2(\mathbf{Z}^d)$ for any fixed $\varphi \in \mathbf{T}^d$.

Let $\lambda_1, \lambda_2 > 0$. Consider the following nonnegative function of $\alpha \in (-\pi, \pi]$

$$r(\alpha) = \frac{\sqrt{\lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2\cos\alpha}}{\lambda_1 + \lambda_2} > 0.$$

Note that the upper bound in the inequality

$$(\lambda_1 - \lambda_2)^2 \le \lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2\cos\alpha \le (\lambda_1 + \lambda_2)^2$$

is attained for $\alpha = 0$, and the lower bound for $\alpha = \pi$.

Then $|\lambda_1 - \lambda_2|/(\lambda_1 + \lambda_2) \le r(\alpha) \le 1$ and $r(\alpha) = 0 \Longleftrightarrow \lambda_1 = \lambda_2 \,\&\, \alpha = \pi$; $r(\alpha) = 1 \Longleftrightarrow \alpha = 0$. Denote

$$c(d, \varphi) = \frac{1}{(2\pi)^d} \int_{\mathrm{T}^d} \frac{d\psi}{\sum_{k=1}^{d} r(\varphi_k)(1 - \cos\psi_k)} \tag{4.6}$$

where $d\psi = d\psi_1 \dots d\psi_d$ and $\varphi = (\varphi_1, \dots, \varphi_d) \in \mathbf{T}^d$. If $\varphi = \overrightarrow{\pi} = (\pi, \pi, \dots, \pi)$ ($\overrightarrow{\pi}$ is d-dimensional vector) and $\lambda_1 = \lambda_2$, then the denominator under the integral equals zero and thus $c(d, \overrightarrow{\pi}) = \infty$.

Let $r = \min_{l=1,\dots,d}\{r(\varphi_l) : r(\varphi_l) \ne 0\} > 0$. Denote $I(\varphi) = \{i_1, \dots, i_s\}$, where $1 \le i_1 < \dots < i_s \le d$ is an array of indices such that $r(\varphi_l) \ne 0 \Longleftrightarrow l \in I(\varphi)$. Here $s = s(\varphi) = \#\{r(\varphi_l), l = 1, \dots, d : r(\varphi_l) \ne 0\}$.

Then

$$\sum_{k=1}^{d} r(\varphi_k)(1-\cos\psi_k) = \sum_{k=1}^{s} r(\varphi_{i_k})(1-\cos\psi_{i_k}) > r\sum_{k=1}^{s}(1-\cos\psi_{i_k})$$

and

$$c(d,\varphi) = \frac{1}{(2\pi)^s}\int\limits_{\mathrm{T}^s}\frac{d\psi_{i_1}\dots d\psi_{i_s}}{\sum_{k=1}^{s} r(\varphi_{i_k})(1-\cos\psi_{i_k})} \leq$$

$$\leq \frac{1}{(2\pi)^s r}\int\limits_{\mathrm{T}^s}\frac{d\psi_{i_1}\dots d\psi_{i_s}}{\sum_{k=1}^{s}(1-\cos\psi_{i_k})} = c(s)$$

Similarly

$$c(d,\varphi) \geq \frac{1}{(2\pi)^s}\int\limits_{\mathrm{T}^s}\frac{d\psi_{i_1}\dots d\psi_{i_s}}{\sum_{k=1}^{s}(1-\cos\psi_{i_k})} = c(s)$$

as $0 \leq r(\varphi_l) \leq 1$.

Then

- for $d = 1, 2$ $c(d,\varphi)$ is divergent for all $\varphi \in \mathbf{T}^d$;
- for $d \geq 3$ the integral $c(d,\varphi)$ diverges iff $s(\varphi) \leq 2$.

The following theorem is a generalization of Theorem 2.1. Put

$$\beta_1(\varphi) = 2(\lambda_1+\lambda_2)\Bigl(-\sum_{k=1}^{d} r(\varphi_k) + d\Bigr),$$

$$\beta_2(\varphi) = 2(\lambda_1+\lambda_2)\Bigl(\sum_{k=1}^{d} r(\varphi_k) + d\Bigr).$$

Note that $0 \leq \beta_1(\varphi) \leq \beta_2(\varphi) \leq 4(\lambda_1+\lambda_2)\,d$ and

$$\beta_1(\varphi) = \beta_2(\varphi) = 2(\lambda_1+\lambda_2)\,d \Longleftrightarrow s(\varphi) = 0 \Longleftrightarrow \lambda_1 = \lambda_2 \,\&\, \varphi = \overrightarrow{\pi}$$

$$\beta_1(\varphi) = 0 \Longleftrightarrow \varphi = \overrightarrow{0}$$

$$\beta_2(\varphi) = 4(\lambda_1+\lambda_2)\,d \Longleftrightarrow \varphi = \overrightarrow{0}$$

where $\overrightarrow{0} \in \mathbf{T}^d$ is the vector consisting of zeros.

Theorem 4.1. *Let $\lambda_1, \lambda_2 > 0$. For all $\mu \in \mathbf{R}$, for all dimensions d and for all $\varphi \in \mathbf{T}^d$ we have $\sigma_{ess}(\hat{H}(\varphi)) = [\beta_1(\varphi),\, \beta_2(\varphi)]$.*

For $\mu = 0$ we have $\sigma_p(\hat{H}(\varphi)) = \emptyset$ for all $\varphi \in \mathbf{T}^d$.

Let $\mu \neq 0$.

For $d = 1, 2$ and for all $\varphi \in \mathbf{T}^d$ the point spectrum $\sigma_p(\hat{H}(\varphi))$ consists of exactly one eigenvalue $\nu = \nu(\varphi, \mu, \lambda_1, \lambda_2)$, where $\nu \notin \sigma_{ess}(\hat{H}(\varphi))$;

For $d = 3, 4$

- *if condition* $\{\lambda_1 = \lambda_2 \,\&\, s(\varphi) \le 2\}$ *holds, then for all* $\varphi \in \mathbf{T}^d$ *the point spectrum* $\sigma_p(\hat{H}(\varphi))$ *consists of exactly one eigenvalue* $\nu = \nu(\varphi, \mu, \lambda_1, \lambda_2)$, *where* $\nu \notin \sigma_{ess}(\hat{H}(\varphi))$;
- *if condition* $\{\lambda_1 \ne \lambda_2 \,\vee\, s(\varphi) \ge 3\}$ *holds, then*

  $\left|\frac{2(\lambda_1+\lambda_2)}{\mu}\right| < c(d,\varphi)$ *implies that* $\sigma_p(\hat{H}(\varphi))$ *consists of exactly one eigenvalue* $\nu = \nu(\varphi, \mu, \lambda_1, \lambda_2)$, *where* $\nu \notin \sigma_{ess}(\hat{H}(\varphi))$;

  $\left|\frac{2(\lambda_1+\lambda_2)}{\mu}\right| \ge c(d,\varphi)$ *implies that* $\sigma_p(\hat{H}(\varphi)) = \emptyset$.

For $d \ge 5$

- *if condition* $\{\lambda_1 = \lambda_2 \,\&\, s(\varphi) \le 2\}$ *holds, then for all* $\varphi \in \mathbf{T}^d$ *the point spectrum* $\sigma_p(\hat{H}(\varphi))$ *consists of exactly one eigenvalue* $\nu = \nu(\varphi, \mu, \lambda_1, \lambda_2)$, *where* $\nu \notin \sigma_{ess}(\hat{H}(\varphi))$;
- *if condition* $\{\lambda_1 = \lambda_2 \,\&\, s(\varphi) = 3, 4\}$ *holds, then*

  $\left|\frac{2(\lambda_1+\lambda_2)}{\mu}\right| < c(d,\varphi)$ *implies that* $\sigma_p(\hat{H}(\varphi))$ *consists of exactly one eigenvalue* $\nu = \nu(\varphi, \mu, \lambda_1, \lambda_2)$, *where* $\nu \notin \sigma_{ess}(\hat{H}(\varphi))$;

  $\left|\frac{2(\lambda_1+\lambda_2)}{\mu}\right| \ge c(d,\varphi)$ *implies that* $\sigma_p(\hat{H}(\varphi)) = \emptyset$.
- *if condition* $\{\lambda_1 \ne \lambda_2 \,\vee\, s(\varphi) \ge 5\}$ *holds, then*

  $\left|\frac{2(\lambda_1+\lambda_2)}{\mu}\right| < c(d,\varphi)$ *implies that for all* $\varphi \in \mathbf{T}^d$ *the point spectrum* $\sigma_p(\hat{H}(\varphi))$ *consists of exactly one eigenvalue* $\nu = \nu(\varphi, \mu, \lambda_1, \lambda_2)$, *where* $\nu \notin \sigma_{ess}(\hat{H}(\varphi))$;

  $\frac{2(\lambda_1+\lambda_2)}{\mu} = c(d,\varphi)$ *implies that* $\sigma_p(\hat{H}(\varphi))$ *consists of exactly one eigenvalue* $\nu = \beta_2(\varphi)$;

  $-\frac{2(\lambda_1+\lambda_2)}{\mu} = c(d,\varphi)$ *implies that the point spectrum consists of exactly one eigenvalue* $\nu = \beta_1(\varphi)$;

  $\left|\frac{2(\lambda_1+\lambda_2)}{\mu}\right| > c(d,\varphi)$ *implies that* $\sigma_p(\hat{H}(\varphi)) = \emptyset$.

In all cases the eigenvalue $\nu \ge \beta_2(\varphi)$, *if* $\mu > 0$ *and* $\nu \le \beta_1(\varphi)$, *if* $\mu > 0$.

Remark 4.1. The eigenvalue $\nu = \nu(\varphi, \mu, \lambda_1, \lambda_2)$ is the unique solution of the equation

$$\frac{2(\lambda_1+\lambda_2)}{\mu} = \frac{1}{(2\pi)^d} \int\limits_{\mathrm{T}^d} \frac{d\psi}{\sum_{k=1}^d r(\varphi_k)\cos\psi_k - d + \frac{\nu}{2(\lambda_1+\lambda_2)}}, \tag{4.7}$$

where $d\psi = d\psi_1 \dots d\psi_d$.

Remark 4.2. If $\varphi = \vec{0}$ then the Hamiltonian $\hat{H}(\vec{0})$, defined in (4.4), coincides with the Hamiltonian H, defined in (2.1)–(2.3) if $\lambda = \lambda_1 + \lambda_2$. Thus, theorem 2.1 follows from theorem 4.1.

Since

$$0 \le \sum_{k=1}^{d} r(\varphi_k)(1-\cos\psi_k) \le \sum_{k=1}^{d}(1-\cos\psi_k),$$

we have

$$\begin{aligned} c(d,\varphi) &= \frac{1}{(2\pi)^d}\int\limits_{\mathrm{T}^d} \frac{d\psi}{\sum_{k=1}^{d} r(\varphi_k)(1-\cos\psi_k)} \ge \\ &\ge \frac{1}{(2\pi)^d}\int\limits_{\mathrm{T}^d} \frac{d\psi}{\sum_{k=1}^{d}(1-\cos\psi_k)} = c(d) \end{aligned}$$

and $c(d,\varphi) = c(d)$ iff $\varphi = \overrightarrow{0}$.

Thus for any $\varphi \in \mathbf{T}^d$ the operator $\hat{H}(\varphi)$ has the only eigenvalue $\nu = \nu(\varphi,\mu,\lambda_1,\lambda_2)$ iff one of the following conditions holds:

- $d = 1,2$ and $\mu \neq 0$
- $d = 3,4$, $\mu \neq 0$ and $\left|\frac{2(\lambda_1+\lambda_2)}{\mu}\right| < c(d)$
- $d \ge 5$, $\mu \neq 0$ and $\left|\frac{2(\lambda_1+\lambda_2)}{\mu}\right| \le c(d)$.

From implicit function theorem it follows that $\nu = \nu(\varphi,\mu,\lambda_1,\lambda_2)$ is a continuous function of $\varphi \in \mathbf{T}^d$.

Now we give the following fundamental definition. A linear subspace $L_1 \subset L$ is called one-particle subspace if

- L_1 is invariant with respect to the translation group U_s and with respect to dynamics e^{itH},
- there exists vector $g_0 \in L$ such that L_1 is generated by the vectors $\{U_s g_0, s \in \mathbf{Z}^d\}$.

Theorem 4.2. *Let* $\mu \neq 0$. *Then:*

For $d = 1,2$ *there always exists unique one-particle subspace.*

For $d = 3,4$ *one-particle space exists iff* $\left|\frac{2(\lambda_1+\lambda_2)}{\mu}\right| < c(d)$. *Then it is unique.*

For $d \ge 5$ *one-particle subspace exists iff* $\left|\frac{2(\lambda_1+\lambda_2)}{\mu}\right| \le c(d)$. *Then it is unique.*

Let $x_1 = (x_1^1,\dots,x_1^d) \in \mathbf{Z}^d$, $x = (x^1,\dots,x^d) \in \mathbf{Z}^d$.

Remark 4.3. One-particle subspace is generated by the vectors $\{U_s g_0, s \in \mathbf{Z}^d\}$, where

$$g_0(x_1, x) = \frac{1}{(2\pi)^d} \int\limits_{\mathrm{T}^{2d}} \frac{\frac{\mu}{2(\lambda_1+\lambda_2)} \cos x_1^1 \varphi_1 \dots \cos x_1^d \varphi_d \cos x^1 \psi_1 \dots \cos x^d \psi_d}{\sum_{k=1}^d r(\varphi_k) \cos \psi_k - d + \frac{\nu(\varphi)}{2(\lambda_1+\lambda_2)}} \, d\varphi \, d\psi$$

belongs to $l_2(\mathbf{Z}^{2d})$, and the function $\nu(\varphi)$ is defined as the unique solution of the equation (4.7).

5. Proofs

5.1. Proof of Theorem 2.1

Consider the unitary transformation $U : l_2(\mathbf{Z}^d) \to L_2(\mathbf{T}^d)$ defined by the one-to-one correspondence between elements of the orthonormal basis δ_x, $x = (x_1, \dots, x_d) \in \mathbf{Z}^d$ in $l_2(\mathbf{Z}^d)$ and the elements of the orthonormal basis

$$\frac{1}{(2\pi)^{d/2}} e^{i\varphi_1 x_1} \dots e^{i\varphi_d x_d}, \ (x_1, \dots, x_d) \in \mathbf{Z}^d,$$

in $L_2(\mathbf{T}^d)$. That is $Uf = F$ where

$$f = \sum_{x \in \mathbf{Z}^d} f(x) \delta_x \in l_2(\mathbf{Z}^d), F = \sum_{x \in \mathbf{Z}^d} f(x) \frac{1}{(2\pi)^{\frac{d}{2}}} e^{i\varphi_1 x_1} \dots e^{i\varphi_d x_d} \in L_2(\mathbf{T}^d).$$

In these terms $\hat{H} = UHU^{-1} : L_2(\mathbf{T}^d) \to L_2(\mathbf{T}^d)$ can be written as follows

$$\hat{H}F = -\lambda(e^{i\varphi_1} + e^{-i\varphi_1} + \dots + e^{i\varphi_d} + e^{-i\varphi_d} - 2d)F + \mu f(0) \frac{1}{(2\pi)^{d/2}}.$$

If for some μ there exists eigenvalue ν, then the corresponding eigenfunction F satisfies the equation

$$-\lambda(e^{i\varphi_1} + e^{-i\varphi_1} + \dots + e^{i\varphi_d} + e^{-i\varphi_d} - 2d)F + \mu f(0) \frac{1}{(2\pi)^{\frac{d}{2}}} = \nu F,$$

whence

$$\begin{aligned} F &= \frac{1}{(2\pi)^{d/2}} \frac{\frac{\mu}{2\lambda} f(0)}{\frac{e^{i\varphi_1}+e^{-i\varphi_1}}{2} + \dots + \frac{e^{i\varphi_d}+e^{-i\varphi_d}}{2} - d + \frac{\nu}{2\lambda}} = \\ &= \frac{1}{(2\pi)^{d/2}} \frac{\frac{\mu}{2\lambda} f(0)}{\gamma(\varphi) - d + \frac{\nu}{2\lambda}}. \end{aligned} \tag{5.1}$$

For $\mu = 0$ from (5.1) it follows that $F \equiv 0$. It means that for $\mu = 0$ there are no eigenvalues.

Note that if $\nu \notin [0, 4\lambda d]$, then the denominator in (5.1) is not zero, and the function F belongs to $L_2(\mathbf{T}^d)$. As it is shown in Lemma A.1 (see section A below), for $d \leq 4$ $F \notin L_2(\mathbf{T}^d)$ if $\nu \in [0, 4\lambda d]$; and for $d \geq 5$ $F \notin L_2(\mathbf{T}^d)$ if $\nu \in (0, 4\lambda d)$.

It follows that in dimension $d \leq 4$ there are no eigenvalues on the segment $[0, 4\lambda d]$. Similarly, in dimension $d \geq 5$ there are no eigenvalues on the interval $(0, 4\lambda d)$.

Let $\mu \neq 0$ and $\nu \notin (0, 4\lambda d)$. If we expand both sides of the equality (5.1) in the basis $(2\pi)^{-d/2} \exp\{i\varphi_1 n_1\} \ldots \exp\{i\varphi_d n_d\}$, then all coefficients of both parts should coincide. In particular,

$$f(0) = \frac{1}{(2\pi)^d} \int\limits_{\mathrm{T}^d} \frac{\frac{\mu}{2\lambda} f(0)}{\gamma(\varphi) - d + \frac{\nu}{2\lambda}} d\varphi.$$

Note that as F is not identically zero, then $f(0) \neq 0$. We get then the equation on ν:

$$\frac{2\lambda}{\mu} = \frac{1}{(2\pi)^d} \int\limits_{\mathrm{T}^d} \frac{d\varphi}{\gamma(\varphi) - d + \frac{\nu}{2\lambda}}. \tag{5.2}$$

Consider the case $d = 1, 2$. Put

$$p\,(\nu) = \frac{1}{(2\pi)^d} \int\limits_{\mathrm{T}^d} \frac{d\varphi}{\gamma(\varphi) - d + \frac{\nu}{2\lambda}}$$

and $\varphi \in T^d$. For $\nu > 4\lambda d$ the integrand

$$\frac{1}{\gamma(\varphi) - d + \frac{\nu}{2\lambda}} > 0$$

is strictly decreasing (if ν increases) and tends to 0 as $\nu \to +\infty$. It follows that the function $p\,(\nu)$ is also strictly decreasing and $p\,(\nu) \to 0$ as $\nu \to +\infty$. As it was mentioned above, for $\nu = 4\lambda d$

$$p\,(4\lambda d) = \frac{1}{(2\pi)^d} \int\limits_{\mathrm{T}^d} \frac{d\varphi}{\gamma(\varphi) + d} = +\infty.$$

Thus, the function $p\,(\nu)$ strictly decreases from $+\infty$ to 0 for $\nu > 4\lambda d$. Then for $\mu > 0$ the equation (5.2) has a unique solution if $\nu > 4\lambda d$.

If $\nu = 0$ then $p\,(0) = -\infty$. Similarly, we can get that, if $\nu < 0$, $p\,(\nu)$ strictly increases from $-\infty$ to zero (as ν decreases to $-\infty$). That is why for $\mu < 0$ the equation (5.2) has unique solution for any $\nu < 0$.

It follows that the function $p\,(\nu)$ takes all values except zero, moreover exactly once. Then for any μ there exists exactly one ν, such that the equation

(5.2) holds and there exists unique (up to multiplicative constant) eigenfunction $F \in L_2(\mathbf{T}^d)$, defined by (5.1), and such that $\hat{H}F = \nu F$.

Thus, there exists unique eigenvalue ν such that $\nu \notin [0, 4\lambda d]$. Moreover, $\nu < 0$ for $\mu < 0$, and $\nu > 4\lambda d$ for $\mu > 0$.

Let now $d \geq 3$. In this case for $\nu = 4\lambda d$ and for $\nu = 0$ the integrals

$$p\,(4\lambda d) = c(d) = \frac{1}{(2\pi)^d} \int\limits_{\mathrm{T}^d} \frac{d\varphi}{\gamma(\varphi) + d} > 0$$

$$p\,(0) = -c(d) = \frac{1}{(2\pi)^d} \int\limits_{\mathrm{T}^d} \frac{d\varphi}{\gamma(\varphi) - d} < 0$$

are finite.

Similarly to above, we come to the conclusion that for $\nu \geq 4\lambda d$ the function $p\,(\nu)$ is strongly decreasing and $p\,(\nu) \to 0$ as $\nu \to +\infty$, and for $\nu \leq 0$, if ν decreases, the function $p\,(\nu)$ strongly increases and $p\,(\nu) \to 0$ as $\nu \to -\infty$. Thus, the function $p\,(\nu)$ takes all values except 0 in the segment $[-c(d), c(d)]$, moreover each value only once. It follows that for any μ such that $\left|\frac{2\lambda}{\mu}\right| \leq c(d)$ there is exactly one ν such that the equation (5.2) holds. For $\left|\frac{2\lambda}{\mu}\right| > c(d)$ the equation (5.2) does not have solutions.

If μ satisfies the condition $\left|\frac{2\lambda}{\mu}\right| < c(d)$, then the solution ν of the equation (5.2) satisfies condition $\nu \notin [0, 4\lambda d]$, and the function F in the formula (5.1) always belongs to $L_2(\mathbf{T}^d)$. Thus, for such μ there exists unique eigenvalue ν.

If $\left|\frac{2\lambda}{\mu}\right| = c(d)$, then the solution of the equation (5.2) will be the following: $\nu = 0$ for $\frac{2\lambda}{\mu} = -c(d)$ and $\nu = 4\lambda d$ for $\frac{2\lambda}{\mu} = c(d)$.

As it follows from lemma A.1, for $\nu = 0, 4\lambda d$ the function F in the formula (5.1) belongs to $L_2(\mathbf{T}^d)$ only in dimension $d \geq 5$. Whence, for μ satisfying the condition $\left|\frac{2\lambda}{\mu}\right| = c(d)$ in dimension $d \geq 5$ there is unique eigenvalue.

5.2. Proof of Theorem 4.1.

Fix some $\varphi \in \mathbf{T}^d$ and let $\nu(\varphi)$ be an eigenvalue of the operator $\hat{H}(\varphi)$. Then

$$\hat{H}(\varphi)\hat{F}_0(\varphi) = \nu(\varphi)\hat{F}_0(\varphi)$$

holds where $\hat{F}_0(\varphi) = \{F_0(\varphi, x)), \, x \in \mathbf{Z}^d\} \in l_2(\mathbf{Z}^d)$ is an eigenvector corresponding to $\nu(\varphi)$.

By (4.3) and (4.5) we get

$$-\sum_{k=1}^{d} \big((\lambda_1 e^{-i\varphi_k} + \lambda_2)F_0(\varphi, x - e_k) + (\lambda_1 e^{i\varphi_k} + \lambda_2)F_0(\varphi, x + e_k)\big) + \qquad (5.3)$$

$$+ 2d(\lambda_1 + \lambda_2)F_0(\varphi, x) + \mu\delta_{x,0}F_0(\varphi, x) = \nu(\varphi)F_0(\varphi, x),\ x \in \mathbf{Z}^d.$$

For any $\varphi \in T^d$ consider the unitary operator $\mathcal{G} :\ l_2(\mathbf{Z}^d) \longrightarrow L_2(\mathbf{T}^d)$ such that

$$\mathcal{G} :\ F(\varphi, x) \longrightarrow G(\varphi, \psi) = \frac{1}{(2\pi)^{d/2}} \sum_{x\in\mathbf{Z}} F(\varphi, x)e^{i(x,\psi)},\ \psi = (\psi_1, \ldots, \psi_d) \in \mathbf{T}^d.$$

The inverse operator coincides with the adjoint and looks as

$$\mathcal{G}^* :\ G(\varphi, \psi) \longrightarrow F(\varphi, x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathrm{T}^d} G(\varphi, \psi)e^{-i(x,\psi)}d\psi$$

where $d\psi = d\psi_1 \ldots d\psi_d$. In particular, for $x = 0$

$$F(\varphi, 0) = \frac{1}{(2\pi)^{d/2}} \int_{\mathrm{T}^d} G(\varphi, \psi)d\psi.$$

For any $\varphi \in \mathbf{T}^d$ the operator $\mathcal{G}\hat{H}\mathcal{G}^*$ acts in $L_2(\mathbf{T}^d)$ as follows:

$$\begin{aligned}
&\big(\mathcal{G}\hat{H}\mathcal{G}^* G\big)(\varphi, \psi) = \\
&= -\Bigg(\sum_{k=1}^{d} \big((\lambda_1 e^{-i\varphi_k} + \lambda_2)e^{i\psi_k} + (\lambda_1 e^{i\varphi_k} + \lambda_2)e^{-i\psi_k} - 2(\lambda_1 + \lambda_2)\big)\Bigg)G(\varphi, \psi) + \\
&\quad + \frac{\mu}{(2\pi)^d} \int_{\mathrm{T}^d} G(\varphi, \psi)d\psi = \\
&= -2\Bigg(\sum_{k=1}^{d} (\lambda_1 \cos(\psi_k - \varphi_k) + \lambda_2 \cos\psi_k - \lambda_1 - \lambda_2)\Bigg)G(\varphi, \psi) + \\
&\quad + \frac{\mu}{(2\pi)^d} \int_{\mathrm{T}^d} G(\varphi, \psi)d\psi.
\end{aligned}$$

Then the system of equations (5.3) in $L_2(\mathbf{T}^d)$ can be reduced to one equation

$$\begin{aligned}
&- 2\Bigg(\sum_{k=1}^{d} \lambda_1 \cos(\psi_k - \varphi_k) + \lambda_2 \cos\psi_k - \lambda_1 - \lambda_2\Bigg)G_0(\varphi, \psi) + \frac{\mu}{(2\pi)^d} \int_{\mathrm{T}^d} G_0(\varphi, \psi)d\psi \\
&= \nu(\varphi)G_0(\varphi, \psi)
\end{aligned}$$

where

$$G_0(\varphi, \psi) = \frac{1}{(2\pi)^{d/2}} \sum_{x\in\mathbf{Z}^d} F_0(\varphi, x)e^{i(x,\psi)}.$$

Then

$$G_0(\varphi,\psi)=\frac{\frac{\mu}{2(2\pi)^d}\int_{\mathrm{T}^d}G_0(\varphi,\psi)d\psi}{\sum_{k=1}^d\left(\lambda_1\cos(\psi_k-\varphi_k)+\lambda_2\cos\psi_k-\lambda_1-\lambda_2\right)+\nu/2}. \tag{5.4}$$

As

$$\begin{aligned}\lambda_1\cos(\psi_k-\varphi_k)+\lambda_2\cos\psi_k &=\\ &=\lambda_1(\cos\varphi_k\cos\psi_k+\sin\varphi_k\sin\psi_k)+\lambda_2\cos\psi_k=\\ &=(\lambda_1\cos\varphi_k+\lambda_2)\cos\psi_k+\lambda_1\sin\varphi_k\sin\psi_k,\end{aligned}$$

we have

$$\lambda_1\cos(\psi_k-\varphi_k)+\lambda_2\cos\psi_k=r'(\varphi_k)\cos(\psi_k-\eta(\varphi_k))$$

where

$$r'(\varphi_k)=\sqrt{(\lambda_1\cos\varphi_k+\lambda_2)^2+(\lambda_1\sin\varphi_k)^2}=\sqrt{\lambda_1^2+\lambda_2^2+2\lambda_1\lambda_2\cos\varphi_k}$$

$$\eta(\varphi_k)=\arccos\Big(\frac{\lambda_1\cos\varphi_k+\lambda_2}{p(\varphi_k)}\Big).$$

From (5.4) we get

$$G_0(\varphi,\psi)=\frac{\frac{\mu}{2(\lambda_1+\lambda_2)(2\pi)^d}\int_{\mathrm{T}^d}G_0(\varphi,\psi)d\psi}{\sum_{k=1}^d r(\varphi_k)\cos(\psi_k-\eta(\varphi_k))-d+\frac{\nu}{2(\lambda_1+\lambda_2)}}, \tag{5.5}$$

where

$$r(\varphi_k)=\frac{r'(\varphi_k)}{\lambda_1+\lambda_2}>0.$$

For $\mu=0$ from (5.5) it follows that $G_0(\varphi,\psi)\equiv 0$. It follows that for $\mu=0$ there are no eigenvalues.

Using the evident inequality

$$-\sum_{k=1}^d r(\varphi_k)\le\sum_{k=1}^d r(\varphi_k)\cos(\psi_k-\eta(\varphi_k))\le\sum_{k=1}^d r(\varphi_k),$$

the denominator in (5.5) is never zero for $\nu\notin[\beta_1(\varphi),\beta_2(\varphi)]$, and the function $G_0(\varphi,\psi)$ evidently belongs to $L_2(\mathbf{T}^d)$. As it is shown in Lemma A.1 (see section A below), $G_0(\varphi,\psi)\notin L_2(\mathbf{T}^d)$ for $d\le 4$ and $\nu\in[\beta_1(\varphi),\beta_2(\varphi)]$; and $G_0(\varphi,\psi)\notin L_2(\mathbf{T}^d)$ when $d\ge 5$ and $\nu\in(\beta_1(\varphi),\beta_2(\varphi))$.

Thus, in dimension $d\le 4$ there are no eigenvalues $\nu\in[\beta_1(\varphi),\beta_2(\varphi)]$, and in dimension $d\ge 5$ there are no eigenvalues such that $\nu\in(\beta_1(\varphi),\beta_2(\varphi))$.

Let now $\nu\notin[\beta_1(\varphi),\beta_2(\varphi)]$. Integrate both parts of the equality (5.5) in the vector variable ψ:

$$\int\limits_{T^d} G_0(\varphi,\psi)\,d\psi = \int\limits_{T^d} G_0(\varphi,\psi)\,d\psi \times$$

$$\times \frac{1}{(2\pi)^d}\int\limits_{T^d} \frac{\frac{\mu}{2(\lambda_1+\lambda_2)}\,d\psi}{\sum_{k=1}^d r(\varphi_k)\cos(\psi_k-\eta(\varphi_k)) - d + \frac{\nu}{2(\lambda_1+\lambda_2)}}.$$

Note that $\int_{T^d} G_0(\varphi,\psi)\,d\psi \neq 0$, otherwise, by (5.5), we had $G_0(\varphi,\psi)\equiv 0$.

After cancellation we get the following equation for ν:

$$1 = \frac{1}{(2\pi)^d}\int\limits_{T^d} \frac{\frac{\mu}{2(\lambda_1+\lambda_2)}\,d\psi}{\sum_{k=1}^d r(\varphi_k)\cos(\psi_k-\eta(\varphi_k)) - d + \frac{\nu}{2(\lambda_1+\lambda_2)}}.$$

From periodicity of the integrand in each variable ψ_k it follows:

$$(2\pi)^d = \int\limits_{-\pi}^{\pi}\cdots\int\limits_{-\pi}^{\pi} \frac{\frac{\mu}{2(\lambda_1+\lambda_2)}\,d\psi_1\ldots d\psi_d}{\sum_{k=1}^d r(\varphi_k)\cos(\psi_k-\eta(\varphi_k)) - d + \frac{\nu}{2(\lambda_1+\lambda_2)}} =$$

$$= \int\limits_{-\pi-\eta(\varphi_1)}^{\pi-\eta(\varphi_1)}\cdots\int\limits_{-\pi-\eta(\varphi_d)}^{\pi-\eta(\varphi_d)} \frac{\frac{\mu}{2(\lambda_1+\lambda_2)}d\psi_1\ldots d\psi_d}{\sum_{k=1}^d r(\varphi_k)\cos\psi_k - d + \frac{\nu}{2(\lambda_1+\lambda_2)}} =$$

$$= \int\limits_{-\pi}^{\pi}\cdots\int\limits_{-\pi}^{\pi} \frac{\frac{\mu}{2(\lambda_1+\lambda_2)}d\psi_1\ldots d\psi_n}{\sum_{k=1}^d r(\varphi_k)\cos\psi_k - d + \frac{\nu}{2(\lambda_1+\lambda_2)}} \tag{5.6}$$

That is why $\nu(\varphi)$ satisfies the equation

$$\frac{2(\lambda_1+\lambda_2)}{\mu} = \frac{1}{(2\pi)^d}\int\limits_{T^d} \frac{d\psi}{\sum_{k=1}^d r(\varphi_k)\cos\psi_k - d + \frac{\nu}{2(\lambda_1+\lambda_2)}} \tag{5.7}$$

Let us study this equation. Denote

$$q(\nu,\varphi) = (2\pi)^{-d}\int\limits_{T^d} \frac{d\psi}{\sum_{k=1}^d r(\varphi_k)\cos\psi_k - d + \frac{\nu}{2(\lambda_1+\lambda_2)}}$$

For any fixed φ the function $q(\nu,\varphi)$ is defined for $\nu \notin (\beta_1(\varphi);\beta_2(\varphi))$, and at the end points of this interval it takes the values

$$q(\beta_2(\varphi),\varphi) = c(d,\varphi),\ q(\beta_1(\varphi),\varphi) = -c(d,\varphi).$$

In fact, for $\nu = \beta_2(\varphi)$ we get

$$q(\beta_2(\varphi),\varphi) = \int\limits_{\mathrm{T}^d} \frac{d\psi}{\sum_{k=1}^d r(\varphi_k)(\cos\psi_k + 1)} =$$
$$= \int\limits_{\mathrm{T}^d} \frac{d\psi}{\sum_{k=1}^d r(\varphi_k)(-\cos(\psi_k+\pi) + 1)} =$$
$$\int\limits_{\mathrm{T}^d} \frac{d\psi}{\sum_{k=1}^d r(\varphi_k)(-\cos\psi_k + 1)} = c(d,\varphi)$$

Similarly, for $\nu = \beta_1(\varphi)$

$$q(\beta_1(\varphi),\varphi) = \int\limits_{\mathrm{T}^d} \frac{d\psi}{\sum_{k=1}^d r(\varphi_k)(\cos\psi_k - 1)} = -c(d,\varphi).$$

For fixed $\varphi \in \mathbf{T}^d$ the function $q(\nu,\varphi)$ is strictly decreasing in ν, when $\nu \notin [\beta_1(\varphi);\beta_2(\varphi)]$. For $\nu > \beta_2(\varphi)$ the function $q(\nu,\varphi)$ is positive and tends to 0 as $\nu \to +\infty$. For $\nu < \beta_1(\varphi)$ the function $q(\nu,\varphi)$ is negative and tends to 0 when $\nu \to -\infty$.

Thus, if $c(d,\varphi) = +\infty$, then the function $q(\nu,\varphi)$ takes all real values, except 0, and moreover only once, due to strict monotonicity of the function $q(\nu,\varphi)$ in ν for any fixed $\varphi \in \mathbf{T}^d$. That is why the equation (5.7) has a unique solution $\nu(\varphi)$.

If $c(d,\varphi)$ is finite, then the function $q(\nu,\varphi)$ takes all values, except 0, from the finite interval $[-c(d,\varphi), c(d,\varphi)]$.

It follows that there exists exactly one eigenvalue $\nu(\varphi)$ for any $\varphi \in \mathbf{T}^d$ iff the left hand part of the equation (5.7) belongs to this interval, that is if $|2(\lambda_1+\lambda_2)\mu^{-1}| \le c(d,\varphi)$.

Also it is true that for $\mu > 0$ the eigenvalue $\nu(\varphi) \ge \beta_2(\varphi)$, and for $\mu < 0$ we have $\nu(\varphi) \le \beta_1(\varphi)$.

It remains to check that the eigenfunction, corresponding to $\nu(\varphi)$, belongs to $L_2(\mathbf{T}^d)$. If the strict inequality $|2(\lambda_1+\lambda_2)\mu^{-1}| < c(d,\varphi)$ holds, then $\nu(\varphi) \notin [\beta_1(\varphi);\beta_2(\varphi)]$ and the eigenfunction G_0, defined in (5.5), belongs to $L_2(\mathbf{T}^d)$, as the denominator in (5.5) cannot vanish. If $|2(\lambda_1+\lambda_2)\mu^{-1}| = c(d,\varphi) < \infty$, then $\nu(\varphi) = \beta_1(\varphi), \beta_2(\varphi)$ and the eigenfunction G_0 belongs to $L_2(\mathbf{T}^d)$ only in dimension $d \ge 5$.

The theorem is proved.

5.3. Proof of Theorem 4.2

Existence of one particle subspace Consider the function $\varphi \to \hat{F}_0(\varphi) = \{F_0(\varphi,x),\, x \in \mathbf{Z}^d\} \in l_2(\mathbf{Z})$, where $\hat{F}_0(\varphi)$ is the eigenvector of the operator

$\hat{H}(\varphi)$, corresponding to the eigenvalue $\nu(\varphi)$. By (5.5),

$$G_0(\varphi,\psi)=\frac{\frac{\mu}{2(\lambda_1+\lambda_2)(2\pi)^{d/2}}F_0(\varphi,0)}{\sum_{k=1}^d r(\varphi_k)\cos(\psi_k-\eta(\varphi_k))-d+\frac{\nu}{2(\lambda_1+\lambda_2)}}$$

as

$$F_0(\varphi,0)=\frac{1}{(2\pi)^{d/2}}\int\limits_{\mathrm{T}^d}G_0(\varphi,\psi)d\psi.$$

Let us find the components $F_0(\varphi,x), x\in\mathbf{Z}^d$, of the eigenvector $\hat{F}_0(\varphi)$, by applying the operator $\mathcal{G}^{-1}$ to the function $G_0(\varphi,\psi)$:

$$\begin{aligned} F_0(\varphi,x) &= \frac{1}{(2\pi)^d}\int\limits_{\mathrm{T}^d}\frac{\frac{\mu}{2(\lambda_1+\lambda_2)}F_0(\varphi,0)e^{-i(x,\psi)}}{\sum_{k=1}^d r(\varphi_k)\cos(\psi_k-\eta(\varphi_k))-d+\frac{\nu}{2(\lambda_1+\lambda_2)}}d\psi \quad (5.8)\\ &= \frac{1}{(2\pi)^d}\int\limits_{\mathrm{T}^d}\frac{\frac{\mu}{2(\lambda_1+\lambda_2)}F_0(\varphi,0)e^{-i(x,\psi)}}{\sum_{k=1}^d r(\varphi_k)\cos\psi_k-d+\frac{\nu}{2(\lambda_1+\lambda_2)}}d\psi \quad (5.9)\\ &= \frac{1}{(2\pi)^d}\int\limits_{\mathrm{T}^d}\frac{\frac{\mu}{2(\lambda_1+\lambda_2)}F_0(\varphi,0)\cos x^1\psi_1\ldots\cos x^d\psi_d}{\sum_{k=1}^d r(\varphi_k)\cos\psi_k-d+\frac{\nu}{2(\lambda_1+\lambda_2)}}d\psi \quad (5.10)\end{aligned}$$

where x^k are the coordinates of the vector $x=(x^1,\ldots,x^d)\in\mathbf{Z}^d$. The equality (5.9) can be deduced similarly to (5.6). The equality (5.10) holds as

$$\int\limits_{\mathrm{T}^d}\frac{\sin x^k\psi_k}{\sum_{k=1}^d r(\varphi_k)\cos\psi_k-d+\frac{\nu}{2(\lambda_1+\lambda_2)}}d\psi=0$$

because the integrand is odd in the variable ψ_k.

Denote

$$K(\varphi,x)=\frac{1}{(2\pi)^d}\int\limits_{\mathrm{T}^d}\frac{\frac{\mu}{2(\lambda_1+\lambda_2)}\cos x^1\psi_1\ldots\cos x^d\psi_d}{\sum_{k=1}^d r(\varphi_k)\cos\psi_k-d+\frac{\nu}{2(\lambda_1+\lambda_2)}}d\psi$$

$$\hat{K}(\varphi)=\{K(\varphi,x),\,x\in\mathbf{Z}^d\}\in\hat{L}. \quad (5.11)$$

Then by (5.10) we have

$$F_0(\varphi,x)=F_0(\varphi,0)K(\varphi,x)\Longleftrightarrow\hat{F}_0(\varphi)=F_0(\varphi,0)\hat{K}(\varphi). \quad (5.12)$$

Introduce the linear subspace $\hat{L}_1$ of the space $\hat{L}$ of functions of two variables $\hat{L}_1=\{F(\varphi)K(\varphi,x),\,F(\varphi)\in L_2(\mathbf{T}^d)\}\subset\hat{L}$. This subspace is isomorphic to

$L_2(\mathbf{T}^d)$ and is invariant with respect to the Hamiltonian $\hat{H}$, where $\hat{H}$ acts in $\hat{L}_1$ as the multiplication on the function $\nu(\varphi)$:

$$\hat{H} : F(\varphi)K(\varphi, x) \longrightarrow \nu(\varphi)F(\varphi)K(\varphi, x).$$

Put $g_0 = \mathcal{F}^* K \in L$ $L_1 = \mathcal{F}^* \hat{L}_1$. Then L_1 is one-particle subspace, according to the definition above, and L_1 is generated by the vectors $\{g_s = U_s g_0, s \in \mathbf{Z}^d\}$.

In fact

$$\mathcal{F}^* : F(\varphi)K(\varphi, x) \longrightarrow \sum_{s\in\mathbf{Z}^d} f(s)g_0(x_1 - s, x) = \sum_{s\in\mathbf{Z}^d} f(s)U_s g_0(x_1, x) \in l_2(\mathbf{Z}^{2d})$$

where

$$f(s) = \frac{1}{(2\pi)^{d/2}} \int\limits_{\mathrm{T}^d} F(\varphi)e^{-i(s,\varphi)}\, d\varphi.$$

By (4.2) and (5.11) we have

$$g_0(x_1, x) = (\mathcal{F}^* K)(x_1, x) = \frac{1}{(2\pi)^{d/2}} \int\limits_{\mathrm{T}^d} K(\varphi, x)e^{-i(x_1,\varphi)}\, d\varphi =$$

$$= \frac{1}{(2\pi)^{3d/2}} \int\limits_{\mathrm{T}^{2d}} \frac{\frac{\mu}{2(\lambda_1+\lambda_2)} e^{-i(x_1,\varphi)} \cos x^1\psi_1 \ldots \cos x^d\psi_d}{\sum_{k=1}^d r(\varphi_k)\cos\psi_k - d + \frac{\nu}{2(\lambda_1+\lambda_2)}}\, d\varphi\, d\psi$$

As the integrand is periodic and odd with respect to each variable ψ_k

$$\int\limits_{\mathrm{T}^d} \frac{\frac{\mu}{2(\lambda_1+\lambda_2)} \sin x^k\varphi_k}{\sum_{k=1}^d r(\varphi_k)\cos\psi_k - d + \frac{\nu}{2(\lambda_1+\lambda_2)}}\, d\varphi = 0$$

Then

$$g_0(x_1, x) =$$

$$= \frac{1}{(2\pi)^{3d/2}} \int\limits_{\mathrm{T}^{2d}} \frac{\frac{\mu}{2(\lambda_1+\lambda_2)} \cos x_1^1\varphi_1 \ldots \cos x_1^d\varphi_d \cos x^1\psi_1 \ldots \cos x^d\psi_d}{\sum_{k=1}^d r(\varphi_k)\cos\psi_k - d + \frac{\nu}{2(\lambda_1+\lambda_2)}}\, d\varphi\, d\psi.$$

Unicity of the one particle subspace Assume the contrary: that there exists another one-particle subspace $L_1' = \{g_s' = U_s g_0', s \in \mathbf{Z}^d\}$. Put $\hat{L}_1' = \mathcal{F}L_1'$. This subspace is generated by the vectors $F_s' = \mathcal{F}g_s' = \mathcal{F}U_s g_0' = e^{i(s,\varphi)}\mathcal{F}g_0'$, where $s \in \mathbf{Z}^d$. It follows that $\hat{L}_1'$ looks like $\hat{L}_1' = \{F(\varphi)K'(\varphi, x), F(\varphi) \in L_2(\mathbf{T}^d)\}$, where $K' = \mathcal{F}g_0'$, and the function K' cannot be presented as

$$K'(\varphi, x) = \nu'(\varphi)K(\varphi, x)$$

for some function $\nu' \in L_2(\mathbf{T}^d)$, where K is defined in (5.11). Due to invariance of $\hat{L}_1'$ with respect to the Hamiltonian $\hat{H}$ we get that for some function $\nu_1(\varphi) \in L_2(\mathbf{T}^d)$ the following holds:

$$(\hat{H}K')(\varphi, x) = \nu_1(\varphi)K'(\varphi, x).$$

By (4.5) this equality is equivalent to

$$\hat{H}(\varphi)\hat{K}'(\varphi) = \nu_1(\varphi)\hat{K}'(\varphi)$$

where $\hat{K}'(\varphi) = \{K'(\varphi, x), x \in \mathbf{Z}^d\}$. Remind that $\nu(\varphi)$ is the unique eigenvalue of the operator $\hat{H}(\varphi)$, as it was shown above. It follows that $\nu_1(\varphi) \equiv \nu(\varphi)$. And then by (5.11), (5.12) for some function $\nu' \in L_2$ we will have $K'(\varphi, x) = \nu'(\varphi)K(\varphi, x)$. This means that subspaces $\hat{L}_1$ and $\hat{L}_1'$ coincide, what contradicts to our initial assumption.

A. Appendix

Put

$$\gamma_v(\varphi) = \sum_{k=1}^{d} v_k \cos\varphi_k$$

where $0 \le v_k \le 1$. Consider the integral

$$b(y) = \int\limits_{\mathrm{T}^d} \left(\frac{1}{\sum_{k=1}^d v_k \cos\varphi_k - y} \right)^2 d\varphi$$

where $y \in [-D, D]$ and $D = \sum_{k=1}^d v_k$. If $y \in [-D, D]$, then the denominator of the integrand can be 0 and the question appears whether the integral $b(y)$ is finite or not.

Let m be the number of nonzero coefficients v_k. Denote $I = \{i_1, \dots, i_m\}$, where $1 \le i_1 < \dots < i_m \le d$ is the array of indices such that $v_l \ne 0 \Longleftrightarrow l \in I$. Without loss of generality we can assume that $I = \{1, \dots, m\}$, where $m \le d$. If $m < d$, then the integrand depends only on the variables $\varphi_1, \dots, \varphi_m$ and

$$b(y) = \int\limits_{\mathrm{T}^d} \left(\frac{1}{\sum_{k=1}^d v_k \cos\varphi_k - y} \right)^2 d\varphi =$$

$$= (2\pi)^{d-m} \int\limits_{\mathrm{T}^m} \left(\frac{1}{\sum_{k=1}^m v_k \cos\varphi_k - y} \right)^2 d\varphi_1 \dots d\varphi_m.$$

Thus it is sufficient to consider the integral

$$\int\limits_{\mathrm{T}^m} \left(\frac{1}{\sum_{k=1}^m v_k \cos\varphi_k - y} \right)^2 d\varphi_1 \dots d\varphi_m$$

where all $v_k > 0$.

Lemma A.1. *1. If $m \le 4$, then $b(\pm D) = +\infty$. If $m \ge 5$, then $b(D) < \infty$.*

2. The integral $b(y)$ for $y \in (-D, D)$ is divergent in any dimension.

Proof of assertion 1
Let $y = D$ and

$$b(D) = \int\limits_{\mathrm{T}^d} \left(\frac{1}{\gamma_v(\varphi) - D} \right)^2 d\varphi_1 \dots d\varphi_m. \tag{A.1}$$

The integrand has singularity only at $\varphi_1 = \dots = \varphi_m = 0$. Consider the integral

$$I_m = \int\limits_{U_\delta} \left(\frac{1}{\gamma_v(\varphi) - D} \right)^2 d\varphi$$

where $U_\delta \subset R^m$ is a neighborhood of the point $\varphi_1 = \ldots = \varphi_m = 0$ of small radius δ.

From the Taylor expansion $\cos\varphi - 1 = -\varphi^2/2 + O(\varphi^4)$ it follows that for sufficiently small neighborhood U_δ of the point $\varphi_1 = \ldots = \varphi_m = 0$ we have

$$\begin{aligned} I_m &= \int\limits_{U_\delta} \left(\frac{1}{\gamma_v(\varphi) - D} \right)^2 d\varphi = \\ &= \int\limits_{U_\delta} \left(\frac{1}{v_1^2\varphi_1^2 + \ldots + v_m^2\varphi_m^2 + O(\varphi_1^4 + \ldots + \varphi_m^4)} \right)^2 d\varphi. \end{aligned}$$

First of all we do the change of variables $\varphi_k := v_k\varphi_k$, keeping the same notation for the new variable, and then use spherical coordinates (see, for example, [9], pp. 313)

$$\begin{aligned} \varphi_1 &= r\cos\alpha_1 \\ \varphi_2 &= r\sin\alpha_1\cos\alpha_2 \\ \varphi_3 &= r\sin\alpha_1\sin\alpha_2\cos\alpha_3 \\ &\ldots\ldots \\ \varphi_{m-1} &= r\sin\alpha_1\sin\alpha_2\ldots\sin\alpha_{m-2}\cos\alpha_{m-1} \\ \varphi_m &= r\sin\alpha_1\sin\alpha_2\ldots\sin\alpha_{m-2}\sin\alpha_{m-1} \end{aligned}$$

where $\alpha_1, \ldots, \alpha_{m-2} \in [0, \pi]$ $\alpha_{m-1} \in [0, 2\pi]$. The Jacobian of this transformation $J = r^{m-1}\Psi(\alpha_1, \ldots, \alpha_{m-1}) = r^{m-1}\sin^{m-2}\alpha_1\sin^{m-3}\alpha_2\ldots\sin\alpha_{m-1}$.

Then

$$I_m = C_v \int_0^\delta dr \int_{S^{m-1}} \frac{r^{m-1}\Psi(\alpha_1, \ldots, \alpha_{m-1}) d\alpha_1 \ldots d\alpha_{m-1}}{\left(r^2 + O(r^4)\right)^2}$$

where S^{m-1} is the $(m-1)$-dimensional sphere of radius 1 and $C_v^{-1} = v_1 \ldots v_m$. Then

$$I_m = \int_0^\delta \frac{r^{m-5}}{1 + O(r^2)} dr \int_{S^{m-1}} \Psi(\alpha_1, \ldots, \alpha_{m-1}) d\alpha_1 \ldots d\alpha_{m-1}.$$

Thus for $m \geq 5$ the integral diverges; but for $m \leq 4$ it is finite.

If $y = -D$, then

$$b(-D) = \int_{\mathrm{T}^m} \left(\frac{1}{\gamma_v(\varphi) + D} \right)^2 d\varphi =$$

$$= \int_{\mathrm{T}^m} \left(\frac{1}{v_1 \cos(\pi + \varphi_1) + \ldots + v_k \cos(\pi + \varphi_d) + D} \right)^2 d\varphi =$$

$$= \int_{\mathrm{T}^m} \left(\frac{1}{-\gamma_v(\varphi) + D} \right)^2 d\varphi = b(D)$$

Proof of assertion 2

Let now $y \in (-D, D)$. We shall prove the divergence of the integral $b(y)$. We shall find point $a \in \mathbf{T}^m$ and its neighborhood $V(a) \subset \mathbf{T}^m$ so that the integral

$$\int_{V(a)} \left(\frac{1}{\gamma_v(\varphi) - y} \right)^2 d\varphi$$

diverges. All the following is not more than a technical exercise but it is useful to do it accurately.

For $d = 1$

$$b(y) = \int_{-\pi}^{\pi} \left(\frac{1}{v_1 \cos \varphi - y} \right)^2 d\varphi.$$

Take point a such that $\cos a = y/v_1$, $-v_1 < y < v_1$. Then $\sin a \neq 0$ and in sufficiently small neighborhood $V(a)$ we have $\cos \varphi - y/v_1 = (-\sin a)(\varphi - a) + O\left((\varphi - a)^2\right)$. At the point a the integrand $(v_1 \cos \varphi - y)^{-2}$ has singularity of the type $(\varphi - a)^{-2}$. That is why the integral

$$\int_{V(a)} \left(\frac{1}{v_1 \cos \varphi - y} \right)^2 d\varphi$$

diverges.

Let $m > 1$. Consider the hypersurface Γ

$$v_1 \cos\psi_1 + \cdots + v_m \cos\psi_m = y, \ -D < y < D.$$

Choose the point $a = (a_1, \ldots, a_m) \in \Gamma$ such that all $a_i \in (0, \pi)$. Then $\nabla\gamma_v(a) \neq 0$, where $\gamma_v(\varphi) = v_1 \cos\varphi_1 + \cdots + v_m \cos\varphi_m$, $\varphi = (\varphi_1, \ldots, \varphi_m)$.

Below we shall use the following notation. Vector $\psi = (\psi_1, \ldots, \psi_m) \in \Gamma$ will denote the corresponding point on the surface, vector $\varphi = (\varphi_1, \ldots, \varphi_m) \in \mathbf{T}^m$ will denote an arbitrary point of the torus $\mathbf{T}^m = (-\pi, \pi] \times \cdots \times (-\pi, \pi]$, vector $\xi = (\psi_1, \ldots, \psi_{m-1})$ – coordinates on the surface Γ. Then ψ_m is a function of ξ so that $(\xi, \psi_m(\xi)) \in \Gamma$.

Let $a' = (a_1, \ldots, a_{m-1}) \in \mathbf{R}^{m-1}$. Without loss of generality we can assume that $v_m = 1$. In sufficiently small neighborhood $U(a') \subset \mathbf{T}^{m-1}$ of a' the surface Γ can be defined by the following equations

$$\psi_m = \psi_m(\xi) = \arccos(y - v_1 \cos\psi_1 - \cdots - v_{m-1} \cos\psi_{m-1})$$

where $\xi = (\psi_1, \ldots, \psi_{m-1}) \in U(a')$.

We shall prove that in some neighborhood $V(a)$ of a the integrand is asymptotically behaves as $c(\varphi)/\rho^2(\varphi)$, where $\rho(\varphi)$ is the distance of point $\varphi \in V(a)$ to Γ and $c(\varphi)$ is some smooth function in $V(a)$. The divergence follows from this.

Let $n(\xi) = (n_1(\xi), \ldots, n_m(\xi))$ be the unit normal to the surface at the point $\psi = (\xi, \psi_m)$, $\xi \in U(a')$ where

$$n_i(\xi) = (C(\xi))^{-1} \frac{v_i \sin\psi_i}{\sqrt{1 - (y - v_1 \cos\psi_1 - \cdots - v_{m-1} \cos\psi_{m-1})^2}} \tag{A.2}$$

where $i = 1, \ldots, m-1$, $n_m(\xi) = (C(\xi))^{-1}$ and

$$C(\xi) = \sqrt{\frac{\sum_{i=1}^{m-1} v_i^2 \sin^2\psi_i}{1 - (y - v_1 \cos\psi_1 - \cdots - v_{m-1} \cos\psi_{m-1})^2} + 1}$$

Let $\psi(\varphi) \in \Gamma$ be the point such that φ belongs to the normal at the point $\psi(\varphi)$.

Define $V(a) = \{\varphi : \rho(\varphi) < \delta, \ \psi(\varphi) \in S(a)\}$, where $S(a) = (\psi = (\xi, \psi_m(\xi)) : \xi \in U(a')) \subset \Gamma$. Then $S(a) = V(a) \cap \Gamma$.

In the integral

$$\int\limits_{V(a)} \left(\frac{1}{\gamma_v(\varphi) - y}\right)^2 d\varphi$$

we do the following change of variables

$$\varphi_i(\xi, r) = \psi_i + r n_i(\xi), \quad i = 1, \ldots, m-1,$$

$$\varphi_m(\xi, r) = \arccos(y - v_1 \cos\psi_1 - \cdots - v_{m-1}\cos\psi_{m-1}) + rn_m(\xi)$$

where $\xi = (\psi_1, \ldots, \psi_{m-1}) \in U(a')$ and $r \in I_\delta$, interval of length 2δ. Then

$$\varphi(\xi, r) = \psi + rn(\xi)$$

where $\varphi(\xi, r) = (\varphi_1(\xi, r), \ldots, \varphi_m(\xi, r))$, $\psi = (\xi, \psi_m(\xi)) \in \Gamma$, and

$$\psi_m = \arccos(y - v_1 \cos\psi_1 - \cdots - v_{m-1}\cos\psi_{m-1}).$$

Denote by $J = J(\xi, r)$ the Jacobian of the transformation

$$\Phi = (\varphi_1(\xi, r), \ldots, \varphi_m(\xi, r)) : U(a') \times I_\delta \to V(a).$$

We have

$$\cos(\psi_i + rn_i) = \cos\psi_i - (\sin\psi_i)\, rn_i + O(r^2).$$

For $y = \sum_{i=1}^m v_i \cos\psi_i$ and sufficiently small r

$$y - \sum_{i=1}^m v_i \cos\varphi_i = \sum_{i=1}^m v_i (\cos\psi_i - \cos(\psi_i + rn_i)) = = r\sum_{i=1}^m v_i n_i \sin\psi_i + O(r^2).$$

From formula (A.2) we have

$$\sum_{i=1}^m v_i n_i \sin\psi_i = \sqrt{\sum_{i=1}^m v_i^2 \sin^2\psi_i} = \|\nabla\gamma_v(\psi)\| = \|\nabla\gamma_v(\xi, \psi_m(\xi))\|.$$

Thus the integrand in $V(a)$ can be represented as

$$\frac{1}{(y - \gamma_v(\varphi))^2} = \frac{1}{r^2 \|\nabla\gamma_v(\xi, \psi_m(\xi))\|^2} + O(r^{-4})$$

and

$$\int\limits_{V(a)} \left(\frac{1}{\gamma_v(\varphi) - y}\right)^2 d\varphi_1 \ldots d\varphi_m = = \int\limits_{-\delta}^{\delta} r^{-2} \left(\int\limits_{U(a')} \|\nabla\gamma_v(\xi, \psi_m(\xi))\|^{-2} J(\xi, r) d\xi \right) dr$$

where $d\xi = d\psi_1 \ldots d\psi_{m-1}$.

As $\|\nabla\gamma_v(a)\| > 0$, we have in this neighborhood of a,

$$\|\nabla\gamma_v(\xi, \psi_m(\xi))\| > \varepsilon > 0.$$

Also

$$\int_{-\delta}^{\delta} r^{-2}\left(\int_{U(a')} \|\nabla\gamma_v(\xi, \psi_m(\xi))\|^{-2} J(\xi, r)d\xi\right)dr \leq$$

$$\leq \varepsilon^{-2}\int_{-\delta}^{\delta} r^{-2}\left(\int_{U(a')} J(\xi, r)d\xi\right)dr$$

and we obtain the desired divergence.

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