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Gibbs and quantum discrete spaces

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Abstract. The Gibbs field is one of the central objects of modern probability theory, mathematical statistical physics, and Euclidean field theory. In this paper we introduce and study a natural generalization of this field to the case in which the background space (a lattice, a graph) on which the random field is defined is itself a random object. Moreover, this randomness is given neither *a priori* nor independent of the configuration; on the contrary, the space and the configuration on it depend on each other, and both objects are given by a Gibbs construction. We refer to the resulting distribution as a Gibbs family because it parametrizes Gibbs fields on different graphs belonging to the support of the distribution. We also study the quantum analogue of Gibbs families and discuss relationships with modern string theory and quantum gravity.

Contents

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§ 1. Introduction

We list various motivations for writing this paper.

The Gibbs field [1], [2] is one of the central objects of modern probability theory, mathematical statistical physics, and Euclidean field theory [3]. We introduce here a natural generalization of this field to the case in which the background space (which can be a lattice or a graph) on which the random field is defined is itself a random object. Moreover, this randomness is given neither *a priori* nor independent of the configuration, but the space and the configuration on it depend on each other, and both objects are given by a Gibbs construction. One obtains a non-trivial object even if the graph is endowed with no configuration, and we refer to this object as a *Gibbs graph*. In the general case the resulting object is called a *Gibbs family* because it parametrizes a set of pairs (G, μ) belonging to the support of the corresponding distribution (this set can be finite or of the cardinality of the continuum), where G is a graph and μ is a Gibbs measure on G.

The quantum analogue of Gibbs families is a generalization of quantum spin systems [4] to the case of a 'quantum' lattice. The resulting object, the so-called quantum discrete space, is of interest from many points of view even if there is no spin.

We present here the mathematical foundations of this theory, which has diverse aspects, from the general notion of locality to the combinatorics of graphs.

Another motivation for this paper involves questions arising in the study of more modern physical theories, including string theory, M-theory, and other approaches to quantum gravity. The first question arises in connection with diverse aspects of locality. In physical theories it is customary that the space is given *a priori* as the scene for all further actions. For a long time, only three-dimensional Euclidean space was treated in physics. In the 20th century, the dimension became variable, and problems for a globally Euclidean space were replaced by those for manifolds. Moreover, the global structure of a manifold is determined by matter according to the general relativity equations, restrictions arising from string theory, and so on.

Nevertheless, the postulate of local smoothness was preserved, and it is preserved in some form also now. However, this postulate has been seriously compromised by divergences in quantum field theory, by quantum properties of the metric, and so on. This state of things leads to natural questions on possible modifications of the property of being locally Euclidean. At present many experts think that one must seek the answer in the notion of discrete space, but it turns out that for discrete spaces the very notion of locality has diverse interpretations discussed below.

On the other hand, the space becomes equal in status with fields. In other words, the space itself must be random or quantum. We note that the concept of a quantum discrete space is often mentioned in connection with non-commutative geometry [5]–[8]. We do not study this connection here, but we stress the aspect of quantum spaces connected with locality. In physics the local property is mainly associated with the Gibbs property. The space is constructed from discrete elements (quanta) via a special (Gibbs) procedure. If we assume that a neighbourhood of a point can contain arbitrarily many quanta, then to obtain the known spaces in this general treatment of locality one needs renormalization procedures similar to those in quantum field theory, whereas if one understands locality in the metric sense, that is, if a quantum corresponds to an entire neighbourhood, then one needs no renormalizations. The latter understanding is not quite satisfactory because the metric need not be fixed.

The structure of the paper is as follows. In § 2.1 we introduce two definitions of finite Gibbs discrete spaces (finite Gibbs families), namely, a local space (for which the energy is equal to the sum of the energies of the subgraphs) and a superlocal space (for which the energy is equal to the sum of the energies of all neighbourhoods of a given radius). This corresponds to a different understanding of the local property if there is no fixed space. It is shown that these two definitions can be reduced to each other only by a procedure similar to renormalization in quantum field theory. In § 2.2 we consider limit Gibbs families and find an analogue of the Dobrushin–Lanford–Ruelle condition and the compactness condition. It is shown that, for a given Gibbs family, the conditional measure on the configurations (under the condition that the graph is fixed) is a Gibbs measure (in the ordinary sense) with the same potential. In § 2.3 we suggest an analogue of probability measures on the set of countable graphs without enumeration of the vertices and without a distinguished vertex for which the Kolmogorov approach of finite-dimensional distributions does not work. We refer to this analogue as an empirical distribution, and we present examples of such distributions arising from discrete quantum gravity.

In § 3 we obtain the logarithmic asymptotics for $\sum_g F_{b,p}(g)$, where $F_{b,p}(g)$ is the number of charts with $p+1$ vertices and $b+p$ edges on a closed compact surface of genus g. We note that a triangulation is a special case of a chart, and attempts to compute the number of charts on a surface of fixed genus show that the character of the asymptotic behaviour is the same and does not depend on the class of charts under consideration.

In § 4.2 we study in detail the simplest model of discrete planar gravity (with boundary) which can be completely controlled, that is, one can prove the existence of the thermodynamic limit, find the explicit form of the free energy, and give the complete phase diagram with three different phases, including the critical point. We also prove the effect of the influence of an observer on the fluctuations of the length of the boundary at a critical point. The corresponding model with two boundaries is studied in the 'high temperature' domain only. In § 4.3 we give an example of a topological phase transition in a locally homogeneous graph such that the 1-neighbourhoods of all vertices are the same. In § 4.4 we prove the uniqueness of the limit Gibbs family for a model with factorial-type growth of the partition function. This also enables us to give a local Gibbs characterization of trees and of free non-commutative groups. In § 4.5 we discuss how substantially different topologies can appear on different scales. The notion of scale generalizes that of thermodynamic limit.

In § 5.1 we introduce quantum analogues of classical Gibbs graphs with local structure, the so-called quantum grammars on graphs. We prove basic results on self-adjointness of the corresponding Hamiltonian and the existence of the automorphism group of the corresponding C^*-algebra. Quantum spin systems form a special case of these constructions. For linear graphs and for high temperatures we prove the existence and analyticity of KMS states. In § 5.2 we give diverse examples of C^*-algebras and Hamiltonians in quantum grammars, including linear grammars and Toeplitz operators, quantum expansion and contraction of the space, the two-dimensional Lorentzian model, and so on.

§ 2. Gibbs Families

2.1. Finite Gibbs families.

Spin graphs. We consider graphs G (finite or countable) with vertex set $V = V(G)$ and edge set $L = L(G)$. It is always assumed that there is at most one edge between any two vertices and that each vertex has finite degree (the number of incident edges). The graphs are assumed to be connected unless otherwise stated.

It is assumed that these graphs are also loop-free unless otherwise stated. However, the general results can readily be extended to graphs with loops and with multiple edges. For these more general graphs, each loop adds the number 2 to the degree of the vertex.

A *subgraph* of a graph G is defined by a set of vertices $V_1 \subset V$ together with some edges that are inherited from L and connect some vertices in V_1. A subgraph $G(V_1)$ of a graph G with vertex set V_1 is said to be *regular* if it contains each edge in $L(G)$ between vertices in V_1.

The set V of vertices forms a metric space in which the distance $d(x,y)$ between any vertices $x, y \in V$ is defined as the minimum length of the paths joining these vertices. The length of an edge is assumed to be equal to 1.

A spin graph $\alpha = (G, s)$ is defined as a graph G together with a function $s \colon V \to S$, where S is some set (of spin values). The graphs and spin graphs are always regarded up to isomorphism, that is, no enumeration of vertices is assumed. A graph isomorphism is a one-to-one correspondence between the vertex sets that respects the edges and spins. Let $\mathcal{G}_N$ ($\mathcal{A}_N$) be the set of equivalence classes (with respect to isomorphism) of connected graphs (spin graphs) with N vertices. The countable graphs correspond to $N = \infty$.

Below we use a simplified terminology if this leads to no confusion. For instance, we speak of a graph or spin graph instead of the corresponding equivalence class, of a subgraph instead of spin subgraph, and so on.

Rooted spin graphs. We also consider graphs with a distinguished vertex (so-called rooted graphs), where the vertex is referred to as the *origin* or *root.* If this is the case, then we consider only isomorphisms preserving the origin. To stress that the graphs under consideration have a distinguished vertex, say v, we write $G^{(v)}$, $\mathcal{G}_N^{(v)}, \dots$.

Let us define an annular neighbourhood $\gamma(\alpha, v; a, b)$ of the vertex v as the regular subgraph of the spin graph α with vertex set $V(\gamma(\alpha, v; a, b)) = \{v' : a \leqslant d(v, v') \leqslant b\}$. The set $O_d(v) = \gamma(\alpha, v; 0, d)$ is called the *d-neighbourhood of* v, and we write $O(v) = O_1(v)$. The number $R_0(G) = \max_{v \in V(G)} d(0, v)$ is referred to as the *radius* of G with respect to the vertex 0. In particular, $R_v(O(v)) = 1$. In this case we mean by the *diameter* of the graph G the number $d(G) = \max_{v,v' \in V(G)} d(v, v') = \max_{v \in V(G)} R_v(G)$.

We denote by $\mathcal{G}_N^{(0)}$ ($\mathcal{A}_N^{(0)}$) the set of equivalence classes of connected graphs (spin graphs) of radius N with respect to 0 that are rooted at 0. We also denote by $\mathcal{A}^{(0)} = \bigcup_{N \neq \infty} \mathcal{A}_N^{(0)}$ the set of all finite spin graphs rooted at 0.

Graphs with local structure. A simple generalization of a spin graph is given by the notion of graph with local structure, which suffices to describe arbitrary multidimensional objects.

Definition 1. Let $s(\gamma)$ ($t(\gamma)$) be a function on $\mathcal{A}_d^{(0)}$ (on the set $\mathcal{A}(d)$ of spin graphs of diameter d, respectively) with range in some set S. A superlocal structure (of radius d) on G is given by the set of values $s(O_d(v))$, $v \in V(G)$. A local structure (of diameter d) on G is given by the set of values $t(\gamma)$ for all regular subgraphs γ of diameter d.

Examples.

- The spin graphs correspond to local (or superlocal) structures with $d = 0$. Many definitions and results mentioned below for spin graphs can be generalized to the case $d > 0$ by appropriate modifications.
- Gauge fields on graphs: to any vertex and any edge we assign a value in a group U; in this case one can set $d = 1$.
- A *simplicial complex* is completely determined by its one-dimensional skeleton if we assign the value 1 to every complete regular subgraph γ of diameter 1 of the one-dimensional skeleton whenever this subgraph defines a simplex of the corresponding dimension, and the value 0 otherwise. Here one can also take $d = 1$.
- Penrose quantum networks [9], [10]: in a finite graph each of whose vertices is of degree 3, we assign to every edge l an integer $p_l = 2s_l$, where the half-integers s_l are interpreted as degrees of irreducible representations of the group $SU(2)$. Moreover, the following condition must be satisfied at any vertex: the sum of the three numbers p_i is even, and any number p_i does not exceed the sum of the other two. In this case there is a unique invariant element (up to a factor) for the tensor product of the three representations, and we assign this element to the vertex.
- However, there are structures on graphs that cannot be characterized as graphs with local structure (for example, trees, *Cayley graphs* of groups,

and so on). Still, we shall see that a Gibbs characterization of these graphs is possible (which is also local in a sense).

σ-algebra and a free measure. Let $\mathcal{A}$ be an arbitrary set of finite spin graphs $\alpha = (G, s)$. Let $\mathcal{G} = \mathcal{G}(\mathcal{A})$ be the set of all graphs G for which there is a function s with $\alpha = (G, s) \in \mathcal{A}$. It is always assumed that if $\alpha = (G, s) \in \mathcal{A}$, then (G, s') also belongs to $\mathcal{A}$ for any s'.

Then $\mathcal{A}$ is a topological space that is a discrete (finite or countable) union of the topological spaces $T_G = S^{V(G)}$, $G \in \mathcal{G}(\mathcal{A})$. The Borel σ-algebra on $\mathcal{A}$ is generated by the cylindrical subsets $A(G, B_v, v \in V(G))$, $G \in \mathcal{G}$, where the B_v are Borel subsets of S. Moreover, $A(G, B_v, v \in V(G))$ is the set of all $\alpha = (G, s)$ such that the graph G is the same for any α and the functions (configurations) $s(v)\colon V(G) \to S$ satisfy the condition $s(v) \in B_v$ for any $v \in V(G)$.

Let a non-negative (not necessary probability) measure λ_0 be given on S. The corresponding free measure is defined to be the non-negative measure $\lambda_{\mathcal{A}}$ on $\mathcal{A}$ given by

$$\lambda_{\mathcal{A}}(A(G, B_v, v \in V(G))) = \prod_{v \in V(G)} \lambda_0(B_v). \tag{1}$$

Potentials. By a (*local*) *potential* we mean a function $\Phi\colon \cup_{N<\infty} \mathcal{A}_N \to R \cup \{+\infty\}$ (that is, a function on the set of finite spin graphs) which is invariant with respect to isomorphisms. If Φ can take the value ∞, then we say that Φ has a *hard core.* We say that Φ has a *finite diameter* if $\Phi(\alpha) = 0$ for any α whose diameter is greater than some constant d. The minimum value d with this property is called the *diameter* of Φ.

For a potential Φ, the energy of the subgraph α is defined as

$$H(\alpha) = \sum_{\gamma \subset \alpha} \Phi(\gamma), \qquad \alpha = (G, s), \tag{2}$$

where the sum is taken over all regular connected subgraphs γ of the spin graph α.

A *chemical potential*, which is a special case of a potential, is given by a function Φ that is equal to some constant μ_0 at any vertex v (which does not depend on the spin at v), to some constant μ_1 for each edge, and to 0 otherwise. In other words, for a chemical potential we have

$$H_N(\alpha) = \mu_0 V(\alpha) + \mu_1 L(\alpha), \tag{3}$$

where $V(\alpha)$ and $L(\alpha)$ are the numbers of vertices and edges of α.

Finite Gibbs families. Let $\mathcal{A}$ be a set of finite spin graphs. By a *Gibbs $\mathcal{A}$-family with potential* Φ we mean a probability measure $\mu_{\mathcal{A}}$ on $\mathcal{A}$ determined by the following density with respect to the corresponding free measure:

$$\frac{d\mu_{\mathcal{A}}}{d\lambda_{\mathcal{A}}}(\alpha) = Z^{-1} \exp(-\beta H(\alpha)), \qquad \alpha \in \mathcal{A}, \tag{4}$$

where $\beta \geqslant 0$ is the inverse temperature. Here it is assumed that $Z \neq 0$, that is, $H(\alpha) < \infty$ for at least one value $\alpha \in \mathcal{A}$. Below we give examples with $Z = 0$. We also assume that if $\mathcal{A}$ is infinite, then the following stability condition holds:

$$Z = Z(\mathcal{A}) = \int_{\mathcal{A}} \exp(-\beta H(\alpha))\, d\lambda_{\mathcal{A}} < \infty. \tag{5}$$

If S is trivial (a singleton), then we obtain a probability distribution on the graphs, which is referred to as a *Gibbs graph.*

Stability. A potential Φ is said to be *stable* on $\mathcal{A}_N^{(0)}$ if the partition function Z_N is finite for any N. The potential $\Phi \equiv 0$ is obviously not stable, nor is the chemical potential with $\mu_0 = 0$.

Lemma 1. *For any finite μ_0 and μ_1 the chemical potential is not stable on $\mathcal{A}_N^{(0)}$.*

Proof. Let $g(n,k)$ be the number of graphs of radius 1 for which the degree of the vertex 0 is equal to n and the number of edges joining the other vertices is equal to k. It suffices to prove that the quantity

$$G_n = \sum_{k=0}^{N} g(n,k)\exp\bigl(-\mu_1(k+n) - \mu_0(n+1)\bigr)$$

increases as n increases, where $N = n(n-1)/2$ is the number of distinct pairs formed by n enumerated vertices.

Indeed, the number of graphs with n enumerated vertices and k edges is equal to $\binom{N}{k}$. At the same time, the number of enumerations of n vertices does not exceed $n! = 2^{n\log_2 n + O(n)}$. Let us prove that for any μ_1 there is a sufficiently small $\delta > 0$ such that $G_n \to \infty$ as $n \to \infty$ for $k = \delta N$. We have

$$\begin{aligned} &g(n,k)\exp\bigl(-\mu_1(k+n) - \mu_0(n+1)\bigr) \\ &\qquad \approx 2^{-N(\delta\log_2\delta + (1-\delta)\log_2(1-\delta)) + O(n\log_2 n)}\exp(-\mu_1(k+n) - \mu_0(n+1)) \\ &\qquad > \exp(-\mu_1\delta N)2^{N(-\delta\log_2\delta+\delta)+O(n\log_2 n)} \to \infty \end{aligned}$$

if δ is sufficiently small.

Instability phenomena can occur because of large degrees of vertices. For this reason, a regularization is needed. One of the possibilities is to introduce regularizing potentials.

An example of a regularizing potential is as follows. Let γ_2 (γ_3) be the graph consisting of three vertices $1, 2, 3$ and two edges $12, 13$ (three edges $12, 13, 23$, respectively). The potential $\Phi_2 = \mu_2 P(\alpha)$, where $P(\alpha)$ is the number of regular subgraphs of α isomorphic to either γ_2 or to γ_3, ensures the stability of the potential $\Phi + \Phi_2$ for many potentials Φ. For example, the following assertion holds (we omit the proof).

Proposition 1. *If $\mu_2 > \log 2$, then the potential $\Phi = \Phi_0 + \Phi_1 + \Phi_2$ is stable for any $\mu_0, \mu_1 \geqslant 0$.*

However, the most interesting potentials are equal to ∞ on graphs of a special kind. For example, a potential can be infinite on all graphs of diameter greater than or equal to 1 or 2 such that at least one of the vertices is of degree $r+1$. Consideration of this potential is equivalent to restriction of our treatment to graphs whose vertices are all of degree not exceeding r.

Another possibility is to understand locality in some other way.

Locality and superlocality. In many papers on discrete quantum gravity it is assumed in advance that an action occurs on a smooth or a piecewise-linear manifold of a given dimension. In this case, the space remains locally classical. A more consistent program might be to abandon this assumption. However, in this case the central question arises: what is locality if there is no space?

It is immediately clear that if the background space is absent, then locality can be understood in many ways. Firstly, an object can be said to be local if it is defined in terms of 'small' connected subgraphs. A substantially stronger notion of locality (superlocality) holds if we know that the subgraph exhausts a neighbourhood of a given vertex. In this connection, we mean by a superlocal potential of radius r a function $\Phi\colon \mathcal{A}_r^{(0)} \to \mathbb{R} \cup \{+\infty\}$ (that is, a function on the set of finite rooted spin graphs of radius r with respect to 0) such that Φ is invariant under isomorphisms.

The energy of a superlocal potential is defined as

$$H(\alpha) = \sum_{v \in V(\alpha)} \Phi(O_r(v)),$$

where the sum is taken over all vertices of the spin graph α. In this case we shall speak of superlocal Gibbs families, and the previously introduced Gibbs families are said to be local.

A superlocal potential can be reduced to a local one; however, this needs an additional limit procedure described below. One can say that, if one wants to describe the world locally (but without *a priori* restrictions on the local structure of the graph), then one must use renormalizations.

To construct examples, the following class of graphs is useful. Let us consider a set of 'small' graphs $G_1, \dots, G_k$ with root 0 and of radius r with respect to 0. We say that a graph G is generated by $G_1, \dots, G_k$ if any vertex of G has a neighbourhood (of radius r) isomorphic to one of the graphs $G_1, \dots, G_k$. We denote by $\mathcal{G}(G_1, \dots, G_k)$ the set of all such graphs and introduce a superlocal potential $\Phi = \Phi_{G_1,\dots,G_k}$ of radius r by setting $\Phi(\Gamma) = 0$ if the rooted subgraph Γ is isomorphic to one of the graphs G_i and $\Phi(\Gamma) = \infty$ otherwise. It is obvious that any Gibbs family with potential Φ is supported by $\mathcal{G}(G_1, \dots, G_k)$.

Example with $Z_N = 0$. A finite Gibbs family with potential $\Phi = \Phi_{G_1,\dots,G_k}$ need not even exist. For the simplest example we consider the case with $k = 1$ and $r = 1$ in which G_1 is a finite complete graph g_m of radius 1 with $m+1$ vertices $0, 1, \dots, m$ and root 0. Then it is 'self-generating', and $\mathcal{G}(G_1)$ consists of a single element, the graph g_m itself.

Let us give another example. We introduce the graph $g_{m,0}$ with $m + 1$ vertices $0, 1, \dots, m$ and m edges $01, 02, \dots, 0m$ and the graph $g_{m,m}$ with $m + 1$ vertices $0, 1, \dots, m$ and $2m$ edges $01, 02, \dots, 0m, 12, 23, \dots, m1$. One can see that the pentagon $G_1 = g_{5,5}$ cannot generate a countable graph.

Renormalization. A superlocal potential can be expressed via local ones as follows (we shall explain this for $k = 1$). Let the radius of G_1 be equal to r. We construct a finite Gibbs family supported by $\mathcal{G}(G_1)$ without using superlocal potentials. Let $F(G_1)$ be the set of all graphs isomorphic to some proper regular subgraph γ of G_1 such that the radius of γ with respect to some vertex $v \in V(\gamma)$ is equal to r, and let $F(\Gamma, \gamma)$ be the set of all proper regular subgraphs of Γ isomorphic to γ.

We introduce the truncated local potential Φ_R by setting

- $\Phi_R(\gamma) = \infty$ for the graphs γ which are of radius r with respect to some vertex of γ and satisfy the conditions $\gamma \neq G_1$ and $\gamma \notin F(G_1)$;

- $\Phi_R(\gamma) = 0$ for the graphs that are not of radius r with respect to any vertex of γ;
- $\Phi_R(\gamma) = R$, where $R > 0$, for any graph $\gamma \in F(G_1)$;
- $\Phi_R(G_1) = 0$.

The limit of the Gibbs family $\mu_{\mathcal{A}}(R)$ with potential Φ_R as $R \to \infty$ need not exist; however, the following theorem holds.

Theorem 1. *Suppose that $\mathcal{G}(G_1) \cap \mathcal{A}_N \neq \varnothing$. In this case one can find a number k and local potentials $\phi_R^{(j)}$, $j = 1, 2, \dots, k$, such that the limit as $R \to \infty$ of the Gibbs family on $\mathcal{A}_N$ with the renormalized potential*

$$\Phi_R^{\text{ren}} = \Phi_R + \sum_{j=1}^{k} \phi_R^{(j)}$$

exists and is supported by $\mathcal{G}(G_1) \cap \mathcal{A}_N$.

Proof. Let us define the potentials (counterterms) $\phi_R^{(j)}$ that vanish everywhere except for the following cases:

- if Γ is isomorphic to G_1, then

$$\phi_R^{(1)}(\Gamma) = \sum_{\gamma \in F(G_1)} \phi_{R,\gamma}^{(1)}(\Gamma), \qquad \phi_{R,\gamma}^{(1)}(\Gamma) = -R \sum_{\Gamma \in F(G_1,\gamma)} 1;$$

- for any $j \geqslant 2$, if Γ is the union of j pairwise distinct subgraphs $\Gamma_i, i = 1, \dots, j$, where every graph Γ_i is isomorphic to G_1, then

$$\phi_R^{(j)}(\Gamma) = \sum_{\gamma \in F(G_1)} \phi_{R,\gamma}^{(j)}(\Gamma), \qquad \phi_{R,\gamma}^{(j)}(\Gamma) = (-1)^j R \sum_{\Gamma \in F(\Gamma_1,\gamma) \cap \dots \cap F(\Gamma_j,\gamma)} 1.$$

It follows from the definition that one has $\phi_R^{(j)} = 0$ starting from some j. Let us consider a vertex v of the graph. If the neighbourhood $O_r(v)$ does not belong to $F(G_1)$, then the potential Φ_R is equal to ∞, which is impossible. If the neighbourhood $O_r(v)$ is not isomorphic to G_1 and belongs to $F(G_1)$, then the potential Φ_R is equal to R on this neighbourhood and the counterterms vanish, and hence $\Phi_R^{(\text{ren})}$ is also equal to R. Let us now consider a subgraph $\gamma \in F(G_1)$ that belongs to exactly m subgraphs Γ_i isomorphic to G_1. Then it follows from the inclusion-exclusion formula that

$$\Phi_R(\gamma) + \sum_{j=1}^{m} \phi_{R,\gamma}^{(j)}(g) = 0$$

for any γ. Therefore, the energy of the graphs in $\mathcal{G}(G_1) \cap \mathcal{A}_N$ is equal to zero, and for the remaining graphs it is not less than R, which gives the result.

We note that the summands $\phi_R^{(j)}(\Gamma)$ are similar to the counterterms of the ordinary renormalization theory. For $k > 1$ we face a more complicated combinatorics. For some models with continuous spin there may be relationships between these counterterms and those of the standard renormalization theory.

2.2. Limit Gibbs families. Let us introduce a topology on the set $\mathcal{A}_\infty^{(0)}$ of countable spin graphs rooted at 0 by defining a basis of open sets. Let C_N be an arbitrary open subset of the set $\mathcal{A}_N^{(0)}$ of all spin graphs with radius N with respect to the origin. Then the open sets of the basis consist of all spin graphs in which the N-neighbourhood of the origin belongs to C_N for some N and C_N. For the σ-algebra in $\mathcal{A}_\infty^{(0)}$ we take the Borel σ-algebra generated by these open sets.

For any spin graph α with root 0 the set $\gamma(\alpha, 0; N, N+d)$ is called an *N-annulus* of width d and the sets $\gamma(\alpha, 0; 0, N)$ and $\gamma(\alpha, 0; N, \infty)$ are referred to as the *N-internal* and *N-external* parts of the graph, respectively. We note that an N-annulus and an N-external part can be disconnected. For given N and d and for some finite graph γ (not necessarily connected) we denote by $\mathcal{A}_{N+d}^{(0)}(\gamma, d) \subset \mathcal{A}_{N+d}^{(0)}$ the set of all finite connected spin graphs α with root 0 and radius $N+d$ with respect to the origin for which γ is the N-annulus of width d.

Let a local potential Φ of diameter $d+1$ be given.

Definition 2. A measure μ on $\mathcal{A}_\infty^{(0)}$ is called a *Gibbs family with local potential* Φ if for any N, γ, and Γ the conditional distribution on the set of spin graphs $\alpha \in \mathcal{A}_\infty^{(0)}$ for which Γ is the N-external part and γ is the N-annulus of width d (this set can be naturally identified with $\mathcal{A}_{N+d}^{(0)}(\gamma, d)$) coincides almost surely with the (finite) Gibbs family with potential Φ on $\mathcal{A}_{N+d}^{(0)}(\gamma, d)$.

In particular, this conditional distribution depends only on γ rather than on the entire N-external part Γ.

Boundary conditions. As in the usual theory, the notion of a finite Gibbs family with given boundary conditions is useful for finding pure phases. The boundary conditions (d-boundary conditions) are given by a sequence $\nu_{N,d}(\gamma)$ of probability measures on the set of finite (not necessarily connected) spin graphs γ (intuitively, on the N-annuli of width d). A Gibbs family on $\mathcal{A}_N^{(0)}$ with the boundary conditions $\nu_{N+1,d}(\gamma)$ is defined as ($\alpha \in \mathcal{A}_N^{(0)}$)

$$\mu_N(\alpha) = Z^{-1}(N, \nu_{N+1,d}) \sum_{\xi \in \mathcal{A}_{N+d+1}^{(0)}(\gamma, d; \alpha)} \int \exp(-\beta H(\xi))\, d\nu_{N+1}(\gamma, d), \tag{6}$$

$$Z(N, \nu_{N+1,d}) = \int_{\mathcal{A}_N^{(0)}} \left[\sum_{\xi \in \mathcal{A}_{N+d+1}^{(0)}(\gamma, d; \alpha)} \int \exp(-\beta H(\xi))\, d\nu_{N+1}(\gamma, d) \right] d\lambda_{\mathcal{A}_N^{(0)}},$$

where $\mathcal{A}_{N+d+1}^{(0)}(\gamma, d; \alpha)$ is the set of all spin graphs in $\mathcal{A}_{N+d+1}^{(0)}$ with $(N+1)$-annulus γ of width d and with N-internal part α. We note that $\mathcal{A}_{N+d+1}^{(0)}(\gamma, d; \alpha)$ is finite for any α and γ.

Compactness. Gibbs families on the set $\mathcal{A}_\infty^{(0)}$ of connected countable spin graphs with root 0 can be obtained as weak limits of Gibbs families on $\mathcal{A}_N^{(0)}$. In principle, there can be three reasons for a limit Gibbs family to be non-existent: finite Gibbs families with this potential can be absent (see examples above), S can be non-compact, and the distribution of finite Gibbs families can be concentrated on graphs that have vertices of large degree. The assumptions of the next proposition correspond to this list; however, compactness can certainly hold under more general conditions.

Let $\mathcal{A}_\infty^{(0)}(r) \subset \mathcal{A}_\infty^{(0)}$ be the set of all countable spin graphs with vertices of degree not greater than r.

Proposition 2. *Let S be compact. Let Φ have finite diameter, and let $\Phi(\gamma) = \infty$ if at least one vertex of γ is of degree $r+1$. Let a Gibbs family with potential Φ on $\mathcal{A}_N^{(0)}$ exist for any N. Then there is a Gibbs family with potential Φ supported by $\mathcal{A}_\infty^{(0)}(r)$.*

Proof. Let us consider the Gibbs families μ_N on $\mathcal{A}_N^{(0)}$ (in fact, on $\mathcal{A}_N^{(0)}(r)$). Let $\widetilde{\mu}_N$ be an arbitrary probability measure on $\mathcal{A}_\infty^{(0)}(r)$ such that its quotient measure on $\mathcal{A}_N^{(0)}(r)$ coincides with μ_N. Then the sequence of measures $\widetilde{\mu}_N$ on $\mathcal{A}_\infty^{(0)}(r)$ is compact, as can be proved by the standard diagonal procedure of enumeration of all N-neighbourhoods. It is clear that any limit point of the sequence $\widetilde{\mu}_N$ is a Gibbs family with potential Φ.

Gibbs families and Gibbs fields. The next simple result explains why we use the term "Gibbs family". Let μ be a Gibbs family with potential Φ. We assume that the conditions of the previous proposition hold. Let us consider the following measurable partition of the set $\mathcal{A}_\infty^{(0)}$: any element S_G of this partition is defined by a fixed graph G (without spins) and consists of all configurations s_G on G.

Proposition 3. *For a given graph G the conditional measure on the set $S_G = \{s_G\}$ of configurations is almost surely a Gibbs measure on G with the same potential Φ.*

We note that the expression "the same" has here the following obvious meaning: if Λ_N is the N-neighbourhood of zero in G and s_{Λ_N} is a configuration on Λ_N, then $H(s_{\Lambda_N}) = \sum \Phi(\gamma)$, where the sum is taken over all subgraphs of the spin graph $(\Lambda_N, s_{\Lambda_N})$. Thus, any Gibbs family is a convex combination (of a very special kind) of Gibbs measures (fields) with the same potential on some fixed graphs. This convex combination is defined by a measure $\nu = \nu(\mu)$ on the quotient space $\mathcal{G}_\infty^{(0)} = \mathcal{A}_\infty^{(0)}/\{S_G\}$, and ν is induced by the measure μ corresponding to the Gibbs family itself.

The following lemma (having a well-known and useful analogue for ordinary Gibbs fields) reduces the proof of the assertion to the corresponding result for Gibbs families on finite graphs, whereas the proof for finite graphs is a direct calculation, and we omit it.

Lemma 2. *Let μ be a Gibbs family on $\mathcal{A}_\infty^{(0)}$ with potential Φ. Let $\nu_{N+1,d}(\gamma)$ be a measure on the N-annuli of width d, and let this measure be induced by μ. Then μ is a weak limit of the Gibbs families on $\mathcal{A}_N^{(0)}$ with potential Φ and boundary conditions $\{\nu_{N+1,d}(\gamma)\}$.*

2.3. Empirical distributions. One can construct probability measures on $\mathcal{A}_\infty^{(0)}$ in the standard way by using the Kolmogorov approach with consistent distributions on cylindrical subsets. However, this approach does not work if we intend to construct probability measures on the set $\mathcal{A}_\infty$ of equivalence classes of countable connected spin graphs. The problem is that there is no natural enumeration of the countable sets of vertices, and thus it is not clear how to introduce finite-dimensional distributions. One can say that the problem is that there is no coordinate system, and no reference point. This is why the Kolmogorov approach does not work. However, there is an analogue of finite-dimensional distributions. A system of numbers thus obtained is called an *empirical distribution* on $\mathcal{A}_\infty$.

Let S be finite or countable, and consider the system of numbers

$$\pi = \{p(\Gamma), \Gamma \in \mathcal{A}^{(0)}\}, \qquad 0 \leqslant p(\Gamma) \leqslant 1, \tag{7}$$

(that is, Γ ranges over the finite spin graphs with root 0). We introduce the consistency property for these numbers for any $k = 0, 1, 2, \dots$ and any fixed spin graph Γ_k of radius k (with respect to 0) as follows:

$$\sum_{\Gamma_{k+1}} p(\Gamma_{k+1}) = p(\Gamma_k), \qquad k = 0, 1, 2, \dots, \tag{8}$$

where the sum is taken over all spin graphs Γ_{k+1} of radius $k+1$ such that the neighbourhood $O_k(0)$ in Γ_{k+1} is isomorphic to Γ_k. We recall that in the last formula it is assumed that the sum is taken over all equivalence classes of spin graphs.

We also assume the normalization condition

$$\sum p(\Gamma_0) = 1, \tag{9}$$

where Γ_0 is the vertex 0 with an arbitrary spin on it.

Definition 3. Any system π of the above form is called an *empirical distribution.*

One can rewrite the consistency condition in terms of conditional probabilities as follows:

$$\sum_{\Gamma_{k+1}} p(\Gamma_{k+1} \mid \Gamma_k) = 1, \qquad k = 0, 1, \dots, \tag{10}$$

where the summation is as above and

$$p(\Gamma_{k+1} \mid \Gamma_k) = \frac{p(\Gamma_{k+1})}{p(\Gamma_k)}. \tag{11}$$

Thus, we regard $\mathcal{A}^{(0)}$ as a tree whose vertices are spin graphs, and Γ_{k+1} and Γ_k are joined by an edge if and only if Γ_k is isomorphic to the k-neighbourhood of 0 in Γ_{k+1}.

It is clear that the system π defines a probability measure on $\mathcal{A}_\infty^{(0)}$. The examples below show that it is natural to interpret this system as a 'measure' on $\mathcal{A}_\infty$.

Examples of empirical distributions can be obtained via the following limit procedure, which we call the empirical limit. Let μ_N be a probability measure on $\mathcal{A}_N$. For any N, k and any spin graphs $\Gamma \in \mathcal{A}_k^{(0)}$ and $\alpha \in \mathcal{A}_N$ we set

$$p^N(\Gamma) = \left\langle \frac{n^N(\alpha, \Gamma)}{N} \right\rangle_{\mu_N}, \tag{12}$$

where $n^N(\alpha, \Gamma)$ is the number of vertices in α whose k-neighbourhoods are isomorphic to the spin graph Γ. We write $\pi_N = \{p^N(\Gamma)\}$. For the measures μ_N one can take some Gibbs families on $\mathcal{A}_N$ with a fixed potential. The system π_N is not itself an empirical distribution. However, the following assertion holds.

Lemma 3. *Let S be finite and let*

$$\mu_N\Big(\min_{\alpha\in\mathcal{A}_N} D(\alpha)\leqslant D\Big)\to 0 \tag{13}$$

for any D as $N\to\infty$, where $D(\alpha)$ is the diameter of α. Then any weak limit point of the system π_N is an empirical distribution.

Proof. The probability μ_N that $R_v(\alpha)>D$ for any vertex v of a random graph tends to 1 as $N\to\infty$. Therefore, for any positive integer n_0 and any number $\delta>0$ there is an $N_0=N_0(n_0,\delta)$ such that for any $N>N_0$ the numbers $p_N(\Gamma)$ satisfy the consistency conditions with accuracy δ for all subgraphs Γ_k with $k<n_0$. Since S is finite, it is compact, that is, there is at least one limit empirical family.

Remark 1. The question arises of how to characterize the empirical distributions which can be obtained in this way, and, in particular, the empirical distributions which can be obtained from themselves, that is, as limits of numbers of the form

$$p^N(\Gamma)=\frac{\langle n^N(\Gamma_N,\Gamma)\rangle_{\pi_N}}{\langle V(\Gamma_N)\rangle_{\pi_N}},$$

where $V(\Gamma_N)$ is the number of vertices in Γ_N and π_N is the restriction of the measure π to $\mathcal{A}_N^{(0)}$.

Remark 2. We note that an empirical measure is automatically 'homogeneous' in an obvious sense, that is, it is the same for any vertex. The reason is that the vertices are not enumerated, or the graph is not embedded in any space. However, one can readily introduce an 'inhomogeneity' on $\mathcal{A}_\infty^{(0)}$, namely, it suffices to take Φ depending on the distance from 0.

Remark 3. There is another possible approach to empirical distributions, based on locality rather than superlocality, and then we 'do not know' any entire neighbourhood of a vertex. For instance, let two graphs $\gamma\subset\Gamma$ be given such that Γ has no other subgraph isomorphic to γ. Then one can study the limits of the ratios (the conditional empirical distributions)

$$\left\langle\frac{n^N(\alpha,\Gamma)}{n^N(\alpha,\gamma)}\right\rangle_{\mu_N}.$$

2.3.1. *Empirical Gibbs families.* We show that limit Gibbs families on $\mathcal{A}_\infty^{(0)}$ can sometimes be obtained by passing to the limit with respect to the number of vertices rather than with respect to the radius. Let $\mathcal{B}_{N,p}^{(0)}$ be the set of spin graphs rooted at 0 with N vertices such that the degree of any vertex is at most p, and consider a finite Gibbs family μ_N on $\mathcal{B}_{N,p}^{(0)}$ with some local potential Φ.

Lemma 4. *Any weak limit point of the sequence μ_N is a limit Gibbs family on $\mathcal{A}_\infty^{(0)}$ with the potential Φ.*

Proof. Let us fix n, d and some $\alpha\in\mathcal{A}_{n+d+1,p}^{(0)}$ with n-annulus γ of width $d+1$. Then for any sufficiently large N there is a $\beta\in\mathcal{B}_{N,p}^{(0)}$ such that $O_{n+d+1}(\beta,0)$ is isomorphic to α. For any such β the conditional distribution on $O_n(\alpha, 0)$ (under the condition γ) is a Gibbs distribution with the potential Φ.

An empirical distribution π on $\mathcal{A}_\infty$ is said to be a *Gibbs distribution* if it can be obtained as an empirical limit of finite Gibbs families π_N on $\mathcal{A}_N$ with some potential Φ.

To justify this definition, let us show that under certain conditions an empirical distribution π on $\mathcal{A}_\infty$ is a Gibbs family as a measure on $\mathcal{A}_\infty^{(0)}$. We assume that the empirical distribution π on $\mathcal{A}_\infty$ is a Gibbs distribution with potential Φ which takes the infinite value if the degree of at least one vertex is greater than p; we also suppose that the probability π_N of the event that the automorphism group of the graph $\alpha \in \mathcal{A}_N$ is trivial tends to 1 as $N \to \infty$. In this case π is a Gibbs family as a measure on $\mathcal{A}_\infty^{(0)}$. Indeed, if the automorphism group of a graph is trivial, then one can position the root at any vertex, which results in different rooted graphs. Then we have Gibbs families on $\mathcal{B}_{N,p}^{(0)}$, and by the previous lemma we obtain a limit Gibbs family on $\mathcal{A}_\infty^{(0)}$.

2.3.2. *Empirical distributions in planar gravity.* It should be stressed that the existence problem for limit correlation functions (which is usually ignored in physics) is far from being trivial. This problem not only is mathematically natural but also has aspects that are also of interest for physics. The point is that, under the definition of a distribution by means of the grand canonical ensemble (as is customary in discrete quantum gravity), the thermodynamic passage to the limit makes sense only at some critical point, which determines the boundary of the domain of convergence of the corresponding series. Moreover, this point depends heavily on geometric details even if the topology of the manifold is fixed. On summation over the topologies, the corresponding series diverges for all values of the parameters already in dimension two because the entropy has factorial asymptotic behaviour. We do not face these difficulties if we use the canonical ensemble related to our definition of Gibbs distributions.

Let us look at common examples of empirical Gibbs distributions. In physics, the two-dimensional planar quantum gravity (or a string in dimension 0) is determined by the grand partition function

$$P(T) = Z^{-1} \exp(-\mu F(T)), \qquad Z = \sum_{T \in \mathcal{A}} \exp(-\mu F(T)) = \sum_N C(N) \exp(-\mu N),$$

where the first sum is taken over all triangulations T of the sphere that belong to some class $\mathcal{A}$ and have N triangles. The exact specification of the class $\mathcal{A}$ is not essential (see [11]), but different classes $\mathcal{A}$ do lead to different empirical distributions. The number $C(N) = C_{\mathcal{A}}(N)$ is called the *entropy* and is equal to the number of these triangulations. As is known,

$$C(N) \sim aN^{-\frac{7}{2}} c^N.$$

One can regard N (if needed) as the number of vertices of the dual graph, which is a 3-regular graph embedded in the two-dimensional sphere. The grand ensemble is normally studied as $\mu \to \mu_{\rm cr} = -\log c$. In the canonical ensemble one need not introduce the parameter μ.

Let us consider triangulations with a distinguished vertex 0 and a fixed neighbourhood $O_d(0) = \Gamma_d$ of radius d. Let $p^N(\Gamma_d)$ be the probability of this neighbourhood in our canonical ensemble.

Theorem 2. *The limit* $\lim p^N(\Gamma_d) = \pi(\Gamma_d)$ *exists and defines an empirical distribution.*

We present a proof using explicit formulae. Let Γ_d have u triangles and k boundary edges (that is, edges for which both vertices belong to the d-annulus). Let $r(d,k,u)$ be the number of such neighbourhoods. The external part is a triangulation of a disc with $N-u$ triangles and k boundary edges. As follows from Tutte's results in [11], the number $D(N-u,k)$ of such triangulations has the following asymptotic behaviour as $N \to \infty$:

$$D(N-u,k) \sim \phi(k) N^{-\frac{5}{2}} c^{N-u}. \tag{14}$$

Therefore, the probability of Γ_d (at the vertex 0) is equal to

$$P^N(\Gamma_d) \sim \frac{\phi(k) N^{-\frac{5}{2}} c^{N-u}}{\sum_{u,k} r(d,k,u)\phi(k) N^{-\frac{5}{2}} c^{N-u}} = \frac{\phi(k) c^{-u}}{\sum_{u,k} r(d,k,u)\phi(k) c^{-u}}. \tag{15}$$

Explicit expressions for the functions ϕ and r and for the constant c are known, but we do not need them. Since almost every graph has trivial automorphism group, it follows that the empirical limit is the same.

We note that if we consider a surface of genus ρ instead of the sphere, then we obtain another empirical distribution depending on ρ and $\mathcal{A}$.

Is it true that this (planar!) empirical distribution is a Gibbs empirical family? One can single out two-dimensional triangulations by means of a potential Φ such that $\Phi(O_1(v)) = \infty$ for $O_1(v) \neq g_{k,k}, k \geqslant 2$. However, in this way we simultaneously obtain triangulations of any two-dimensional surface. Therefore, only two approaches are possible. The first is to obtain triangulations of the sphere as a pure phase of some Gibbs family. A corresponding example is constructed below. The second is to consider non-local potentials (of infinite radius). The latter characterization is well-known in graph theory. Let $\Phi(K_5) = \Phi(K_{3,3}) = \infty$, where K_5 is a (non-regular) subgraph homeomorphic to the complete graph with 5 vertices and $K_{3,3}$ is a (non-regular) subgraph homeomorphic to the graph with 6 vertices $1,2,3,4,5,6$ and edges of the form (i,j) with $i = 1,2,3$ and $j = 4,5,6$ (by the Pontryagin–Kuratowski theorem, this condition distinguishes planar graphs). Set $\Phi = 0$ otherwise. Thus, planarity must be regarded as an *a priori* condition of global nature. However, without restrictions on the genus of the surface we obviously have an empirical Gibbs distribution.

§ 3. Entropy

The famous van Hove theorem in statistical physics claims that the partition function increases exponentially as the volume increases. In quantum gravity, this result on the number of triangulations (the entropy) has been the subject of many investigations; see [12]. The only case treated has been the case in which the topology is either fixed or subjected to strong restrictions (see [13]). However, one can readily see that the partition function grows faster than any exponential function even in the two-dimensional case if the genus is not fixed. Therefore, the grand canonical ensemble, in the form customary in physical papers, does not exist at all if the canonical partition function has superexponential growth. The double scaling limit is a (non-rigorous) attempt to overcome this difficulty.

On the other hand, the problem of the limits within which the entropy can vary is of interest. Below we shall see an interesting example of power-law growth of the entropy for a case in which a phase transition occurs. At the same time, the number p-regular graphs with N vertices has growth of order $N^{\frac{pN}{2}}$. We note that the additional consideration of spin for a Gibbs graph can have an effect only on the typical topologies (which differ from the maximal entropy only by an exponential factor). Therefore, we must know a rough (for instance, logarithmic) asymptotic estimate for the entropy.

3.1. Arbitrary genus in dimension two. We recall that the asymptotic behaviour of the number of triangulations of a surface of fixed genus g is given by

$$C(N,g) \sim f(g)N^{ag+b}c^N, \qquad a = \frac{5}{2}, \qquad b = -\frac{5}{2} - 1, \tag{16}$$

where c does not depend on g. The author does not know a rigorous formal proof of this result, but it is similar to those in [14] obtained for other classes of charts by the direct (but cumbersome) development of Tutte's method. We note that the reduction of the problem to a matrix model leads to the same result but gives no rigorous proof. The case in which the genus is not fixed is much more complicated. We present here some early results in this direction; see also [15].

Here are the necessary definitions (see [16]). A *chart* is a triple (S, G, ϕ), where S is a closed oriented connected two-dimensional manifold, G is a connected graph, and ϕ is a homeomorphic embedding of G in S such that the complement to ϕG is a union of connected open sets homeomorphic to an open disc. A *rooted chart* is a chart with distinguished *sprout* (that is, an end of one of the edges) of some vertex of G. Two rooted charts are (combinatorially) equivalent if there is an orientation-preserving homeomorphism of S that also preserves the set of vertices, the set of edges, and the root. There is an equivalent purely combinatorial definition of chart as an ordered fat graph. A graph is said to be *fat* if the set of sprouts from any vertex is endowed with a cyclic order. This definition is based on the following theorem.

Theorem 3. *For any connected oriented graph there is a chart for which the order of sprouts corresponds to a clockwise motion, and this chart is unique up to equivalence.*

Let E be the set of edges. In this case the set $2E = E \times \{-1, 1\}$ can be identified with the set of sprouts, and the vertices are defined by a partition of the set $2E$. The partition, together with the cyclic order inside each of its elements, is defined by a permutation P on $2E$. Another permutation, which we denote by $(-)$, transposes the sprouts of the same edge, that is, defines the partition of the set $2E$ into edges.

A face (disc) is defined by a sequence of sprouts e, $(-)e$, $P(-)e$, $(-)P(-)e$, ... that corresponds to a circuit of the boundary of the disc. Thus, the number of faces is equal to the number of cycles of the permutation $P(-)$. This explanation enables us to give the following definition.

Definition 4. A *combinatorial chart* is a triple $(2E, P, (-))$ consisting of a set $2E$ with an even number of elements and two permutations of it such that the group generated by these permutations is transitive on $2E$ (this corresponds to the condition that the graph is connected). It is also assumed that the permutation $(-)$ consists of disjoint cycles of length 2. The root is a distinguished element of $2E$.

Let $F_{b,p}(g)$ be the number of rooted charts with $p+1$ vertices and $b+p$ edges on a surface of genus g. We set $F_{b,p} = \sum_g F_{b,p}(g)$.

Theorem 4. *Let $p(b)$ be a non-decreasing sequence of integers such that $\frac{p(b)}{b} \to \alpha$ for some constant $0 < \alpha < \infty$. Then*

$$b^{-1} \log \frac{F_{b,p(b)}}{b!} \to c = c(\alpha), \qquad 0 < c(\alpha) < \infty.$$

In [16] the following recursion equations are obtained for the numbers $F_{b,p}$:

$$F_{b,p} = \sum_{j=0}^{p-1} \sum_{k=0}^{b} F_{k,j} F_{b-k,p-j-1} + (2(b+p)-1) F_{b-1,p}, \qquad b, p \geqslant 0, \tag{17}$$

except for the case $b = p = 0$. Moreover,

$$F_{0,0} = 1, \qquad F_{-1,p} = F_{b,-1} = 0.$$

One should separately verify the cases $b = 0$ or $p = 0$. The case $b = p = 0$ corresponds to an embedding of a single vertex in the sphere. The cases in which $p \geqslant 1$ and $b = 0$ correspond to a tree embedded in the sphere, and $F_{0,p} = \dfrac{(2p)!}{p!(p+1)!}$. The numbers $F_{b,0}$, $b \geqslant 1$, are equal to $(2b-1)(2b-3)\cdots = \dfrac{(2b)!}{b! 2^b}$, that is, to the number of partitions of the set $\{1, 2, \dots, 2b\}$ into pairs.

Lemma 5. *The following estimates hold*:

$$c_1^{b+p} < G_{b,p} = \frac{F_{b,p}}{b!} < c_2^{b+p}, \qquad 0 < c_1 < c_2 < \infty.$$

Proof. It follows from (17) that

$$G_{b,p} = \sum_{j=0}^{p-1} \sum_{k=0}^{b} \frac{k!\,(b-k)!}{b!} G_{k,j} G_{b-k,p-j-1} + b^{-1}(2(b+p)-1) G_{b-1,p}, \quad b, p \geqslant 1. \tag{18}$$

A lower bound can be obtained from the following minorizing recursion relation:

$$E_{b,p} > 2E_{b-1,p}, \quad b \geqslant 1, \quad p \geqslant 1, \qquad E_{0,p} = G_{0,p} = \frac{(2p)!}{p!\,(p+1)!}.$$

To find an upper bound, we need a certain technique, the so-called tree expansion, which corresponds to iteration of the following majorizing relation:

$$G_{b,p} = \sum_{j=0}^{p-1} \sum_{k=0}^{b} \frac{k!\,(b-k)!}{b!} G_{k,j} G_{b-k,p-j-1} + \left(2 + \frac{2p}{b}\right) G_{b-1,p} \tag{19}$$

with the same boundary conditions $G_{0,0} = 1$ and with $G_{b,p} = 0$ if either $p = -1$ or $b = -1$. We consider pairs (T, ϕ), where T is a finite rooted tree and ϕ is a map of the set $V(T)$ of vertices into the positive quadrant of the plane, and we write $\phi(v) = (b = b(v), p = p(v))$ for $v \in V(T)$. Let $\Phi(b,p)$ be the set of all pairs (T, ϕ) for which the root corresponds to the point (b,p). Corresponding to any tree of this kind is an iteration of the relation (19) up to some stopping point. The construction of a new vertex (new vertices are assumed to be below the old one) corresponds to an iteration step as follows: we either choose one of the terms of the right-hand side of (19) or stop for a given vertex. Any vertex $v = (b,p)$ of the tree (except for the root) can have two outgoing edges, and if this is the case, then the vertex is said to be *binary*, and the other two vertices of these edges are at the points (k,j) and $(b-k, p-j-1)$. We assign the number $f(v) = \dfrac{k!(b-k)!}{b!}$ to this vertex. If a vertex has only one outgoing edge, then this vertex is said to be *unary*, and the other vertex of the edge is at the point $(b-1,p)$. To the vertex $v = (b,p)$ we then assign the number $f(v) = 2 + 2p/b$. A vertex $v = (b,p)$ that has no outgoing edges is said to be *final*, or *terminal*, and we assign the number $f(v) = G_{b,p}$ to this vertex. The *contribution of the tree* T is defined as the product

$$f(T) = \prod_{v \in V(T)} f(v).$$

A tree is said to be *full* if the point $(0,0)$ corresponds to each final vertex of it.

Lemma 6. *The following expansion holds*:

$$G_{b,p} = \sum_T f(T),$$

where the sum is taken over all full trees.

The proof is obtained by the complete iteration of the recursion relation (19), whose application is terminated only if there are no final vertices distinct from $(0,0)$.

We note that the number of binary vertices of an arbitrary tree with root at (b,p) is not greater than p. Indeed, let us write $p(T) = \sum p(v)$, where the sum is taken over all final vertices v of the tree T. If a tree T' is obtained from T by joining two edges to some final vertex of T, then $p(T') = p(T) - 1$. If T' is obtained from T by joining one edge to some final vertex of T, then $p(T') = p(T)$. This implies the desired result.

Lemma 7. *The number $K_{b,p}$ of pairs (T,ϕ), where T is a complete tree with root at (b,p), does not exceed C^{b+p} for some constant C.*

Proof. We have the following recursion relation:

$$K_{b,p} = \sum_{j=0}^{p-1} \sum_{k=0}^{b} K_{k,j} K_{b-k,p-j-1} + K_{b-1,p}.$$

The equation for the generating function

$$K = K(x,y) = \sum K_{b,p} x^b y^p$$

is of the form

$$K = xK^2 + yK + 1,$$

and hence

$$K = -\frac{y-1}{2x} + \frac{y-1}{2x}\sqrt{1 - \frac{4x}{(y-1)^2}}.$$

The result follows because K is analytic at the point $x = y = 0$.

By the previous lemma, it suffices to prove that the contribution of any pair (T,ϕ) does not exceed C^{b+p} for some constant C independent of the pair. Estimation of the product of $2p/b$ over the unary pairs presents some difficulties. Let us prove by induction that the number of unary vertices does not exceed b. We begin with the root vertex (b,p). Suppose that there are exactly b_1 unary vertices between the root and the next binary vertex. The binary vertex $(b-b_1,p)$ creates two other vertices (b',p') and (b'',p'') with $b'+b'' = b-b_1$ and $p'+p'' = p-1$. By the induction hypothesis, there are at most $b'+b''$ unary vertices below these two new vertices. Therefore, the total number of binary vertices does not exceed $b'+b''+b_1 = b$.

There is a set V' of vertices with the following properties: 1) no vertex of V' is below another vertex of V', 2) any vertex (b'',p'') that is not below a vertex in V' satisfies the condition $b'' \geqslant 2p''$, 3) $b' < 2p'$ for any vertex (b',p') in V'.

The contribution of the unary vertices of type 2) can be estimated as 1. Let us choose some vertex of V' with $b' < p'/\log p'$. The total number of unary vertices below the chosen vertex does not exceed b', and thus their joint contribution does not exceed $(2p')^{p'/\log p'} < C^{p'}$.

Let $p/\log p \leqslant b < 2p$ for some vertex (b,p). We write $g(T) = b!f(T)$ if (b,p) is the root of T. If the root is a unary vertex with contribution $2p/b$, then $g(T) < 2pg(T')$, because $(b-1,p)$ is a vertex below this unary vertex. If the root is a binary vertex with contribution $\frac{k!(b-k)!}{b!}$, then $g(T) = g(T_1)g(T_2)$, where T_1 and T_2 are trees with roots (k,j) and $(b-k, p-j-1)$, respectively, for some j. It follows that $g(T) < (2p)^b$, and thus

$$f(T) < \frac{(2p)^b}{b!} < e^{2p}.$$

We note that $\sum_{v' \in V'} p(v') \leqslant p$, and this proves the result.

Let us now complete the proof of the theorem. We consider the function of one complex variable

$$f(z) = \sum_{b=0}^{\infty} z^{b+p(b)} G_{b,p(b)}.$$

By the previous lemma, this series has finite radius of convergence, say R. Then

$$(R^{-1} - \varepsilon)^b < G_{b,p(b)} < (R^{-1} + \varepsilon)^b$$

for any $\varepsilon > 0$ and any sufficiently large $b = b(\varepsilon)$. This proves the theorem.

Let us present the result in a more invariant form. Let $\rho(M)$ be the minimal genus in which the chart M can be embedded and let $\mathfrak{M}(b,p)$ be the set of charts with given parameters b and p. We set $\rho(b,p) = \max_{M \in \mathfrak{M}(b,p)} \rho(M)$. The theorem claims that $\dfrac{|\mathfrak{M}(b,p)|}{(2\rho(b,p))!}$ has exponential growth.

In other words, the factorial multiple is of order $(-\chi_{\min})!$, where $\chi_{\min}$ is the minimum of the Euler characteristics of the manifolds thus obtained.

This statement can be seen from the formula

$$-\chi = 2\rho - 2 = -V + L - F = b - 1 - F = -z(P,-) + b - 1,$$

where $z(P,-)$ is the number of orbits of the group generated by the two permutations. One can readily see that, if the orbits of P are given, then one can always choose the permutation $(-)$ in such a way that the orbits of $(P,-)$ are as long as desired, that is, the contribution of $z(P,-)$ is much less than b.

This result enables us to propose the following reasonable canonical distribution on the set $\mathcal{A}(b)$ of all charts M with fixed characteristic $b = -\chi_{\min}$:

$$P_b(M) = Z_b^{-1} \exp\bigl(-\mu p(M) - \lambda \rho(M)\bigr) = Z_b^{-1} \exp\Bigl(-\mu p(M) + \frac{\lambda}{2} F(M)\Bigr).$$

3.2. Dimension three and higher. In quantum gravity, for a fixed triangulable d-dimensional manifold M (a simplicial complex) the potential is assumed to be linear in the number $N_i(T)$ of i-dimensional simplices of the triangulation T for $i = 0, \dots, d$, that is, the partition function is formally equal to

$$Z = \sum_T \exp\Bigl(-\sum_i k_i N_i(T)\Bigr) = \sum_{\{N_i\}} \exp\Bigl(-\sum_i k_i N_i\Bigr) C(\{N_i\}),$$

where $C(\{N_i\})$ is the number of triangulations with given numbers $\{N_i\}$. It follows from the Dehn–Sommerfeld relations (see [12]) that all the numbers $N_i(T)$ can be expressed linearly in terms of two of them. Therefore, the distribution can be rewritten as follows:

$$\sum_T \exp\bigl(-\lambda_d N_d(T) + \lambda_{d-2} N_{d-2}(T)\bigr),$$

where the sum is taken over all triangulations T of the given manifold. Moreover, the last formula corresponds to the Hilbert–Einstein action

$$S = \frac{1}{16\pi G}\int dx\,\sqrt{g}\,(2\Lambda - R),$$

where R is the scalar curvature, Λ is the cosmological constant, and G is the Newton constant. To prove convergence of the previous series in some domain of parameters, one needs exponential estimates for the numbers $C(\{N_i\})$. For a discussion of the necessary exponential estimates, see [12]. Boulatov [13] presents the following result. Let $C(N,\mathbf{M})$ be the number of three-dimensional triangulations with N simplices of a three-dimensional manifold $\mathbf{M}$. Then

$$\sum C(N,\mathbf{M}) < C\exp(\gamma N),$$

where the sum taken is over all manifolds $\mathbf{M}$ with a fixed homology group $H_1(\mathbf{M},Z)$, and $\gamma = \gamma(H_1(\mathbf{M},Z))$.

In the general case (in which the topology is arbitrary), the entropy is not exponential, and, as far as the author knows, the only way to introduce a probability distribution is given by Gibbs families.

Corresponding to any d-dimensional simplicial complex is a dual graph whose vertices are in one-to-one correspondence with d-simplices. Each vertex of this graph has d sprouts (legs) that correspond to d faces of this graph of dimension $(d-1)$. Corresponding to the pairing of sprouts is the gluing together of the appropriate faces by means of a linear map. In dimension two this gluing always gives a manifold. In higher dimensions, the gluing together of faces of d-dimensional cells gives a pseudomanifold with probability tending to 1. Moreover, in the thermodynamic limit, the pseudomanifold thus obtained has no vertex with a neighbourhood isomorphic to a ball. This follows from the results in § 4.4.

The condition singling out manifolds among pseudomanifolds is local, and it can be given by a superlocal potential (with hard core). However, the restrictions on the topology are not local in general. Below we consider the simplest examples in which the condition on the topology follows from the Gibbs property.

§ 4. Phase transitions

A Gibbs family (on $\mathcal{A}_\infty^{(0)}$ or on $\mathcal{A}_\infty$) with potential Φ is said to be *pure* if it is not a convex combination of other Gibbs families with the same potential. Thus, the non-uniqueness of Gibbs families means that there are at least two pure Gibbs families with the same potential Φ. A pure Gibbs family can be of two types: it can be supported either by a fixed graph or by several graphs (or even by a family of pairwise distinct graphs that has the cardinality of the continuum). Thus, in the first case the Gibbs family reduces to an ordinary Gibbs field.

In this connection, the non-uniqueness of Gibbs families can be of a different nature. First, on each graph in the support of a Gibbs family, the pure phases can differ by the structure of typical configurations (this is a phase transition inherited from ordinary Gibbs fields on a fixed graph). Second, pure phases can differ by the typical structure of the graph in itself (this is a metric or topological phase transition). Roughly speaking, a topological phase transition occurs if there are at least two Gibbs families with disjoint supports that have graphs (with local structure defining a topological space) with essentially distinct topological properties.

The problem of describing the phases is rather complicated. It is simpler to single out some pure phases, as is done below for some examples. To this end, the following remark can be useful: if a pure Gibbs field is a Gibbs family, then it is a pure Gibbs family.

4.1. Mean-field model. In discrete quantum gravity, there is a simple model [17] in which one can analytically explain the appearance of different phases in simplicial gravity with spin under numerical simulation. This is an urn model with $M = \lambda N$ balls in N urns. It can be interpreted as a random graph if the urns are referred to as vertices and the balls as sprouts, with the sprouts randomly paired. Thus, there are N enumerated vertices $i = 1, \dots, N$ with $q_i \geqslant 1$ sprouts at any vertex i. The canonical ensemble is given by the partition function

$$Z_N = \sum_{\{q_i\}: \sum q_i = M} \exp\biggl(-\sum_{i=1}^{N} \beta\Phi(q_i)\biggr).$$

We note that, in our terminology, this is a model with local potential (which is unstable, as was shown above) and non-local restriction $\sum q_i = M = \lambda N$ of the mean-field type. This restriction eliminates the instability of these local potentials. We note that the factorial multiple $A(M)$, which is responsible for the random pairing of sprouts in the asymptotic expression for the partition function, can be regarded as a common multiple, after which the asymptotic expression for the partition function becomes

$$Z_N = cN^{\alpha} \exp(\gamma N).$$

For a sufficiently broad class of potentials, including the potential $\Phi(q) = \log q$, there are two phases with phase transition with respect to λ for a fixed β. One phase (which is called elongated or fluid) corresponds to small densities for which the balls are uniformly distributed with respect to the urns, and the other (crumpled, or Bose–Einstein) to the case in which the number of balls at some single vertex is of order M. A modification of the model can lead to a mixed phase (which is also said to be condensed or crinkled), where there is a vertex with cM balls, where $c < 1$, and the other urns are filled with balls uniformly.

The numerical simulation in [18] shows the appearance of similar phases in two-dimensional simplicial gravity for the case in which spins are added. Therefore, diverse explanations and analogues of these phases are of interest. As noted above, this is a smoothed transition between the stability and instability of the potential, where the smoothing is realized by a restriction of mean-field type.

In statistical physics an analogue is given by the Bose–Einstein condensation. In computer science the choice $\Phi = cq$ gives a stationary distribution in Jackson networks in the mean-field model [19], where quite similar phase transitions are observed. The classical theory of random graphs deals in fact with the chemical potential under the same restriction of mean-field type (see [20]).

4.2. Phase transitions in the model with boundaries. We consider the two-dimensional triangulations T which are called quasi-triangulations in [21] and triangulations in [11]. In other words, we consider embeddings of a graph in a two-dimensional surface, where edges are disjoint smooth arcs, and the complement consists of disjoint open domains homeomorphic to a disc, with each of the domains bounded by exactly three arcs.

Let $T(\rho,N,k;m_1,\dots,m_k)$ be the set of all (equivalence classes of) such triangulations of a sphere with ρ handles, k holes, N triangles, and m_i edges on the boundary of the ith hole. We assume that $m_i \geqslant 2$. Let $C(\rho,N;m_1,\dots,m_k)$ be the number of triangulations of this kind. For fixed ρ and k and for a given N we define a distribution $\mu_{\rho,N,k}$ (the canonical ensemble) on the set

$$T(\rho,N,k)=\bigcup_{m_1,\dots,m_k} T(\rho,N,k;m_1,\dots,m_k)$$

of triangulations by the following four equivalent rules.

- If the interaction is proportional to the total number of edges, then

$$\mu_{\rho,N,k}(T)=Z_N^{-1}\exp(-\mu_1 L(T)).$$

- If the interaction is proportional to the sum of the degrees of the vertices, then we use the relation

$$\exp(-\mu_1 L(T))=\exp\biggl(-\frac{\mu_1}{2}\sum_{v\in V(T)}\deg v\biggr),$$

  where $L(T)$ stands for the number of edges of T, including the boundary edges. Therefore,

$$Z_N=Z_N(\rho,k)=\sum_{(m_1,\dots,m_k)}\ \sum_{T\in T(\rho,N;m_1,\dots,m_k)}\exp\biggl(-\frac{\mu_1}{2}\sum_{v\in V(T)}\deg v\biggr).$$

  Using the discrete analogue of the Gauss–Bonnet theorem, one can show that this is a discrete analogue of the Einstein–Hilbert action. We also note that this is a special case (with the parameters $t_q=t$) of the interaction treated in [22]:

$$\prod_{q>2} t_q^{n(q,T)},$$

  where $n(q,T)$ is the number of vertices of degree q.

- If the interaction is proportional to the number of boundary edges, then for any ρ and k we define the correlation functions of the vectors $(m_1,\dots,m_k)$ by the rule

$$\begin{aligned}P_{N,\rho,k}(m_1,\dots,m_k)&=\sum_{T\in T(\rho,N;m_1,\dots,m_k)}\mu_{\rho,N,k}(T)\\&=\Theta_N^{-1}(\rho,k)\exp\biggl(-\mu_1\frac{\sum_{i=1}^k m_i}{2}\biggr)C(\rho,N;m_1,\dots,m_k),\end{aligned}$$

where we have used the formula

$$L(T) = \frac{3N - \sum m_i}{2} + \sum m_i = \frac{3N}{2} + \frac{\sum m_i}{2},$$

which implies that

$$\Theta_N(\rho, k) = \sum_{m_1,\dots,m_k=2}^{\infty} \exp\Big(-\mu_1 \frac{\sum m_i}{2}\Big) C(\rho, N; m_1, \dots, m_k).$$

- Let us introduce at the vertices v of the triangulation fictitious spins σ_v that take values in an arbitrary compact set and are such that the corresponding potential of the Gibbs family is

$$f(\sigma_v, \sigma_{v'}) \equiv 1$$

for any two neighbouring vertices v and v'.

We first consider the case $k = 1$, $\rho = 0$. Moreover, we assume that a vertex on the boundary and an edge incident to this vertex are singled out, thus defining the origin and the orientation. Denote the class of all triangulations of this kind by $T_0(N, m)$ and the number of these triangulations by $C_0(N, m)$. Here the probability of the triangulations is equal to

$$P_{0,N}(T) = Z_{0,N}^{-1} \exp\Big(-\frac{\mu_1}{2} \sum_{v \in V(T)} \deg v\Big).$$

Let us prove the existence of a phase transition with respect to μ_1. We set $\mu_{1,\text{cr}} = \log 12$. Let $0 < \beta_0 = \beta_0(\mu_1) < 1$ be determined by the equation

$$\frac{1 + \frac{4\beta_0}{3(1-\beta_0)}}{(1 + \frac{2\beta_0}{1-\beta_0})^2} \exp(-\mu_1 + \log 12) = 1.$$

Theorem 5. *The free energy* $\lim_N \frac{1}{N} \log Z_{0,N} = F$ *is equal to* $-\frac{3}{2}\mu_1 + c$ *with* $c = 3\sqrt{\frac{3}{2}}$ *for* $\mu_1 > \mu_{1,\text{cr}}$, *and to*

$$-\frac{3}{2}\mu_1 + c + \beta_0(-\mu_1 + \log 12) + \int_0^{\beta_0} \log \frac{1 + \frac{4\beta}{3(1-\beta)}}{(1 + \frac{2\beta}{1-\beta})^2}\, d\beta \tag{20}$$

for $\mu_1 < \mu_{1,\text{cr}}$.

We note that $\mu_1 \to \mu_{1,\text{cr}}$ as $\beta_0 \to 0$.

Let $m(N)$ be the random length of the boundary for a fixed N. The distribution of $m(N)$ is of the following form (we use the relation $|L(T)| = 3N/2 + m/2$):

$$P_{0,N}(m(N) = m) = \Theta_{0,N}^{-1} \exp\Big(-\mu_1 \frac{m}{2}\Big) C_0(N, m),$$

$$\Theta_{0,N} = \sum_m \exp\Big(-\mu_1 \frac{m}{2}\Big) C_0(N, m).$$

Theorem 6. *There are three phases, for which the distribution of* $m(N)$ *has quite different asymptotic behaviour:*

- *the subcritical domain* $12\exp(-\mu_1) < 1$, *in which case* $m(N) = O(1)$, *or, more precisely, the distribution of* $m(N)$ *has a limit* $\lim_N P_N(m(N) = m) = p_m$ *for any fixed* m *as* $N \to \infty$;
- *the supercritical domain (the elongated phase)* $12\exp(-\mu_1) > 1$, *in which case the length of the boundary is of order* $O(N)$, *or, more precisely, there is an* $\varepsilon > 0$ *such that* $\lim P_{0,N}(m_N/N > \varepsilon) = 1$;
- *at the critical point, that is, for* $12\exp(-\mu_1) = 1$, *the length of the boundary is of order* $\sqrt{N}$, *or, more precisely, the distribution of* $m_N/\sqrt{N}$ *converges in probability.*

Proof. We use the formula (for $N = m + 2j$)

$$C_0(N,m) = \frac{2^{j+2}(2m+3j-1)!\,(2m-3)!}{(j+1)!\,(2m+2j)!\,((m-2)!)^2}$$

(see [21]). The direct calculation, in which we use the fact that the change of variables of the form $N \to N$ and $m \to m+2$ corresponds to the change of indices $j \to j-1$ and $m \to m+2$, gives

$$\begin{aligned}\frac{P_{0,N}(m+2)}{P_{0,N}(m)} &= f(N,m) = \frac{C_0(N,m+2)\exp(-\frac{\mu_1}{2}(m+2))}{C_0(N,m)\exp(-\frac{\mu_1}{2}m)}\\ &= \exp(-\mu_1+\log 12)\frac{(1+\frac{2}{N-m})(1+\frac{4m}{3(N-m)})}{(1+\frac{2m}{N-m}+\frac{2}{N-m})(1+\frac{2m}{N-m}+\frac{1}{N-m})}\,\frac{(1-\frac{1}{4m^2})}{(1-\frac{1}{m})}.\end{aligned} \tag{21}$$

In the subcritical case we obtain

$$\frac{P_{0,N}(m+2)}{P_{0,N}(m)} \sim \exp(-\mu_1+\log 12)\biggl(1+\frac{1}{m}+O\Bigl(\frac{1}{m^2}\Bigr)\biggr)$$

for a fixed m as $N \to \infty$, and thus we see that as $m \to \infty$, for instance, for even m,

$$\lim_N P_{0,N}(2m) \sim Cm\exp\bigl(m(-\mu_1+\log 12)\bigr).$$

At the same time, the second factor in (21) does not exceed 1. Therefore, it follows from

$$Z_{0,N} = \exp\Bigl(-\mu_1\frac{3N}{2}\Bigr)\Theta_{0,N} = \exp\Bigl(-\mu_1\frac{3N}{2}\Bigr)\sum_m \exp\Bigl(-\mu_1\frac{m}{2}\Bigr)C_0(N,m)$$

that $F = -\frac{3}{2}\mu_1 + c$, where $c = 3\sqrt{\frac{3}{2}}$, because for a fixed m we have

$$C_0(N,m) \sim \phi(m)N^{-\frac{5}{2}}c^N.$$

The formula (21) implies (20) in the supercritical case as well if we set $m = \beta N$ and $0 < \beta < 1$. The two expressions coincide at the critical point; however, the free energy is not differentiable at this point.

To prove the second assertion of the theorem, we set $12\exp(-\mu_1) = 1 + r$ and estimate separately the three factors in (21). Then we see that there exist δ and ε with $0 < \delta \ll \varepsilon \ll 1$ such that

$$\frac{P_{0,N}(m)}{P_{0,N}(\varepsilon N)} < \left(1 + \frac{r}{2}\right)^{-\frac{\varepsilon}{2}N}$$

for any m and N with $m \leqslant \delta N$. This implies the desired result.

In the critical case we have similarly

$$\frac{P_{0,N}(m+2)}{P_{0,N}(2)} \sim \prod_{k=1}^{m/2}\left(1 + \frac{1}{2k}\right)\left(1 - \frac{4}{3}\frac{k}{N}\right) \sim C\sqrt{m}\exp\left(-\frac{1}{3}\frac{m^2}{N}\right).$$

Hence, for any α and β with $0 < \alpha < \beta < \infty$ we obtain

$$\lim\left(\frac{P_{0,N}(m(N) < \varepsilon\sqrt{N})}{P_{0,N}(\beta\sqrt{N} < m(N) < \beta\sqrt{N})} + \frac{P_{0,N}(m(N) > \varepsilon^{-1}\sqrt{N})}{P_{0,N}(\beta\sqrt{N} < m(N) < \beta\sqrt{N})}\right) = 0$$

as $\varepsilon \to 0$.

Let us now consider the case in which there is no distinguished vertex on the boundary; in other words, we extend the automorphism group. Both the free energy and the behaviour away from the critical point are unchanged.

Theorem 7. *At the critical point, without any coordinate system on the boundary, the length of the boundary is of order N^α for $0 < \alpha < 1/2$. More precisely, the distribution of $\dfrac{\log m_N}{\log\sqrt{N}}$ converges to the uniform distribution on the unit interval, that is,* $P_{0,N}\left(\dfrac{\alpha}{2} \leqslant \dfrac{\log m_N}{\log\sqrt{N}} \leqslant \dfrac{\beta}{2}\right) \to \beta - \alpha$ *for any α and β such that $0 \leqslant \alpha < \beta \leqslant 1$.*

Lemma 8. $C(N,m) \sim m^{-1}C_0(N,m)$ *as $N, m \to \infty$.*

Proof of the lemma. Let us enumerate the edges of the boundary $1, 2, \dots, m$ in cyclic order, starting from the root edge. An automorphism ϕ of the disc is uniquely determined if one knows the edge $j = \phi(1)$ to which the edge with index 1 is taken. Indeed, in this case the triangles abutting on the edge 1 are mapped to the triangles abutting on j, and so on by connectedness.

We consider the strip of width 1 abutting on the boundary, that is, the set of triangles of three types depending on the set common with the boundary, where this set can contain one edge (type 1), two edges (type 2), or a point only (type 0). Thus, the strip can be represented as a word $\alpha = x_1 \dots x_n$, $n > m$, where $x_i = 0, 1, 2$. Let us consider the set $W(m, n_0, n_1, n_2)$ of words with a given m and given numbers n_i of each of the letters i. An automorphism of the disc gives a cyclic automorphism of the word α. Moreover, the sets $W(m, n_0, n_1, n_2)$ are invariant. We note that

$$m = n_1 + 2n_2,$$

and the length of the other boundary of the strip is equal to $m' = n_0$. Therefore, if there is no cyclic automorphism of the word α, then there are no automorphisms of the disc. One can readily see that the set of words in $\bigcup_{n_1,n_2} W(m,n_0,n_1,n_2)$ that have non-trivial cyclic automorphisms is small compared with the total number $|\bigcup_{n_1,n_2} W(m,n_0,n_1,n_2)|$ for a given n_0 as $m \to \infty$.

This gives the factor $1-1/m$ in $f(N,m)$ instead of $1+1/m$ in the previous case. Similar calculations prove that, asymptotically, the distribution coincides with the family ν_N of distributions on the set $\{1,\dots,\sqrt{N}\}$ with $\sqrt{N}$ elements,

$$\nu_N(i) = Z_{\sqrt{N}}\, i^{-1}, \qquad Z_{\sqrt{N}} = \sum_{i=1}^{\sqrt{N}} i^{-1}.$$

It can readily be seen that

$$\nu_N\biggl(\frac{\alpha}{2} \leqslant \frac{\log i}{\log\sqrt{N}} \leqslant \frac{\beta}{2}\biggr) = \nu_N(N^{\alpha/2} \leqslant i \leqslant N^{\beta/2}) \to \beta-\alpha$$

for $0 \leqslant \alpha \leqslant \beta \leqslant 1$.

4.2.1. *Relationship with topological field theory.* The introduction of a system of coordinates on the boundary recalls the passage from topological field theory to conformal field theory. It is customary to introduce topological field theory (TFT) axiomatically in the framework of category theory [23], either heuristically (with the help of path integrals) or algebraically (as a Frobenius algebra). For a path integral to become meaningful, it seems to be natural to use the idea of discretization (see, for example, [24]–[26]). In the last chapter of [27] a topological field theory is constructed that satisfies all axioms already on finite triangulations. However, it is possible in principle that such a theory satisfies these axioms only after passing to the limit as $N \to \infty$. The study of this limit is of interest in itself. We solve this problem here for the case of two boundaries only (that is, for $k=2$), for zero genus, and for large μ_1 in the previous model. It is of interest that a non-trivial joint distribution of the lengths of the boundaries appears.

Theorem 8. *For $k=2$ there are correlation functions $P_{0,2}(m_1,m_2) = \lim_N P_{N,0,2}(m_1,m_2)$ for sufficiently large μ_1 and for fixed m_1 and m_2.*

The proof below is complete. Apparently, a similar method works for any k and ρ. We first obtain the following recursion relations for the numbers $C_0(N,m,m_2)$ with $m=m_1$:

$$\begin{aligned} C_0(N,m,m_2) &= C_0(N-1,m+1,m_2) + C_0(N-1,m+m_2+1) \\ &\quad + \sum_{N_1+N_2=N-1}\ \sum_{k_1+k_2=m+1} C_0(N_1,k_1,m_2)C_0(N_2,k_2). \end{aligned}$$

These relations can be proved by deleting a root edge as in Tutte's method [28]; see Figure 1.

Three cases are now possible, corresponding to the three terms on the right-hand side: the graph can become disconnected, two holes can coalesce into one hole,

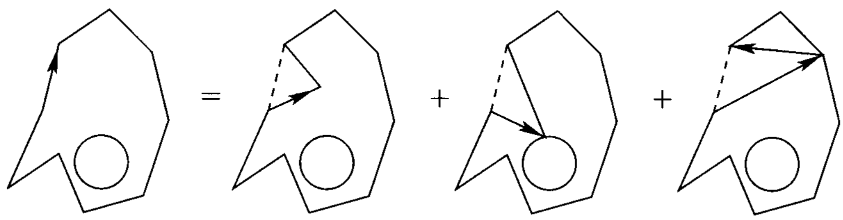

Figure 1

and two holes can be preserved in such a way that the boundary of the first hole is elongated and the number of its triangles is reduced. The boundary conditions are as follows. For a given number N there are only finitely many non-zero numbers $C_0(N,m,m_2)$, and it is convenient to assume that these numbers $C_0(N,m,m_2)$ are known for $N < N_0$ (we are not interested in their explicit form). Moreover, $C_0(N,m,m_2) = 0$ if either $m < 2$ or $m_2 < 2$.

Lemma 9. *The asymptotic behaviour of the numbers $C_0(N,m,m_2)$ is given by*

$$C_0(N,m,m_2) \sim \phi(m,m_2)N^{-5/2+1}c^N \tag{22}$$

for fixed m and m_1 as $N \to \infty$, where $\phi(m,3) = \phi(m)$.

Proof. As is well known,

$$C_0(N,m) \sim \phi(m)N^{-5/2}c^N. \tag{23}$$

Let us find the numbers $C_0(N,m,2)$, $C_0(N,m,3)$, $C_0(N,2,m_2)$, and $C_0(N,3,m_2)$. We first assume that $m_2 = 3$. To obtain a triangulation in $T(0,N;m,3)$, we take a triangulation in $T(0,N+1;m)$ and choose one of the N triangles in such a way that it is not tangent to the boundary, which can be done in $N - O(m)$ ways. Hence, $C_0(N,m,3) \sim C_0(N+1,m)N$, and thus $\phi(m,3) = c\phi(m)$. Removing the root on the boundary of length m and placing it on the boundary of length 3, we see that $\phi(3,m) = \frac{3c}{m}\phi(m)$ because the automorphism group is trivial for almost all graphs. To obtain a triangulation in $T(0,N;m,2)$, we take a triangulation in $T(0,N;m)$, choose one of the N triangles not tangent to the boundary, choose a side of this triangle, and double this side. Then $\phi(m,2) = \frac{3}{2}\phi(m)$, and we can obtain $\phi(2,m) = \frac{3}{m}\phi(m)$ similarly.

Let us rewrite the recursion relations:

$$\begin{aligned} C_0(N-1,m+1,m_2) &= C_0(N,m,m_2) \\ &\quad - \sum_{N_1+N_2=N-1} \sum_{k_1+k_2=m+1} C_0(N_1,k_1,m_2)C_0(N_2,k_2) \\ &\quad - C_0(N-1,m+m_2+1). \end{aligned}$$

Since $C_0(N,m,2)$, $C_0(N,m,3)$, $C_0(N,2,m_2)$, and $C_0(N,3,m_2)$ are known, the assertion can be obtained by induction on m for a fixed m_2, and conversely. Indeed, multiply the above relation by $c^{-N+1}(N-1)^{5/2-1}$. By the induction hypothesis, the right-hand side tends to

$$c\phi(m,m_2) - 2\sum_{k_1+k_2=m+1}\phi(k_1,m_2)\sum_{n=1}^{\infty}C_0(n,k_2)c^{-n}$$

in this case. Therefore, the limit of the left-hand side $c^{N-1}(N-1)^{5/2-1}C_0(N-1, m+1,m_2)$ is finite; we denote this limit by $\phi(m+1,m_2)$.

Let us proceed to the proof of the theorem. Substituting (23) into the recursion formula for $C_0(N,m)$, we obtain the following recursion formula for $\phi(m)$:

$$\phi(m+1) = c\phi(m) - 2\sum_{k_1+k_2=m+1}\phi(k_1)\sum_{n}C_0(n,k_2)c^{-n}.$$

In particular,

$$\phi(3) = c\phi(2), \qquad \phi(4) = c\phi(3) - 2\phi(2)\sum_{n}C_0(n,2)c^{-n}.$$

Let us now find the asymptotic behaviour for $\phi(m,m_2)$ by using the relations

$$\phi(m+1,m_2) = c\phi(m,m_2) - 2\sum_{k_1+k_2=m+1}\phi(k_1,m_2)\sum_{N_2=1}^{\infty}C_0(N_2,k_2)c^{-N_2}.$$

Hence, $\phi(m,m_2) < \phi(2,m_2)c^m$. However, $\phi(2,m_2) = \frac{2}{m_2}\phi(m_2,2) < \phi(2,2)c^{m_2}$. This proves the theorem.

4.3. Non-uniqueness in the model with low entropy. Let U_2 be a graph isomorphic to the 2-neighbourhood of the origin in the lattice $\mathbb{Z}^2$. Then among the graphs belonging to $\mathcal{G}(U_2)$ we have the lattice $\mathbb{Z}^2$ itself and its factor groups $\mathbb{Z}(k_1,0)$ and $\mathbb{Z}(k_1,k_2)$ with respect to the subgroups $\{(nk_1,0)\}$ and $\{(n_1k_1,n_2k_2)\}$, respectively $(n,n_1,n_2\in\mathbb{Z})$, that is, cylinders and tori, where $k_i \geqslant 4$. We note that there are other cylinders, for instance, twisted ones, which can be obtained from the strip $\mathbb{Z}\times\{0,1,2,\dots,k\}\subset\mathbb{Z}^2$ by identifying the points $(n,0)$ and $(n+j,k)$ for any n. If there is no spin, then the growth of the entropy is subexponential.

Let us consider a Gibbs family with $S=\{-1,1\}$ and with potential Φ equal to the sum of $\Phi_{U_{d,2}}$ and the Ising potential of nearest-neighbour interaction, $\Phi_{\mathrm{Is}} = \sum_{\langle i,j\rangle}\sigma_i\sigma_j$.

Theorem 9. *For any β there are infinitely many pure Gibbs families with potential Φ that are Gibbs fields on the above graphs (cylinders).*

Proof. We construct a finite Gibbs family with boundary conditions determined by a graph γ_N defined as follows (for any sufficiently large N). Consider the subgraph α_n of $\mathbb{Z}^2$ that is the strip consisting of the points (i,j), $i\in\mathbb{Z}$, $j=-n,\dots,n$, $n\geqslant 2$, with the root $(0,0)$, and with the points $(i,-n)$ and (i,n) identified. The distance from a point (i,n) to $(0,0)$ is defined as $|i|+|n|$. We set $\gamma_N = \gamma(\alpha_n,(0,0);N+1,N+3)$. If $n\ll N$, then γ_N consists of two isomorphic connected components. Here we use Lemma 2 to assume that ν_N is the unit measure on the spin graph γ_N and that all spins on γ_N are equal to 1. We claim that the (finite) Gibbs family on $\mathcal{G}_N^{(0)}$ is a Gibbs (Ising) measure on the graph $\gamma(\alpha_n,(0,0);0,N)$.

For the proof, we first assume that S is trivial and use only the part $\Phi_{U_{d,2}}$ of the potential. Let us prove that the desired finite Gibbs family with the above boundary conditions is the unit measure on $\gamma(\alpha_n,(0,0);0,N)$, that is, a random graph G is the graph $\gamma(\alpha_n,(0,0);0,N)$ with probability 1. We introduce the method of 'analytic' continuation, which is an inductive proof that the annuli $\gamma(N) = \gamma(G,0;N,N), \gamma(N-1), \dots$ of a random graph G coincide with the corresponding annuli of the graph α_n. Let us first construct the annulus $\gamma(N)$ and prove that it is unique and coincides with the corresponding annulus $\gamma(\alpha_n,(0,0);N,N)$ of α_n.

Consider the point $(N+2,0)$ of γ_N together with its 2-neighbourhood in γ_N. This neighbourhood is lacking an edge l_0 from the point $(N+1,0)$ to a new vertex v_0, which we denote by $(N,0)$ and which must be at the distance N from the origin because all the points at the distances $N+1$, $N+2$, and $N+3$ are already given.

We now consider the point $(N+1,1)$. It follows from the shape of its neighbourhood in $\gamma(\alpha_n,(0,0);N+1,N+3)$ that two new edges, say l_1 and l_2, must issue from $(N,1) \in \gamma(\alpha_n;N+1,N+3)$ to the annulus N. One of these edges must be joined with the vertex v_0.

We next construct the entire annulus $\gamma(N)$ (and only this annulus) by induction, beginning with the points with positive first coordinate and then proceeding to the symmetric points. In this way all points of the annulus $\gamma(N+2)$ obtain the desired neighbourhoods, and thus can be excluded from our considerations below.

Next, we construct $\gamma(N-1)$, $\gamma(N-2)$, ... by induction on the annuli, using the neighbourhoods of the annuli $N+1, -N-1, N, -N, \dots$ constructed above. These induction steps can be repeated until two connected components combine into a single component. We prove that this event occurs on the line $y = \pm n$ of the annulus n. Let us show that the following cases are impossible: 1) the combination first occurs on another line, 2) this combination occurs on the line $y = \pm n$ but on an annulus whose index differs from n. The second case is excluded by the following argument. After combination occurs on the line $y = \pm n$, the strip thus formed can be filled in exactly n steps of the induction. Therefore, the combination must happen exactly on the annulus n. The first case is eliminated as follows. Suppose that the combination first occurs on a line $y = k$. In this case the line $y = k+1$ must have an obstacle to being filled at the same step of induction.

Thus, independently of the spins on γ_N, the Gibbs family is supported by a single graph, and hence the Gibbs family is a Gibbs field. This proves the theorem, because the number n can be chosen arbitrarily.

4.4. Gibbs characterization of structures.

4.4.1. *Uniqueness in the model with high entropy.* The set of countable trees cannot be locally characterized, that is, this set cannot be distinguished in the class of all graphs by a restriction on the subgraphs with diameter less than some constant. However, it is of interest that a local characterization of countable trees can be given by means of Gibbs families with a local potential. More precisely, we can introduce

Gibbs families with potential of diameter 2 such that the support of these families belongs to the set of countable trees. We consider the case of p-regular trees only.

This construction also gives an example in which the partition function has factorial growth and nevertheless the corresponding Gibbs family is unique.

We consider the set $\mathcal{A}_{N,p}$ of all p-regular graphs with N vertices, that is, the graphs with all vertices of degree $p \geqslant 3$, and we introduce a finite Gibbs family on $\mathcal{A}_N$ with the following superpotential Φ: $\Phi = 0$ on the graphs of radius 1 if 0 is of degree p, and $\Phi = \infty$ for the other graphs of radius 1. In other words, we consider a finite Gibbs family μ_N on $\mathcal{A}_{N,p}$ with potential $\Phi \equiv 0$.

Theorem 10. $\lim p^N(\Gamma_k) = p(\Gamma_k) = 1$ *if* Γ_k *is an arbitrary* p*-regular tree in which any path from* 0 *to a final vertex is of length* k, *and the limit vanishes for the other* Γ_k.

Proof. We note that all graphs in $\mathcal{A}_{N,p}$ are equiprobable with respect to the measure μ_N, which reduces the problem to the classical theory of random graphs; see [20]. The combinatorial proof of this theorem consists of several steps.

1. A graph is said to be *enumerated* if its vertices are enumerated. Let $L_N(p)$ be the number of enumerated p-regular graphs with N vertices. This number is equal to

$$L_N(p) \sim C(p)\frac{(pN-1)(pN-3)\cdots}{(p!)^N}$$

(see Theorem II.16 in [20]), where $C(p)$ is a constant whose explicit form is known, but we do not need it. Similarly, for given values $p, k, d_1, \dots, d_k$ we denote by $L_N(p; d_1, \dots, d_k)$ the number of enumerated graphs that have $N-k$ vertices of degree p and $N-k$ other vertices of degrees $d_1, \dots, d_k$. This number is equal to

$$L_N(p; d_1, \dots, d_k) \sim C(p, k; d_1, \dots, d_k)\frac{(2m-1)(2m-3)\cdots}{(p!)^N}$$

(see Theorem II.16 in [20] again), where $C(p, k; d_1, \dots, d_k)$ is a constant and $2m = p(N-k) + d_1 + \dots + d_k$.

2. Let us consider the probability that the neighbourhood $O_1(v)$ of some vertex (for instance, of the vertex 0) has no cycles and prove that this probability tends to zero as $N \to \infty$. The number of graphs for which $O_1(v)$ has no cycles is equal to $L_{N-1}(p; d_1, \dots, d_p)(N-1)\cdots(N-p)$, where $d_1 = \dots = d_p = p-1$. At the same time, the number of graphs for which $O_1(v)$ contains exactly one cycle, that is, a single additional edge, say between the edges 1 and 2, is equal to $L_{N-1}(p; e_1, \dots, e_p)(N-1)\cdots(N-p)$, where $d_1 = d_2 = p-2$ and $d_3 = \dots = d_p = p-1$. It can readily be seen that

$$\frac{L_{N-1}(p; e_1, \dots, e_p)}{L_{N-1}(p; d_1, \dots, d_p)} \to 0$$

as $N \to \infty$.

3. This implies that the mean number of vertices v for which the neighbourhoods $O_1(v)$ have no cycles is of order $o(N)$. Since almost all p-regular graphs have no automorphisms (by Theorem IX.7 in [20]), the same result holds for non-enumerated graphs as well.

4. The neighbourhoods of greater radius can be treated similarly.

The following remark is of importance: for a given N the probability of the event that a graph in $\mathcal{A}_{N,p}$ is a tree need not be close to 1. Moreover, the probability that a given graph in $\mathcal{A}_{N,p}$ is a tree tends to zero as $N \to \infty$. A non-trivial topology (with cycles) occurs at a larger scale, and the typical length of a cycle tends to ∞ as $N \to \infty$; see below.

Remark 4. If the above set S is finite, if there is a non-trivial spin, and if the interaction is trivial, then the limit is an independent field on a p-regular tree. There is a question: what can happen if there is an additional interaction potential, for example, a two-particle nearest-neighbour potential?

4.4.2. *Gibbs varieties of groups.* The Gibbs approach can distinguish not only trees but also other mathematical objects that admit no local characterization. We consider cases in which the passage to the limit gives an algebraic object, namely, a group, with probability 1.

Let us consider the class $\mathcal{H}_{N,2p}$ of $2p$-regular graphs with local structure and assume that all sprouts of any vertex are enumerated by the symbols $a_1, \dots, a_p$, $a_1^{-1}, \dots, a_p^{-1}$. Moreover, if the symbol corresponding to one end of an edge is, say, a_i, then a_i^{-1} corresponds to the other end, and conversely. This class can be singled out in the class $\mathcal{A}_N$ by the local potential Φ on the edges that is equal to 0 if the sprouts of the edge are endowed with symbols of the form a_i and a_i^{-1} and equal to ∞ otherwise, and also by the superlocal potential Φ_1 on neighbourhoods of radius 1 such that Φ_1 vanishes only if the vertex is of degree $2p$ and its sprouts are enumerated by different symbols.

In particular, the Cayley diagrams of countable groups (see [29]) belong to this class of graphs with local structure. The graph of a group with finitely many generators $a_1, \dots, a_p$ and finitely many defining relations $\alpha_1 = \dots = \alpha_n = 1$, where $\alpha_1, \dots, \alpha_n$ are some words, is distinguished by the following non-local condition: any cycle in the graph is generated by $\alpha_1, \dots, \alpha_n$, and conversely (we recall that to any path on the graph one can assign a word, that is, an element of the group). A free non-commutative group corresponds to the case in which $n = 0$, that is, there are no defining relations.

Theorem 11. *The limit Gibbs measure on $\mathcal{A}_\infty^{(0)}$ with potential $\Phi + \Phi_1$ is concentrated on the tree corresponding to the free group with n generators.*

We do not present the proof; however, we note that it differs from the above argument for p-regular graphs in that the set of sprouts is partitioned into $2p$ subsets $A(a_i)$, $A(a_i^{-1})$, and the only restriction on the joining of the sprouts is that any sprout in $A(a_i)$ can be joined only with sprouts in $A(a_i^{-1})$.

Apparently, this result admits the following sweeping generalization. Let us consider the set $\mathcal{H}_{N,2p}(\alpha_1, \dots, \alpha_n)$ of graphs corresponding to the groups in the variety determined by the relations $\alpha_1 = \dots = \alpha_n = 1$ (and possibly some other relations). Then the limit Gibbs measure is concentrated on the free group relative to this variety (that is, the group in which only the relations $\alpha_1 = \dots = \alpha_n = 1$ are satisfied), because the formation of additional short cycles is a rare event. Results of this kind can play some role in physics in explaining how non-local objects can appear with high probability on the basis of local conditions.

4.5. Scales. Let us briefly discuss an important generalization that enables us to go beyond the framework of thermodynamic limit leading to limit Gibbs families.

Let Gibbs families μ_N on the sets $\mathcal{B}_N$ of finite graphs with local structure be given, where N is a parameter which can be the number of vertices, the radius, and so on. These families induce measures ν_N on the corresponding sets $\mathcal{G}_N$ of finite graphs. We also assume that a non-decreasing function $f\colon \mathbb{Z}_+ \to \mathbb{R}_+$ is given. Let us define the macrodimension of a random finite graph (with respect to ν_N) on the scale $f(N)$. Let $O_{f(N)}(v)$ be the neighbourhood of a vertex v of radius $f(N)$. We set

$$D_N(f) = \left\langle \frac{1}{|V(G)|} \sum_{v \in V(G)} \frac{\log O_{f(N)}(v)}{\log f(N)} \right\rangle_{\nu_N}.$$

If the limit $\lim_{N\to\infty} D_N(f)$ exists, then the value $D(f)$ of this limit is called the macrodimension on the scale f.

For a countable Gibbs family one can define corresponding invariants obtained by passing to the limit as $N \to \infty$. See, for instance, the definition of the macrodimension of a countable complex in [30]. However, this definition of macrodimension corresponds to a minimal scale f, for which the growth of $f(N)$ is as slow as desired (as N increases).

We can define other topological characteristics on diverse scales similarly. Let $\mathcal{B}_N = \mathcal{A}_N$, that is, $V(G) = N$. For instance, let $h(O_d(v))$ be the number of independent cycles in the d-neighbourhood of a vertex v. The exponent for the number of independent cycles on the scale f is defined as follows:

$$b(f) = \lim_{N\to\infty} \frac{1}{N} \left\langle \sum_{v} \frac{h(O_{f(N)}(v))}{|O_{f(N)}(v)|} \right\rangle,$$

if the limit exists.

Proposition 4. *Under the assumptions of Theorem* 10 *one has* $b(f) = 0$ *for any scale* $f(N) \leqslant (1-\varepsilon)\log_{p-1} N$, *where* $\varepsilon > 0$ *is arbitrary, and* $b(f) = p/2 - 1$ *for any scale* $f(N) \geqslant (1+\varepsilon)\log_{p-1} N$.

Proof. Let us consider the neighbourhood $O_{(1-\varepsilon)\log_{p-1} N}(v)$ of some vertex v. This neighbourhood can have at most $(p-1)^{(1-\varepsilon)\log_{p-1} N} = N^{1-\varepsilon}$ vertices. Each of the edges issuing from these vertices is incident to another vertex of this neighbourhood with probability not exceeding $N^{-\varepsilon}$. Therefore, the number of these edges is at most $N^{-\varepsilon} p N^{1-\varepsilon} = pN^{1-2\varepsilon}$. Any such edge involves the creation of at most one new cycle. This implies the first assertion.

To prove the other assertion, we denote by $D(N)$ the diameter of the random graph. As is known [20],

$$P_N\biggl(1-\varepsilon < \frac{D(N)}{\log_{p-1} N} < 1+\varepsilon\biggr) \to 0 \quad \text{as} \quad N \to \infty$$

for any $\varepsilon > 0$. The number of independent cycles in the entire graph is equal to $(p/2-1)N - 1$, which follows from the Euler formula $V - L + M = 1$, where M is the number of linearly independent cycles. Therefore, $M = L - V = (pN/2) - N$.

4.6. On the discrete quantum gravity. In conclusion it is appropriate to discuss the present state of development of quantum gravity. The answer to the question "What is quantum gravity?" cannot be exactly formulated. There are many approaches to the formulation and solution of this problem (the unrestrained optimism of some physics papers has always been quickly replaced by more temperate views). As we shall see now, many approaches can be kept within the scheme developed in the present paper. The main difficulty, aside from the need for fundamental ideas, is in the specification of the model on the basis of numerous physical requirements which also remain unclear.

It is assumed in physics that the so-called standard model of quantum field theory meets practical needs and describes all interactions in nature except for gravity (that is, the electromagnetic, weak, and strong interactions). Although the standard model in four-dimensional space-time has not yet been given in the framework of constructive quantum field theory (that is, there is no mathematically rigorous construction in the ordinary sense), this model at least admits clearly formulated definitions and assertions-conjectures. The latter can be formulated both in the framework of the Wightman axioms and in the form of the scaling limit of lattice approximations. However, even the exact definitions disappear under any attempt to create a unified theory including the gravitational interaction. All we have now are pieces of a 'future theory' which are based on various physical, methodological, and even philosophical principles. I shall try to make some classification of these pieces while remaining in the framework of mathematics, that is, using exact definitions and formulations.

Let us note first that there are no apparent ways to directly generalize the Wightman axioms nor, in the Euclidean version, the Osterwalder–Schrader axioms (although the axioms of S-matrix type, in which space-time plays a small role, can be generalized in various ways, for instance, as axioms of topological field theory or diagrams of string theory). The main notions are based on the classical and fixed space-time. For instance, locality uses the Minkowski metric, and unitarity is based on the very existence of classical time. If, in the framework of the Wightman axioms, we add the metric (regarded as a new field) to the field of matter, then (as proved by the experience of constructive field theory) this metric need not be a continuous function, which can modify the topology of space-time in the small and make the notion of locality unclear in itself. In this case one says that the quantization of the metric leads to a 'quantization' of space-time.

A lattice approximation is a much more convenient object for generalizations. Indeed, the existing theories admit discrete reformulations. For example, the ordinary Euclidean quantum field theory (for simplicity, we restrict ourselves to boson fields) deals with the lattice $\mathbb{Z}_\varepsilon^d$ with step ε, with the volume $\Lambda = [-N, N]^d \subset \mathbb{R}^d$, with fields $\phi(x) \in \mathbb{R}^m$, $x \in \mathbb{Z}_\varepsilon^d$, and with the partition function

$$Z_{\Lambda,\varepsilon} = \left[\prod_{x\in\Lambda\cap\mathbb{Z}_\varepsilon^d} \int_{\mathbb{R}^m} d\phi(x) \right] \exp\Bigl(- \sum_{\langle x,x'\rangle} \Phi(\phi(x), \phi(x'); \varepsilon) \Bigr),$$

where x and x' is an arbitrary pair of nearest neighbours in $\Lambda \cap \mathbb{Z}_\varepsilon^d$ and Φ is a real function on $\mathbb{R}^m \times \mathbb{R}^m$ that also depends on the parameter ε. The main mathematical problem is the existence of limits of diverse quantities as $\varepsilon \to 0$.

In the framework of the present paper we can only briefly list some approaches to the construction of discrete models. We claim that these models are Gibbs families, and we shall explicitly indicate the corresponding class of graphs with local structure and the potential. For physical surveys of diverse directions in quantum gravity, see [31]–[33].

It is of importance that almost all the theories listed below involve quantization of the metric rather than the topology because the class of graphs under consideration is related to a unique (and fixed) manifold or to a class of manifolds whose topology satisfies rather strong conditions. For this reason, and also because a part of the theory seems to be the same for all models, it seems that a general theory of Gibbs families is needed.

Regge theory. Chronologically, the Regge theory is the earliest discrete version of classical general relativity (see [34]). In contrast to the model of dynamical triangulations in which the lengths of all edges are assumed to be equal, these lengths are arbitrary in the Regge theory. Let us choose a triangulation T of a smooth manifold M of dimension d, and let $S_k(T)$, $k = 0,\dots,d$, be the set of k-dimensional simplices of T. Let l_i be the length of the edge $i \in S_1(T)$ and let $\mathcal{L}(T) = \{l_i,\ i \in S_1(T)\}$ be the family of sets of lengths from which a positive-definite Euclidean metric can be recovered on any simplex. Then the partition function becomes

$$Z_N = \sum_{T:|T|=N} \exp(-S(T))\,d\Lambda(T), \quad S(T) = \lambda \sum_{B\in S_{d-2}(T)} v(B)\varepsilon(B) + \mu \sum_{\Delta\in S_d(T)} v(\Delta),$$

where $d\Lambda(T)$ is the restriction of the measure $\prod_{i\in S_1(T)} dl_i$ to $\mathcal{L}(T)$, $v(\,\cdot\,)$ stands for the volume, and $\varepsilon(B)$ is the defect angle of the $(d-2)$-dimensional face B, which we do not define here (see [34]). These quantities can be expressed via the values l_i. Thus, in the quantum Euclidean approach the quantities l_i are random variables satisfying the corresponding inequalities, which complicate the introduction of a free measure.

Dynamical triangulations. Apparently, the so-called model of 'dynamical triangulations', which arose in string theory, is the most rich in results.

We note that, among all directions in quantum gravity, only string theory includes the standard model and tries to develop the theory 'to numbers'. Although string theory follows the formal apparatus of quantum field theory, any axioms are out of the question. String theory develops the theory of the scattering matrix rather than our fundamental concepts concerning space-time.

In the Euclidean approach, a discrete string is a Gibbs family on some subsets $\mathcal{A}_N$ of the set $\bigcup_{\rho,k}\mathcal{A}_N(\rho,k)$ of triangulations T of a surface of genus ρ with k holes. The spin σ_i with values in $\mathbb{R}^d$ is defined on every triangle i. For $\mathcal{A}_N$ one usually takes $\mathcal{A}_N = \mathcal{A}_N(0,0)$ or $\bigcup_\rho \mathcal{A}_N(\rho,0)$. In this case the canonical partition function is of the form

$$Z_N = \sum_T \int_{\mathcal{A}_N} \exp\Bigl(-\lambda\rho(T) - \sum_{\langle i,j\rangle}(\sigma_i-\sigma_j)^2\Bigr)\prod_{i\in T} d\sigma_i.$$

However, other approaches to string theory are more popular. For instance, the spectrum of a free boson string can be found explicitly in the Hamiltonian approach, while this has not been done (and is seemingly hard to do) in the approach of dynamical triangulations. This enigmatic circumstance shows that relationships between different approaches to the same physical object can be quite subtle.

Matrix theories. The matrix models are used to enumerate charts (and more complicated objects) on surfaces. By a *smooth chart* we mean a triple (S, G, ϕ), where S is a smooth compact orientable surface, G is a connected graph (one-dimensional complex), and $\phi\colon G \to S$ is an embedding for which the images of the edges of the graph G are smooth arcs, and the complement consists of domains homeomorphic to a disc. A *chart* (or a *combinatorial chart*) is an equivalence class of smooth charts, where two smooth charts (S, G, ϕ_1) and (S, G, ϕ_2) are said to be *equivalent* if there is a homeomorphism $f\colon S \to S$ that takes the vertices and the arcs of $\phi_1(G)$ to the vertices and the arcs of $\phi_2(G)$, respectively.

A model of random matrices is a probability distribution μ on the set of self-adjoint $n \times n$-matrices $\phi = (\phi_{ij})$ with a density of the form

$$\frac{d\mu}{d\nu} = Z^{-1} \exp\biggl(-\operatorname{tr}\biggl(\frac{\phi^2}{2h}\biggr) - \operatorname{tr}(V)\biggr),$$

where $V = \sum a_k \phi^k$ is a polynomial in ϕ that is bounded below and ν stands for the Lebesgue measure on the real n^2-dimensional space of vectors of the form $(\phi_{ii}, \operatorname{Re}\phi_{ij}, \operatorname{Im}\phi_{ij}, i < j)$. If $V = 0$, then $\mu = \mu_0$ is a Gaussian measure with the covariations $\langle \phi_{ij}, \phi^*_{kl} \rangle = \langle \phi_{ij}, \phi_{lk} \rangle = h\delta_{ik}\delta_{jl}$. Then the density of μ with respect to the Gaussian measure μ_0 is equal to

$$\frac{d\mu}{d\mu_0} = Z_0^{-1} \exp(-\operatorname{tr}(V)).$$

For the existence of a measure μ it is necessary that the leading coefficient a_p of the polynomial V be positive and that p be even. In this case there is a well-developed theory of such models; see [35].

The fundamental relationship (due to 't Hooft) between the matrix models and the calculation of the number of combinatorial charts on surfaces is given by the following formal series in the semi-invariants or in the diagrams (see [36]):

$$\log Z = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \langle \operatorname{tr}(V), \dots, \operatorname{tr}(V) \rangle = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \sum_{D_k} I(D_k),$$

where $\sum_{D_k}$ is the sum over all connected diagrams D_k with k vertices and $L = L(D_k)$ edges. For instance, let $V = a\phi^4$. Then every diagram has enumerated vertices $1, \dots, k$ and $L = 2k$ edges, and each vertex has enumerated sprouts $1, 2, 3, 4$ corresponding to the factors of the product $\phi_{ij}\phi_{jk}\phi_{kl}\phi_{li}$. For instance, the sprout corresponding to ϕ_{ij} can be represented as a two-sided strip (thus defining a 'ribbon' (or a 'fat') graph) whose sides have the matrix indices i and j, respectively. Moreover, in a neighbourhood of a vertex, the strips are arranged on the plane according to our figure. Let us agree that the sprouts-strips are paired in such a way that paired sides have equal indices.

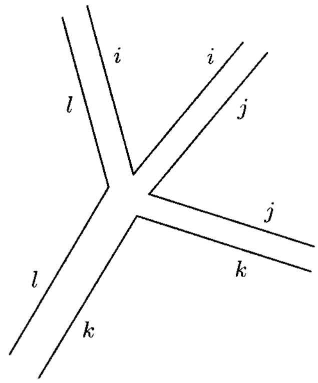

Figure 2. A vertex of a ribbon graph

Since any index occurs an even number of times at every vertex, it follows that for a given edge l with index i there is a unique connected path in the diagram along the edges with index i that passes through l. All these paths are closed; we refer to them as *index loops*. Summing over the indices, we obtain the factor n^N, where $N = N(D_k)$ is the number of index loops. This results in the formal series

$$\sum_k \frac{(-a)^k}{k!} \sum_{D_k} h^{2k} n^{N(D_k)} = \sum_k (-4ah^2)^k \sum_{E_k} n^{N(E_k)} = \sum_{k,N} (-4ah^2)^k n^N M(k,N),$$

where the sum over E_k is taken over all graphs with unordered vertex set such that the set of sprouts of any vertex is cyclically ordered. In the passage from D to E we introduce the factor $\frac{k!\,4^k}{A(E)}$, where $A(E)$ is the cardinality of the automorphism group of E. We omit $A(E)$, assuming that $A(E) = 1$ for 'almost all' graphs (as $k \to \infty$).

In the last sum, $M(k,N)$ stands for the number of charts with k vertices and N cells. Indeed, one can readily see that for any graph E there is an embedding $f(E)$ of this graph in a compact orientable surface S_ρ of some genus ρ such that any index loop bounds an open domain of S_ρ that is homeomorphic to a disc, and this embedding is unique (up to combinatorial equivalence). This chart has k vertices, $2k$ edges, and N faces. Using the Euler formula $k = N + 2\rho - 2$, we obtain the following expansion by setting $h = 1$ and $a = b/n$:

$$\log Z = \sum_{N,\rho} (-4b)^{N+2\rho-2} n^{-2\rho+2} M_\rho(N),$$

where $M_\rho(N)$ stands for the number of charts of genus ρ with N cells. This implies, for instance, that for $\rho = 0$ the terms $M_\rho(N)$ are formally singled out as follows:

$$\lim_{n\to\infty} \frac{\log Z}{n^2} = \sum_N (-4b)^{N-2} M_0(N).$$

Generally, it should be noted that the application of the matrix method to combinatorial problems is rigorous only for the calculation of the number of triangulations of a surface with a single cell (see [37]) or with finitely many cells (see [38], [39]). For mathematicians a quite clear introduction to the matrix method for charts can be found in [40], and complete (but less mathematical) expositions can be found in many physics papers; see [41]. The case of a fixed genus ρ (which corresponds to the calculation of the number of diagrams in string theory by using the perturbation theory) is well developed only for the dimension $d = 0$ of the enveloping space, or, at the physical level of rigor, for the next generalization, which is a model with Q matrices M_q and with action of the form

$$\operatorname{Tr}\biggl(\sum_{q=1}^{Q} V(M_q) + \sum_{q=1}^{Q-1} M_q M_{q+1}\biggr),$$

and also for its continuous analogue; see the survey [42].

It should be noted that the case of large ρ (that is, if ρ increases together with N), which corresponds in the physical language to the investigation of the space-time 'foam' in the sense of Hawking, has hardly been studied at all even without spin, that is, for $d = 0$.

Spin foam. By a *spin foam* we mean a graph Γ together with a local structure of a two-dimensional complex (such that any vertex and any edge belong to at least one two-dimensional cell) and with functions $a(f)$ (on the set $\Gamma^{(2)}$ of two-dimensional cells f) and $b(l)$ (on the set of edges $\Gamma^{(1)}$). It is assumed that the function $a(f)$ determines an irreducible representation of a (Lie or quantum) group G; let $H_{a(f)}$ be the Hilbert space of this representation. If an edge l is incident to the cells $f_1, \dots, f_k$, then we set $H_l = H_{a(f_1)} \otimes \dots \otimes H_{a(f_k)}$. Let H_l^0 be an invariant subspace of H_l. Once and for all, let us choose orthonormal bases in the invariant subspaces of the tensor product of arbitrary irreducible representations. It is assumed that the function $b(l)$ determines one of the elements of this basis.

On some class $\mathcal{A}$ of finite spin graphs of this kind (for instance, on those with N vertices) endowed with a structure of a two-dimensional complex we define the partition function

$$Z_{\mathcal{A}} = \sum_{\Gamma\in\mathcal{A}} Z(\Gamma), \qquad Z(\Gamma) = \sum_{\{a(\,\cdot\,),b(\,\cdot\,)\}} \prod_{f\in\Gamma^{(2)}} \dim a(f) \prod_{v\in\Gamma^{(0)}} \Phi(O_1(v)),$$

where Φ is a function of the neighbourhood (of the vertex v) of radius 1. If Φ is positive, then this defines a Gibbs family.

As far as I know, partition functions of this kind have been regarded until now only as formally algebraic objects. Generally, there are many partition functions in physics that are not positive, and the problem of attaching some meaning to quantities is quite open in this case. The quantum discrete spaces introduced below in § 5 can be more useful here than Gibbs spaces.

Let us present some examples; for details, see [43]. For a vertex v we write $H_v = H_{l_1} \otimes \dots \otimes H_{l_m}$, where $l_1, \dots, l_m$ are the 1-cells incident to v. Since every 2-cell f_i is incident to exactly two edges issuing from v, it follows that the corresponding space $H_{a(f_i)}$ occurs in H_v also twice. Therefore, the space H_v can be represented naturally as a tensor square, and the trace Tr is well defined for any element of H_v. In this case the Turaev–Ooguri–Crane–Yetter model is determined by the function

$$\Phi = \mathrm{Tr}(b(l_1) \otimes \cdots \otimes b(l_m)).$$

This partition function is finite if for the group we take the quantum group $SU(2)_q$. The main result is that $Z(\Gamma)$ does not depend on Γ if Γ ranges over the two-dimensional skeleton of a triangulation of a given four-dimensional manifold. The three-dimensional version of this model is the Turaev–Viro model, and the three-dimensional version with the group $SU(2)$ is the Ponzano–Regge model (see [44]), which develops the Regge model and assumes that the lengths of the edges vary discretely, namely, $l = \delta j$, where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$. Corresponding to any j is a representation of the group $SU(2)$ of spin j.

Models of spin foam enable one to use the Gibbs approach to the so-called loop quantum gravity, which was studied earlier in the framework of the Hamiltonian approach. This model was based on new variables introduced by Ashtekar in general relativity, and it used the Penrose networks as a convenient basis in the function space on the set of all connections in the space at a fixed instant.

Topological quantum field theory. The axioms of topological quantum field theory in [23] resemble S-matrix axioms rather than give models of the structure of space-time. Gibbs families are closer to the physical representation of topological field theory in terms of functional integrals. The known models of this theory deal with finite-dimensional Hilbert spaces related to boundary manifolds. For the classification of such cases in dimension two, see the last chapter of [27]. Here we can only mention the corresponding problems from the viewpoint of Gibbs families.

Let us consider triangulations of a two-dimensional compact orientable surface of genus ρ with k holes, and let S be a spin space. We consider the set $\mathcal{T}(N, m_1, \dots, m_k)$ of triangulations with a given number N of triangles and with given numbers $m_1, \dots, m_k$ of edges on the k boundaries. For a given potential one can consider the conditional partition function $Z_{m_1 \dots m_k}(s_{\mathrm{boundary}})$ for given values s_{boundary} of the spin on the boundary, which yields a positive (non-normalized) measure $\mu(m_1, \dots, m_k)$ on the configurations $S^{m_1} \times \cdots \times S^{m_k}$ and a measure on

$$D(k) = \bigcup_{m_1, \dots, m_k} S^{m_1} \times \cdots \times S^{m_k}$$

for a given k, or even a measure on $\bigcup_k D(k)$. In particular, there are the following questions: in which cases are these measures finite, and for what scalings does the weak limit of these measures exist? This can give examples in which the Hilbert spaces are infinite-dimensional.

Causal sets. By a causal set V (see the brief survey [45]) one means a partially ordered set with a relation $\leqslant$ that is transitive, reflexive, and antisymmetric (which means that if $x \leqslant y$ and $y \leqslant x$, then $x = y$). A given (finite or countable) causal set determines an oriented graph G (the Hasse diagram of V) in such a way that V becomes the set of vertices of G, and there is a directed edge from x to y if $x \leqslant y$, $x \neq y$, and there is no other z such that $x \leqslant z \leqslant y$. Conversely, any oriented cycle-free graph defines a causal set V. Therefore, causal sets reduce to graphs with local structure. Causal sets are of interest because discrete analogues of the light cone and other notions used in general relativity can be defined for these sets.

Non-commutative lattices. Corresponding to a non-commutative C^*-algebra is its dual object which can be interpreted as a discrete non-commutative space. This topic belongs to the next (quantum) section of the paper, and here we say only a few words.

A topological space X can be approximated by discrete sets in two ways: one can either take an expanding sequence of dense subsets of X or use an embedded sequence of coverings $\mathfrak{U}^n = \{\mathfrak{U}_i^n\}$ (and consider the set of all closed points of the projective limit of this sequence). An arbitrary covering of a compact space X generates a finite partition of X, and the set of blocks of this partition is a T_0-space in the quotient topology. Defining a T_0-topology on a finite or countable set is equivalent to defining a relation of partial order on this set, namely, $x \leqslant y$ if and only if y belongs to the closure of $\{x\}$. As we saw above, to a partially ordered set we can assign a natural oriented graph, namely, the Hasse diagram of this graph.

Every finite partially ordered set V is the set $\widehat{A}$ of irreducible representations of some C^*-algebra A (which is not unique in general); see [46]. In this case, if A is separable, then the set $\widehat{A}$ is homeomorphic to the space $\operatorname{prim} A$ of primitive ideals of A, which is called a non-commutative lattice. For an application of non-commutative lattices in a simple problem of quantum mechanics, see the last chapter in [46].

§ 5. Quantum discrete spaces

5.1. Quantum graphs. We first assume that S is at most countable. We denote by $\mathcal{H} = l_2(\mathcal{A}^{(0)})$ the Hilbert space with an orthonormal basis e_α enumerated by finite spin graphs $\alpha \in \mathcal{A}^{(0)}$, that is, $(e_\alpha, e_\beta) = \delta_{\alpha\beta}$, where $e_\alpha(\beta) = \delta_{\alpha\beta}$. Every vector ϕ in $\mathcal{H}$ is a function on the set $\mathcal{A}^{(0)}$ and can be represented in the form

$$\phi = \sum \phi(\alpha) e_\alpha \in \mathcal{H}, \qquad \|\phi\|^2 = \sum |\phi(\alpha)|^2.$$

The states of the system are wave functions, that is, vectors ϕ with unit norm $\|\phi\|^2 = \sum |\phi(\alpha)|^2 = 1$.

The definition of quantum 'Gibbs' states differs from the classical Gibbs states in that in the quantum case one automatically obtains a dynamics, or the corresponding Hamiltonian. On quantum discrete spaces, the dynamics is related to substitutions. Roughly speaking, a substitution is the replacement of a regular subgraph γ by another subgraph δ, but the exact definition is more complicated because one must be careful about the rule for joining the subgraph δ to the complement of γ.

A vertex $v \in V(\gamma) \subset V(G)$ of a regular subgraph γ of G is said to be *interior* if any edge incident to this vertex joins it to another vertex γ, and it is said to be a *boundary* vertex otherwise. The set of all boundary vertices is called the *boundary* $\partial\gamma$ of the regular subgraph γ.

Definition 5. The substitution rule (the production) $\mathrm{Sub} = (\Gamma, \Gamma', V_0, V_0', \varphi)$ is determined by two 'small' spin graphs Γ and Γ', by subsets $V_0 \subset V = V(\Gamma)$ and $V_0' \subset V' = V(\Gamma')$, and by a one-to-one map $\varphi\colon V_0 \to V_0'$ preserving the spins. Suppose that there is an isomorphism $\psi\colon \Gamma \to \gamma$ onto a *regular* spin subgraph γ of a spin graph α such that $\psi(V_0) \supset \partial\gamma$. The transformation (substitution) $T = T(\text{Sub}, \psi)$ of α corresponding to a given substitution rule Sub and to an isomorphism ψ is defined as follows: delete all edges of the subgraph $\gamma = \psi(\Gamma)$ and all vertices in $V(\gamma) \setminus \psi(V_0)$; in the disjoint union of α and Γ' identify any $\psi(v)$, where $v \in V_0$, with the vertex $\varphi(v) \in V_0'$. We denote by $T(\text{Sub}, \psi)\alpha$ the resulting spin graph. The function s is inherited from α on $V(\alpha) \setminus V(\gamma)$, and from Γ' on $V(\gamma)$.

Examples of substitutions: deleting an edge, adding an edge with another new vertex at a given vertex, joining two vertices by an edge, modifying the value of the spin at a single vertex. The mere possibility of such a substitution depends on some neighbourhood of the place of substitution.

For arbitrary N and p we denote by $\mathcal{H}_{N,p} \subset \mathcal{H}$ the finite-dimensional subspace spanned by all e_α with $\alpha \in \bigcup_{n=0}^{N} \mathcal{A}_{n,p}^{(0)}$. Let P_N be the orthogonal projection onto $\mathcal{H}_{N,p}$. For a given p there are natural embeddings $\mathcal{H}_{N,p} \subset \mathcal{H}_{N+1,p}$. In what follows we fix some p and omit this subscript, bearing in mind that the degrees of the vertices do not exceed p. We write $\mathcal{H}_N = \mathcal{H}_{N,p}$.

Definition 6. A grammar (to be more exact, a grammar on graphs) is defined to be a finite set of substitutions $\text{Sub}_i = (\Gamma_i, \Gamma_i', V_{i,0}, V_{i,0}', \varphi_i)$, where $i = 1, \dots, |\text{Sub}|$. A grammar is said to be *local* if the graph Γ_i corresponding to Sub_i is connected for any i. A grammar is said to be *locally bounded* if the set $\bigcup_N \mathcal{A}_{N,p}^{(0)}$ is invariant with respect to all substitutions of this grammar for sufficiently large p.

Let us define the main operators $a_i(j)$. The index i corresponds to a substitution of the grammar. For a given i we consider all possible pairs (ψ, ξ), where $\xi \in \mathcal{A}^{(0)}$ and ψ is a map from Γ_i onto a regular subgraph of the spin graph ξ. In this case a pair (ψ, ξ) is said to be *minimal* if $\psi(\Gamma_i)$ does not belong to a neighbourhood of 0 of radius less than the radius $R(\xi) = R_0(\xi)$ of the graph ξ itself. For a given i we enumerate all regular pairs by indices $j = 1, 2, \dots$:

$$(\psi_{i1}, \xi_{i1}), (\psi_{i2}, \xi_{i2}), \dots, (\psi_{ij}, \xi_{ij}), \dots\ .$$

If ξ contains two regular subgraphs isomorphic to Γ_i that are taken to each other under a non-trivial automorphism of the graph ξ, then corresponding to these subgraphs are distinct indices j in our enumeration.

We set

$$a_i(j)e_\alpha = e_\beta, \qquad \beta = T(\text{Sub}_i, \rho\psi_{ij})\alpha,$$

for any α such that there is an isomorphism ρ of the spin graph ξ_{ij} onto the neighbourhood $O_{R(\xi)}(0)$ of the origin in the graph α. We also set $a_i(j)e_\alpha = 0$ otherwise. It is clear that this definition does not depend on the choice of ρ. We also note that $\|a_i(j)\| = 1$.

Let us introduce the linear operator $H = H(\{\text{Sub}_i,\ i = 1, \dots, k\})$ corresponding to the grammar $\text{Sub}_i = (\Gamma_i, \Gamma_i', V_{i,0}, V_{i,0}', \varphi_i)$, $i = 1, \dots, k$, as the formal sum

$$H = \sum_{i=1}^{k} \sum_{j} \lambda_i a_i(j)$$

for some complex constants λ_i. The operator H is defined on the linear space $\mathcal{H}^0$ of finite linear combinations of the e_α. It should be noted that this operator does not depend on the enumeration $\psi_{i,j}$ because of the following equivalent definition:

$$He_\alpha = \sum_{i=1}^{k} \sum_{\psi} e_{T(\mathrm{Sub}_i, \psi)\alpha}$$

for any α, where for a given i the sum is taken over all embeddings $\psi\colon \Gamma_i \to \alpha$.

The *adjoint* substitution $\mathrm{Sub}^* = (\Gamma^*, \Gamma'^*, V_0^*, V_0'^*, \varphi^*)$ to a substitution Sub is determined by the following properties: $\Gamma^* = \Gamma'$, $\Gamma'^* = \Gamma$, $V_0^* = V_0'$, $V_0'^* = V_0$, and $\varphi^* = \varphi^{-1}$. If substitutions i and i_1 are adjoint, then for any j there is a j_1 such that $(a_i(j))^* = a_{i_1}(j_1)$.

The Hamiltonian is formally symmetric, $H = H^*$, if for any i there is a j such that Sub_j is adjoint to Sub_i and $\lambda_i = \overline{\lambda}_j$.

One-dimensional grammars. If $p = 2$, then the graph is linear, and it can be cyclic or non-cyclic. Equivalently, one can speak of a word $\alpha = x_1 \dots x_n$, where $x_i \in S$. In this case, the substitution rule is a pair $\gamma \to \delta$, and the substitution is the transformation

$$\alpha\gamma\beta \to \alpha\delta\beta$$

for any α and β, where the concatenation of the words $\alpha = x_1 \dots x_n$ and $\beta = y_1 \dots y_m$ is defined as

$$\alpha\beta = x_1 \dots x_n y_1 \dots y_m.$$

The operator $a_i(j)$ permits one to make the substitution i in the word α only for a subword γ_i starting with the symbol with index j. The substitution adjoint to $\gamma \to \delta$ is $\delta \to \gamma$.

Self-adjointness. Let H be symmetric on the linear space $\mathcal{H}^0$ of finite linear combinations of the elements e_α. All these vectors are C^∞ elements for H (see [47]), that is, $He_\alpha \in \mathcal{H}^0$. We note that H is unbounded in general.

Theorem 12. *Let a grammar be locally bounded and let H be symmetric. Then H is essentially self-adjoint on $\mathcal{H}^0$.*

The proof follows from the next theorem and from the Nelson criterion; see [47]. This defines the group $\exp(itH)$. The lemma does not assume that the Hamiltonian is symmetric.

Lemma 10. *For any grammar an arbitrary vector $\phi \in \mathcal{H}^0$ is analytic for the corresponding Hamiltonian, that is,*

$$\sum_{k=0}^{\infty} \frac{\|H^k\phi\|}{k!} t^k < \infty$$

for some $t > 0$.

It suffices to take $\phi = e_\alpha$ for some α. In this case the number of pairs (i,j) with $a_i(j)e_\alpha \neq 0$ does not exceed CV, $V = V(\alpha)$, for some constant C. Indeed, the number of regular subgraphs isomorphic to a given 'small' subgraph depends on the number of vertices and on p. Let us represent the expansion of H in the form

$$H = \sum_a V_a,$$

where V_a is equal to some $\lambda_i a_i(j)$. Then

$$H^n e_\alpha = \sum_{a_n,\dots,a_1} V_{a_n} \cdots V_{a_1} e_\alpha = \sum C_\beta e_\beta. \tag{24}$$

The maximal number of vertices of the graphs β in the expansion $V_{a_n} \cdots V_{a_1} e_\alpha$ does not exceed $V(\alpha) + C_1 n$, because every factor adds a number of vertices that cannot exceed some constant C_1. Therefore, for any given $e_\alpha, a_1, \dots, a_n$, the number of operators $V_{a_{n+1}}$ that have a non-zero contribution to $V_{a_{n+1}} V_{a_n} \cdots V_{a_1} e_\alpha$ does not exceed $C(V(\alpha) + C_1 n)$ for some constant C. Thus, the number of non-zero terms $V_{a_n} \cdots V_{a_1} e_\alpha$ does not exceed the expression

$$C^n \prod_{j=1}^{n} (|V(\alpha)| + C_1 j) = (CC_1)^n \prod_{j=1}^{n} \left(\frac{|V(\alpha)|}{C_1} + j \right)$$
$$< (CC_1)^n n! \prod_{j=1}^{n} \left(1 + \frac{|V(\alpha)|}{C_1 j} \right) < (CC_1)^n n! \, n^{2\frac{|V(\alpha)|}{C_1}},$$

and the norm of any term is bounded by $(\max \lambda_i)^n$. This proves the convergence of the series for $|t| < t_0$, where t_0 does not depend on α, and with it the lemma.

5.1.1. *C^*-algebras.* There are some useful C^*-algebras related to grammars on graphs: two universal C^*-algebras $\mathbf{B}$ and $\mathbf{C}$ and a grammar-dependent C^*-algebra $\mathbf{A}(Gr)$. Let us introduce these objects.

Let $\mathbf{C}_N$ be the algebra of all operators in the finite-dimensional space $\mathcal{H}_N = \mathcal{H}_{N,p}$. The operator $C \in \mathbf{C}_N$ can be assumed to act on $\mathcal{H}$ if we set $Ce_\alpha = 0$ for $e_\alpha \notin \mathcal{H}_{N,p}$. Then $\mathbf{C}_N \subset \mathbf{C}_{N+1}$, and we set $\mathbf{C} = \overline{\bigcup_N \mathbf{C}_N}$. The algebra $\mathbf{C}$ consists of compact operators. It was used in [48].

Let $\mathbf{A}_N = \mathbf{A}_N(\mathrm{Sub}_i, i = 1, \dots, k)$ be the C^*-algebra of operators on $\mathcal{H}$ generated by all operators $a_i(j)$ such that $R(\xi_{ij}) \leqslant N$. Then the natural embeddings $\phi_N \colon \mathbf{A}_N \to \mathbf{A}_{N+1}$ are defined. The inductive limit $\bigcup_N \mathbf{A}_N = \mathbf{A}^0$ of the C^*-algebras $\mathbf{A}_N$ is called the *local algebra*, and its norm closure $\mathbf{A} = \mathbf{A}(\mathrm{Sub}_i, i = 1, \dots, k)$ is called the *quasi-local algebra*. The structure of the C^*-algebra $\mathbf{A} = \mathbf{A}(\mathrm{Sub}_i, i = 1, \dots, k)$ depends on the grammar and on the chosen class of graphs. We present below a number of examples of Hamiltonians and C^*-algebras.

The universal C^*-algebra $\mathbf{B}$ is generated by the operators $a_i(j)$ for all possible substitutions, and $\mathbf{B}$ differs from the algebra of all bounded operators. For instance, many operators diagonal in the basis e_α do not belong to $\mathbf{B}$.

5.1.2. *Automorphisms.* In what follows we always assume that the grammar is locally bounded and that some p is fixed in such a way that the set $\bigcup_N \mathcal{A}^{(0)}_{N,p}$ is invariant with respect to all substitutions of this grammar.

A formal Hamiltonian defines a derivation of the local algebra $\mathbf{A}^0$. Let us consider the operator

$$H_N = \sum_{i=1}^{|\mathrm{Sub}|} \sum_{j:\, R(\xi_{ij}) \leqslant N} \lambda_i a_i(j).$$

We define a group of automorphisms of $\mathbf{A}$ as follows. We take a local element $A \in \mathbf{A}^0$, find a number N such that $A \in \mathbf{A}_N$, and set

$$\alpha_t^{(N)}(A) = \exp(iH_N t) A \exp(-iH_N t).$$

Theorem 13. *There is a* $t_0 > 0$ *such that for any local element* A *and any number* t *with* $|t| < t_0$ *the limit*

$$\lim_{N\to\infty} \alpha_t^{(N)}(A)$$

exists with respect to the operator norm. This defines a unique automorphism of the quasi-local algebra.

Proof. Let us consider the Dyson–Schwinger series

$$A_t^{(N)} = A + \sum_{n=1}^{\infty} \frac{(it)^n}{n!} [H_N, \dots, [H_N, [H_N, A]] \dots].$$

We can restrict ourselves to local elements $A = a_i(k)$ for some i and k.

Lemma 11. *The norm of the operator* $\dfrac{(it)^n}{n!}[H_N, \dots, [H_N, [H_N, A]] \dots]$ *is bounded above by* $(Ct)^n$ *independently of* N.

We note that the commutator belongs to $\mathbf{A}$ and is equal to a sum of commutators of the form

$$[a_{i_n}(j_n), \dots, [a_{i_2}(j_2), [a_{i_1}(j_1), A]] \dots]$$

multiplied by $\lambda_{i_1} \cdots \lambda_{i_n}$. Each of the commutators is equal to a sum of 2^n terms of the form

$$\pm a_{q_n}(p_n) \cdots a_{q_2}(p_2) a_{q_1}(p_1).$$

We prove that, after collecting like terms, the number of terms of this kind in the sum under consideration is at most $\prod_{j=1}^n Cj = C^n n!$ (independently of N). To this end, we must use the other definition of the Hamiltonian. Let us introduce the following notation: if $a_i(j)e_\alpha = e_{T(\mathrm{Sub},\psi)}$, then we write $a_i(j)e_\alpha = a(T(\mathrm{Sub},\psi))e_\alpha$. Then for any α, any two substitution rules i_1 and i_2, and any two maps $\psi_1 : \Gamma_{i_1} \to \gamma_1$ and $\psi_2 : \Gamma_{i_2} \to \gamma_2$ onto disjoint regular subgraphs γ_1 and γ_2 of the graph α the corresponding transformations in terms of operators $a_i(j)$ give terms that cancel,

$$\begin{aligned} &a(T(T(\mathrm{Sub}_1, \psi_1)\alpha, \mathrm{Sub}_2, \psi_2)) a(T(\alpha, \mathrm{Sub}_1, \psi_1)) e_\alpha \\ &\quad - a(T(T(\mathrm{Sub}_2, \psi_2)\alpha, \mathrm{Sub}_1, \psi_1)) a(T(\alpha, \mathrm{Sub}_2, \psi_2)) e_\alpha = 0, \end{aligned}$$

and so on by induction.

One can prove similarly that $A_t^{(N)}$ converges termwise as $N \to \infty$ (for any n) to the series

$$A_t = A + \sum_{n=1}^{\infty} \frac{(it)^n}{n!}[H, \dots, [H, [H, A]] \dots].$$

Each term of the last series is defined, and it follows from the above estimate that the series is norm convergent for any sufficiently small t. The existence of the group of automorphisms for every t can now be proved as in the Robinson theorem for quantum spin systems, see [4].

5.1.3. *KMS-states on* **A**. We consider here the KMS-states on **A**, which are a natural generalization of KMS-states on a lattice. However, sometimes it is useful to consider KMS-states on the algebra **C** (see [48]), where the situation is different.

We restrict ourselves to the case $p = 2$, that is, to the one-dimensional linear case, and assume that $2 \leqslant |S| < \infty$. Generally (if the length of a word is changed under substitutions), the algebra $\mathbf{A}_N$ is not finite-dimensional, in contrast to quantum spin systems. We can introduce the finite-dimensional algebras $\mathbf{F}_N$ generated by all the operators $a_i^{(N)}(j) = P_N a_i(j) P_N$ and assume that these operators act on $\mathcal{H}_N$. Let us define states on $\mathbf{F}_N$ by setting

$$\langle A_N \rangle_N = Z_N^{-1} \operatorname{Tr}_N [A_N \exp(-\beta P_N H P_N)],$$
$$Z_N = \operatorname{Tr}_N \exp(-\beta P_N H P_N),$$

for any $A_N \in \mathbf{F}_N$, where Tr_N is the trace on $\mathcal{H}_N$. Any limit point of this sequence defines a state on **A** such that

$$\langle A \rangle = \lim_{N \to \infty} \langle P_N A P_N \rangle_N$$

if A is local.

Theorem 14. *If β is sufficiently small, then $\log Z_N \sim fN$ for some constant $f > 0$.*

Proof. We first note that the assertion is obvious for $H = 0$, because $\operatorname{Tr}_H 1 = \dim \mathcal{H}_N = 2^{N+1}$.

We can write

$$Z_N = \sum_{\alpha:\, R_0(\alpha) \leqslant N} z(\alpha), \qquad z(\alpha) = (e_\alpha, \exp(-\beta P_N H P_N) e_\alpha),$$

or, which is the same,

$$Z_N = \sum_{k=0}^{\infty} \sum_{I_k, J_k} \sum_{\alpha:\, |\alpha| \leqslant N} \frac{(-\beta)^k}{k!} z(\alpha, (I_k, J_k)), \tag{25}$$

where

$$z(\alpha, (I_k, J_k)) = \lambda_{i_1} \cdots \lambda_{i_k} (e_\alpha, P_N a_{i_k}(j_k) P_N \cdots P_N a_{i_1}(j_1) P_N e_\alpha)$$

for ordered sets $I_k = (i_1, \dots, i_k)$ and $J_k = (j_1, \dots, j_k)$, that is, $z(\alpha, (I_k, J_k))$ is equal to either $\lambda_{i_1} \cdots \lambda_{i_k}$ or 0. Our objective is to obtain a cluster representation for

$$z(N) \doteq \sum_{\alpha:|\alpha|=N} z(\alpha)$$

of the form

$$z(N) = \sum c_{U_1} \cdots c_{U_n}, \tag{26}$$

whose precise definition is given below, after which we can apply the general theory of cluster expansions (see [36]). To derive the expansion (26), we use a resummation of the expansion (25) and a recursive construction.

In (26) the sum is taken over the tuples $(U_1, \dots, U_n)$ of disjoint subsets. Let $U_0 = \{1, \dots, N\} \setminus \bigcup_{i=1}^n U_i$. We first define U_0 and then the 'clusters' U_i and the numbers c_{U_i}. We do this separately for each term

$$\frac{(-\beta)^k}{k!} z(\alpha, (I_k, J_k))$$

corresponding to a word α of length N and to some (I_k, J_k), and then we sum the results.

Definition of $U_0 = U_0(\alpha, (I_k, J_k))$. We say that a substitution $T = T(\mathrm{Sub}, \psi)$ preserves a symbol at a position v of a word α if the image of ψ does not contain v. It is convenient to denote by T_p the substitution corresponding to the operator $b(p) = a_{i_p}(j_p)$, $p = 1, \dots, k$. A symbol v of a word α is said to be *preserved* for a given (I_k, J_k) if any substitution T_p, $p = 1, \dots, k$, preserves this symbol. Similarly, a symbol v of a word $b(s) \dots b(1)e_\alpha$ is said to be *preserved* for a given (I_k, J_k) if every T_l with $s < l \leqslant k$ preserves v. For a given pair $(\alpha, (I_k, J_k))$ the set $U_0 = U_0(\alpha, (I_k, J_k))$ is defined as the set of all symbols of the word α that belong to all the words $b(s) \dots b(1)e_\alpha$ and are preserved for any s.

Definition of the clusters $U_i(\alpha, (I_k, J_k))$. We write $\alpha_0 = \alpha$ and $\alpha_s = T_s \cdots T_1 \alpha$, where $s \geqslant 1$. Let us introduce a system of partitions g_{sr}, $r, s = 0, \dots, k$, of the set of symbols of the word α_s into subsets. The partition $g_0 = g_{0,0}$ is defined as the partition of α into separate symbols, that is, the partition of the set $\{1, \dots, N\}$ into N blocks.

For a given substitution T transforming the word α into a word β and for a given partition $g = g(\alpha)$ of the set $V(\alpha)$ of symbols of the word α we define a partition $g(\beta)$ of the word β as follows. Let $V_1(\alpha, T)$ be the set of symbols of α preserved by the substitution T. Suppose that, in the new word β, the substitution T deletes the set of symbols $V(\alpha) \setminus V_1(\alpha)$ and adds a new set of symbols V_2. Thus, $V(\beta) = V_1(\alpha) \cup V_2$. We say that a partition $g(\beta) = g(\beta, g(\alpha), T)$ of the word β is *induced* by the partition $g = g(\alpha)$ and the substitution T if the following conditions hold:

1) if a block I of the partition $g(\alpha)$ belongs to $V_1(\alpha)$, then this block is also a block of the partition $g(\beta)$;
2) the vertices of V_2 form a single block together with all vertices (not belonging to $V_1(\alpha)$) of the other blocks I of the partition $g(\alpha)$ that intersect $V(\alpha) \setminus V_1(\alpha)$.

Let us now define the partitions $g_s = g_{s,0} = g_s(\alpha_s)$ of the word α_s by induction. If g_s is already defined, then g_{s+1} is the partition of α_{s+1} induced by the substitution T_{s+1} and the partition g_s of the word α_s. We next use induction on r. If a partition $g_{s+1,r}$ of α_{s+1} is defined, then we define $g_{s,r+1}$ as the partition of α_s induced by the adjoint substitution T_s^*.

Let us consider the partition $g_{0,k}$. The blocks of this partition either are single preserved symbols or contain more than one symbol. Let us denote the latter blocks by $U_i = U_i(\alpha, (I_k, J_k))$ and call them *clusters* with respect to the given pair $\alpha, (I_k, J_k)$. We note that the number $L = L(\alpha, (I_k, J_k))$ of clusters does not exceed k. Moreover, the set $\{1, \dots, k\}$ is uniquely partitioned into subsets $M_1 = M_1(\alpha, (I_k, J_k))$, ..., $M_L = M_L(\alpha, (I_k, J_k))$ of substitutions relating to one of the clusters $1, \dots, L$, respectively. Let (I_{m_i}, J_{m_i}), where $I_{m_i} = I_{m_i}(I_k, J_k)$ and $J_{m_i} = J_{m_i}(I_k, J_k)$, be an order-preserving subset of (I_k, J_k) corresponding to M_i.

Definition of expansion: expansion into connected groups. Let us consider a set $B \subset \{1, \dots, N\}$. If there is a cluster corresponding to this set, then the set is an interval.

Let $M_1, \dots, M_L$ be subsets of $\{1, \dots, k\}$ such that $\bigcup_{i=1}^{L} M_i = \{1, \dots, k\}$. We denote by m_i the cardinality of the set M_i. To obtain the multiplicative expansion (26), we take the sum of all terms $\frac{(-\beta)^k}{k!} z(\alpha, (I'_k, J'_k))$, where the sets (I'_k, J'_k) differ from (I_k, J_k) only by the order of the terms, and the order inside each of the sets M_i is preserved. This sum is equal to

$$\begin{aligned} \frac{(-\beta)^k}{m_1! \cdots m_L!} z(\alpha, (I_k, J_k)) &= \prod_{i=1}^{L} \frac{(-\beta)^{m_i}}{m_i!} z(\alpha, (I_{m_i}((I_k, J_k)), J_{m_i}((I_k, J_k)))) \\ &= \prod_{i=1}^{L} \frac{(-\beta)^{m_i}}{m_i!} \prod_{j \in M_i} \lambda_j. \end{aligned}$$

To define the expansion (26), we use a resummation. For given subsets $U_0, U_1, \dots$, U_n we consider $z(\alpha, (I_k, J_k))$ such that $U_i(z(\alpha, (I_k, J_k))) = U_i$, $i = 0, 1, \dots, n$, and we set

$$c_U = \sum \frac{(-\beta)^k}{k!} z(\alpha, (I_k, J_k))$$

for a given connected set U, where the sum is taken over all words α with the set of symbols $V(\alpha) = U$ and over all sets (I_k, J_k) for which U is a single connected cluster. We have the cluster estimate

$$c_U < c_1 (C\beta)^{|U|}.$$

Therefore, the standard technique of cluster expansions gives $\log z(N) \sim fN$ with some constant $f > 0$. Hence,

$$\log Z_N \sim \log \left[z(N) \Big(1 + \frac{z(N-1)}{z(N)} + \cdots \Big) \right] \sim cN.$$

Theorem 15. *There is a* $\beta_0 > 0$, *such that the limit*

$$\langle A\rangle = \lim_{N\to\infty}\langle P_N A P_N\rangle_N$$

exists for any local element A, *and this limit is analytic with respect to* β *for* $\beta < \beta_0$.

Proof. For instance, let $A = P_{j,\delta}$ be the projection onto the subspace generated by the words that contain the word δ at the jth position. We have

$$\begin{aligned}\mathrm{Tr}_N\left[\exp(-\beta P_N H P_N)A\right] &= \sum_{k=0}^{\infty}\frac{(-\beta)^k}{k!}\,\mathrm{Tr}_N((P_N H P_N)^k A)\\ &= \sum_{k=0}^{\infty}\frac{(-\beta)^k}{k!}\sum_{I_k,J_k}\mathrm{Tr}_N(a_{i_k}(j_k)\dots a_{i_1}(j_1)A),\end{aligned}$$

where the sum is taken over all $I_k = (i_1,\dots,i_k)$ and $J_k = (j_1,\dots,j_k)$. The rest of the proof is quite similar and reduces to an application of the general cluster expansion technique.

Remark 5. The case of spin graphs with $p \geqslant 3$ must be similar to the theory developed in § 4.3.

5.2. Examples and structure of Hamiltonians. Let us pass from the general theory to the simplest examples in the one-dimensional case. However, we first indicate the relationship with the 't Hooft quantization.

5.2.1. *'t Hooft's quantization.* The idea that the physical laws on the Planck scale can become transformations of a finite set with discrete time goes back to Feynman [49]. In the series of papers [50] and [51] 't Hooft discusses the quantization of discrete single-valued (deterministic) maps.

If $f\colon S \to S$ is a single-valued map of a finite set S into itself, then there is a subset S_0 which is invariant with respect to f such that the map $f\colon S_0 \to S_0$ is one-to-one. Corresponding to this map is a permutation matrix U defined in $l_2(S_0)$ by

$$(U\phi)(s) = \phi(f^{-1}s).$$

By definition, the transformation U of the Hilbert space $l_2(S)$ is the ('t Hooft) quantization of the deterministic map f. Moreover, U is equivalent to a direct sum of cyclic shifts on discrete circles. In fact, examples of physical systems admitting the above representation are based on unitary equivalence of the Hamiltonian to a multiplication operator or to a shift operator (via the Fourier transform).

It should be noted that this is a kind of quantization of a 'first-order system', in which case there is a matrix H such that

$$U_t = \exp(itH), \qquad U_1 = U.$$

At non-integral moments of time U_t does not define a deterministic map of S into itself. Moreover, H is defined only up to multiplication of every eigenvalue by a root of unity of a certain degree (which is equal to the length of the cycle).

The simplest physical example is a particle with spin in a magnetic field. Namely, in a finite-dimensional linear space with basis e_m, $m = -l, \dots, l$, let the diagonalization of a Hamiltonian be of the form $He_m = \mu m e_m$. However, in another basis given by

$$f_n = \frac{1}{\sqrt{N}} \sum_m \exp\left(-\frac{2\pi i m n}{N}\right) e_m, \qquad n = 0, 1, \dots, N-1, \quad N = 2l+1,$$

we have the deterministic dynamics of the form $\exp\left(i\frac{2\pi k}{\mu N} H\right) : f_n \to f_{n+k} \pmod N$ on discrete time intervals $t - t' = \dfrac{2\pi k}{\mu N}$. In the papers mentioned above, 't Hooft presents less banal examples as well. However, the most general examples of this kind can be constructed with the help of quantum grammars.

Quantum grammars on graphs were originally introduced independently, but they can also be treated in the framework of the above ideas of 't Hooft with the following new elements: the substitution operation is not single-valued on many graphs (in the terminology of computer science, this operation is non-deterministic), and the set S can be countable (the growth can be unlimited). Although there are many ways to deal with non-deterministic maps, the choice of quantization becomes unique up to a technical realization (sequential realization of any such map, synchronous parallelism, or asynchronous parallelism) due to homogeneity (independence of the location in the graph). We choose asynchronous parallelism. If there are several deterministic maps, then one can assign coefficients (weights) to these maps.

5.2.2. *Quantum spin systems on a fixed graph.* If a Hamiltonian changes only the configurations rather than the graph itself, then we obtain a quantum spin system on this (fixed) graph. Conversely, any quantum spin system can be represented as a grammar on a graph; this can be attained by an appropriate choice of the constants λ_i. In this case the embedding $\phi_N \colon \mathbf{A}_N \to \mathbf{A}_{N+1}$ is given by $A \to A \otimes 1 \otimes \cdots \otimes 1$. Algebras of this kind have been intensively studied (see [4]).

5.2.3. *Linear grammars and Toeplitz operators.* In some cases one obtains interesting examples of C^*-algebras which arose earlier from other considerations. For instance, the C^*-algebra $\mathbf{A}_N$ generated by the substitution $a \to aa$ is isomorphic to the C^*-algebra of Toeplitz operators on $l_2(\mathbb{Z}_+)$, which is generated by shifts. The investigation of the structure of a C^*-algebra is often a simpler problem than the study of the spectrum of the Hamiltonian; however, the solution of the former problem can already give information on the spectral decomposition.

Let us consider the case in which $S = \{a, w\}$ consists of two letters and the substitutions are $aw \to w$ and $w \to aw$. We set $\lambda(aw \to w) = \lambda(w \to aw) = \lambda$. In computer science this grammar is called a *right linear grammar.* The subspaces $\mathcal{H}(L_k)$, where L_k is the set of words with exactly k symbols w (for instance, $L_1 = \{a^m w = aa\dots aw,\ m = 0, 1, 2, \dots\}$), are invariant. We refer to $\mathcal{H}(L_k)$ as the *k-particle space.* Let H_k be the restriction of H to $\mathcal{H}(L_k)$.

Theorem 16 (one-particle spectrum). *The operator H_1 on $\mathcal{H}(L_1)$ is unitarily equivalent to the operator of multiplication by $\lambda(z + z^{-1})$ in $L_2(S^1, d\nu)$, where $d\nu$ is Lebesgue measure on the unit circle S^1 in the complex plane $\mathbb{C}$.*

The space $\mathcal{H}(L_1)$ is isomorphic to $l_2(\mathbb{Z}_+)$ by means of the map $aa\dots aw \to m$. Under this isomorphism H becomes a Toeplitz operator on $l_2(\mathbb{Z}_+)$ given by

$$\lambda b + \lambda b^*,$$

where b is the left shift in $l_2(\mathbb{Z}_+)$, that is,

$$(bf)(m) = f(m-1), \quad m = 1, 2, \dots, \qquad (bf)(0) = 0.$$

Moreover, the C^*-algebra $\mathbf{A}$ is taken to the C^*-algebra $\mathbf{W}_1$ of Toeplitz operators, that is, to the C^*-algebra of operators on $l_2(\mathbb{Z}_+)$ that is generated by b. We note that the commutator $[b, b^*]$ is finite-dimensional. More generally, we have the exact sequence of algebras

$$0 \to K \to^j W_1 \to^\pi W \to 0,$$

where K is the closed two-sided ideal in W_1 formed by all compact operators on $l_2(\mathbb{Z}_+)$, j is the natural embedding, and π is the natural projection of W_1 onto $W_1/K \sim W$. Indeed, the commutator of any two elements of W_1 is compact. This assertion can readily be verified for the monomials of the form $b_1 b_2 \dots b_k$ with each b_i either b or b^*. Passing to the limit, we prove this for any elements. Therefore, W_1/K is commutative. We set

$$y = \pi b, \qquad y^* = y^{-1} = \pi b^*.$$

One can prove that all the elements

$$\sum_{i=-k}^{k} c_i y^i$$

are distinct and that the norm of such an element is equal to $\sum |c_i|$ (it suffices to consider the action of these elements on the functions of the form $f(m)$ that are equal to 1 for some sufficiently large m and vanish otherwise). Hence, $W_1/K \sim W$. Thus, the continuous part of the spectrum coincides with the spectrum of the multiplication operator.

To prove that the discrete spectrum is empty, we rewrite the equation $Af - \lambda f = \phi$ as the well-studied difference equation

$$\begin{aligned} \phi_n &= -\lambda f_n + f_{n-1} + f_{n+1}, \quad n \geqslant 1, \\ \phi_0 &= -\lambda f_0 + f_1. \end{aligned}$$

This equation has no solutions in l_2 for $\phi_n \equiv 0$.

Let us now consider the operator H on the entire Hilbert space, and let us restrict ourselves to the words ending with w. Every word of this kind can be represented in the form

$$\alpha = a^{m_1} w a^{m_2} w \dots a^{m_k} w, \qquad m_1, m_2, \dots, m_k = 0, 1, 2, \dots .$$

Accordingly, every basis vector in $\mathcal{H}(L_k)$ can be written as the tensor product

$$e_\alpha = e_{\alpha(1)} \otimes e_{\alpha(2)} \otimes \dots \otimes e_{\alpha(k)}, \qquad \alpha(i) = a^{m_i} w.$$

In other words, $\mathcal{H}(L_k)$ is isomorphic to the kth tensor power of $\mathcal{H}(L_1)$, and the evolution is

$$\exp(itH_1) \otimes \dots \otimes \exp(itH_1),$$

or

$$H_k = H_1 \otimes 1 \otimes \dots \otimes 1 + \dots + 1 \otimes 1 \otimes \dots \otimes H_1.$$

We have proved the following result.

Theorem 17 (many-particle spectrum). *The Hilbert space has the following decomposition into invariant subspaces:*

$$\mathcal{H}(\Sigma^*) = \bigoplus_{k=0}^{\infty} \mathcal{H}(L_k), \qquad \mathcal{H}(L_k) = \mathcal{H}(L_1) \otimes \cdots \otimes \mathcal{H}(L_1).$$

The Hamiltonian H_k on $\mathcal{H}(L_k)$ is unitarily equivalent to multiplication by the function

$$\sum_{j=1}^{k} \lambda(z_j + z_j^{-1})$$

in the space $L_2(S^k, d\nu)$ of functions $f(z_1, \dots, z_k)$.

5.2.4. *Expansion and contraction of the space.* Let the alphabet consist of the symbol a alone. We consider two substitutions $a \to aa$ and $aa \to a$, $\lambda_1 = \lambda_2 = \lambda$, and the Hamiltonian

$$H = \lambda \sum_{j=1}^{\infty} (a_1(j) + a_2(j))$$

with real λ. In this case, the Hilbert space $\mathcal{H}$ is isomorphic to $l_2(\mathbb{Z}_+)$ because $aa \dots a$ can be identified with its length minus one. The Hamiltonian is unitarily equivalent to the Jacobi matrix

$$(Hf)(n) = \lambda(n-1)f(n-1) + \lambda n f(n+1).$$

We refer to this operator as a one-particle operator, keeping in mind that a 'particle' is associated with a quantum of the space. Here the evolution of the space consists of (quantum) expansion and contraction which takes place at each point independently.

The generalized eigenfunctions are found from the equation

$$(H_1 - \lambda)f = \phi,$$

or

$$\phi_n = n f_{n+1} + (n-1) f_{n-1} - \lambda f_n.$$

Introducing the generating functions

$$F(z) = \sum_{n=1}^{\infty} f_n z^n, \qquad \Phi(z) = \sum_{n=1}^{\infty} \phi_n z^n,$$

we obtain the following equation for $F(z)$:

$$F' - \frac{\lambda + \frac{1}{z}}{1+z^2} F = \frac{\Phi}{1+z^2}.$$

The homogeneous equation has the solution

$$\exp\left(\int \frac{\lambda + \frac{1}{z}}{1+z^2} dz\right) = z \left(\frac{z+i}{z-i}\right)^{\frac{1}{2} + \frac{i\lambda}{2}}.$$

This implies, for instance, that there is no discrete spectrum. This model is studied in more detail in [52].

5.2.5. *Lorentzian models and grammars.* The Lorentzian model (see [53], [54]) can be introduced as follows. Let us first construct a graph. On each of the lines (fibres) $t = 0, \dots, N$ on the cylinder $\mathbb{R} \times S^1 = \{(x,t)\}$ we have $l(t)$ points $x_1(t)$, $\dots$, $x_{l(t)}(t)$ that are ordered clockwise. Each pair of neighbours is joined by an edge on the line $t = \text{const}$. Every point $x_i(t)$ is joined by edges to $k(i,t) = 1, 2, \dots$ consecutive points of the fibre $t+1$; we denote by $K(i, t+1)$ the set of these points. It is assumed that $K(i,t+1) \cap K(i+1, t+1)$ consists of exactly one point that is simultaneously the rightmost point for $K(i,t+1)$ and the leftmost for $K(i+1,t+1)$. The enumeration of points plays no role, and the graph is regarded up to a fibre-preserving isomorphism. To any graph G we assign the 'amplitude'

$$z(G) = \exp\biggl(\lambda \sum_{i,t} k(i,t)\biggr).$$

We note that the sum $\sum_i k(i,t)$ is equal to the number of triangles between the fibres t and $t+1$. At the same time, $\sum_{i,t} k(i,t) = 2\sum_t l(t)$. Here the numbers k_i are positive (and arbitrary), and it is unclear whether or not this transformation is unitary. If $k_i \leqslant \text{const}$, then we can really construct the unitary and Euclidean analogues of this transformation.

The unitary analogue coincides with the above quantum grammar for the expansion and contraction of the space. Indeed, let us consider a triangulation of the disc with m edges on the boundary at the instant 0 and identify the sequence of edges on the boundary with the word $aa\dots a$ of length m. The action of the operator related to the substitution $a \to aa$ consists of attaching a triangle (with two edges outside the disc) to the corresponding edge of the boundary, and the action of the operator related to the substitution $aa \to a$ consists of attaching a triangle (with an edge outside the disc) to two consecutive edges of the boundary.

A random analogue of this quantum grammar was studied in [11]. It was shown that this leads to another universal class. To obtain the class corresponding to dynamical triangulations with exponent $\frac{7}{2}$, one must study non-Markovian dynamics with quadratic transformation of measures.

Lorentzian models regarded as Gibbs families were studied in detail in [52]. The relationship of these models with causal sets is of importance for physics (see the surveys [45] and [53]).

5.2.6. *A particle on a one-dimensional quantum space.* Let us consider the grammar with $S = \{a, w\}$ and with the substitutions

$$a \to aa, \qquad aa \to a, \qquad aw \to wa, \qquad wa \to aw.$$

The subspace $\mathcal{H}_1$ spanned by all the words $a^k w a^l$, $k, l = 0, 1, 2, \dots$, containing exactly one symbol w is invariant. The Hamiltonian is a mixture of the above operator H and the discrete Laplacian H_1 for a free Schrödinger particle in l_2 on a finite set.

5.2.7. *Two types of quanta of the space.* Let $S = \{a, b\}$, and let us consider the four substitutions

$$1) \quad a \to aa, \qquad 2) \quad aa \to a, \qquad 3) \quad b \to bb, \qquad 4) \quad bb \to b.$$

Here we have two invariant one-particle spaces, $\mathcal{H}_a$ and $\mathcal{H}_b$. For instance, $\mathcal{H}_a$ is spanned by the words a, $a^2 = aa, \dots, a^n, \dots$. There are two invariant two-particle spaces $\mathcal{H}_{ab} = \mathcal{H}_a \otimes \mathcal{H}_b$ and $\mathcal{H}_{ba} = \mathcal{H}_b \otimes \mathcal{H}_a$. For instance, $\mathcal{H}_{ab}$ is spanned by the words $a^k b^l$, $k, l > 0$. In the general case there are two invariant $2n$-particle spaces for any n: the space $\mathcal{H}_{(ab)_n}$ spanned by the words $a^{k_1} b^{l_1} \dots a^{k_n} b^{l_n}$, $\mathcal{H}_{(ab)_n} = \mathcal{H}_a \otimes \mathcal{H}_b \otimes \dots \otimes \mathcal{H}_a \otimes \mathcal{H}_b$, and the space $\mathcal{H}_{(ba)_n}$ defined similarly. In the same way, there are two invariant $(2n+1)$-particle spaces for any n: $\mathcal{H}_{b(ab)^n} = \mathcal{H}_b \otimes \mathcal{H}_a \otimes \mathcal{H}_b \otimes \dots \otimes \mathcal{H}_a \otimes \mathcal{H}_b$ and the space $\mathcal{H}_{(ab)^n a}$ defined similarly. Due to the decomposition

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_a \oplus \mathcal{H}_b \oplus \mathcal{H}_{ab} \oplus \mathcal{H}_{ba} \oplus \dots \oplus \mathcal{H}_{(ab)_n} \oplus \mathcal{H}_{(ba)_n} \oplus \mathcal{H}_{b(ab)^n} \oplus \mathcal{H}_{(ab)^n a} \cdots,$$

the spectrum of the Hamiltonian reduces here to the spectrum of two one-particle Hamiltonians discussed above. For example, the evolution on $\mathcal{H}_{ab}$ is given by the Hamiltonian $\mathcal{H}_a \otimes 1 + 1 \otimes \mathcal{H}_b$.

5.2.8. *Symmetries and quantum grammars.* We note here that to a given quantum grammar one can assign certain representations of the Lie algebra $\mathcal{D}$ of diffeomorphisms of a circle. We denote by D_n and D_{-n} the Hamiltonians corresponding to the substitutions $a \to a^{n+1}$ and $a^{n+1} \to a$, $n \geqslant 0$, respectively. Then in the basis $\nu_k = e_{a^k}$ we have $D_n \nu_k = k\nu_{k+n}$ for $n \geqslant 0$. However, for $n < 0$ we have the relation $D_n \nu_k = (k + n + 1)\nu_{k+n}$ only. For large k and for fixed m and n we can prove that

$$(D_m D_n - D_n D_m)\nu_k = (m - n + o(1))D_{n+m}\nu_k.$$

In this case we say that an asymptotic representation of the algebra $\mathcal{D}$ is given. We can obtain more for cyclic words (for periodic boundary conditions), for which the relation $D_n \nu_k = k\nu_{k+n}$ holds for any $|n| < k$, and hence the relation

$$(D_m D_n - D_n D_m)\nu_k = (m - n)D_{n+m}\nu_k$$

holds for any $|n|, |m| \ll k$. In other words, the last relation holds for any m and n except for a finite-dimensional subspace $\mathcal{H}(m, n)$. We note that the spectrum of D_0 coincides with $\mathbb{Z}_+$ in both cases.

To obtain a faithful representation of the algebra $\mathcal{D}$, we must consider the cyclic words in two symbols a and a^{-1} with the relations $aa^{-1} = a^{-1}a = e$, where e is the empty word, along with the substitutions D_n given by $a \to a^{n+1}$ and $a^{-1} \to a^{n+1}$ for any $n \in \mathbb{Z}$. We note that we are referring to the representation $-V_{0,0}$ in the notation of [55].

Let us consider linear cyclic graphs (that is, words with periodic boundary conditions) and the grammars on these graphs with the substitutions

$$a \to aa, \qquad aa \to a, \qquad aw \to wa, \qquad wa \to aw.$$

We single out the words with exactly one symbol w. We remark that all cyclic words of the same length and with a single symbol w are equivalent. Let $\mathcal{H}_1$ be the Hilbert space spanned by these words; this space is isomorphic to $l_2(\mathbb{Z}_+)$. We note that an asymptotic representation of the Virasoro algebra acts on $\mathcal{H}_1$.

If we pass from the equivalence classes of cyclic words to the enumerated words, that is, to the basis $e_k \otimes j$, $j \in \mathbb{Z}_k$, then the map $e_k \otimes j \to e_k \otimes (j + n)$ defines an action of the spin group $\mathbb{Z}$ for any $n \in \mathbb{Z}$, where $j + n$ is regarded modulo k. Thus, the symmetry (which was trivial above) shows itself in the added enumeration. The action of $\mathbb{Z}$ can be represented with the help of fermion variables in accordance with the 't Hooft quantization discussed above (see [50], [51]).

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