Mathematics and Computer Science II

Algorithms, Trees, Combinatorics and Probabilities

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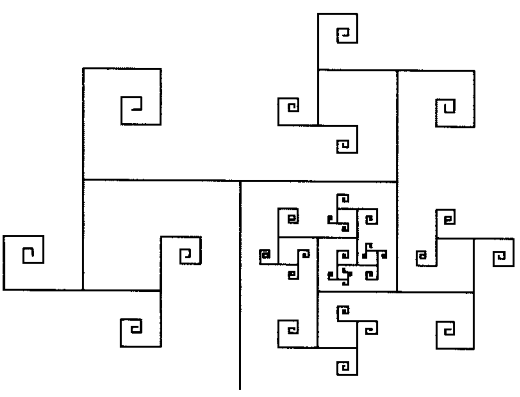

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The logo on the cover is a binary search tree in which the directions of child nodes alternate between horizontal and vertical, and the edge lengths decrease as 1 over the square root of 2. The tree is a Weyl tree, which means that it is a binary search tree constructed from a Weyl sequence, i.e., a sequence (na) mod 1, $n = 1,2,\ldots$, where a is an irrational real number.
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Random Boundary of a Planar Map

Maxim Krikun, Vadim Malyshev

ABSTRACT: *We consider the probability distribution P_N on the class of near-triangulations T of the disk with N triangles, where each T is assumed to have the weight y^m, $m = m_N = m_N(T)$ is the number of boundary edges of T. We find the limiting distribution of the random variable $m_N(T)$ as $N \to \infty$: in the critical point $y = y_{cr} = 6^{-\frac{1}{2}}$ the random variables $N^{-\frac{1}{2}} m_N$ converge to a non-gaussian distribution, for $y > y_{cr}$ for some constant c the random variables $N^{-\frac{1}{2}}(m_N - cN)$ converge to a gaussian distribution.*

1 Introduction

Enumeration of maps is an important part of the art of combinatorics. It started in sixties with the papers by W. Tutte. He invented powerful "deleting a rooted edge" and analytic "quadratic" methods, that have been exploited and developped in hundreds of subsequent papers, until nowadays. Unfortunately since then, no essentially new analytic methods for enumeration of maps appeared in combinatorics itself. This lack of essentially new ideas was compensated by two breakthroughs in other fields of mathematics and physics, where maps played an important role. One breakthrough occured in theoretical physics in eighties. Maps provided a discrete approximation to the string theory and two-dimensional quantum gravity. To deal with maps new powerful matrix methods were invented. Second one was initiated by A. Grothendieck in his program devoted to algebraic geometry and Galois theory. Some connections between these two breakthroughs were understood in nineties as having essential physical interpretation. We do not give references here, see a detailed introduction and references in [5]. For several reasons enumerative combinatorics of maps has been developping all this period in a stand alone way.

We study here some probabilistic problems for maps. Enumeration of maps deals in fact with the uniform distribution on some finite class $\mathcal{A}$ of maps. If this class has $|\mathcal{A}|$ elements then the probability of each map T is $P(T) = |\mathcal{A}|^{-1}$. In physics one is interested in the probability when maps $T \in \mathcal{A}$ have non-negative weights $w(T)$, the weights have a special Gibbs form, derived from physics. We use one below. Then the probabilities are $P(T) = Z^{-1}w(T)$, where $Z = \sum_{T\in\mathcal{A}} w(T)$ is called a partition function. We hope that rigorous probability approach can establish interconnections between differents applications of maps clearer.

As a particular case of probability for maps, we consider classes $\mathcal{T}_0(N,m)$ of rooted maps of a disk, called rooted near-triangulations in [2], with N triangles and m edges on the boundary. Enumeration problem for the number $C_0(N,m) = |\mathcal{T}_0(N,m)|$ was completely solved by Tutte [1], see also [2]. We remind that this class of maps is defined by the following restrictions: the boundary of each cell consists exactly of three edges, moreover the map is assumed to be nonseparable, thus multiple edges are allowed but no loops.

In this paper we consider the probability distribution P_N on a class $\mathcal{T}_0(N) = \cup_{m=2}^{\infty}\mathcal{T}_0(N,m)$ of maps with fixed N but variable boundary length, given by the formula

$$P_N(T) = Z_N^{-1} y^{m(T)}.$$

Here y is a positive parameter, that corresponds to $y = e^{-\mu/2}$ according to [4], and $m(T) = m_N(T)$ is the number of the boundary edges of the triangulation T. We will be interested with asymptotic propeties of the random variable $m_N = m_N(T)$. Its distribution is given by

$$P_N(m_N = m) = Z_N^{-1} y^m C_0(N, m), m \geq 2$$

where we use the normalization factor (canonical partition function)

$$Z_N(y) = \sum_{T:F(T)=N} \exp(-\frac{\mu}{2} m(T)) = \sum_{m=2}^{\infty} y^m C_0(N, m)$$

Note that N and m are always of one parity, because $m + 3N$ equals twice the number of edges, consequently $P_N(m_N = m) = 0$ if $N + m$ is odd.

In [4] relations with quantum gravity are explained, and several equivalent definitions of the distribution P_N are given, showing its naturalness, also in [4] the phase transition phenomena for m_N is described.

Here we essentially strengthen the results of section 4.2 of [4] and get explicit expressions for the limiting distributions for all three phases. Moreover, complex analytic methods we use here are quite different from [4], where the explicit combinatorial formula for $C_0(N, m)$ by Tutte was used. The method used here seems to be more adequate also in more general situations.

In the subcritical region a finite limit of m_N exists. In the critical point and the supercritical region by choosing an appropriate scaling we get a limiting distribution, which is non-gaussian or gaussian correspondingly. This is summarized in the following three theorems.

Here and further the critical parameter value is $y_{cr} \equiv \frac{1}{\sqrt{6}}$.

Theorem 1.1 (subcritical). *If $y < y_{cr}$ then for any $z, |z| < 1$, the generating function of $(m_N - 2)$,*

$$f_N(z) = \sum_{m=2}^{\infty} (m-2) P_N(m_N = m) z^{m-2},$$

for even N tends as $N \to \infty$ to

$$f_{even}(z) = \frac{(1 - \sqrt{6}yz)^{-3/2} + (1 + \sqrt{6}yz)^{-3/2}}{(1 - \sqrt{6}y)^{-3/2} + (1 + \sqrt{6}y)^{-3/2}},$$

and for odd N to

$$f_{odd}(z) = \frac{(1 - \sqrt{6}yz)^{-3/2} - (1 + \sqrt{6}yz)^{-3/2}}{(1 - \sqrt{6}y)^{-3/2} - (1 + \sqrt{6}y)^{-3/2}}.$$

Theorem 1.2 (critical). *If $y = y_{cr}$ then $\xi_N = \frac{m_N}{\sqrt{N}}$ tends in probability to the random variable ξ with the density*

$$p_\xi(x) = \frac{2}{3^{\frac{3}{2}}} \sqrt{x} e^{-\frac{x^2}{2}}, \quad x \geq 0.$$

Theorem 1.3 (supercritical). *If* $y > y_{cr}$ *then*

$$Em_N = c_1 N(1 + O(\frac{1}{N})), \qquad \frac{m_N - Em_N}{\sqrt{N}} \xrightarrow{Pr} \mathcal{N}(0, \sigma^2),$$

where

$$c_1 = \frac{24y^3 + 8y - (12y^2+1)\sqrt{4y^2+2}}{\sqrt{4y^2+2}(1+4y^2-2y\sqrt{4y^2+2})},$$

$$\sigma^2 = 4y\frac{32y^4 + 16y^2 + 1 - (16y^3+4y)\sqrt{4y^2+2}}{(2y^2+1)\sqrt{4y^2+2}(1+4y^2-2y\sqrt{4y^2+2})^2}.$$

2 The generating function

It is known [1, 2] that the generating function

$$U_0(x,y) = \sum_{N=0}^{\infty}\sum_{m=2}^{\infty} C_0(N,m)x^N y^{m-2} \tag{1}$$

is analytic in $(0,0)$ and satisfies the following equation (in a neighborhood of $(0,0)$)

$$U_0(x,y) = xy^{-1}(U_0(x,y) - U_0(x,0)) + xyU_0^2(x,y) + 1, \tag{2}$$

which also can be rewritten as

$$(2xy^2U_0(x,y) + x - y)^2 = (x-y)^2 - 4xy^3 + 4x^2y^2S(x), \tag{3}$$

where $S(x) = U_0(x,0)$. We will need some analytic techniques which slightly differs from the original method by Tutte.

Denote by $D(x,y)$ the righthand side of (3) and consider the analytic set $\mathcal{D} = \{(x,y) : D(x,y) = 0\}$ in a small neighbourhood of $(0,0)$. This set is not empty as it contains the point $(0,0)$, and it defines the branch of the function $y = y(x)$ such that $y(x) = x + O(x^2)$ in a neighbourhood of $x = 0$, we denote it further mostly by $h(x)$. In particular, it will be shown that $h(x)$ and $S(x)$ are algebraic functions. Because $D(x,y)$ is a square of an analytic function, we have two equations valid at the points of $\mathcal{D}$

$$D(x,y) = 0, \qquad \frac{\partial D(x,y)}{\partial y} = 0$$

or

$$4x^2y^2S(x) + (x-y)^2 - 4xy^3 = 0, \tag{4}$$

$$8x^2yS(x) - 2(x-y) - 12xy^2 = 0.$$

One can exclude the function $S(x)$ by multiplying the second equation (4) on $\frac{y}{2}$ and subtracting it from the first equation, then

$$y = x + 2y^3 \tag{5}$$

or

$$y = \frac{x}{1-2y^2}. \tag{6}$$

We have exposed the quadratic method belonging to Tutte. Now we have to get more information about analytic properties of the solution.

By the theorem on implicit functions equation (6) gives the unique function $h(x) = y(x)$, analytic for small x with $h(0) = 0$. It is evident from (6) that the convergence radius of $h(x)$ is finite, and its series have nonnegative coefficients. Moreover, $y(x)$ is an algebraic function satisfying the equation $y^3+py+q=0$ with $p = -\frac{1}{2}, q = \frac{x}{2}$. The polynomial $f(y) = y^3+py+q$ has multiple roots only when $f = f'_y = 0$, which gives $x_\pm = \pm\sqrt{\frac{2}{27}}$. These roots are double roots because $f''_y \neq 0$ at these points. From $f'_y = 3y^2 - \frac{1}{2} = 0$ and $f = 0$ it follows that $y(x_\pm) = \pm\frac{1}{\sqrt{6}}$. From (6) it also follows that $x(-y) = -x(y)$ and thus $y(x)$ is odd. It follows that $y(x)$ has both $x_\pm = \pm\sqrt{\frac{2}{27}}$ as its singular points.

From (4) we know $S(x)$ explicitely. The unique branch $y(x) = h(x)$, defined by equation (6), is related to the unique branch of $S(x)$ by the equation

$$S(x) = \frac{1-3h^2(x)}{(1-2h^2(x))^2} = x^{-2}h^2(1-3h^2) \tag{7}$$

that is obtained by substituting $x = h - 2h^3$ to the first equation (4).

We know that $S(x)$ has positive coefficients, that is why $x_+ = \sqrt{\frac{2}{27}}$ should be among its first singularities. Then $x_- = -\sqrt{\frac{2}{27}}$ should also be a singularity of both $h(x)$ and $S(x)$. We proved also that the generating functions are algebraic. The principal part of the singularity at the root x_+ is $h(x) = A(x-x_+)^{d+\frac{1}{2}}$ for some integer d (as the sigularity is algebraic and the root is a double root). As $y_+ = h(x_+)$ is finite then $d \geq 0$. At the same time $h'(x) = \frac{1}{1-6h^2(x)}$ that is ∞ for $x = x_+$. It follows that $d = 0$. For $S(x)$ we have the same type of singularity $A(x-x_+)^{d+\frac{1}{2}}$ but here $d = 1$ as $S(x_+)$ and $S'(x_+)$ are finite but $S''(x_+)$ is infinite. As $y = h(x)$ is a double root of the main equation, we have by substituting (7) to (3)

$$\begin{aligned} D(x,y) &= 4y^2h^2(1-3^2h^2) + (h(1-2h^2)-y)^2 - 4y^3h^2(1-2h^2) \\ &= (y-h)^2\left(\frac{x^2}{h^2} - 4xy\right) \end{aligned} \tag{8}$$

Remember that $D(x,y) = (2xy^2U_0(x,y) + x - y)^2$, so

$$U_0(x,y) = \frac{-(x-y)+(h-y)\sqrt{d(x,y)}}{2xy^2}, \qquad d(x,y) = \frac{x^2}{h^2} - 4xy. \tag{9}$$

In the last equality we have chosen the sign appropriately, that is the sign + should be chosen so that for $x = y > 0$ the value $U_0(x,y)$ were positive.

Singularities of $U_0(x,y)$ Let us prove that for any fixed $y \in (0, y_{cr})$ the minimal singularities of $U_0(x,y)$ (as a function of x) coincide with the minimal singularities of $h(x)$ that is with $x_\pm = \pm\sqrt{\frac{2}{27}}$. Consider the right hand side of (9). All singularities of $U_0(x,y)$ that do depend on y are described by the equation $d(x,y) = 0$, which is equivalent to $\frac{4h^2(x)}{x} = y^{-1}$. The series of the function $\frac{4h^2(x)}{x}$ has all coefficients nonnegative, that's why for $|y| < y_{cr}$

$$\max_{|x|\le x_+} \left|\frac{4h^2(x)}{x}\right| = \sqrt{6} = y_{cr}^{-1} < |y^{-1}|.$$

Thus for $y < y_{cr}$ the minimal singularities are at $x_\pm$. Moreover, the equation

$$\frac{x^2}{h^2} = 4xy \tag{10}$$

becomes, as $x = h - 2h^3$,

$$\frac{h-2h^3}{h^2} - 4y = 0.$$

Its solutions are

$$h_{1,2} = -y \pm \frac{1}{2}\sqrt{4y^2+2}, \qquad x_{1,2} = 2y + 8y^3 \mp 4y^2\sqrt{4y^2+2}.$$

In particular this means that for every real y the solution of (10) is real too. As we are interested only in $y > 0$, a minimal singularity is unique and is given by choosing minus in the latter equation,

$$x_1(y) = 2y + 8y^3 - 4y^2\sqrt{4y^2+2}. \tag{11}$$

For each $y \ge \frac{1}{\sqrt{6}}$ this gives $x_1(y) \le x_{cr} = \sqrt{\frac{2}{27}}$, equalities are achieved simultaneously. This can be easy checked by plotting a graph of $(h-2h^3)/h^2$ and using the fact that the function $h(x)$ is strictly increasing, we omit this construction.

3 Subcritical region

The canonical partition function is the coefficient in the expansion

$$U_0(x,y) = \sum_{N=0}^{\infty} Z_N(y)x^N.$$

$U_0(x,y)$ is algebraic, and we will prove that for any fixed $y, 0 < y < y_{cr}$, in the vicinity of $x_\pm$

$$U_0(x,y) = f_{\pm,0}(x,y) + f_{\pm,1}(x,y)(1-\frac{x}{x_\pm})^{\frac{3}{2}}$$

where for fixed y the functions $f_{\pm,0}, f_{\pm,1}$ are analytic near $x_\pm$ correspondingly, the values of $f_{\pm,1}$ at $x_\pm$ are nonzero, namely

$$f_{+,1}(x_+,y) = \frac{6^{\frac{3}{4}}3}{(1-\sqrt{6}y)^{3/2}}, \quad f_{-,1}(x_-,y) = \frac{6^{\frac{3}{4}}3}{(1+\sqrt{6}y)^{3/2}}.$$

Expand $h(x)$ near $x_+ = \sqrt{\frac{2}{27}}$ in $t = x_+ - x$

$$h(x) = \frac{1}{\sqrt{6}} - \frac{1}{\sqrt[4]{6}} t^{1/2} - \frac{1}{6} t - \frac{5\sqrt[4]{6}}{72} t^{3/2} + O(t^2) \quad . \tag{12}$$

Substitute (12) together with $x = x_+ - t$ to the expression (9) for $U_0(x, y)$ and expand in $t^{1/2}$

$$U_0(x_+ - t, y) = a_+(y) + b_+(y)t + \frac{6^{3/4}3}{(1-\sqrt{6}y)^{3/2}} t^{3/2} + O(t^2).$$

Similary we find

$$U_0(x_- + t, y) = a_-(y) + b_-(y)t + \frac{6^{3/4}3}{(1+\sqrt{6}y)^{3/2}} t^{3/2} + O(t^2),$$

Then as $N \to \infty$

$$Z_N(y) \sim 6^{3/4}3\Big(\frac{1}{(1-\sqrt{6}y)^{3/2}} + \frac{(-1)^N}{(1+\sqrt{6}y)^{3/2}}\Big)[x^N](x_+ - x)^{\frac{3}{2}} \tag{13}$$

This is known under different names (for example, as Darboux theorem in [3]). However, it can be proved elementarily, using the following expansion for $a = \frac{3}{2}$

$$t^a = (x_0 - x)^a = \sum_{N=0}^{\infty} \frac{\Gamma(N-a)}{N!\Gamma(-a)} x_0^{a-N} x^N \tag{14}$$

where $[x^N]F(x)$ stands for the N-th coefficient in the $F(x)$ power series. Secondly, subtracting this main term and proving that the rest is asymptotically negligible. In fact, (13) should be read as two separate equations,

$$Z_N(y) \sim 6^{3/4}3\Big(\frac{1}{(1-\sqrt{6}y)^{3/2}} \pm \frac{1}{(1+\sqrt{6}y)^{3/2}}\Big)[x^N](x_+ - x)^{\frac{3}{2}},$$

with a plus sign standing for even values of N and a minus sign for odd.
Finally for given y the generating function for $m_N - 2$ is obtained from the partition function $Z_N(y)$ by normalization, that is

$$f_N(z) = \sum_{m=2}^{\infty} P\{m_N = m\} z^{m-2} = \frac{Z_N(yz)}{Z_N(y)},$$

and after taking limits in N (by even an odd values separately) we come to the assertion of Theorem 1.1.

4 Critical point

In a critical point the expectation of m_N has no finite limit. To describe the limiting distribution we shall calculate the asymptotics (as $N \to \infty$) of the factorial

moments of m_N and find the appropriate scaling. That is we have to study the singularities of all the partial derivatives $\frac{\partial^n}{\partial y^n} U_0(x,y)$ at $y = y_{cr}$, as we have done in the previous section for $U_0(x,y)$ only.
From the previous analysis we know that for $y = y_{cr}$ the singularity defined by $d(x,y) = 0$ is among the minimal ones. According to (11) is equal to $\sqrt{\frac{2}{27}}$ and coinsides to x_+ singularity of $h(x)$, so there are two minimal singularities at points x_+ and x_-.

Lemma 4.1. *Put* $t = x - x_0$. *Then there exist functions* $\varphi_{n,i}(t) = \varphi_{n,i}(t,y), i = 0,1,2,$ *analytic in the vicinity of* $t = 0$ *such that*

$$U_0^{(n)}(x, y_{cr}) = \varphi_{n,0}(t) + \varphi_{n,1}(t)\, t^{3/4-n/2} + \varphi_{n,2}(t)\, t^{5/4-n/2}, \quad \varphi_{n,1}(0) \neq 0$$

Proof. Instead of calculating the y-derivatives of $U_0(x,y)$ we calculate them for $2xy^2U_0(x,y)$, which is much simpler, but keeps all information on $C_0(N,m)$. We have

$$xy^2U_0(x,y) = y - x + (h-y)\sqrt{4x}\Big(\frac{x}{4h^2} - y\Big)^{1/2}, \quad x \geq 0,$$

$$xy^2U_0(x,y) = y - x + (h-y)\sqrt{-4x}\Big(-\frac{x}{4h^2} + y\Big)^{1/2}, \quad x \leq 0.$$

To get the derivatives put $y = y_{cr} + u$ and consider the formal series in u:

$$\begin{aligned} 2xy^2U_0(x,y)\Big|_{y=y_{cr}+u} &= (y_{cr} - x) + u + \Big((h - y_{cr}) - u\Big) \\ &\times\sqrt{4x}\sum_{n=0}^{\infty} \frac{\Gamma(n-\frac{1}{2})}{n!\Gamma(-\frac{1}{2})}\Big(\frac{x}{4h^2} - y_{cr}\Big)^{1/2-n} u^n, \quad x \geq 0, \end{aligned}$$

$$\begin{aligned} 2xy^2U_0(x,y)\Big|_{y=y_{cr}+u} &= (y_{cr} - x) + u + \Big((h - y_{cr}) - u\Big) \\ &\times\sqrt{-4x}\sum_{n=0}^{\infty} \frac{\Gamma(n-\frac{1}{2})}{n!\Gamma(-\frac{1}{2})}\Big(-\frac{x}{4h^2} + y_{cr}\Big)^{1/2-n} (-u)^n, \quad x \leq 0. \end{aligned}$$

For $n > 1$, $x \geq 0$ the n-the coefficient (we denote it $[u^n]$) is equal to

$$\begin{aligned} [u^n]\Big(2xy^2U_0(x,y)\Big|_{y=y_0+u}\Big) &= \sqrt{4x}\frac{\Gamma(n-\frac{3}{2})}{(n-1)!\Gamma(-\frac{1}{2})}\Big(\frac{x}{4h^2} - y_{cr}\Big)^{3/2-n} \\ &\quad -(h - y_{cr})\sqrt{4x}\frac{\Gamma(n-\frac{1}{2})}{n!\Gamma(-\frac{1}{2})}\Big(\frac{x}{4h^2} - y_{cr}\Big)^{1/2-n} \\ &= \sqrt{4x}\frac{\Gamma(n-\frac{3}{2})}{(n-1)!\Gamma(-\frac{1}{2})}\Big(\frac{x}{4h^2} - y_{cr}\Big)^{3/2-n} \\ &\quad \times\Big(1 - (h - y_{cr})\frac{n-\frac{3}{2}}{n}\Big(\frac{x}{4h^2} - y_{cr}\Big)^{-1}\Big), \qquad (15) \end{aligned}$$

and similarly for $n > 1$, $x \leq 0$

$$
\begin{aligned}
[u^n]\Big(2xy^2U_0(x,y)\Big|_{y=y_0+u}\Big) &= \sqrt{-4x}\frac{\Gamma(n-\frac{3}{2})}{(n-1)!\Gamma(-\frac{1}{2})}\Big(-\frac{x}{4h^2}+y_{cr}\Big)^{3/2-n}(-1)^{n-1} \\
&\quad -(h-y_{cr})\sqrt{4x}\frac{\Gamma(n-\frac{1}{2})}{n!\Gamma(-\frac{1}{2})}\Big(-\frac{x}{4h^2}+y_{cr}\Big)^{1/2-n}(-1)^n \\
&= \sqrt{4x}\frac{\Gamma(n-\frac{3}{2})}{(n-1)!\Gamma(-\frac{1}{2})}\Big(-\frac{x}{4h^2}+y_{cr}\Big)^{3/2-n}(-1)^{n-1} \\
&\quad \times\Big(1+(h-y_{cr})\frac{n-\frac{3}{2}}{n}\Big(-\frac{x}{4h^2}+y_{cr}\Big)^{-1}\Big). \qquad (16)
\end{aligned}
$$

Next we need the following auxiliary expansions

$$
\begin{aligned}
(h-y_{cr})\Big(\frac{x}{4h^2}-y_{cr}\Big)^{-1}\Big|_{x=x_+-t} &= -\frac{1}{2}+\frac{3}{8}6^{1/4}t^{1/2}+O(t), \\
\Big(\frac{x}{4h^2}-y_{cr}\Big)\Big|_{x=x_+-t} &= \frac{1}{3}6^{3/4}t^{1/2}+O(t), \\
(h-y_{cr})\Big(-\frac{x}{4h^2}+y_{cr}\Big)^{-1}\Big|_{x=x_-+t} &= -1+\frac{3}{2}6^{1/4}t^{1/2}+O(t), \\
\Big(-\frac{x}{4h^2}+y_{cr}\Big)\Big|_{x=x_-+t} &= \frac{1}{3}\sqrt{6}+\frac{1}{3}6^{3/4}t^{1/2}+O(t).
\end{aligned}
$$

(note that the second one has no constant term). Using these expansions we obtain from (15) and (16) the behaviour of the $U_0(x,y)$ derivatives near $x_\pm$, namely

$$
\begin{aligned}
\frac{\partial^n}{\partial y^n}U_0(x,y)\Big|_{x=x_+-t} &= \text{const } t^{3/4-n/2}(1+O(t^{1/2})), \\
\frac{\partial^n}{\partial y^n}U_0(x,y)\Big|_{x=x_-+t} &= \text{const } +O(t^{1/2}).
\end{aligned}
$$

Lemma is proved.

The factorial moments of m_N are

$$
M_1(N) \sim 2^{-2}3^2\frac{\Gamma(-\frac{3}{4})}{\Gamma(-\frac{1}{4})}N^{\frac{1}{2}}, \qquad M_2(N) \sim 2^{-4}3^4\frac{-\Gamma(-\frac{3}{4})}{\Gamma(\frac{1}{4})}N,
$$

$$
M_n(N) = \frac{[x^N]U_n}{[x^N]U_0} \sim 2^{-2n}3^{n+1}(2n-1)(2n-5)!!\frac{-\Gamma(-\frac{3}{4})}{\Gamma(\frac{n}{2}-\frac{3}{4})}N^{n/2},
$$

Consequently the moments of a random variable $\xi = \lim_{N\to\infty} m_N/\sqrt{N}$ are

$$
E\xi = 3(3/4)\frac{\Gamma(\frac{3}{4})}{\Gamma(-\frac{1}{4})}, \qquad E\xi^2 = 3(3/4)^2\frac{-\Gamma(-\frac{3}{4})}{\Gamma(\frac{1}{4})} = \frac{9}{4},
$$

$$
E\xi^n = 2^{-2n}3^{n+1}(2n-1)(2n-5)!!\frac{-\Gamma(-\frac{3}{4})}{\Gamma(\frac{n}{2}-\frac{3}{4})} = \frac{\Gamma(\frac{n}{2}+\frac{3}{4})3^n}{\Gamma(\frac{3}{4})}.
$$

The moment generating function for ξ^2 is uniquely defined by this sequence (by classical uniqueness criteria, see sections VII.3 and VIII.6(b) of [6]), as they grow slower than $C^n n!$ for some C) and is equal to

$$\varphi_{\xi^2}(s) = \sum_{n=0}^{\infty} E\xi^{2n} \frac{(-s)^n}{n!} = (1+9s)^{-3/4}$$

Using the Laplace transform we get the density of ξ^2

$$p_{\xi^2}(t) = 3^{-\frac{3}{2}} \frac{1}{\Gamma(\frac{3}{4})} e^{-\frac{t}{9}} t^{-1/4}$$

5 Supercritical region

We shall prove that $Em_N \sim cN$ and all the semiinvariants (coefficients in the Taylor expansion of the logarithm of the generating function) of m_N are of order N. Then it follows that the semiinvariants of order greater than two of a scaled random variable $(m_N - Em_N/\sqrt{N}$ tend to zero as $N \to \infty$, which means the imiting distribution is uniquely defined by its moments (see above), and moreover it is gaussian (as the log of its generating function is a quadratic polynomial).

The semiinvariants of m_N are given by the formula

$$s_k(N) = (\frac{\partial}{\partial\lambda})^k \ln \varphi_N(\lambda)|_{\lambda=0}, \quad k \geq 1,$$

where

$$\varphi_N(t) = Ee^{\lambda m_N} = \frac{[x^N]U_0(x, ye^\lambda)}{[x^N]U_0(x,y)}$$

ined thing is the characteristic function of m_N.

We saw that for fixed $y > y_{cr}$ the minimal singularity of $U_0(x,y)$ (as the function of x) is unique and is given by (11). The expansion of $U_0(x,y)$ (as the function of x) at the singular point $x_{cr}(y)$ is

$$U_0(x,y) = a(y) + b(y)(x_{cr}(y) - x)^{1/2} + O(|x_{cr}(y) - x|)$$

for some constants $a(y), b(y)$. Then

$$[x^N]U_0(x,y) \sim b(y)[x^N](x - x_{cr}(y))^{1/2} = b(y)\frac{\Gamma(N-\frac{1}{2})}{N!\Gamma(-\frac{1}{2})} x_{cr}(y)^{\frac{1}{2}-N},$$

$$\ln \varphi_N(t) = \ln[x^N]U_0(x, ye^\lambda) - \ln[x^N]U_0(x,y) \sim N\Big(-\ln x_{cr}(ye^\lambda) + \ln x_{cr}(y)\Big).$$

It follows that all semiinvariants of m_N are $O(N)$.

6 Some remarks

Equivalent presentations of the model The factor $y^m = \exp(-\frac{\mu}{2}m)$ is quite natural: it is derived from the Hilbert-Einstein action in two-dimensional pure quantum gravity, see introductory exposition in [5]. The case $y = 1$ that could be natural for combinatorics seems to have no special interest for physics, where the critical point is of most interest. We could assign weights to maps as $\exp(-\mu L(T)$, where $L(T)$ is the number of all edges of the map T. This would give the same probability distribution because of the formula $|L(T)| = \frac{3N}{2} + \frac{m(T)}{2}$.

Second kind phase transition The free energy for this model is defined as

$$F(\mu) = \lim_{N\to\infty} \frac{1}{N} \log Z_{0,N}, Z_{0,N} = \sum_T \exp\{-\mu L(T)\}$$

The next theorem gives an explicit formula for the free energy, it corrects a calculational mistake in the corresponding result in [4]. It shows also that the phase transtion is a second order phase transition, as in the critical point the free energy is differentiable but not twice differentiable.

Theorem 6.1. *The free energy is equal to* $-\frac{3}{2}\mu+\ln\left(\sqrt{\frac{27}{2}}\right)$ *if* $y \leq y_{cr}$ *and is equal to* $-\frac{3}{2}\mu + \ln x_{cr}(y)$ *if* $y > y_{cr}$.

Proof. It easily follows from the proofs in the preceding sections. We have

$$Z_{0,N} = \sum_T \exp\{-\mu L(T)\} = \sum_T \exp\{-\frac{\mu}{2}(3N+m)\} = \exp\{-\frac{3}{2}\mu N\}[x^N]U_0(x, e^{-\mu/2}),$$

$$\frac{1}{N}\log Z_{0,N} = -\frac{3}{2}\mu + \frac{1}{N}\log\left([x^N]U_0(1, e^{-\mu/2})\right)$$

Put $y = e^{-\mu/2}$. Following section 3, as $y < y_{cr}$:

$$[x^N]U_0(1, e^{-\mu/2}) = f(y)[x^N](x_0 - x)^{3/2},$$

$$\frac{1}{N}\log Z_{0,N} \to -\frac{3}{2}\mu + \ln x_0 = -\frac{3}{2}\mu + \ln\left(\sqrt{\frac{27}{2}}\right).$$

When $y = y_{cr}$:

$$[x^N]U_0(1, e^{-\mu/2}) = f(y)[x^N](x_0 - x)^{3/4}$$

$$\frac{1}{N}\log Z_{0,N} \to -\frac{3}{2}\mu + \ln\left(\sqrt{\frac{27}{2}}\right).$$

Following section 5, as $y > y_{cr}$:

$$[x^N]U_0(1, e^{-\mu/2}) = b(y)[x^N](x_{cr}(y) - x)^{1/2},$$

$x_{cr}(y) = 2y + 8y^3 - 4y^2\sqrt{4y^2+2}$ being defined as in (11) we get

$$\frac{1}{N}\log Z_{0,N} \to -\frac{3}{2}\mu + \ln x_{cr}(y)$$

Further problems The similar problem for two holes in the sphere could be the next solvable problem, that is consider a ring (or cylinder) with two boundaries of lengths m_1, m_2. Joint distribution of these two random variables is to be found. Not that if for one boundary there is the combinatorial formula for $C_0(N, m)$

$$C_0(N, m) = \frac{2^{j+2}(2m+3j-1)!(2m-3)!}{(j+1)!(2m+2j)!((m-2)!)^2}$$

by Tutte (used in [4]). Nothing similar is known for the number $C_0(N, m_1, m_2)$ of rooted near triangulations of a ring with N triangles and the lengths m_1, m_2 of the boundaries, where only analytic methods can be of use.

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