From N-body problem to Euler equations

Lykov A. A., Malyshev V. A. *

Abstract

This paper contains a rigorous mathematical example of direct derivation of the system of Euler hydrodynamic equations from Hamiltonian equations for N point particle system as $N \to \infty$. Direct means that the following standard tools are not used in the proof: stochastic dynamics, thermodynamics, Boltzmann kinetic equations, correlation functions approach by N. N. Bogolyubov.

Key words: N-body problem, continuum mechanics, intersection of particle trajectories, Euler equations.

*Lomonosov Moscow State University, Faculty of Mechanics and Mathematics, Vorobyevy Gory 1, Moscow, 119991, Russia

1 Introduction

Classical mechanics, from mathematical point of view, is mostly developed in cases, where two extremely idealized forms of material objects are assumed - point particles (ordinary differential equations) and continuum media (partial differential equations). However, big difference exists in the ideology of these two theories: for point particles the model is defined by the choice of the interaction potential between particles, which is supposed to be known, but in the continuum mechanics the interaction is defined by the pressure, which is one of the unknown functions in the equations. Many papers - both mission proposals [1, 2] and concrete results for concrete models [3, 4, 5, 6] - discussed the connections between these two fields. We do not give here review of these papers as we do not use neither their results nor methods. Moreover, our approach is direct that is we do not use any of the following approaches: stochastic dynamics, thermodynamics, Boltzmann kinetic equations, correlation functions approach by N. N. Bogolyubov.

Now we make the above claims more precise. Hamiltonian finite particle system is defined by the system of equations for the particle trajectories $x_i(t) \in R^d, i = 1, ..., N$

$$m_i \frac{d^2 x_i}{dt^2} = -\frac{\partial H}{\partial x_i}$$

with the Hamiltonian H.

We define the continuum (d-dimensional) media as a bounded open subset $\Lambda \subset R^d$, the dynamics of this media is given by the system of such domains $\Lambda_t, t \in [0, T), 0 < T \leq \infty$, together with the system of diffeomorphisms $S^t : \Lambda = \Lambda_0 \to \Lambda_t, t \in [0, T)$, smooth also in t. The trajectory of the point (particle) $x \in \Lambda_0$ of the continuum media is the function $y(t, x) = S^t x$. The main unknown variable in the Euler equations is the velocity $u(t, y)$ of the particle, which at time t is at point y. This definition of u has sense iff such particle is unique, that is iff for any t and any $x_1 \neq x_2$

$$y(t, x_1) \neq y(t, x_2)$$

that is iff the trajectories (particles) $y(t, x)$ do not collide.

This property obviously should be related to the similar property for N particle system, if we want to obtain continuum media trajectories in the limit $N \to \infty$ (one could call this the ultralocal limit).

We say that the N-particle system has no collisions, if for all $1 \leq j < k \leq N$ and all $t \in [0, \infty)$

$$x_j(t) \neq x_k(t),$$

and has strong property of absence of collisions if

$$\inf_{t \geqslant 0} \inf_{j,k: j \neq k} |(x_j(t) - x_k(t))| > 0 \quad (1)$$

It is evident that there will not be any collisions if the repulsion between particles is sufficiently strong. However, for general Hamiltonian systems the following question is completely non trivial: for which initial conditions $x_k(0),\ \dot{x}_k(0), k = 1, ..., N$, the system enjoys the absence of collisions property. In this paper the property (1) plays the central role. It is surprising that we did not find papers where this property is discussed in the derivation of continuum media equations. However, it was widely discussed in Celestial Mechanics (gravitation potential), see for example [8].

We consider the particle system on the real line with a particular Lennard-Jones type potential and prove that the particle trajectories of the N-particle system, for $N \to \infty$, converge, in the sense defined below, to the trajectories of the continuum particle system. Moreover, we get the system of 3 equations of the Euler type (which is considered in [7]) for the functions: $u(t,x)$ - the velocity, $p(t,x)$ - the pressure and $\rho(t,x)$ - the density

$$\rho_t + u\rho_x + \rho u_x = 0, \tag{2}$$

$$u_t + uu_x = -\frac{p_x}{\rho}, \tag{3}$$

$$p = p(\rho), \tag{4}$$

In continuum mechanics these equations correspond to the conservation laws of mass, momentum and to the thermodynamic equation of state. In physics the first two equations are quite general. But the third one depends on the matter type and thermodynamic situation and should be given separately. In our derivation, all these equations and functions obtain simple and intuitive mechanical meaning (without probability theory and thermodynamics) for the N-particle system. In particular, the pressure can be considered as an analog of interaction potential in Hamiltonian mechanics.

2 Main Results

The model We consider Hamiltonian system of N particles (of unit mass) with coordinates $x_1, ..., x_N$ on R and the Hamiltonian

$$H = \sum_{k=1}^{N} \frac{v_k^2}{2} + U$$

The potential energy U of the particle system with the coordinates $x_1, x_2, , ..., x_N$ is defined by the interaction potential

$$U = \sum_{1 \le k < l \le N} \frac{\omega^2}{2} I(|x_k - x_l|), I \in \mathbf{I}(a, a_1)$$

where $\mathbf{I}(a, a_1), 0 < a_1 < a$, is the class of functions $I(x)$ on R_+ with the following two properties

1) $I(x) = (x-a)^2$ for $a - a_1 < x < a + a_1$ with some constant $0 < a_1 < a$.
2) $I(x) = const$ for $x \ge a + a_1$
3) $I(x)$ is arbitrary for $0 < x \le a - a_1$

Scaling Our system contains three parameters: ω, a and a_1. We could add also mass but the scaling of mass and/or time could be reduced to the scaling of ω.

If N is large and all particles are situated on some finite interval then a should be of order N^{-1}. We put $a = \frac{1}{N}$. Then the system will be in equilibrium (zero force on each particle) iff for all k $x_{k+1} - x_k = \frac{1}{N}$. Correspondingly, we put $a_1 = \frac{r}{N}$ for some $0 < r < 1$ not depending on N. The remaining parameter we choose as

$$\omega = \omega' N \tag{5}$$

for some $\omega' > 0$, not depending on N,

Initial conditions We always assume the following initial conditions

$$x_1(0) = 0,\ \dot{x}_1(0) = v, \tag{6}$$

$$x_{k+1}(0) - x_k(0) = \frac{1}{N} X\left(\frac{k}{N}\right) > 0, \quad \dot{x}_{k+1}(0) - \dot{x}_k(0) = \frac{1}{N} V\left(\frac{k}{N}\right),\ k = 1, \dots, N-1 \tag{7}$$

for some $v \in \mathbb{R}$, and for some functions $X, V \in C^4([0,1])$, where $X > 0$. Thus, the functions X and V define smooth profile of the initial conditions. Then the kinetic and potential energies of the system will be of the order $O(N)$.

It is convenient to assume also that

$$X(0) = X(1) = 1,\ V(0) = V(1) = 0, \tag{8}$$

The second condition (8) means that two leftmost (two rightmost) particles initially have almost (up to $O(N^{-2})$) equal velocities, and the first condition (8) means that both boundary particles are subjected to almost zero force.

Let $\Omega_N = \Omega_N(\gamma)$ be the domain of $R^N = \{(x_1, ..., x_N)\}$, defined for some $0 < \gamma < 1$ by the estimates

$$\frac{1-\gamma}{N} \leqslant x_{k+1} - x_k \leqslant \frac{1+\gamma}{N}$$

uniformly in $k = 1, ..., N$. We want to prove that if initially our system is in this region, then under certain conditions it will stay in Ω_N forever. The obvious corollary is that there will never be collisions between particles. It might seem that under such conditions there will not be interesting dynamics, but this is wrong, see pictures at the end of the paper.

Absence of collisions for N-particle case The condition below allows to estimate distances between particles at any time moment. Denote

$$\alpha = \int_0^1 |X''(y)|\, dy,\ \ \beta = \int_0^1 |V''(y)|\, dy.$$

Further on we will use the concrete value of γ, defined in terms of the main parameters of the system

$$\gamma = \gamma(\alpha, \beta, \omega') = 2\alpha + \frac{\beta N}{\omega} = 2\alpha + \frac{\beta}{\omega'} \tag{9}$$

Further on the condition $\gamma = \gamma(\alpha, \beta, \omega') < \min(r, \frac{1-r}{2})$ is always assumed.

Theorem 1 *Assume that initially our system is in Ω_N. Then it stays in Ω_N forever. That is for any $t \geqslant 0$ and any $k = 1, 2, \dots, N-1$ the following inequalities hold:*

$$\frac{1-\gamma}{N} \leqslant x_{k+1}(t) - x_k(t) \leqslant \frac{1+\gamma}{N} \tag{10}$$

It follows that the particles never collide in the strong sense (1).

Note that the scaling of ω is crucial to create the repulsion necessary for the particles did not collide.

To understand the importance of the choice of α, note that $X(y) - 1$ characterizes the deviation of the chain from the equilibrium, $X'(y)$ characterizes the speed of change of this equilibrium, and α can be considered as the full variation. Thus the following simple statement is useful to estimate such deviation at initial time moment.

Lemma 1 *For any $y \in [0,1]$*

$$1 - \alpha \leqslant X(y) \leqslant 1 + \alpha$$

Strategy of the proof If we could prove Theorem 1 for some potential in the class $\mathbf{I}(a, a_1)$, then the γ-bounds (10) indicate that it will also hold for any potential $I \in \mathbf{I}(a, a_1)$. In the following proofs we will use the simplest of such potential - the quadratic potential

$$I(x_{k+1} - x_k) = (x_{k+1} - x_k - a)^2$$

Even more, we assume the nearest neighbour interaction for this potential . Then the following system of linear differential equations holds

$$\ddot{x}_1 = \omega^2(x_2 - x_1 - a), \tag{11}$$

$$\ddot{x}_k = \omega^2(x_{k+1} - x_k - a) - \omega^2(x_k - x_{k-1} - a), \; k = 2, 3, \ldots, N-1, \tag{12}$$

$$\ddot{x}_N = \; -\omega^2(x_N - x_{N-1} - a), \tag{13}$$

In this case we will also prove γ-bounds (10).

Now from these γ-bounds we want to show how from this the Theorem 1 follows for any class of potentials $\mathbf{I}(\frac{1}{N}, \frac{r}{N})$. It is sufficient to show that for any t and k the following inequalities hold

$$\frac{1-r}{N} \leqslant x_{k+1}(t) - x_k(t) \leqslant \frac{1+r}{N}, \tag{14}$$

$$x_{k+2}(t) - x_k(t) > \frac{1+r}{N} \tag{15}$$

Then (14) obviously holds if $\gamma < r$. From the γ-bounds (10) we have the estimate

$$x_{k+2}(t) - x_k(t) \geq 2\frac{1-\gamma}{N},$$

This estimate implies (15) if $\gamma < \frac{1-r}{2}$.

Convergence to continuous chain dynamics Denote $q(t, x)$ the solution of the wave equation

$$q_{tt} = (\omega')^2 q_{xx},$$

(here and below the lower indices define the derivatives in the corresponding variables) with fixed boundary conditions

$$q(t, 0) = q(t, 1) = 0, \tag{16}$$

and with the initial conditions:

$$q(0, x) = X(x) - 1, \; q_t(0, x) = V(x). \tag{17}$$

Let $x_k^{(N)}(t)$ be the solution of the main system (11)-(13) for given N. For any fixed t we want to define two functions (algorithms) which map the set of points Λ_t of the continuous media to the set of particles $\{1, 2, ..., N\}$, that is to the set of particle coordinates $\{x_k^{(N)}(t)\}$ of the N-particle approximation. To do this, we will use two coordinate systems on the real intervals: x and z, where $z(x)$, for $x \in (0, L), L = L(0) = \int_0^1 X(y)dy$, is uniquely defined from the equation:

$$\int_0^{z(x)} X(x')dx' = x. \tag{18}$$

In the first algorithm to any $z \in (0, 1]$ correspond the particle with number $[zN]$ (the integer part of $zN \in \mathbb{R}$). Note that for any $0 < z \leq 1$ there exists $N(z)$ such that $N > N(z)$ we have $1 < [zN] \leq N$.

In the second algorithm to any point $x \in [0, L]$ corresponds the particle with number $k(x, N) = k(x, N, 0)$ so that

$$x^{(N)}_{k(x,N)}(0) \leqslant x < x^{(N)}_{k(x,N)+1}(0) \tag{19}$$

Due to positivity of $X(y)$ such number is uniquely defined. Then it is natural to call the function $x^{(N)}_{k(x,N)}(t)$ the N-particle approximation of the trajectory of the particle $x \in [0, L]$ of the continuum media. By definition we put $x^{(N)}_{N+1} = \infty$.

Theorem 2 *Let the conditions of the theorem 1 hold, and assume also (5). Then:*

1) for any $0 < T < \infty$ uniformly in $t \in [0, T]$ and in $z \in (0, 1]$

$$\lim_{N\to\infty} x^{(N)}_1(t) = G(t, 0) = vt + (\omega')^2 \int_0^t (t - s)q_x(s, 0)\ ds, \tag{20}$$

$$\lim_{N\to\infty} x^{(N)}_{[zN]}(t) = G(t, z) = G(t, 0) + z + \int_0^z q(t, x')\ dx', \tag{21}$$

2) Let $0 \leqslant z_1 < z_2 \leqslant 1$. Then for any $t \geqslant 0$

$$(1 - \gamma)(z_2 - z_1) \leqslant G(t, z_2) - G(t, z_1) \leqslant (1 + \gamma)(z_2 - z_1),$$

where γ is defined in (9). Otherwise speaking, the continuum media particles do not collide;

3) for any $T > 0$ uniformly in $x \in [0, L]$ and in $t \in [0, T]$

$$\lim_{N\to\infty} x^{(N)}_{k(x,N)}(t) = y(t, x) = G(t, z(x)), \tag{22}$$

4) the function $G(t, z)$ satisfies the wave equation

$$\frac{d^2G(t, z)}{dt^2} = (\omega')^2 \frac{d^2G(t, z)}{dz^2} \tag{23}$$

with boundary conditions $G_z(t, 0) = G_z(t, 1) = 1$ and initial conditions

$$G(0, z) = \int_0^z X(x')dx', \quad G_t(0, z) = v + \int_0^z V(x')dx'.$$

An obvious corollary is that for any $T > 0$, uniformly in $t \in [0, T]$, we have the following asymptotic limit for the length $L_N(t) = x_N(t) - x_1(t)$ of the chain

$$\lim_{N\to\infty} L_N(t) = L(t) = 1 + \int_0^1 q(t, x')dx'$$

Continuity equation (mass conservation law) Further on, the function $y(t, x)$ will be called the trajectory of the particle $x \in [0, L]$. Then the particles do not collide and one can unambiguously define the function $u(t, y)$ as the speed of the (unique) particle situated at time t at the point y , that is

$$u(t, y(t, x)) = \frac{dy(t, x)}{dt}. \tag{24}$$

Also we will need the notation:

$$Y_0(t) = y(t, 0), \quad Y_L(t) = y(t, L).$$

For given N we define the distribution function at time t:

$$F^{(N)}(t, y) = \frac{1}{N} |\{k \in \{1, 2, \ldots, N\} :\ x^{(N)}_k(t) \leqslant y\}|,\ y \in \mathbb{R}$$

where $| \cdot |$ is the number of particle in the set.

Lemma 2 *Denote $x(t,y) \in [0,L]$ the (unique) particle, which reached the point y at time t, that is*

$$y(t, x(t,y)) = y. \tag{25}$$

Then uniformly in $y \in [Y_0(t), Y_L(t)]$ and in $t \in [0,T]$, for any $T < \infty$, we have

$$\lim_{N\to\infty} F^{(N)}(t,y) = z(x(t,y)) = F(t,y),$$

where the (smooth) function $z(x)$ is the same as the one introduced in (18).

In connection with Lemma 2 define the density by the formula

$$\rho(t,y) = \frac{dF(t,y)}{dy} = \frac{dz(x(t,y))}{dy} \tag{26}$$

Theorem 3 *For any $t \geqslant 0,\ y \in [Y_0(t), Y_L(t)]$*

$$\frac{\partial \rho(t,y)}{\partial t} + \frac{d}{dy}\left(u(t,y)\rho(t,y)\right) = 0. \tag{27}$$

Euler equation (momentum conservation) and equation of state

Theorem 4 *For any $t \geqslant 0,\ y \in [Y_0(t), Y_L(t)]$ we have:*

$$\frac{\partial u(t,y)}{\partial t} + u(t,y)\frac{\partial u(t,y)}{\partial y} = -(\omega')^2 \frac{1}{\rho(t,y)}\frac{d}{dy}\frac{1}{\rho(t,y)} = -\frac{1}{\rho(t,y)}\frac{dp(t,y)}{dy}, \tag{28}$$

if we put

$$p(t,y) = -\frac{(\omega')^2}{\rho(t,y)} + C. \tag{29}$$

Constant C can be chosen as

$$C = (\omega')^2,$$

so that at equilibrium (when $\rho = 1$) the pressure were zero.

Right side of the Euler equation as the limit of interaction forces

For given y and t define the number $k(y,N,t)$ so that

$$x^{(N)}_{k(y,N,t)}(t) \leqslant y < x^{(N)}_{k(y,N,t)+1}(t) \tag{30}$$

Consider the point $y \in [Y_0(t), Y_L(t)]$ and the force acting on the particle with number $k(y,N,t)$:

$$R^{(N)}(t,y) = \omega^2\left(x^{(N)}_{k(y,N,t)+1} - x^{(N)}_{k(y,N,t)} - \frac{1}{N}\right) - \omega^2\left(x^{(N)}_{k(y,N,t)} - x^{(N)}_{k(y,N,t)-1} - \frac{1}{N}\right)$$

Theorem 5 *Let the conditions of the theorem 2 hold. Then for any $0 < T < \infty$, uniformly in $y \in [Y_0(t), Y_L(t)]$ and in $t \in [0,T]$ the following equality holds:*

$$\lim_{N\to\infty} R^{(N)}(t,y) = R(t,y) = -\frac{1}{\rho(t,y)}\frac{dp(y)}{dy},$$

where the functions p, ρ are the same as in theorem 4.

Thus, the pressure can be considered as a continuous interaction potential for continuum media, an analog of interaction potentials in Hamiltonian particle mechanics.

Limit of the energy Define the potential and kinetic energy of the particle with number $k(y, N, t)$ at time t for N-particle approximation correspondingly as:

$$
\begin{aligned}
U^{(N)}(t, y) =& \frac{1}{4}\omega^2 \left(x^{(N)}_{k(y,N,t)+1} - x^{(N)}_{k(y,N,t)} - \frac{1}{N} \right)^2 + \frac{1}{4}\omega^2 \left(x^{(N)}_{k(y,N,t)} - x^{(N)}_{k(y,N,t)-1} - \frac{1}{N} \right)^2, \\
T^{(N)}(t, y) =& \frac{1}{2} \left(\dot{x}^{(N)}_{k(y,N,t)} \right)^2.
\end{aligned}
$$

Theorem 6 *For any t, y (uniformly as in the previous theorem) the following limits hold:*

$$
\begin{aligned}
U(t, y) &= \lim_{N\to\infty} U^{(N)}(t, y) = \frac{1}{2}(\omega')^2 \left(\frac{1}{\rho(t, y)} - 1 \right)^2 = \frac{1}{2(\omega')^2} p^2(t, y), \\
T(t, y) &= \lim_{N\to\infty} T^{(N)}(t, y) = \frac{1}{2} u^2(t, y)
\end{aligned}
$$

3 Proofs

As we explained previously, we shall always use the system (11)-(13).

3.1 Proof of Theorem 1

Proof of Lemma 1 Put $f(y) = X(y) - 1$ and use the following Lemma 3.

Lemma 3 *Assume that $f \in C^2([0, 1])$ and $f(0) = f(1) = 0$. Then the following inequality holds:*

$$
\sup_{y\in[0,1]} |f(y)| \leqslant \int_0^1 |f''(y)| \, dy.
$$

In fact,

$$
f(y) = \int_0^y f'(x) \, dx.
$$

It follows that

$$
\sup_{y\in[0,1]} |f(y)| \leqslant \int_0^1 |f'(x)| \, dx.
$$

As $f(0) = f(1) = 0$, then there exists point $x^* \in (0, 1)$ such that

$$
f'(x^*) = 0.
$$

Thus we have:

$$
\int_0^1 |f'(x)| \, dx = \int_0^1 \left| \int_x^{x^*} f''(u) du \right| \, dx \leqslant \sup_{x\in[0,1]} \left| \int_x^{x^*} f''(u) du \right| \leqslant \int_0^1 |f''(u)| \, du.
$$

This proves the Lemma.

Remark 1 *The set of functions Σ, satisfying the conditions of Lemma 3 is a linear space. Moreover,*

$$
\alpha(f) = \int_0^1 |f''(y)| \, dy.
$$

defines a norm on this space. Lemma 3 states that the uniform norm does not exceed the norm $\alpha(\cdot)$. However, we want to note that these two norms are not equivalent. In fact, assume the contrary, i. e. that there exists constant $a > 0$ such that for any function $f \in \Sigma$

$$\alpha(f) \leqslant a \sup_{y\in[0,1]} |f(y)|.$$

Then put $f_k(y) = \sin \pi k y$. Then $\alpha(f_k) \to \infty$ as $k \to \infty$, but $\sup_{y\in[0,1]} |f_k(y)| = 1$. This is a contradiction.

Deviation variables Define the deviation variables $q_k(t) = x_{k+1}(t) - x_k(t) - a,\ k = 1, \ldots, N - 1$, and put by definition $q_0 = q_N = 0$, Then the functions q_k satisfy the equations:

$$\ddot{q}_k = \omega^2(q_{k+1} - q_k) - \omega^2(q_k - q_{k-1}) = \omega^2(q_{k+1} - 2q_k + q_{k-1}),\ k = 1, 2, \ldots, N - 1 \tag{31}$$

with initial conditions which follow from (7)

$$q_k(0) = \frac{1}{N} X\left(\frac{k}{N}\right) - \frac{1}{N}, \quad \dot{q}_k(0) = \frac{1}{N} V\left(\frac{k}{N}\right),\ k = 0, 1, \ldots, N$$

In fact, from equations (11)-(13) we have

$$\ddot{x}_k = \omega^2(q_k - q_{k-1}),\ \ k = 1, ..., N,$$

$$\ddot{x}_k = \ddot{x}_{k+1} - \ddot{q}_k, k = 1, ..., N - 1,$$

Then for $k = 1, \ldots, N - 1$

$$\omega^2(q_{k+1} - q_k) - \ddot{q}_k = \omega^2(q_k - q_{k-1}).$$

The last equality is equivalent to (31).

Remark 2 *The inverse transformation is given by*

$$x_k(t) = x_1(t) + \frac{k-1}{N} + \sum_{i=1}^{k-1} q_i(t), \quad k = 2, 3, \ldots, N. \tag{32}$$

and for $x_1(t)$, by definition of q_1, we have the equation:

$$\ddot{x}_1 = \omega^2 q_1.$$

It follows

$$x_1(t) = x_1(0) + \dot{x}_1(0)t + \omega^2 \int_0^t ds \int_0^s q_1(s')\ ds' = x_1(0) + \dot{x}_1(0)t + \omega^2 \int_0^t (t-s)q_1(s)\ ds \tag{33}$$

The last equality follows from comparison of derivatives of both sides.

One can rewrite the system (31) in the matrix form:

$$\ddot{q} = -Wq,$$

where the matrix W is a three diagonal non negative definite $(n \times n)$-matrix with $n = N - 1$, and $q = (q_1(t), \ldots, q_n(t))^T$ is a column vector.

Spectrum of the matrix W We will show that W is positive definite, and will find the basis $v_1, \dots, v_n$ of eigenvectors of W, with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$: $Wv_j = \lambda_j v_j,\ j = 1, \dots, n$.

Let e_k be the standard unit coordinate vectors in $\mathbb{R}^n$. For $j = 1, \dots, N-1$ define vectors v_j by

$$y_j(k) = (v_j, e_k) = \sqrt{\frac{2}{N}} \sin \frac{\pi j k}{N},\ \ k = 1, 2, \dots, N-1. \tag{34}$$

and the numbers λ_j by

$$\lambda_j = 4\omega^2 \sin^2 \frac{\pi j}{2N}.$$

Let us prove that v_j are eigenvectors of W with eigenvalues λ_j. Note that if we define $y_j(k)$ from (34) also at the points $k = 0$ and $k = N$, then we get $y_j(0) = y_j(N) = 0$. Then for all $k = 1, \dots, N-1$

$$\begin{aligned}(Vv_j, e_k) &= -\omega^2(y_j(k+1) - y_j(k)) + \omega^2(y_j(k) - y_j(k-1)) = \\ &= 2\omega^2\sqrt{\frac{2}{N}}\left(-\sin\frac{\pi j}{2N}\cos\frac{\pi j(2k+1)}{2N} + \sin\frac{\pi j}{2N}\cos\frac{\pi j(2k-1)}{2N}\right) \\ &= -2\omega^2\sqrt{\frac{2}{N}}\sin\frac{\pi j}{2N}\left(\cos\frac{\pi j(2k+1)}{2N} - \cos\frac{\pi j(2k-1)}{2N}\right) = \\ &= 4\omega^2\sqrt{\frac{2}{N}}\sin\frac{\pi j}{2N}\sin\frac{\pi j k}{N}\sin\frac{\pi j}{2N} = \lambda_j y_j(k).\end{aligned}$$

where $(,)$ is the standard scalar product in $\mathbb{R}^n$. As all λ_j positive and different, then W is positive definite.

Dynamics of deviations

Lemma 4 *Let for any* $j = 1, \dots, N-1$

$$Q_j = \sqrt{\frac{2}{N}}\sum_{i=1}^{N-1} q_i(0)\sin\frac{\pi i j}{N}, \quad P_j = \sqrt{\frac{2}{N}}\sum_{i=1}^{N-1} \dot{q}_i(0)\sin\frac{\pi i j}{N}, \quad \omega_j = 2\omega\sin\frac{\pi j}{2N}.$$

Then

$$q_k(t) = \sqrt{\frac{2}{N}}\sum_{j=1}^{N-1}\left(Q_j\cos\omega_j t + P_j\frac{\sin\omega_j t}{\omega_j}\right)\sin\frac{\pi j k}{N},$$

Proof. Using the expansion of $q(t)$ in the basis

$$q(t) = \sum_{j=1}^{n} Q_j(t) v_j, \quad Q_j(t) = (q(t), v_j),$$

we will get equations for $Q_j(t)$:

$$\ddot{Q}_j = -\lambda_j Q_j,$$

This gives

$$Q_j(t) = Q_j(0)\cos\omega_j t + \dot{Q}_j(0)\frac{\sin\omega_j t}{\omega_j}, \quad \omega_j = \sqrt{\lambda_j}.$$

Then

$$q_k(t) = (q(t), e_k) = \sum_{j=1}^{n} Q_j(t)(v_j, e_k) = \sum_{j=1}^{n}\left(Q_j(0)\cos\omega_j t + \dot{Q}_j(0)\frac{\sin\omega_j t}{\omega_j}\right)(v_j, e_k),$$

and the Lemma is proved.

Estimate of the coefficients Q_j, P_j We have

$$Q_j = \sqrt{\frac{2}{N}} \sum_{i=1}^{N-1} q_i(0) \sin \frac{\pi ij}{N} = (q(0), v_j) = \frac{1}{\lambda_j}(q(0), Vv_j) = \frac{1}{\lambda_j}(Vq(0), v_j) =$$

$$= -\frac{\omega^2}{\lambda_j}\sqrt{\frac{2}{N}} \sum_{i=1}^{N-1} ((q_{i+1}(0) - q_i(0)) - (q_i(0) - q_{i-1}(0))) \sin \frac{\pi ij}{N}.$$

Then let us estimate the sum

$$S_N = \sum_{i=1}^{N-1} ((q_{i+1}(0) - q_i(0)) - (q_i(0) - q_{i-1}(0))) \sin \frac{\pi ij}{N}$$

where

$$q_i(0) = \frac{1}{N} X(\frac{i}{N}) - \frac{1}{N}, \quad i = 0, 1, ..., N$$

We have

$$q_{i+1}(0) - q_i(0) = \frac{1}{N}\left(X\left(\frac{i+1}{N}\right) - X\left(\frac{i}{N}\right)\right) = \frac{1}{N^2} X'(\theta_i),$$

for some point $\theta_i \in (\frac{i}{N}, \frac{i+1}{N})$. This gives

$$|S_N| = \frac{1}{N^2}\left|\sum_{i=1}^{N-1} (X'(\theta_i) - X'(\theta_{i-1})) \sin \frac{\pi ij}{N}\right| \leqslant \frac{1}{N^2} \sum_{i=1}^{N-1} |X'(\theta_i) - X'(\theta_{i-1}))| \leqslant \frac{1}{N^2} \int_0^1 |X''(y)|\, dy.$$

and thus

$$|Q_j| \leqslant \frac{1}{4 \sin^2 \frac{\pi j}{2N}} \sqrt{\frac{2}{N}} \frac{\alpha}{N^2}, \quad \alpha = \int_0^1 |X''(y)|\, dy.$$

Taking into account the inequality $\omega_j = \omega \sin \frac{\pi j}{2N} \geqslant \omega \frac{j}{N}$, we get

$$|Q_j| \leqslant \frac{1}{4}\sqrt{\frac{2}{N}} \frac{\alpha}{j^2}.$$

Similar estimates holds for P_j:

$$|P_j| \leqslant \frac{1}{4}\sqrt{\frac{2}{N}} \frac{\beta}{j^2}, \quad \beta = \int_0^1 |V''(y)|\, dy.$$

This gives the final estimate

$$|q_k(t)| \leqslant \frac{1}{4} \frac{2}{N} \sum_{k=1}^{N-1} (\frac{\alpha}{j^2} + \frac{\beta N}{2\omega j^3}) \leqslant \frac{\gamma}{N}, \quad \gamma = 2\alpha + \frac{\beta N}{\omega}.$$

3.2 Proof of Theorem 2

We will denote now $q_k(t) = q_k^{(N)}(t)$, emphasizing the dependence on N.

Lemma 5 *Assume the conditions of Theorem 2. Then for any $T > 0$*

$$\max_{t \in [0,T]} \max_{k=1,\dots,N-1} \left| q_k^{(N)}(t) - \frac{1}{N} q(t, \frac{k}{N}) \right| \leqslant \frac{c \ln N}{N^3},$$

for some constant $c > 0$ not depending on N.

Proof. Consider the difference

$$\Delta_k^{(N)}(t) = q_k^{(N)}(t) - \frac{1}{N} q(t, \frac{k}{N}), \ k = 0, \dots, N$$

For any $k = 1, \dots, N-1$ we have

$$\ddot{\Delta}_k^{(N)}(t) = \omega^2 (q_{k+1}^{(N)} - q_k^{(N)}) - \omega^2 (q_k^{(N)} - q_{k-1}^{(N)}) - \frac{1}{N} (\omega')^2 q_{xx}(t, \frac{k}{N})$$

Note that for all $k = 1, \dots, N-1$

$$(q(t, \frac{k+1}{N}) - q(t, \frac{k}{N})) - (q(t, \frac{k}{N}) - q(t, \frac{k-1}{N})) = \frac{1}{N^2} q_{xx}(t, \frac{k}{N}) + r_k^{(N)}(t),$$

and moreover the remainder term can be estimated as

$$|r_k^{(N)}(t)| \leqslant \frac{1}{12N^4} \max_{t \in [0,T]} \max_{x \in [0,1]} \left| \frac{d^4 q(t,x)}{dx^4} \right| = \frac{c_1}{N^4}.$$

Then we have the equations

$$\ddot{\Delta}_k^{(N)}(t) = \omega^2 (\Delta_{k+1}^{(N)} - \Delta_k^{(N)}) - \omega^2 (\Delta_k^{(N)} - \Delta_{k-1}^{(N)}) + r_k^{(N)}(t) (\omega')^2 N$$

with initial conditions

$$\Delta_k^{(N)}(0) = 0, \quad \dot{\Delta}_k^{(N)}(0) = 0, \ \ k = 0, ..., N$$

Introduce the vectors

$$\Delta^{(N)}(t) = (\Delta_1^{(N)}(t), \dots, \Delta_{N-1}^{(N)}(t))^T, \quad r^{(N)}(t) = (r_1^{(N)}(t), \dots, r_{N-1}^{(N)}(t))^T.$$

Then we have the equation

$$\ddot{\Delta}^{(N)} = -W \Delta^{(N)} + r^{(N)}(t) (\omega')^2 N,$$

where the matrix W was introduced in the proof of Theorem 1. It is easy to see that the solution of this equation is

$$\Delta^{(N)}(t) = (\omega')^2 N \int_0^t (\sqrt{W})^{-1} \sin \sqrt{W} (t-s) r^{(N)}(s) \ ds,$$

where $\sqrt{W}$ is the positive definite square root of the matrix W. Thus

$$\Delta_k^{(N)}(t) = (\Delta^{(N)}(t), e_k) = (\omega')^2 N \sum_{j=1}^{N-1} \frac{(v_j, e_k)}{\omega_j} \int_0^t \sin \omega_j (t-s) (r^{(N)}(s), v_j) \ ds.$$

For all $s \in [0,t], \ j = 1, \dots, N-1$ we have the inequality:

$$|(r^{(N)}(s), v_j)| \leqslant \sqrt{\frac{2}{N}} \frac{c_1}{N^3}$$

The consequence is that for all $t \in [0,T], \ k = 1, \dots, N-1$ the following estimate holds:

$$|\Delta_k^{(N)}(t)| \leqslant (\omega')^2 N \frac{2}{N} \frac{c_1}{N^3} \sum_{j=1}^{N-1} \frac{T}{2N\omega' \sin \frac{\pi j}{2N}} \leqslant (\omega')^2 N \frac{1}{N} \frac{T c_1}{\omega' N^3} \sum_{j=1}^{N-1} \frac{1}{j} \leqslant \frac{c_2 \ln N}{N^3},$$

for some constant $c_2 > 0$, not depending on N. The Lemma is proved.

Proof of the assertion 1) of Theorem 2 Note that for any $t \in [0,T]$

$$q(t,\frac{1}{N}) = \frac{1}{N}q_x(t,0) + \frac{r(t)}{N^2},$$

where $|r(t)| \leqslant c$ for some constant $c>0$, not depending on N. That is why from the equality (33) and Lemma 5, we get that uniformly in $t \in [0,T]$ the following limiting equality holds

$$\lim_{N\to\infty} x_1^{(N)}(t) = vt + (\omega')^2 \int_0^t (t-s)q_x(s,0)\, ds.$$

Using the equality (32) and Lemma 5, we get:

$$x_{[zN]}^{(N)}(t) = x_1(t) + z + \frac{1}{N}\sum_{k=1}^{[zN]-1} q(t,\frac{k}{N}) + r^{(N)}(t,z),$$

and moreover, there exists constant $C>0$ such that $|r^{(N)}(t,z)| \leqslant \frac{C}{N}$ for all $z \in [0,1],\ t \in [0,T]$. Taking the limit in this equality we get the assertion of the Theorem.

Proof of assertion 2) From evident equality

$$x_{[z_2N]}(t) - x_{[z_1N]}(t) = \sum_{k=[z_1N]}^{[z_2N]-1} (x_{k+1}(t) - x_k(t))$$

and from Theorem 1 we get the estimate:

$$\frac{1-\gamma}{N}([z_2N]-[z_1N]) \leqslant x_{[z_2N]}(t) - x_{[z_1N]}(t) \leqslant \frac{1+\gamma}{N}([z_2N]-[z_1N]).$$

Taking the limit here we get the assertion.

Proof of assertion 3) Firstly, let us prove that for some constant $c>0$, not depending on N, for all $x \in [0,L]$

$$\left|\frac{k(x,N)}{N} - z(x)\right| \leqslant \frac{c}{N} \tag{35}$$

Denote

$$f(z) = \int_0^z X(x')dx'.$$

Then we have $f(z(x)) = x$. On the other side, the integral can be calculated as follows

$$f(\frac{k(x,N)}{N}) = \frac{1}{N}\sum_{i=1}^{k(x,N)} X(\frac{i}{N}) + r_N(x) = x_{k(x,N)+1}(0) + r_N(x),$$

where the remainder term enjoys the following estimate:

$$|r_N(x)| \leqslant \frac{1}{N}\max_{y\in[0,1]} |X'(y)| = \frac{c_1}{N}.$$

By definition of $k(x,N)$ we have:

$$x - \frac{c_1}{N} \leqslant f(\frac{k(x,N)}{N}) < x + \frac{1}{N}X(\frac{k(x,N)}{N}) + \frac{c_1}{N} < x + \frac{c_2}{N},\ c_2 = c_1 + \max_{y\in[0,1]} X(y).$$

The following inequality follows:

$$|f(\frac{k(x,N)}{N}) - f(z(x))| \leqslant \frac{c_2}{N}.$$

But also for some point $\theta \in [0,1]$

$$f(\frac{k(x,N)}{N}) - f(z(x)) = (\frac{k(x,N)}{N} - z(x))f'(\theta),$$

This gives

$$\left|\frac{k(x,N)}{N} - z(x)\right| \leqslant \frac{1}{\min_{y\in[0,1]} X(y)}\frac{c_2}{N} = \frac{c}{N}.$$

From the proved inequality (35) it follows that

$$|k(x,N) - [z(x)N]| \leqslant c', \ c' = c+1 \tag{36}$$

Then by Theorem 1

$$|x_{[z(x)N]}(t) - x_{k(x,N)}(t)| \leqslant c\frac{1+\gamma}{N}$$

Taking the limit in the last inequality we get the assertion.

Proof of Lemma 2 We will use the particle numbers $k(y,N,t)$, introduced in (30). By definition we take $x_0^{(N)}(t) = Y_0(t),\ x_{N+1}^{(N)}(t) = Y_L(T)$. It is clear that

$$F^{(N)}(t,y) = \frac{k(y,N,t)}{N}.$$

Further on for given N we consider particle trajectories for the initial points $x_{k(y,N,t)}^{(N)}(0)$ and $x_{k(x(t,y),N)}^{(N)}(0)$. We want to prove that at time t the distance between them does not exceed c/N. Using theorem 1, we will show that $k(y,N,t)$ differs from $k(x(t,y),N)$ not more than on some constant. Lemma will follow from this. Now we give the formal proof. We use the inequalities:

$$|x_{k(x(t,y),N)}^{(N)}(t) - x_{k(y,N,t)}^{(N)}(t)| \leqslant |x_{k(x(t,y),N)}^{(N)}(t) - y(t,x(t,y))| + |x_{k(y,N,t)}^{(N)}(t) - y|$$

By assertions 1), 2), 3) of Theorem 2, and its proof, we can conclude, that the following inequality holds:

$$|x_{k(x(t,y),N)}^{(N)}(t) - y(t,x(t,y)))| \leqslant \frac{c_1}{N},$$

for some constant $c_1 > 0$ not depending on N and y. Then by definition of $k(y,N,t)$ and Theorem 1 we have the estimate for $0 < k(y,N,t) < N$:

$$|x_{k(y,N,t)}^{(N)}(t) - y| \leqslant |x_{k(y,N,t)}^{(N)}(t) - x_{k(y,N,t)+1}^{(N)}(t)| \leqslant \frac{c_2}{N},$$

for some constant $c_2 > 0$ not depending on $N, y,$. In cases $k(y,N,t) = N$ and $k(y,N,t) = 0$ the latter inequality follows from Theorem 2. Then

$$|x_{k(x(t,y),N)}^{(N)}(t) - x_{k(y,N,t)}^{(N)}(t)| \leqslant \frac{c}{N}, \quad c = c_1 + c_2.$$

From this inequality and Theorem 1 we have

$$|k(x(t,y),N) - k(y,N,t)| \leqslant c', \tag{37}$$

for some constant $c' > 0$, not depending on N, y. We can conclude that

$$\lim_{N\to\infty} \frac{k(y,N,t)}{N} = \lim_{N\to\infty} \frac{k(x(t,y),N)}{N} = z(x(t,y)),$$

where the latter equality follows from the proof of Theorem 2, assertion 3. The Lemma is thus proved.

Proof of assertion 4) The simple calculation gives with (16),(21)

$$\frac{d^2G(t,z)}{dz^2} = q_z(t,z).$$

$$\frac{d^2G(t,z)}{dt^2} = \frac{d^2G(t,0)}{dt^2} + \int_0^z q_{tt}(t,z')dz' = (\omega')^2 q_z(t,0) + (\omega')^2 \int_0^z q_{zz}(t,z')dz' =$$
$$= (\omega')^2(q_z(t,z) - q_z(t,0)) + (\omega')^2 q_z(t,0) = (\omega')^2 q_z(t,z)$$

The boundary and initial conditions can be easily found from the corresponding conditions on the function $q(t,x)$

3.3 Proof of Theorem 3

By definition (26) we have

$$\rho(t,y) = z'(x(t,y))x_y(t,y).$$

On the other side, differentiation in y of the equality (25) gives:

$$x_y(t,y) = \frac{1}{y_x(t,x(t,y))}. \tag{38}$$

Hence,

$$y_x(t,x(t,y)) = \frac{z'(x(t,y))}{\rho(t,y)}. \tag{39}$$

By definition

$$\frac{\partial \rho(t,y)}{\partial t} = \frac{d}{dy}\frac{dz(x(t,y)}{dt} = \frac{d}{dy}\left(x_t(t,y)z'(x(t,y))\right).$$

Differentiating in t the equality (25) we get:

$$x_t(t,y) = -\frac{\frac{\partial y(t,x(t,y))}{\partial t}}{y_x(t,x(t,y))} = -\frac{u(t,y)}{y_x(t,x(t,y))} = -\frac{u(t,y)\rho(t,y)}{z'(x(t,y))}.$$

The theorem is thus proved.

3.4 Proof of Theorem 4

We need the following Lemma.

Lemma 6 *For all* $t \geqslant 0,\ x \in [0,L]$

$$\frac{\partial u(t,y(t,x))}{\partial t} + u(t,y(t,x))\frac{\partial u(t,y(t,x))}{\partial y} =$$

$$= (\omega')^2 G_{zz}(t,z(x)) = (\omega')^2 \frac{y_{xx}(t,x) - \frac{z''(x)}{z'(x)} y_x(t,x)}{[z'(x)]^2}. \tag{40}$$

Proof of the Lemma. The left hand side of the formula (40) is the complete derivative of $u(t,y(t,x))$ in t, that is $\frac{du(t,y(t,x))}{dt}$. On the other side, we have by definition:

$$\frac{du(t,y(t,x))}{dt} = \frac{d^2y(t,x)}{dt^2} = \frac{d^2G(t,z(x))}{dt^2} = (\omega')^2\frac{d^2G(t,z(x))}{dz^2}.$$

Moreover¸ the following formulas hold:

$$y_x(t,x) = \frac{dy(t,x)}{dx} = \frac{dG(t,z(x))}{dx} = z'(x)\frac{dG(t,z(x))}{dz} = z'(x)G_z(t,z(x)), \tag{41}$$

$$y_{xx}(t,x) = \frac{d^2y(t,x)}{dx^2} = [z'(x)]^2\frac{d^2G(t,z(x))}{dz^2} + z''(x)\frac{dG(t,z(x))}{dz} = [z'(x)]^2G_{zz}(t,z(x)) + z''(x)G_z(t,z(x)).$$

Then

$$G_{zz}(t,z(x)) = \frac{y_{xx}(t,x) - \frac{z''(x)}{z'(x)}y_x(t,x)}{[z'(x)]^2}.$$

and the Lemma is proved.

Let us prove now theorem 4. Putting $x = x(t,y)$ in the equation (40) gives the following equation:

$$\frac{\partial u(t,y)}{\partial t} + u(t,y)\frac{\partial u(t,y)}{\partial y} = R(t,y),$$

where we introduced the function:

$$R(t,y) = (\omega')^2\frac{y_{xx}(t,x(t,y)) - \frac{z''(x(t,y))}{z'(x(t,y))}y_x(t,x(t,y))}{[z'(x(t,y))]^2}. \tag{42}$$

Differentiating the equality (39) in y and using (38), we get:

$$y_{xx}(t,x(t,y)) = \frac{1}{x_y(t,y)}\frac{d}{dy}\frac{z'(x(t,y))}{\rho(t,y)} = \frac{z'(x(t,y))}{\rho(t,y)}\frac{d}{dy}\frac{z'(x(t,y))}{\rho(t,y)} =$$

$$= \frac{z'(x(t,y))}{\rho(t,y)}\left(\frac{z''(x(t,y))x_y(t,y)}{\rho(t,y)} - \frac{z'(x(t,y))\rho_y(t,y)}{\rho^2(t,y)}\right) =$$

$$= \frac{z'(x(t,y))}{\rho(t,y)}\left(\frac{z''(x(t,y))}{z'(x(t,y))} - \frac{z'(x(t,y))\rho_y(t,y)}{\rho^2(t,y)}\right) = \frac{1}{\rho(t,y)}\left(z''(x(t,y)) - \frac{[z'(x(t,y))]^2\rho_y(t,y)}{\rho^2(t,y)}\right).$$

That is why the function $R(t,y)$ can be written in terms of the density

$$R(t,y) = (\omega')^2\frac{1}{\rho(t,y)}\left(\frac{z''(x(t,y)) - \frac{[z'(x(t,y))]^2\rho_y(t,y)}{\rho^2(t,y)} - z''(x(t,y))}{[z'(x(t,y))]^2}\right) =$$

$$= -(\omega')^2\frac{1}{\rho(t,y)}\frac{\rho_y(t,y)}{\rho^2(t,y)} = (\omega')^2\frac{1}{\rho(t,y)}\frac{d}{dy}\frac{1}{\rho(t,y)}$$

Thus all assertions of the theorem are proved.

3.5 Proof of the theorem on the force and energy

Proof of the Theorem 5 Write down the force $R^{(N)}(t,y)$ in terms of q variables, introduced in the proof of Theorem 1

$$R^{(N)}(t,y) = \omega^2(q_{k(y,N,t)} - q_{k(y,N,t)-1}).$$

By Lemma 5 we have:

$$R^{(N)}(t,y) = N(\omega')^2\left(q\left(t,\frac{k(y,N,t)}{N}\right) - q\left(t,\frac{k(y,N,t)-1}{N}\right)\right) + O(\frac{\ln N}{N}) =$$

$$= (\omega')^2 q_x\left(t, \frac{k(y,N,t)}{N}\right) + O(\frac{\ln N}{N})$$

Using inequalities (37) and (36) we have the following estimate:

$$\left|\frac{k(y,N,t)}{N} - z(x(t,y))\right| \leqslant \frac{c}{N}.$$

for some constant c, not depending on N. Then we can conclude that

$$\lim_{N\to\infty} R^{(N)}(t,y) = R(t,y) = (\omega')^2 q_x(t, z(x(t,y))) = (\omega')^2 \frac{d^2G(t,z(x(t,y)))}{dz^2}.$$

Using formula (42), we get the proof.

Proof of Theorem 6 Let us check the first equality. Rewrite the potential energy $U^{(N)}(t,y)$ in terms of the q variables, which were introduced in the proof of Theorem 1

$$U^{(N)}(t,y) = \frac{1}{4}\omega^2(q^2_{k_N(t,y)} + q^2_{k_N(t,y)-1}).$$

The same arguments as in the proof of Theorem 5 give

$$U(t,y) = \lim_{N\to\infty} U^{(N)}(t,y) = \frac{1}{2}(\omega')^2 q^2(t, z(x(t,y))) = \frac{1}{2}(\omega')^2\left(\frac{dG(t,z(x(t,y)))}{dz} - 1\right)^2.$$

Using formulas (41) and (39), we get:

$$U(t,y) = \frac{1}{2}(\omega')^2\left(\frac{y_x(x(t,y))}{z'(x(t,y))} - 1\right)^2 = \frac{1}{2}(\omega')^2\left(\frac{1}{\rho(t,y)} - 1\right)^2$$

The formula for the kinetic energy is obvious.

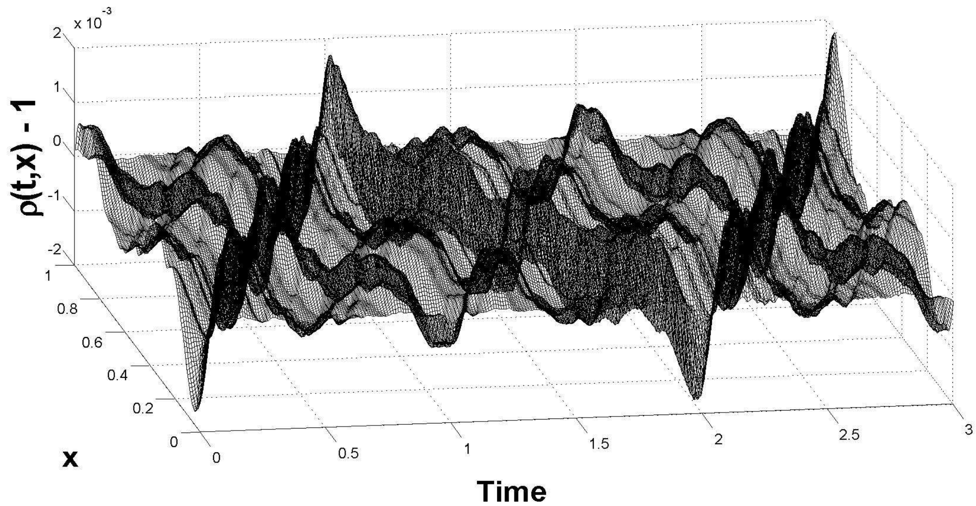
4 The density dynamics

On the three-dimensional $(t,x,z) \in R^3$ graph the surface $z = \rho(t,x) - 1$ is presented, as the result of computer modelling with $N = 200, \omega' = 1$. Initial data were chosen as:

$$X(x) = 1 + \epsilon S_n(x), \quad S_n(x) = \sum_{k=4}^{100} \frac{s_k}{k^2}\sin(\pi k x), V(x) = 0$$

with random numbers $s_k \in [0,1]$. ϵ is chosen so that there were no particle collisions, namely as

$$\epsilon < \frac{1}{2s}, \quad s = \int_0^1 |S_n''(x)|dx.$$



x 10^-3
2
1
0
-1
-2
ρ(t,x) - 1
1
0.8
0.6
0.4
0.2
0
x
0
0.5
1
1.5
2
2.5
3
Time

References

[1] Bogolyubov N. N. On some statistical methods in mathematical physics. 1845. Kiev. Acad of Science of USSR, 1945.

[2] Morrey C. On the derivation of the equations of hydrodynamics from statistical mechanics. Comm. Pure Appl. Math., 1955, 8, 279-326.

[3] Marchioro, M. Pulvirenti. Mathematical Theory of Incompressible Nonviscous Fluids. Springer. 1993.

[4] R. Esposito, J. Lebowitz, R. Marra. On the derivation of hydrodynamics from the Boltzmann equation. Phys. of Fluids, 1999, v. 11, 8, pp. 2354-2366.

[5] R. Esposito, R. Marra. Incompressible fluids on three levels: hydrodynamic, kinetic, microscopic. Mathematical Analysis of Phenomena in Fluid and Plasma Dynamics , RIMS, Kyoto (1993).

[6] Boldrighini C., Dobrushin R. L., Sukhov Yu. M. One-dimensional hard rod caricature of hydrodynamics. Journal of Statistical Physics, 1983, 31, No, 3.

[7] Chorin A., Marsden J. A mathematical introduction to fluid mechanics. Third Ed. Springer. 2000.

[8] Siegel C,. Moser J. Lectures on Celestial Mechanics. Springer-Verlag. 1971.