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Invited paper

Random walks in two-dimensional complexes

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A 2-dimensional complex is a union of a finite number of quarter planes $\mathbb{Z}_+^2$ having some boundaries in common. The most interesting example is the union of all 2-dimensional faces of $\mathbb{Z}_+^N$. We consider maximally homogeneous random walks on such complexes and obtain necessary and sufficient conditions for ergodicity, null recurrence and transience up to some "non-zero" assumptions which are of measure 1 in the parameter space.

The problem we address in this paper is of theoretical range. However, the results can be applied to performance evaluation of some telecommunication systems (e.g. local area networks) viewed as interacting queues. To enforce this assertion, a detailed example of coupled queues in different *regimes* is presented.

Keywords: Random walks; two-dimensional complexes; ergodicity; transience; hedgehog.

1. Introduction and preliminary results

The problem we address in this paper is of theoretical range. However, the results can be applied to performance evaluation of some telecommunication systems (e.g. local area networks) viewed as interacting queues. Consider for example N queues, with lengths $k_1, k_2, \ldots, k_N$, served by a single server (bus, circuit, etc.) visiting them according to the following polling strategy: only one pair of queues, say (i, j), can be active at a time. As soon as one of the two active queues becomes empty, for instance i, the server, using some specified protocol, chooses a queue k and the new pair (j, k) becomes active.

A 2-dimensional complex is a union of a finite number of quarter planes $\mathbb{Z}_+^2$ having some boundaries in common. An example can be the union of all 2-dimensional faces of $\mathbb{Z}_+^N$. We consider maximally homogeneous random walks on such complexes and obtain necessary and sufficient conditions for ergodicity, null recurrence and transience up to some "non zero" assumptions which are of measure 1 in the parameter space.

The main reason for studying this problem is that it is a step in advance comparatively to the paper Malyshev and Menshikov [1], towards obtaining classification of maximally homogeneous random walks in $\mathbb{Z}_+^N$. In [1] the vector field was constructed in terms of which a complete classification was obtained for $N = 2, 3$ and sufficient conditions for ergodicity and transience were derived for $N > 3$. One of the main features of the vector field in question was that it was deterministic. Our advancement, here, is that certain vector fields appear, which are deterministic inside 2-dimensional faces, but give rise to random scattering on 1-dimensional faces. We shall see that the calculation of the exit boundary of some countable Markov chain (one-dimensional) is necessary because of this phenomenon. This is a new phenomenon, which is common also for $\mathbb{Z}_+^N$. The second new phenomenon is that null recurrence exists for a set of parameters which has a positive measure in parameter space.

We give an explicit solution to our problem: in fact it reduces to finding stationary probabilities for finite Markov chains with n states, where n is the number of 2-dimensional faces in the simplex considered, and to calculation of the maximal eigenvalue of some $(n \times n)$ matrix with positive entries.

The paper is organized as follows:

After the main definitions and preliminary results in sections 1 and 2, we formulate the main theorems 3 in section 3. The proof of the ergodicity conditions, using a method of pasting local Lyapounov functions as in [1], is given in section 5. Transience is proved in section 6, by using a simpler method to construct more global Lyapounov functions. Proofs of recurrence and non-ergodicity are presented in sections 7 and 8. An example of interacting queues, submitted to different regimes, is worked out in detail in section 9. Possible generalizations are briefly described in section 10.

We give now the main definitions and quote some preliminary results. We call a 2-dimensional complex $\tilde{T}$ any union of finite number of copies of $\mathbb{R}_+^2$:

$$(\mathbb{R}_+^2)_i = \{(x_1, x_2)_i \colon x_j \geqslant 0\}, \quad i = 1, \dots, n.$$

We assume that all origins are identified, i.e. $(0, 0)_i \stackrel{\text{def}}{=} \mathbf{0}$ for all i. Some pairs $(\tilde{\Lambda}_{ji}^{(1)}, \tilde{\Lambda}_{j_1 i_1}^{(1)})$ of its one dimensional faces

$$\tilde{\Lambda}_{1i}^{(1)} = \{(x_1, 0)_i \colon x_1 > 0\}, \; \tilde{\Lambda}_{2i}^{(1)} = \{(0, x_2)_i \colon x_2 > 0\}$$

can be identified as well. This means that the points of $\tilde{\Lambda}_{ji}^{(1)}$ and $\tilde{\Lambda}_{j_1 i_1}^{(1)}$ lying at the same distance from the origin are identified. We shall consider discrete time homogeneous Markov chains $\mathscr{L} = \mathscr{L}_T$ with state space T, the integer points of $\tilde{T}$, i.e. the union of all

$$(\mathbb{Z}_+^2)_i = \{(x_i, x_2)_i \colon x_j \geqslant 0 \text{ integers}\} \subset (\mathbb{R}_+^2)_i$$

taking into consideration the above identifications between them. We denote the inner part of a generic 2-dimensional face by $\tilde{\Lambda}^{(2)}$, e.g.

$$\tilde{\Lambda}_1^{(2)} = \{(x_1, x_2)_1 : x_1 x_2 > 0\}.$$

Introduce also $\overline{\Lambda}^{(1)}$, $\overline{\Lambda}^{(2)}$ – the closures of $\tilde{\Lambda}^{(1)}$, $\tilde{\Lambda}^{(2)}$, and $\Lambda^{(1)}$, $\Lambda^{(2)}$ – the sets of integer points of $\tilde{\Lambda}^{(1)}$ and $\tilde{\Lambda}^{(2)}$ respectively.

Examples

(1) $\tilde{T}$ (or T) is called planar if any 1-dimensional face is identified with at most one other 1-dimensional face. This means that $\tilde{T}$ can be topologically imbedded into $\mathbb{R}^2$. Examples are:
(a) $\mathbb{Z}_+^2$,
(b) the union of 4 quadrants of $\mathbb{Z}^2$,
(c) the union of three 2-dimensional faces of $\mathbb{Z}_+^3$.
No new ideas are necessary here and in fact these cases were studied 20 years ago.

(2) The union of all $N(N-1)/2$ 2-dimensional faces of $\mathbb{Z}_+^N$, $N \geqslant 4$, is an example of a non-planar complex. The simplest example to have in mind is the union of five 2-dimensional faces of $\mathbb{Z} \times \mathbb{Z}_+^2$, i.e. the union of two $\mathbb{Z}_+^3$ with one two-dimensional face in common.

(3) $\tilde{T}$ is called strongly connected if it is not the union of two complexes $\tilde{T}_1$ and $\tilde{T}_2$ which have only $\mathbf{0}$ as a common point.

Let us note that in $\tilde{T}$ any line has a length. Whence, $\tilde{T}$ is the metric space endowed with the distance $\rho(\alpha, \beta)$ between $\alpha, \beta \in \tilde{T}$, equal to the minimal length of a line between them. We assume that one-step transition probabilities $p_{\alpha\beta}(\alpha \to \beta)$ satisfy the following conditions:

A1. BOUNDEDNESS OF JUMPS
(a) $p_{\alpha\beta} = 0$ if α, β do not belong to the same $\overline{\Lambda}^{(2)}$;
(b) $p_{\alpha\beta} = 0$ if $\rho(\alpha, \beta) > d$ for some fixed $d < \infty$;
(c) $p_{\alpha\beta} = 0$ if at least one component of the vector $\beta - \alpha$ is less than -1.

A2. SIMPLEST SPACE HOMOGENEITY

Let α, α' belong to the same (open) face Λ which can be 1 or 2-dimensional. If $\Lambda \subset \overline{\Lambda}^{(2)}$ (i.e. $\Lambda = \Lambda^{(2)}$ or Λ is a one-dimensional face of $\Lambda^{(2)}$) and, moreover,

$$\beta' - \alpha' = \beta - \alpha,$$

then

$$p_{\alpha\beta} = p_{\alpha'\beta'} \stackrel{\text{def}}{=} p_{\beta-\alpha}^{\Lambda}.$$

Thus, our Markov chain is uniquely specified by a finite number of parameters p_γ^Λ, with $\gamma \in \overline{\Lambda}^{(2)}$ such that $\Lambda \subset \overline{\Lambda}^{(2)}$. We also shall make some assumptions

$0_1, \ldots, 0_5$, which we call "*non-zero assumptions*" and which exclude from our consideration some hypersurfaces in the parameter space (in particular they are of measure zero in the parameter space). Some of these assumptions are made just for economy of space–time but others are very essential. They will appear thereafter.

ASSUMPTION 0_1

The Markov chain $\mathscr{L}_T$ is supposed to be irreducible and aperiodic.

Then, e.g., this chain is ergodic iff any of its strongly connected components is ergodic. Therefore, we shall consider only strongly connected complexes T. For any 2-dimensional face Λ and any $\alpha \in \Lambda$, we define the vectors $M(\alpha)$, the one-step mean jumps from α. They are all equal to

$$M_\Lambda = \sum_{\beta \in \bar{\Lambda}} (\beta - \alpha) p^\Lambda_{\beta-\alpha} = M(\alpha), \quad \forall \alpha \in \Lambda.$$

If $\alpha \in \Lambda^{(1)} \equiv \Lambda$ then we define $M_\Lambda = M(\alpha)$ to be the collection of vectors $M_{\Lambda,\Lambda}$ and $M_{\Lambda,\Lambda^{(2)}}$, for all $\overline{\Lambda^2} \supset \Lambda$, where

$$M_{\Lambda,\Lambda} = \sum_{\beta \in \bar{\Lambda}} (\beta - \alpha) p^\Lambda_{\beta-\alpha},$$

$$M_{\Lambda,\Lambda^{(2)}} = \sum_{\beta \in \Lambda^{(2)}} (\beta - \alpha) p^\Lambda_{\beta-\alpha}.$$

If one can imbed $\tilde{T}$ into $\mathbb{R}^N$ for some N, so that all $\Lambda^{(2)}$ are orthogonal, then $M(\alpha)$ for $\alpha \in \Lambda^{(1)}$ can be defined as the usual vector of mean jumps.

THEOREM 1.1

If at least for one $\Lambda = \Lambda^{(2)}$ the vector M_Λ has both components positive then $\mathscr{L}_T$ is transient.

ASSUMPTION 0_2

For any Λ the vector M_Λ can have no zero component.

DEFINITION 1

Let $\Lambda^{(1)}$ be a 1-dimensional face and let $S(\Lambda^{(1)})$ be the set of all 2-dimensional faces $\Lambda^{(2)}$ such that $\Lambda^{(1)} \subset \overline{\Lambda^{(2)}}$; $S_+(\Lambda^{(1)}) \subset S(\Lambda^{(1)})$ be the set of all $\Lambda^{(2)}$ such that $M_{\Lambda^{(2)}}$ looks onto $\Lambda^{(1)}$, i.e. its component perpendicular to $\Lambda^{(1)}$ is negative. Accordingly, $S_-(\Lambda^{(1)}) = S(\Lambda^{(1)}) - S_+(\Lambda^{(1)})$. We call $\Lambda^{(2)} \in S_+(\Lambda^{(1)})$ (resp. $S_-(\Lambda^{(1)})$) an *ongoing* (resp. *outgoing*) face for $\Lambda^{(1)}$. If $S_+(\Lambda^{(1)}) = S(\Lambda^{(1)})$, then $\Lambda^{(1)}$ is called *ergodic*.

DEFINITION 2

Let us consider a 1-dimensional face $\Lambda^{(1)}$ and a point $\alpha \in \Lambda^{(1)}$. For any 2-dimensional $\Lambda \in S(\Lambda^{(1)})$, let us consider the half-line $C^\Lambda_{\Lambda^{(1)}}$ which belongs to $\bar{\Lambda}$

and is perpendicular to $\Lambda^{(1)}$ at the point α. We call the following 1-dimensional complex

$$H_{\Lambda^{(1)}} = \bigcup_{\Lambda \in S(\Lambda^{(1)})} C^{\Lambda}_{\Lambda^{(1)}}$$

a *hedgehog*. For various $\alpha \in \Lambda^{(1)}$, these hedgehogs are congruent in the obvious sense. Let us consider the Markov chain $\mathscr{L}_{\Lambda^{(1)}}$, with set of states $H_{\Lambda^{(1)}}$ (we call it the induced Markov chain for $\Lambda^{(1)}$) and one-step transition probabilities which are the following projections

$$q^{\Lambda^{(1)}}_{\alpha\beta} = \sum_{\beta'} p_{\alpha\beta'}, \quad \alpha, \beta \in H_{\Lambda^{(1)}},$$

where the summation is over all β', such that β' belongs to the same face as β and (if this face is $\Lambda = \Lambda^{(2)}$) the straight line connecting β and β' is perpendicular to $C^{\Lambda}_{\Lambda^{(1)}}$. From the homogeneity conditions, it follows that the induced chain for $\Lambda^{(1)}$ does not depend on the choice of $\alpha \in \Lambda^{(1)}$.

ASSUMPTION 0_3

For any $\Lambda^{(1)}$, the induced chain $\mathscr{L}_{\Lambda^{(1)}}$ is irreducible and aperiodic.

Then $\mathscr{L}_{\Lambda^{(1)}}$ is ergodic iff $\Lambda^{(1)}$ is ergodic. This explains the word. Let $\pi_{\Lambda^{(1)}}(h)$ be the stationary probabilities of $\mathscr{L}_{\Lambda^{(1)}}$, $h \in H_{\Lambda^{(1)}}$, in the ergodic case. Let us define, for any ergodic $\Lambda^{(1)}$, a number $v_{\Lambda^{(1)}}$ – the "second vector field" on 1-dimensional ergodic faces

$$v_{\Lambda^{(1)}} = \sum_{h \in H_{\Lambda^{(1)}}} \pi_{\Lambda^{(1)}}(h) Pr_{\Lambda^{(1)}} M(h),$$

where $Pr_{\Lambda^{(1)}}$ means "orthogonal projection of $M(h)$ onto $\Lambda^{(1)}$". If $h \in \Lambda^{(1)}$ this means

$$Pr_{\Lambda^{(1)}} M(h) = M_{\Lambda^{(1)},\Lambda^{(1)}} + \sum_{\Lambda^{(2)}} Pr_{\Lambda^{(1)}} M_{\Lambda^{(1)},\Lambda^{(2)}}.$$

THEOREM 1.2

If $v_{\Lambda^{(1)}} > 0$ for at least one ergodic $\Lambda^{(1)}$, then $\mathscr{L}_T$ is transient.

ASSUMPTION 0_4

$v_{\Lambda^{(1)}} \neq 0$ for all $\Lambda^{(1)}$.

Now we want to show that the sign of $v_{\Lambda^{(1)}}$ is easy to calculate.

LEMMA 1.3

$$sgn v_{\Lambda^{(1)}} = sgn\left(M_{\Lambda^{(1)},\Lambda^{(1)}} + \sum_{\Lambda^{(2)}:\Lambda^{(1)}\subset\bar{\Lambda}^{(2)}} Pr_{\Lambda^{(1)}}\left[M_{\Lambda^{(1)},\Lambda^{(2)}} + M_{\Lambda^{(2)}} \frac{Q^{\Lambda^{(2)}}_{\Lambda^{(1)}} M_{\Lambda^{(1)},\Lambda^{(2)}}}{Q^{\Lambda^{(2)}}_{\Lambda^{(1)}} M_{\Lambda^{(2)}}} \right]\right),$$

where $Q^{\Lambda^{(2)}}_{\Lambda^{(1)}}$ denotes the projection of a vector in $\Lambda^{(2)}$ onto the axis of $\Lambda^{(2)}$ other than $\Lambda^{(1)}$.

2. Random walks on hedgehogs

For a given hedgehog $H_{\Lambda^{(1)}}$, we call $C^{\Lambda^{(2)}}_{\Lambda^{(1)}}$ its bristle. A bristle $C^{\Lambda^{(2)}}_{\Lambda^{(1)}}$ is called ingoing if $\Lambda^{(2)}$ is ingoing, i.e. if the following number (representing the mean jump along the bristle)

$$m_{\Lambda^{(2)}} = \sum_{h'} (h' - h) q^{\Lambda^{(1)}}_{hh'}, \tag{2.1}$$

which does not depend on the position of $h \in C^{\Lambda^{(2)}}_{\Lambda^{(1)}}$, is negative.

When $\Lambda^{(1)}$ is not ergodic, we shall define a "*scattering*" probability $p_{sc}(\Lambda^{(1)}, \Lambda^{(2)})$, for $\Lambda^2 \in S_-(\Lambda^{(1)})$, which is the probability that the random walk will go to infinity along $C^{\Lambda^{(2)}}_{\Lambda^{(1)}}$. Under our simplest homogeneity assumptions, this definition does not depend on the initial position, provided that this latter is either at the origin of the hedgehog or on some ingoing bristle. Thus, we can assume that it is at the origin $0 \in H_{\Lambda^{(1)}}$.

COMPUTATION OF THE SCATTERING PROBABILITIES

Let us fix $\Lambda^{(1)}$, $\Lambda^{(2)}$ and put $q_{hh'} = q^{\Lambda^{(1)}}_{hh'}$. It is clear that

$$p_{sc}(\Lambda^{(1)}, \Lambda^{(2)}) = \frac{\sum_{h\in C^{\Lambda^{(2)}}_{\Lambda^{(1)}}} q_{0h}\, p(h)}{\sum_{0\neq h\in H_{\Lambda^{(1)}}} q_{0h}\, p(h)}, \tag{2.2}$$

where $p(h) \equiv p_{\Lambda^{(2)}}(h)$ is the probability that, starting from h, the particle on the hedgehog will never return to 0. Formula (2.2) follows from the fact that

$$p_{sc}(\Lambda^{(1)}, \Lambda^{(2)}) = const \sum_{h\in C^{\Lambda^{(2)}}_{\Lambda^{(1)}}} q_{0h}\, p(h),$$

where *const* does not depend on the outgoing $\Lambda^{(2)}$. We shall show now that

$$p(h) = 1 - (1-\gamma)^h, \tag{2.3}$$

where $\gamma = p(1)$ is the unique root inside the unit disc of the equation

$$(1-\gamma)^h = \sum_{h'} q_{hh'}(1-\gamma)^{h'}, \tag{2.4}$$

with $h \in C_{\Lambda^{(1)}}^{\Lambda^{(2)}}/\{0\}$ (e.g. we can take $h = 1$), and $h' \in C_{\Lambda^{(1)}}^{\Lambda^{(2)}}$. The proof is easily obtained from the recursive relationship

$$p(h+1) = p(1) + [1 - p(1)]\,p(h)$$

and standard generating function method.

Proof of lemma 1.3

Let $H_{\Lambda^{(1)}}$ be a hedgehog such that all $m_{\Lambda^{(2)}}$ are negative, for all its bristles $C_{\Lambda^{(1)}}^{\Lambda^{(2)}}$. Let

$$\pi_{\Lambda^{(2)}} = \sum_{h \in C_{\Lambda^{(1)}}^{\Lambda^{(2)}} \cap \Lambda^{(2)}} \pi_{\Lambda^{(1)}}(h).$$

We claim that

$$\frac{\pi_{\Lambda^{(2)}}}{\pi_0} = \frac{Q_{\Lambda^{(1)}}^{\Lambda^{(2)}} M_{\Lambda^{(1)},\Lambda^{(2)}}}{Q_{\Lambda^{(1)}}^{\Lambda^{(2)}} M_{\Lambda^{(2)}}}.$$

To prove this, we note first that, for computing the above quantity, it suffices to consider a modified random walk on the bristle $C_{\Lambda^{(1)}}^{\Lambda^{(2)}}$, i.e. on $\mathbb{Z}_+^1$, after slightly "updating" the transition probabilities. More exactly, we define $\tilde{q}_{hh'} = q_{hh'}$, for all h, $h' \in C_{\Lambda^{(1)}}^{\Lambda^{(2)}}$, except for $\tilde{q}_{00}$ which is taken equal to

$$\tilde{q}_{00} = 1 - \sum_{0 \neq h' \in C_{\Lambda^{(1)}}^{\Lambda^{(2)}}} \tilde{q}_{0h'}.$$

Then $\pi_{\Lambda^{(2)}}/\pi_0$ does not depend on this modification and its value in the case of $\mathbb{Z}_+^1$ is a well-known result, yielding in particular exact ergodicity conditions for random walks in $\mathbb{Z}_+^2$. (The point is that, due to the homogeneity, it is not necessary to compute the exact values of the $\pi_{\Lambda^{(1)}}(h)$'s: only the drifts are needed.) □

Remark

We have solved an *exit boundary* problem, using Martin's theory terminology. See Feller [4], for instance.

3. Formulation of the main result

DEFINITION 3

For given T and $\mathscr{L}_T$, we define the following associated Markov chain $\mathscr{M}$ having a finite number of states $n = |T|$, equal to the number of 2-dimensional faces of T. (It is thus natural to denote these states by $\Lambda^{(2)}$.) The one-step transition probabilities $p(\Lambda_i^{(2)}, \Lambda_j^{(2)})$ of $\mathscr{M}$ are equal to

$$p\left(\Lambda_i^{(2)}, \Lambda_j^{(2)}\right) = \begin{cases} p_{sc}\left(\Lambda^{(1)}, \Lambda_j^{(2)}\right), & \text{if } \Lambda_i^{(2)} \in S_+(\Lambda^{(1)}),\ \Lambda_j^{(2)} \in S_-(\Lambda^{(1)}), \\ 1, & \text{if } \Lambda_i^{(2)} = \Lambda_j^{(2)} \in S_+(\Lambda^{(1)}) \text{ for some ergodic } \\ & \quad \Lambda^{(1)}, \\ 0, & \text{otherwise}. \end{cases}$$

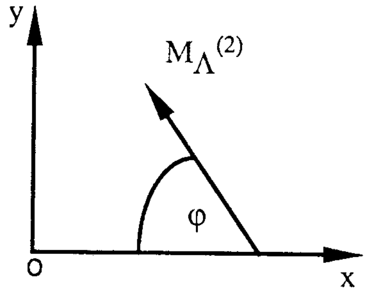

Fig. 1.

We do not exclude that the associated chain be reducible or periodic. Let $\mathscr{A}_1, \ldots, \mathscr{A}_k$ be irreducible classes of essential states. Let us consider some class $\mathscr{A}_i$ with $|\mathscr{A}_i| \geqslant 2$. We define the following function f on $\mathscr{A}_i$: If $\Lambda^{(2)} \in \mathscr{A}_i$ and $\phi_{\Lambda^{(2)}}$ is the angle between $M_{\Lambda^{(2)}}$ and the negative axis from which $M_{\Lambda^{(2)}}$ goes away (see fig. 1), then we put

$$f(\Lambda^{(2)}) = \log \operatorname{tg} \phi_{\Lambda^{(2)}}.$$

Heuristically, if we are, e.g., on the x-axis of $\Lambda^{(2)}$ at a point $(x, 0)$ and we move along the constant vector field $M_{\Lambda^{(2)}}$ to a point $(0, y)$ of the y-axis, then $\exp f(\Lambda^{(2)})$ represents the dilatation coefficient $\alpha = y/x$.

- If $\mathscr{A}_i$ is aperiodic and $\pi_i(\Lambda^{(2)})$ denotes its stationary probabilities, we define

$$M(\mathscr{A}_i) = \sum_{\Lambda^{(2)} \in \mathscr{A}_i} \pi_i(\Lambda^{(2)}) f(\Lambda^{(2)}). \tag{3.1}$$

- If $\mathscr{A}_i$ is periodic, $\pi_i(\Lambda^{(2)})$ is then taken to be the stationary probability in the aperiodic subclass containing $\Lambda^{(2)}$.

The vector $M(\mathscr{A}_i)$ is defined by the same formula (3.1). Perhaps it will be more convenient to normalize it, multiplying by $N(\mathscr{A}_i)^{-1}$ where $N(\mathscr{A}_i)$ is the number of aperiodic subclasses in $\mathscr{A}_i$.

ASSUMPTION 0_5

For all i, $M(\mathscr{A}_i) \neq 0$.

Our main result is the following

THEOREM 3.1

Under the assumptions $0_1, \ldots, 0_5$ and if the assumptions of theorems 1 and 2 are not fulfilled, then $\mathscr{L}_T$ is recurrent iff, for any $\mathscr{A}_i$ with $|\mathscr{A}_i| \geqslant 2$,

$$M(\mathscr{A}_i) < 0. \tag{3.2}$$

It appears that (3.2) is not sufficient for the chain to be ergodic and we find both ergodicity and null recurrence regions. For this we have to define a number

$L(\mathscr{A}_i)$ connected with $\mathscr{A}_i$. Let us denote $\Lambda_0^{(2)}, \dots, \Lambda_t^{(2)}, \dots$ the random states of $\mathscr{M}$ with $\Lambda_0^{(2)} \in \mathscr{A}_i$.

LEMMA 3.2

The following limit exists

$$\lim_{t \to \infty} \frac{1}{t} \log\left(E \prod_{i=1}^{t} \operatorname{tg} \phi_{\Lambda_i} \right) \doteq L(\mathscr{A}_i) \tag{3.3}$$

and does not depend on $\Lambda_0^{(2)} \in \mathscr{A}_i$. Moreover

$$L(\mathscr{A}_i) = \log \lambda_1(\mathscr{A}_i), \tag{3.4}$$

where λ_1 is the maximal eigenvalue of the $n_i \times n_i$-matrix, $n_i = |\mathscr{A}_i|$, with matrix elements

$$A\left(\Lambda_i^{(2)}, \Lambda_j^{(2)}\right) = p\left(\Lambda_i^{(2)}, \Lambda_i^{(2)}\right) \sqrt{\operatorname{tg} \phi_{\Lambda_i^{(2)}} \operatorname{tg} \phi_{\Lambda_j^{(2)}}}, \quad \Lambda_i^{(2)}, \Lambda_j^{(2)} \in \mathscr{A}_i. \tag{3.5}$$

Proof of lemma 3.2

Let us note first that for any two vectors l_1, l_2 with positive components

$$\lim_{N \to \infty} \frac{1}{N} \log\left(l_1, A^N l_2\right) = \log \lambda_1$$

and then we notice that (3.3) can be represented in such a way. □

ASSUMPTION 0_6

For all i, $|\mathscr{A}_i| \geqslant 2$,

$$L(\mathscr{A}_i) \neq 0. \tag{3.6}$$

In the following theorem we assume also that all states of $\mathscr{M}$ are essential. In general, the limit (3.3) can depend on the initial nonessential state. All the proofs, however, are completely the same.

THEOREM 3.3

Under the assumptions $0_1, \dots, 0_6$ and if the assumptions of theorems 1 and 2 are not fulfilled then $\mathscr{L}_T$ is ergodic iff, for any $\mathscr{A}_i$ with $|\mathscr{A}_i| \geqslant 2$

$$M(\mathscr{A}_i) < 0$$

and

$$L(\mathscr{A}_i) < 0. \tag{3.7}$$

This implies that, if for all i we have (3.2), but at least for one i we have

$$L(\mathscr{A}_i) > 0,$$

then $\mathscr{L}_T$ is null recurrent.

Remark 1

Due to the inequality

$$\log M\xi \geqslant M \log \xi$$

for positive r.v. ξ, we have always

$$M(\mathscr{A}_i) \leqslant L(\mathscr{A}_i), \tag{3.8}$$

and so in particular if

$$L(\mathscr{A}_i) < 0,$$

then

$$M(\mathscr{A}_i) < 0.$$

The practical computation of ergodicity conditions can be achieved according to the following sequence:

(1) Calculate the vectors of mean jumps;

(2) calculate $sgn v_{\Lambda^{(1)}}$ for all ergodic $\Lambda^{(1)}$, using lemma 1;

(3) calculate the scattering probabilities, using formulas (2.2)–(2.4);

(4) calculate the stationary probabilities of the associated chain, which in the general case give rise to a system of $|T|$ linear equations;

(5) calculate $L(\mathscr{A}_i)$ using (3.4);

(6) use theorems 1 to 3.

So we get a complete classification up to the assumptions $0_1, \ldots, 0_6$.

4. Quasideterministic process

Here we introduce an auxiliary process η_t on $\tilde{T}$ with $t \in \mathbb{R}_+$. Consider a particle moving along the constant vector field $M_{\Lambda^{(2)}}$ (with velocity $M_{\Lambda^{(2)}}$) on any face $\tilde{\Lambda}^{(2)} \subset \tilde{T}$. When it reaches a 1-dimensional face $\tilde{\Lambda}^{(1)}$, it chooses with probability $p(\Lambda^{(1)}, \Lambda_i^{(2)})$ a face $\Lambda_i^{(2)} \in S_-(\Lambda^{(1)})$ and continues its way along $\tilde{\Lambda}_i^{(2)}$ and so on. So η_t is deterministic outside 1-dimensional faces. Let $\eta_0 = x \in \tilde{T}$ and let

$$\tau_1(x) < \tau_2(x) < \cdots$$

be all times when η_t is on a 1-dimensional face of $\tilde{T}$. Let us define discrete time process (imbedded)

$$\chi_n = \chi_n(x) \equiv \eta_{\tau_n(x)}.$$

We shall find the conditions of ergodicity, null recurrence and transience for the process η_t which will appear to be the same for the corresponding random walk.

LEMMA 4.1

Under the assumptions of theorem 3.1 the following conditions are equivalent:

(i) (3.2) holds;

(ii) for any x, $\chi_n(x)$ reaches a.s. any neighbourhood of 0;

(iii) for any x, the time for $\eta_t(x)$ to reach any neighbourhood of 0 is a.s. finite.

Proof

If (3.2) holds then $\log \chi_n \to -\infty$ a.s. by the strong law of large numbers. So (i) $\to$ (ii) $\Leftrightarrow$ (iii). Vice versa, if we have

$$M(\mathscr{A}_i) > 0,$$

then $\log \chi_n \to \infty$ a.s. and so η_t cannot reach any neighbourhood with probability 1. □

LEMMA 4.2

Under the assumptions of theorem 3.2 the following conditions are equivalent:

(i) (3.7) holds;

(ii) $E\tau_n$ is uniformly bounded, i.e. $\sum_{i=1}^{\infty} E(\tau_i - \tau_{i-1}) < \infty$, $\tau_0 = 0$;

(iii) $\eta_t(x)$ reaches zero with a finite mean time.

(iv) $\eta_t(x)$ reaches any neighbourhood of 0 with a finite mean time.

Proof

Obviously we have

$$\text{(ii)} \Leftrightarrow \text{(iii)} \Rightarrow \text{(iv)}$$

Let us prove (i) $\Rightarrow$ (ii). In fact if (3.5) holds then $E\chi_n$ exponentially converges to zero:

$$E\chi_n \leqslant C\,\mathrm{e}^{-\alpha n}$$

for some $C > 0$, $\alpha > 0$ and all n. But also

$$E\,|\,\tau_n - \tau_{n-1}\,| \leqslant CE\chi_n.$$

So we proved that (i) implies (ii). Let us now prove that (iv) $\Rightarrow$ (i), i.e. if

$$L(\mathscr{A}_i) > 0,$$

then the mean time of reaching a neighbourhood of 0 by $\eta_t(x)$ is infinite. Let

$$\tilde{T}(\mathscr{A}_i) = \bigcup_{\Lambda \in \mathscr{A}_i} \bar{\Lambda}.$$

For any $x \in \tilde{T}(\mathscr{A}_i)$ and any $\epsilon > 0$, let $\tau^{\epsilon}(x)$ be the first reaching time of the set $\{y \in \tilde{T}(\mathscr{A}_i)\colon \|y\| \leqslant \epsilon\}$ by the quasideterministic process $\eta_t(x)$ (we take $\tau^{\epsilon}(x) = 0$ if $\|X\| \leqslant \epsilon$).

PROPOSITION 1

Let $x \in \tilde{T}(\mathscr{A}_i)$ and for any $\epsilon > 0$

$$E\tau^{\epsilon}(x) < \infty.$$

Then for any $r > 0$ and for any $\epsilon > 0$

$$\frac{1}{r} E\tau^{r\epsilon}(x) = E\tau^{\epsilon}(x). \tag{4.1}$$

This is the obvious scaling property of the process η_t.

PROPOSITION 2

If there exists $x \in \tilde{T}(\mathscr{A}_i)$ such that for any $\epsilon > 0$

$$E\tau^{\epsilon}(x) < \infty,$$

then this property holds for any x.

Proof

Let us fix such x. Then for any $y \in \tilde{T}(\mathscr{A}_i)$ there exist $t \in \mathbb{R}_+$ and $r \in \mathbb{R}_+$ such that

$$\eta_t(x) = ry$$

with positive probability. It follows that for any $\epsilon > 0$

$$E\tau^{\epsilon}(ry) < \infty$$

and so by proposition 1

$$E\tau^{\epsilon}(y) < \infty$$

for any $\epsilon > 0$. □

PROPOSITION 3

If there exists $x \in \tilde{T}(\mathscr{A}_i)$ such that for any $\epsilon > 0$

$$E\tau^{\epsilon}(x) < \infty, \tag{4.2}$$

then for any $x \in \tilde{T}(\mathscr{A}_i)$

$$E\tau^{0}(x) < \infty,$$

where $\tau^0(x)$ is the first time of reaching 0 by $\eta_t(x)$.

Proof

By proposition 2 (4.2) holds for all x and $\epsilon > 0$. As the number of 1-dimensional faces is finite then due to proposition 1 for any $\epsilon > \epsilon' > 0$ we have

$$\sup_{x \in \tilde{T}(\mathscr{A}_i): \| x \| = \epsilon} E\tau^{\epsilon'}(x) < \infty.$$

Let us consider the sequence $\epsilon_n = 1/2^n$, $n = 0, 1, 2, \ldots$. Let

$$\sup_{x \in \tilde{T}(\mathscr{A}_i): \|x\| = \epsilon_0} E\tau^{\epsilon_1}(x) = C_1 < \infty.$$

Then for any n

$$\sup_{x: \|x\| = \epsilon_n} E\tau^{\epsilon_{n+1}}(x) = C_1\left(\tfrac{1}{2}\right)^n \tag{4.3}$$

by proposition 1. Let now $x \in \tilde{T}(\mathscr{A}_i)$, $\|x\| = 1$, and let us put

$$t^1(x) = \tau^{\epsilon_1}(x), \ldots, t^{k+1}(x) = t^k(x) + \tau^{\epsilon_{k+1}}(\eta_{t^k(x)}(x)) \ldots.$$

But we have first

$$t^k(x) \uparrow \tau^0(x) \quad \text{a.s. for } k \to \infty \tag{4.4}$$

(in fact $\tau^0(x)$ is defined in this way) and second by (4.3)

$$Et^k(x) \leqslant \sum_{j=1}^{k} \sup_{y \in \tilde{T}(\mathscr{A}_i): \|y\| = \epsilon_j} E\tau^{\epsilon_{j+1}}(y) \leqslant C_1 \sum_{j=1}^{k} \left(\tfrac{1}{2}\right)^k.$$

It follows that

$$E\tau^0(x) < \infty$$

for all x (as in proposition 1). Proposition 3 is proved. □

But at the same time we see as above that if $L(\mathscr{A}_i) > 0$ then $E_{\chi_n}(x)$ increases exponentially fast and so $\tau^0(x) = \infty$. We came to a contradiction which proves (iv) → (i). □

5. Proof of ergodicity in theorem 3

Let us first consider the case when there is a single essential class $\mathscr{A}$ with at least two essential states. So, all 1-dimensional faces are not ergodic. The process η_t is called ergodic if the mean time f_x of reaching 0 starting from a point $x \in \tilde{T}$ is finite (for all x).

We should like (as in [1]) to use f_x as a Lyapounov function for $\mathscr{L}_T$ in the following criterion for ergodicity [1]: $\mathscr{L}_T$ is ergodic iff there exists a positive integer valued function $m(\alpha)$ and $\epsilon > 0$ such that

$$\sum_{\beta} \tilde{p}_{\alpha\beta}^{(m(\alpha))} f_\beta - f_\alpha < -\epsilon m(\alpha), \tag{5.1}$$

for $\alpha \in T - T_0$ and some finite set $T_0 \subset T$. Let $\tilde{p}_{\alpha\beta}^{(t)}$ be the transition probabilities in t steps of the process η_t for α, $\beta \in \tilde{T}$. For α and t given, they differ from zero only for a finite number of points β. Let us suppose $L(\mathscr{A}) < 0$. Then η_t is ergodic and so, for α not very close to the origin,

$$f_\alpha = 1 + \sum_\beta \tilde{p}^{(1)}_{\alpha\beta} f_\beta,$$

or

$$\sum_\beta \tilde{p}^{(1)}_{\alpha\beta} f_\beta - f_\alpha = -1. \tag{5.2}$$

Let us note that f_α has the following properties:

(1) It is continuous everywhere except on the 1-dimensional faces where scattering occurs;

(2) $C_1 |x| \leqslant f_x \leqslant C_2 |x|$, for some $C_1, C_2 > 0$;

(3) for any 2-dimensional face $\Lambda^{(2)}$, the function f_x has a linear decrease along any line parallel to $M_{\Lambda^{(2)}}$ in the direction of $M_{\Lambda^{(2)}}$;

(4) due to the space homogeneity, f_x satisfies (4.1) with $m(\alpha) = 1$ everywhere, except for neighbourhoods of 1-dimensional faces. More exactly, let us take some nonergodic 1-dimensional face $\tilde{\Lambda}^{(1)}$ and put for any $\tilde{\Lambda}^{(2)}$

$$\mathscr{D}_{\Lambda^{(1)},\Lambda^{(2)}}(\rho) = \left\{\alpha\colon \alpha \in \overline{\Lambda^{(2)}},\ \rho(\alpha, \tilde{\Lambda}^{(1)}) = \rho\right\}.$$

Then, if $\Lambda^{(2)} \in S_-(\Lambda^{(1)})$, $\alpha \in \mathscr{D}_{\Lambda^{(1)},\Lambda^{(2)}}(1)$, (5.1) can not be satisfied. For this reason, we modify our Lyapounov functions as follows. Define

$$\tilde{f}_\alpha = \begin{cases} f_\alpha, & \alpha \notin \mathscr{D}(\rho_0), \\ (C_2 + 1)|\alpha|, & \alpha \in \mathscr{D}(\rho_0), \end{cases}$$

where

$$\mathscr{D}(\rho_0) = \bigcup_{\Lambda^{(1)}} \bigcup_{\Lambda^{(2)} \in S_-(\Lambda^{(1)})} \bigcup_{\rho=0}^{\rho_0} \mathscr{D}_{\Lambda^{(1)},\Lambda^{(2)}}(\rho)$$

and ρ_0 is a constant to be specified below.

LEMMA 5.1

There exist ρ_0, m, $\delta > 0$ such that (5.1) holds for the new Lyapounov function $\tilde{f}_\alpha$ with

$$m(\alpha) = \begin{cases} m, & \alpha \in \mathscr{D}(\rho_0 + 1), \\ \delta |\alpha|, & \alpha \in \bigcup_{\Lambda^{(1)}} \bigcup_{\Lambda^{(2)} \in S_+(\Lambda^{(1)})} \mathscr{D}_{\Lambda^{(1)},\Lambda^{(2)}}(1), \\ 1, & \text{in other cases}. \end{cases}$$

Proof

We choose $\delta > 0$ sufficiently small; then we take ρ_0 sufficiently large and then $m = m(\rho_0)$ sufficiently large. When $m(\alpha) = 1$, it is easy to verify (5.1), due to the linearity property 4.

Let us now take a point $\alpha \in \mathscr{D}(\rho_0 + 1)$. One can prove that starting from α after m steps, we shall be outside $\mathscr{D}(\rho_0)$ with probability $1 - \epsilon_1$, where $\epsilon_1 = \epsilon_1(m) \to 0$ when $m \to \infty$, uniformly in α, with $|\alpha| > a_0$, for some a_0 sufficiently large. This follows just from the transience of the corresponding hedgehog.

It follows that (5.1) holds since, for large m, we can take ϵ_1 arbitrarily small with

$$\tilde{f}(\beta) = f(\beta) < C_2(|\alpha| + md),$$

where $\beta \notin \mathscr{D}(\rho_0)$ is the final point after m steps.

Let now $\alpha \in \mathscr{D}_{\Lambda^{(1)},\Lambda^{(2)}}(1)$, $\Lambda^{(2)} \in S_+(\Lambda^{(1)})$ and $\xi_t(\alpha)$ be the position of the random walk starting from α.

Let $\Lambda_i \in S_-(\Lambda')$. Let us denote $\eta_t(\alpha, \Lambda_i)$ the point of $\mathbb{R}_+^2 = \Lambda_i$ which is the unique point of Λ_i, where process η_t is to be found at time t, after having started from α.

LEMMA 5.2

Let $t = \delta|\alpha|$. Then, for any $\epsilon_2, \epsilon_3 > 0$ sufficiently small, there exists $a_0 > 0$ such that, for any $|\alpha| > a_0$ and for any $\Lambda_i \in S_-(\Lambda^{(1)})$,

$$|Prob\{\xi_t(\alpha) \in \Lambda_i,\ |\xi_t(\alpha) - \eta_t(\alpha, \Lambda_i)| < \epsilon_3|\alpha|\} - p(\Lambda, \Lambda_i)| < \frac{\epsilon_2}{l}, \quad (5.3)$$

where $l = |S_-(\Lambda^{(1)})|$.

Proof

From the point $\alpha \in \mathscr{D}_{\Lambda',\Lambda}(1)$, we make first $\epsilon_3|\alpha|/(2(d+1))$ jumps. Then, for $|\alpha|$ large enough, with the probability p_i such that

$$|p_i - p(\Lambda, \Lambda_i)| < \frac{\epsilon_2}{3l},$$

we shall be in some point $\alpha_i \in \Lambda_i$ satisfying

$$\rho(\alpha_i, \eta_t(\alpha, \Lambda_i)) < \tfrac{1}{2}\epsilon_3|\alpha|.$$

After starting from α_i, we perform the remaining $(\delta - \epsilon_3/(2(d+1)))|\alpha|$ jumps. This will be in fact a translation invariant random walk in $\mathbb{Z}^2$ and, using Kolmogorov's inequality, we prove that, for $|\alpha|$ large, it will never go out of Λ_i with probability $1 - \epsilon_2/3l$. But, moreover, by the Law of Large Numbers, its final point $\xi_t(\alpha_i, \alpha)$ will satisfy the inequality

$$\xi_t(\alpha_i, \alpha) - \left(\alpha_i + M_{\Lambda_i}\left(\delta - \frac{\epsilon_3}{2(d+1)}\right)|\alpha|\right) < \tfrac{1}{2}\epsilon_3|\alpha|,$$

with probability $1 - \epsilon_2/3l$. Putting together all these estimates, we get (5.3), concluding the proof of lemma 5.2. □

We can finish now the proof of lemma 5.1. From (5.2), we have

$$\sum_{\beta} \tilde{p}_{\alpha\beta}^{(t)} \tilde{f}_{\beta} - \tilde{f}_{\alpha} = -t. \tag{5.4}$$

Comparing (5.1) and (5.4) yields

$$\sum p_{\alpha\beta}^{(t)} \tilde{f}_{\beta} - \tilde{f}_{\alpha} = \sum_{\beta} \tilde{p}_{\alpha\beta}^{(t)} \tilde{f}_{\beta} - \tilde{f}_{\alpha} + \Delta,$$

where

$$\Delta < \epsilon_2 (C_2 + 1) |\alpha| (1 + d\delta) + \epsilon_3 |\alpha|. \tag{5.5}$$

So for ϵ_2, ϵ_3 small we get

$$\sum p_{\alpha\beta}^{(t)} \tilde{f}_{\beta} - \tilde{f}_{\alpha} < -\tfrac{1}{2}\delta |\alpha|,$$

concluding the proof of lemma 5.1 and the ergodicity. □

If there are several essential classes, we use the same Lyapounov function as before inside 2-dimensional faces and in a vicinity of nonergodic 1-dimensional faces. We define it in a neighbourhood of ergodic faces, exactly as it was done in [1] for $\mathbb{Z}_+^3$.

6. Proof of the transience

Assume that, for the class $\mathscr{A}_i$,

$$M(\mathscr{A}_i) > 0. \tag{6.1}$$

We shall prove then that $\mathscr{L}_T$ is transient.

Let $\xi_t(\alpha)$ be the position of the random walk corresponding to $\mathscr{L}$, corresponding to the initial condition α, i.e. $\xi_0(\alpha) = \alpha$. Choosing $\alpha \neq \mathbf{0}$ belonging to some $\Lambda^{(1)}$, we define the following sequence of random times $0 = \tau_0 < \tau_1 < \cdots < \tau_n < \cdots$ by induction: if $\xi_{\tau_{n-1}}(\alpha) \neq 0$ then τ_n is the first after τ_{n-1} hitting time of $\mathbf{0}$ or some 1-dimensional face $\Lambda^{(1)} = \Lambda^{(1)}$ different from the face $\Lambda_{n-1}^{(1)}$ to which $\xi_{\tau_{n-1}}(\alpha)$ belongs. Of course, if $\xi_{\tau_n}(\alpha) = 0$ for some τ_n, then $\tau_{n+1} = \tau_n + 1$. Let us consider the new Markov chain $\zeta_n(\alpha) = \xi_{\tau_n}(\alpha)$, $\zeta_0(\alpha) = \alpha$, the state space of which is the union of all 1-dimensional faces $\bigcup \Lambda^{(1)}$ and $\mathbf{0}$. The probability of sometimes hitting $\mathbf{0}$ are equal for $\xi_n(\alpha)$ and $\zeta_n(\alpha)$. So it is sufficient to prove the non-recurrence of $\zeta_n(\alpha)$.

LEMMA 6.1

Let us consider two 1-dimensional faces $\Lambda^{(1)}$, $\Lambda_1^{(1)}$, and the corresponding 2-dimensional ones

$$\Lambda_1^{(2)} \in S_+(\Lambda^{(1)}), \quad \Lambda^{(2)} \in S_-(\Lambda^{(1)}) \cap S_+(\Lambda_1^{(1)}).$$

Then, for any $\epsilon > 0$, one can find $D_1 > 0$ such that, for any $\beta \in \Lambda^{(1)}$,

$$\left| P\left\{\zeta_1(\beta) \in \Lambda_1^{(1)},\ \left| |\zeta_1(\beta)| - \operatorname{tg} \phi_{\Lambda_1^{(2)}} \right| \leqslant \epsilon |\beta| \right\} - p\left(\Lambda_1^{(2)}, \Lambda^{(2)}\right) \right| \leqslant \frac{D_1}{|\beta|}. \tag{6.2}$$

Proof

Let us note first that, for any $\Lambda_2^{(2)} \in S_+(\Lambda^{(1)})$, we have $p(\Lambda_1^{(2)}, \Lambda^{(2)}) = p(\Lambda_2^{(2)}, \Lambda^{(2)})$. As the jumps are bounded, we have $\tau_1 \geqslant [|\beta|/d]$ a.s. It follows (see lemma 1.1 [1]) that there exist $\delta_1, \delta_2, D_2 > 0$, such that, for any $\beta \in \Lambda^{(1)}$ and any $t \in \mathbb{Z}_+$, $t < [|\beta|/d]$,

$$P\left\{\rho\left(\check{\xi}_t(\beta), \Lambda^{(1)}\right) < \delta_1 t\right\} \leqslant D_2\, \mathrm{e}^{-\delta_2 t}.$$

We also have

$$\left| P\left\{\rho\left(\check{\xi}_t(\beta), \Lambda^{(1)}\right) > \delta_1 t,\ \check{\xi}_t(\beta) \in \Lambda^{(2)}\right\} - p\left(\Lambda_1^{(2)}, \Lambda^{(2)}\right) \right| \leqslant C_3\, \mathrm{e}^{-\delta_2 t}, \tag{6.3}$$

for any $t < |\beta|/d$, where C_3 is a positive constant.

For any ϵ_1, $0 < \epsilon_1 < 1$, and $t_1 = [\epsilon_1 |\beta|/d]$,

$$\left|\check{\xi}_{t_1}(\beta) - \beta\right| \leqslant \epsilon_1 |\beta|. \tag{6.4}$$

To prove (6.2), it is sufficient by (6.3), (6.4) to prove that, for $\check{\xi}_{t_1}(\beta)$ such that

$$\rho\left(\check{\xi}_{t_1}(\beta), \Lambda^{(1)}\right) > \delta_1 t_1,$$

and for any $\epsilon_2 > 0$, there exists a constant $C_4 = C_4(\epsilon_1, \epsilon_2) > 0$ (not depending on β) such that

$$P\left\{\check{\xi}_{\tau_1}(\beta) \in \Lambda_1^{(1)},\ \rho\left(\check{\xi}_{\tau_1}(\beta) + (\tau_1 - t_1) M_{\Lambda^{(2)}}\right) > \epsilon_2 |\beta| / \rho\left(\check{\xi}_{t_1}(\beta), \Lambda^{(1)}\right) > \delta_1 t_1\right\} \leqslant \frac{C_4}{\beta}. \tag{6.5}$$

To prove (6.5), we consider $\Lambda^{(2)}$ imbedded into $\mathbb{Z}^2$ and the space homogeneous r.w. ξ_t', $t \in \mathbb{Z}_+$, on this $\mathbb{Z}^2$, with initial value

$$\xi_0' = \check{\xi}_{t_1}(\beta)$$

and one step transition probabilities

$$p'(\alpha, \beta) = p'(\alpha + \alpha', \beta + \alpha') = p(\alpha, \beta),$$

for all $\alpha, \beta \in \Lambda^{(2)}$, $\alpha' \in \mathbb{Z}^2$.

Let τ' be the first hitting time $\Lambda_1^{(1)}$ by ξ_t'. We choose $T \in \mathbb{Z}_+$ and $\epsilon_3 > 0$, so that

$$\rho\left(\xi_0', \Lambda^{(1)}\right) \geqslant \delta_1 t = \delta_1 \left[\epsilon_1 \frac{|\beta|}{d}\right] > \epsilon_3 |\beta|$$

and

$$\rho\left(\xi_0', \Lambda_1^{(1)}\right) + TPr_{\Lambda^{(1)}} M_{\Lambda^{(2)}} + \epsilon_3 |\beta| \leqslant \beta |\beta| (1+\epsilon_1) + TPr_{\Lambda^{(1)}} M_{\Lambda^{(2)}} + \epsilon_3 |\beta| < 0.$$

Let us note that, if

$$\max_{t \in [0,T]} |\xi_t' - \xi_0' - M_{\Lambda^{(2)}} t| \leqslant \epsilon_3 |\beta|,$$

then $\tau' < T$ and $\xi_t' \in \Lambda^{(2)}$ for all $t < \tau'$. From this remark and Kolmogorov's inequality, we get

$$\begin{aligned} &P\left\{\check{\xi}_{\tau_1}(\beta) \in \Lambda_1^{(1)}, \rho\left(\check{\xi}_{\tau_1}(\beta), \beta_1 + M_{\Lambda^{(2)}}(\tau_1 - t_1)\right) \leqslant \epsilon_3 |\beta| / \check{\xi}_{t_1}(\beta)\right\} \\ &\geqslant P\left\{\max_{t \in [t_1, \tau_1]} \left|\check{\xi}_t(\beta) - \check{\xi}_{t_1}(\beta) - (t - t_1) M_{\Lambda^{(2)}}\right| \leqslant \epsilon_3 |\beta| / \check{\xi}_{t_1}(\beta)\right\} \\ &= P\left\{\max_{t \in [0, \tau']} |\xi_t' - \xi_0 - t M_{\Lambda^{(2)}}| \leqslant \epsilon_3 |\beta|\right\} \\ &\geqslant P\left\{\max_{t \in [0, T]} |\xi_t' - \xi_0 - t M_{\Lambda^{(2)}}| \leqslant \epsilon_3 |\beta|\right\} \\ &\geqslant 1 - \frac{C_5 T}{\epsilon_3^2 |\beta|^2}, \end{aligned}$$

for some constant $C_5 > 0$.

The last step to derive (6.5) is now achieved by choosing T such that $T \leqslant const\,|\beta|$. Lemma 6.1 is proved. □

It follows from lemma 6.1 that, for any sequence $\Lambda_0, \ldots, \Lambda_n \in \mathscr{A}_i$ of 2-dimensional faces, for any $\alpha \in \overline{\Lambda}_0 \cap \overline{\Lambda}_1$ and any $\epsilon > 0$, $n \in \mathbb{Z}_+$, there exist $C_6 = C_6(\epsilon, n)$ not depending on $\alpha \in \overline{\Lambda}_0 \cap \overline{\Lambda}_1$, such that

$$\begin{aligned} &\Big| P\left\{\zeta_k(\alpha) \in \overline{\Lambda}_k \cap \overline{\Lambda}_{k+1}, |\zeta_k(\alpha)| \geqslant (\operatorname{tg} \phi_{\Lambda_k} - \epsilon) |\zeta_{k-1}(\alpha)|, k = 1, \ldots, n\right\} \\ &\quad - p(\Lambda_0, \Lambda_1) p(\Lambda_1, \Lambda_2) \ldots p(\Lambda_{n-1}, \Lambda_n)\Big| \leqslant \frac{C_6}{|\alpha|}. \end{aligned} \tag{6.6}$$

Let us choose $\epsilon > 0$, $\delta > 0$ and $n \in \mathbb{Z}_+$ so that, for any $\Lambda_0 \in \mathscr{A}_i$,

$$\sum_{k=1}^{n} \sum_{\Lambda^{(2)}} p^{(k)}\left(\Lambda_0, \Lambda^{(2)}\right) \log\left(\operatorname{tg} \phi_{\Lambda^{(2)}} - \epsilon\right) > \delta > 0. \tag{6.7}$$

Due to (6.1), it is always possible to satisfy (6.7) since

$$\frac{1}{n} \sum_{k=1}^{n} p^{(k)}\left(\Lambda_0, \Lambda^{(2)}\right) \to \pi\left(\Lambda^{(2)}\right), \text{ as } n \to \infty,$$

for any Λ_0, $\Lambda^{(2)}$, whenever Λ_0 is an essential state in the associated Markov chain.

To prove the transience of $\zeta_k(\alpha)$, $k \in \mathbb{Z}_+$, it is sufficient to prove the transience of the chain $\eta_k = \zeta_{nk}(\alpha)$, $k \in \mathbb{Z}_+$. We shall use Foster's criterion [5] recalled thereafter: if for a Markov chain with state space X and transition probabilities p_{ij}, $i, j \in X$, there exists a positive function f defined on the state space X and a set $A \subset X$, such that

$$\sum_{j \in X} p_{ij} f_j \leqslant f_i, \quad \forall i \in X - A,$$

and

$$\inf_{i \in A} f_i > \sup_{j \in X - A} f_j,$$

then the Markov chain is transient.

We define f on the state space of η_k.

$$f(\alpha) = \begin{cases} \dfrac{1}{\log 3|\alpha|}, & \text{if } \alpha \neq 0 \text{ and } \alpha \in \overline{\Lambda}^{(2)}, \text{ for some } \Lambda^{(2)} \in \mathscr{A}_i, \\ 1 & \text{in the other cases.} \end{cases}$$

Let us prove that, if $|\eta_0|$ is sufficiently large, then

$$E(f(\eta_1)) \leqslant f(\eta_0). \tag{6.8}$$

In fact, if $f(\eta_0) = 1$, then (6.8) evidently holds. Let

$$f(\eta_0) = \frac{1}{\log 3|\eta_0|} \quad \text{and} \quad \eta_0 \in \Lambda^{(1)},$$

for some 1-dimensional face $\Lambda^{(1)}$. Then, by (6.6),

$$E(f(\eta_1)) \leqslant \sum_{\Lambda_1, \ldots, \Lambda_n} p(\Lambda_0, \Lambda_1) \ldots p(\Lambda_{n-1}, \Lambda_n) \times \frac{1}{\log\left[3|\eta_0| \prod_{j=1}^n \left(\operatorname{tg} \phi_{\Lambda_j} - \epsilon\right)\right]} + \frac{C_6}{|\eta_0|}, \tag{6.9}$$

where $\Lambda_0 \in S_-(\Lambda^{(1)})$, $\Lambda_0 \in \mathscr{A}_i$. But, for $|\eta_0|$ sufficiently large,

$$\begin{aligned} &\left(\log\left[3|\eta_0| \prod_{j=1}^n \left(\operatorname{tg} \phi_{\Lambda_j} - \epsilon\right)\right]\right)^{-1} \\ &= \frac{1}{\log 3|\eta_0|}\left[1 + \frac{1}{\log 3|\eta_0|}\left(\sum_{j=1}^n \log\left(\operatorname{tg} \phi_{\Lambda_j} - \epsilon\right)\right)\right]^{-1} \\ &\leqslant \frac{1}{\log 3|\eta_0|} - \frac{\sum_{j=1}^n \log\left(\operatorname{tg} \phi_{\Lambda_j} - \epsilon\right)}{\left(\log 3|\eta_0|\right)^2} + o\left(\frac{1}{\left(\log 3|\eta_0|\right)^2}\right). \end{aligned} \tag{6.10}$$

From (6.9), (6.10), (6.7), we get

$$E(f(\eta_1)) \leqslant \frac{1}{\log 3|\eta_0|} - \frac{\delta}{(\log 3|\eta_0|)^2} + o\left(\frac{1}{(\log 3|\eta_0|)^2}\right).$$

Transience is proved. □

Theorems 1.1 and 1.2 are in fact implicitly contained in the results of [1]. We shall briefly explain how they could be proved.

The proof of theorem 1.1 is based on the following criterion of transience:

LEMMA 6.2

Assume that, for the irreducible Markov chain with state space S, there exists a function $f: S \to \mathbb{R}_+$, such that

(i) for some $d > 0$, $|f_\alpha - f_\beta| > d \Rightarrow p_{\alpha\beta} = 0$, $\alpha, \beta \in S$;

(ii) $\Sigma_\beta p_{\alpha\beta} f_\beta - f_\alpha \geqslant \epsilon$, for some $\epsilon > 0$ and all $\alpha \in A_C = \{\alpha : f_\alpha > C\} \neq \emptyset$, for some $C > 0$.

Then the chain is transient.

Proof

See [1]. □

Let $\Lambda^{(2)}$ be a two-dimensional face, with $M_{\Lambda^{(2)}}$ having both components positive. Then we can construct a Lyapounov function f, taking it equal to 0 outside $\Lambda^{(2)}$ and ϵ-linear on $\Lambda^{(2)}$, as in [2]. To prove theorem 1.2, we could use condition B' and theorem 2.1 of [1].

7. Proof of recurrence

Here for simplicity we also assume that the associated chain has a single essential class with at least two essential states. Let

$$M = M(\mathscr{A}) < 0. \tag{7.1}$$

We shall show that $\mathscr{L}_T$ is recurrent. As in the proof of transience for any 1-dimensional chain $\Lambda^{(1)}$ and any $\alpha \in \Lambda^{(1)}$, $\alpha \neq 0$, we consider Markov process $\zeta_n(\alpha)$, $n \in \mathbb{Z}_+$. See the definition of this process in section 6. To prove recurrence of $\mathscr{L}_T$ it is sufficient to prove recurrence of $\zeta_n(\alpha)$.

Let us choose $\tilde{\epsilon} > 0$, $\tilde{\delta} > 0$, $\tilde{n} \in \mathbb{Z}_+$ so that for any state $\Lambda_0^{(2)} \in \mathscr{A}$ the following inequality holds

$$\sum_{k=1}^{\tilde{n}} \sum_{\Lambda^{(2)} \in \mathscr{A}} p^{(k)}(\Lambda_0^{(2)}, \Lambda^{(2)}) \log(\operatorname{tg} \alpha_{\Lambda^{(2)}} + \tilde{\epsilon}) < -\tilde{\delta}. \tag{7.2}$$

By (7.1) this is always possible as for any states $\Lambda_0^{(2)}$, $\Lambda^{(2)} \in \mathscr{A}$ we have

$$\frac{1}{n}\sum_{k=1}^{n} p^{(k)}\left(\Lambda_0^{(2)}, \Lambda^{(2)}\right) \to \pi\left(\Lambda^{(2)}\right).$$

To prove recurrence for the chain $\zeta_k(\alpha)$ it is sufficient to prove recurrence of the chain $\eta_k(\alpha) = \zeta_{\tilde{n}k}(\alpha)$.

To prove recurrence of $\eta_k(\alpha)$ we use the well known recurrence criteria [6]: a Markov chain with the state space $\mathbb{Z}_+$ and transition probabilities p_{ij}, $i, j \in \mathbb{Z}_+$, is recurrent if there exists a nonnegative function f on $\mathbb{Z}_+$ and a finite set $A \subset \mathbb{Z}_+$ such that for any $i \in \mathbb{Z}_+ - A$

$$\sum_{j \in \mathbb{Z}_+} p_{ij} f_j - f_i \leqslant 0$$

and $f_j \to \infty$ as $j \to \infty$. We define the function on the state space of η_k putting for any $\alpha \neq 0$

$$f(\alpha) = \log 3|\alpha| \quad \text{and} \quad f(0) = 0.$$

Let us show that there exists $\mathscr{D} > 0$ such that for any α belonging to the state space of η_k and such that $|\alpha| > \mathscr{D}$, the following inequality holds

$$Ef(\zeta_{\tilde{n}}(\alpha)) \leqslant f(\alpha). \tag{7.3}$$

After this the proof of recurrence of η_k and so ζ_t will be finished.

LEMMA 7.1

Let $\Lambda^{(1)}$ be some 1-dimensional non-ergodic face and let $\beta \in \Lambda^{(1)} \cap T$. Let $\tau_1(\beta)$ be the first time the process $\xi_t(\beta)$ hits $\mathbf{0}$ or a 1-dimensional face different from $\Lambda^{(1)}$. Then one can find constants $q_1 > 0$, $C_1 > 0$, $\kappa_1 > 0$ which do not depend on β and are such that for any $t > q_1|\beta|$

$$P\{\tau_1(\beta) > t\} \leqslant C_1\, \mathrm{e}^{-\kappa_1 t}.$$

Proof

It easily follows from lemma 1.2 of [1] that for any $t_1 \in \mathbb{Z}_+$

$$P\{\tau_1(\beta) > t_1, \exists t \in \mathbb{Z}_+, t_1 \leqslant t < \tau_1(\beta), \rho(\xi_t(\beta), \Lambda^{(1)}) < \delta' t_1\} \leqslant c'\, \mathrm{e}^{-\alpha' t_1}, \tag{7.4}$$

where $\delta' > 0$, $c' > 0$, $\kappa' > 0$ do not depend on $\beta \in \Lambda^{(1)}$ and on $t_1 \in \mathbb{Z}_+$. Moreover, due to boundedness of jumps for $\xi_t(\beta)$ we have for any $t_1 \in \mathbb{Z}_+$

$$|\xi_{t_1}(\beta)| \leqslant |\beta| + dt_1. \tag{7.5}$$

Let us note that for any 2-dimensional chain $\Lambda^{(2)}$ the vector $M_{\Lambda^{(2)}}$ has at least one negative component and therefore by lemma 1.2, [1], for any $\Lambda^{(2)}$, any $\alpha \in \Lambda^{(2)} \cap T$ and any $t_2 \in \mathbb{Z}_+$

$$P\{\xi_t(\alpha) \in \Lambda^{(2)}, t = 1, \ldots, t_2\} \leqslant c''\, \mathrm{e}^{h|\alpha| - \kappa'' t_2}, \tag{7.6}$$

where the constants $c' > 0$, $\kappa'' > 0$, $h > 0$ do not depend on $\Lambda^{(2)}$, $\alpha \in \Lambda^{(2)}$ and on $t_2 \in \mathbb{Z}_+$. Let now some $t \in \mathbb{Z}_+$ be given. Let

$$t_1 = [\gamma t], \quad t_2 = t - t_1.$$

Then it easily follows from (7.4)–(7.6) that

$$P\{\tau_1(\beta) > t\} \leqslant c' \, e^{-\alpha'[\gamma t]} + c'' \exp\{h\,|\beta| + + hd\gamma t - \kappa'' t(1-\gamma)\}.$$

Lemma 7.1 will follow from this if we take $\gamma > 0$ sufficiently small. □

LEMMA 7.2

Let us choose $\Lambda_1^{(1)}$, $\Lambda_2^{(1)}$ and

$$\Lambda_1^{(2)} \in S_-\big(\Lambda_1^{(1)}\big), \quad \Lambda_2^{(2)} \in S_-\big(\Lambda_1^{(1)}\big) \cap S_+\big(\Lambda_2^{(1)}\big).$$

Then there exist constants $q_2 > 0$, $\kappa_2 > 0$, $c_2 > 0$ such that for any $\beta \in \Lambda_1^{(1)} \cap T$ and any $r > q_2$

$$P\Big\{\zeta_1(\beta) \in \Lambda_2^{(2)},\ |\zeta_1(\beta)| \geqslant |\beta|\big(\mathrm{tg}\ \phi_{\Lambda_2^{(2)}} + r\big)\Big\} \leqslant c_2 \, e^{-\kappa_2 r |\beta|}.$$

Proof

Let $\beta \in \Lambda_1^{(1)} \cap T$. Let $\tau_1(\beta)$ be the first time the process $\xi_t(\beta)$ hits some 1-dimensional face different from $\Lambda_1^{(1)}$ (we recall that $\zeta_1(\beta) = \xi_{\tau_1(\beta)}(\beta)$). By lemma 7.1 for any $t > q_1 |\beta|$, $t \in \mathbb{Z}_+$,

$$P\{\tau_1(\beta) > t\} \leqslant c_1 \, e^{-\alpha_1 t}. \tag{7.7}$$

On the other hand, by the boundedness of jumps for $\xi_t(\beta)$

$$P\Big\{\xi_{\tau_1(\beta)}(\beta) \in \Lambda_2^{(1)},\ |\xi_{\tau_1(\beta)}(\beta)| - |\beta|\ \mathrm{tg}\ \phi_{\Lambda_2^{(2)}} > 2d\tau_1(\beta)\Big\} = 0. \tag{7.8}$$

From (7.7), (7.8) for any $t > q_1 |\beta|$ we get

$$P\Big\{\xi_{\tau_1(\beta)}(\beta) \in \Lambda_2^{(1)},\ |\xi_{\tau_1(\beta)}(\beta)| > |\beta|\ \mathrm{tg}\ \phi_{\Lambda_2^{(2)}} + 2dt\Big\} \leqslant c_1 \, e^{-\alpha t}.$$

Lemma 7.2 follows from this. □

Now we shall prove the inequality (7.3). Let us choose some $\Lambda^{(1)}$, $\alpha \in \Lambda^{(1)} \cap T$ and $\Lambda_0^{(2)} \in \mathscr{A}$, $\Lambda_0^{(2)} \in S_+(\Lambda^{(1)})$. Let $\tilde{n} \in \mathbb{Z}_+$, $\tilde{\epsilon} > 0$, $\tilde{\delta} > 0$ be the constant of (7.2). It follows from lemma 6.1 that there exists a constant $\tilde{c} > 0$ such that for any sequence of 2-dimensional faces $\Lambda_1, \dots, \Lambda_{\tilde{n}} \in \mathscr{A}$, where $\alpha \in \bar{\Lambda}_0 \cap \bar{\Lambda}_1$, the following inequality holds

$$\left| P\left\{\zeta_k(\alpha) \in \bar{\Lambda}_k \cap \bar{\Lambda}_{k+1},\ k = 1, \dots, \tilde{n},\ |\zeta_{\tilde{n}}(\alpha)| \leqslant |\alpha| \prod_{j=1}^{\tilde{n}} \big(\mathrm{tg}\ \phi_{\Lambda_j} + \tilde{\epsilon}\big)\right\} \right.$$
$$\left. - p(\Lambda_0, \Lambda_1) \dots p(\Lambda_{\tilde{n}-1}, \Lambda_{\tilde{n}}) \right| \leqslant \frac{\tilde{c}}{|\alpha|}. \tag{7.9}$$

Moreover, from lemma 7.2 it follows that there exist constants $c > 0$, $\kappa > 0$, $q > 0$ such that for any sequence of 2-dimensional faces $\Lambda_1, \ldots, \Lambda_{\tilde{n}} \in \mathscr{A}$ such that $\alpha \in \overline{\Lambda}_0 \cap \overline{\Lambda}_1$, and for any $k > q$

$$P\left\{\zeta_k(\alpha) \in \overline{\Lambda}_k \cap \overline{\Lambda}_{k+1},\ k = 1, \ldots, \tilde{n},\ |\zeta_{\tilde{n}}(\alpha)| \geqslant |\alpha| \prod_{j=1}^{\tilde{n}} \left(\operatorname{tg} \phi_{\Lambda_j} + k\right)\right\}$$
$$\leqslant c\, \mathrm{e}^{-\kappa k |\alpha|} \tag{7.10}$$

From (7.9) and (7.10) it follows that

$$Ef(\zeta_{\tilde{n}}(\alpha)) \leqslant f(\alpha) + \sum_{\Lambda_1, \ldots, \Lambda_{\tilde{n}} \in \mathscr{A}} p(\Lambda_0, \Lambda_1) \ldots p(\Lambda_{\tilde{n}-1}, \Lambda_{\tilde{n}})$$
$$\times \log \prod_{j=1}^{\tilde{n}} \left(\operatorname{tg} \phi_{\Lambda_j} + \tilde{\epsilon}\right) + \theta(\alpha),$$

where $\theta(\alpha) \to 0$ when $|\alpha| \to \infty$. From the latter inequality using (7.2) one gets (7.3). Recurrence is proved. □

8. Proof of nonergodicity

Let there exist an irreducible class of essential states $\mathscr{A}_i$ for which

$$L(\mathscr{A}_i) > 0. \tag{8.1}$$

As in the proof of transience we consider (for any 1-dimensional face $\Lambda^{(1)}$ and any $\alpha \in \Lambda^{(1)} \cap T$) random walk $\xi_t(\alpha)$, $\xi_0(\alpha) = \alpha$, corresponding to $\mathscr{L}_T$. Define the sequence of stopping times

$$\tau_0(\alpha) = 0, \quad \tau_1(\alpha), \ldots, \tau_n(\alpha), \ldots,$$

where, for $\xi_{\tau_n}(\alpha) \neq 0$, $\tau_{n+1}(\alpha)$ is defined as the next (after $\tau_n(\alpha)$) time the process $\xi_t(\alpha)$ hits either 0 or some 1-dimensional face different from that of $\xi_{\tau_n}(\alpha)$. If $\xi_{\tau_n}(\alpha) = 0$ we put $\tau_{n+1}(\alpha) = \tau_n(\alpha)$, etc. Let

$$\zeta_n(\alpha) = \xi_{\tau_n(\alpha)}(\alpha), \quad n \in \mathbb{Z}_+,$$
$$\nu(\alpha) = \inf\{n \in \mathbb{Z}_+ \colon \xi_{\tau_n(\alpha)} = 0\}.$$

From the definition of $\tau_n(\alpha)$, $n \in \mathbb{Z}_+$, evidently follows that $\tau_{\nu(\alpha)}(\alpha)$ is the first time when $\xi_t(\alpha)$ hits zero, so $\tau_{\nu(\alpha)+n}(\alpha) = \tau_{\nu(\alpha)}(\alpha)$, $n \in \mathbb{Z}_+$. So to prove nonergodicity of $\xi_t(\alpha)$ it is sufficient to show that

$$E\tau_n(\alpha) \to \infty, \quad n \to \infty.$$

LEMMA 8.1

For any 1-dimensional face $\Lambda^{(1)}$, any $\alpha \in \Lambda^{(1)}$ and any $n \in \mathbb{Z}_+$

$$\tau_{n+1}(\alpha) - \tau_n(\alpha) \geqslant |\xi_{\tau_n(\alpha)}| \text{ a.s.}$$

To prove this lemma it is sufficient to note that for each 1-dimensional face $\Lambda^{(1)}$ and any $\alpha \in \Lambda^{(1)}$,

$$\tau_1(\alpha) \geqslant |\alpha| \text{ a.s.},$$

and to use that

$$\tau_{n+1}(\alpha) = \tau_n(\alpha) + \tau_1\big(\xi_{\tau_n(\alpha)}(\alpha)\big) \text{ a.s.} \quad \square$$

Now let us denote the set of all 1-dimensional faces $\Lambda^{(1)}$ for which $S_-(\Lambda^{(1)}) \cap \mathscr{A}_i \neq \emptyset$ by $\Theta(\mathscr{A}_i)$. We define a function f on the set of states of $\zeta_n(\alpha) = \xi_{\tau_n(\alpha)}(\alpha)$:

$$f(\beta) = \begin{cases} |\beta|; & \text{if } \beta \in \Lambda^{(1)} \cap T \text{ with } \Lambda^{(1)} \in \Theta(\mathscr{A}_i), \\ 0, & \text{otherwise.} \end{cases}$$

LEMMA 8.2

There exist constants $\tilde{n} \in \mathbb{Z}_+$, $\tilde{\gamma} > 1$, $\tilde{c} > 0$ such that for any 1-dimensional face $\Lambda^{(1)}$ and any $\alpha \in \Lambda^{(1)} \cap T$

$$Ef(\zeta_{\tilde{n}}(\alpha)) \geqslant \tilde{\gamma} f(\alpha) - \tilde{c}. \tag{8.2}$$

Proof

For any $\Lambda^{(1)} \in \Theta(\mathscr{A}_i)$ and any $x \in \overline{\Lambda^{(1)}} \cap T$ the inequality (8.2) evidently holds for all $\tilde{\gamma} > 1$, $\tilde{c} > 0$, $\tilde{n} \in \mathbb{Z}_+$. Now let us consider the case $\Lambda^{(1)} \in \Theta(\mathscr{A}_i)$. Let us choose $\tilde{n} \in \mathbb{Z}_+$, $\tilde{\epsilon} > 0$ and $\tilde{\gamma} > 1$ so that for any $\Lambda_0^{(2)} \in \mathscr{A}_i$.

$$\sum_{\Lambda_1^{(2)}, \ldots, \Lambda_{\tilde{n}}^{(2)} \in \mathscr{A}_i} p\big(\Lambda_0^{(2)}, \Lambda_1^{(2)}\big) \ldots p\big(\Lambda_{\tilde{n}-1}^{(2)}, \Lambda_{\tilde{n}}^{(2)}\big) \prod_{j=1}^{\tilde{n}} \big(\text{tg } \phi_{\Lambda_j^{(2)}} - \tilde{\epsilon}\big) > \tilde{\gamma}. \tag{8.3}$$

This is always possible by (8.1) and lemma 3.2. Let us consider some $\Lambda^{(1)} \in \Theta(\mathscr{A}_i)$, $\Lambda_0^{(2)} \in S_+(\Lambda^{(1)})$, $\alpha \in \Lambda^{(1)} \cap T$. From lemma 6.1 easily follows that

$$Ef(\zeta_{\tilde{n}}(\alpha)) \geqslant |\alpha| \sum_{\Lambda_1^{(2)} \ldots \Lambda_{\tilde{n}}^{(2)} \in \mathscr{A}_i} p\big(\Lambda_0^{(2)}, \Lambda_1^{(2)}\big) \ldots p\big(\Lambda_{\tilde{n}-1}^{(2)}, \Lambda_{\tilde{n}}^{(2)}\big) \times \prod_{j=1}^{\tilde{n}} \big(\text{tg } \phi_{\Lambda_j^{(2)}} - \tilde{\epsilon}\big) - \tilde{c}, \tag{8.4}$$

where $c = c(\tilde{n}, \tilde{\epsilon})$ is a positive constant which does not depend on $\Lambda^{(1)}$ and on $\alpha \in \Lambda^{(1)}$. Then (8.3) follows from (8.4), (8.5). Lemma 8.2 is thus proved.

Let us prove now (8.2). From lemma 8.1 it follows that for any 1-dimensional face $\Lambda^{(1)}$, any $\beta \in \Lambda^{(1)} \cap T$ and any $n \in \mathbb{Z}_+$

$$E(\tau_{n+1}(\beta) - \tau_n(\beta)) \geqslant Ef(\zeta_n(\beta)). \tag{8.5}$$

From lemma 8.2 it follows that for any $\Lambda^{(1)}$, any $\beta \in \Lambda^{(1)} \cap T$ and $k \in \mathbb{Z}_+$

$$Ef(\zeta_{k\tilde{n}}(\beta)) \geqslant \tilde{\gamma}^k \left(f(\beta) - \frac{\tilde{c}}{\tilde{\gamma} - 1} \right), \tag{8.6}$$

where $\tilde{n} \in \mathbb{Z}_+$, $\tilde{\gamma} > 1$, $\tilde{c} > 0$ are the constants from lemma 8.2. Let $\Lambda^{(1)} \in \theta_1(\mathscr{A}_i)$, $\beta \in \Lambda^{(1)} \cap T$ and $f(\beta) = |\beta| > \tilde{c}/(\tilde{\gamma} - 1) + 1$. Then by (8.6) and (8.7) for any $k \in \mathbb{Z}_+$

$$E\tau_{k\tilde{n}}(\beta) \geqslant \sum_{j=0}^{k} E\big(\tau_{j\tilde{n}+1}(\beta) - \tau_{j\tilde{n}}(\beta)\big)$$

$$\geqslant \sum_{j=0}^{k} Ef\big(\zeta_{j\tilde{n}}(\beta)\big) \geqslant \sum_{j=0}^{k} \tilde{\gamma}^k, \tag{8.7}$$

and so

$$E\tau_n(\beta) \to \infty \quad \text{if } n \to \infty.$$

So nonergodicity is proved. □

9. Queueing applications

First we shall describe two models which in some sense are dual.

MODEL 1

A single service station has two infinite buffers 1 and 2; $\xi_i(t)$, $i = 1, 2$, being the number of customers in buffer i at time t. There are two independent Poisson arrival streams with intensities λ_i, $i = 1, 2$, to the buffers $i = 1, 2$ respectively.

The station can serve the buffers according to several regimes numbered from 1 to N. For a given regime j, the station serves the buffers, according to independent exponential service times with rates μ_{1j} and μ_{2j} respectively, provided that both buffers are non-empty, i.e. $\xi_1(t) \neq 0$, $\xi_2(t) \neq 0$. When one of the buffers is empty, the service rate is taken to be equal to μ. One can imagine changing algorithms, prophylaxis or commuting context, etc.

Let $r(t)$ denote the regime of the station at time t. Let us assume that $r(t) = k$, $\xi_1(t) = 0$, $\xi_2(t) \neq 0$ and let $\tau > t$ be the first arrival of a customer to the buffer i. Our assumption is that, at time τ, *the regime suddenly changes to* $r(\tau) = l$, *with probability* p_{kl}^1, $\sum_{l=1}^{N} p_{kl}^1 = 1$. Similarly, we define the probabilities p_{kl}^2.

If $\xi_1(t) = \xi_2(t) = 0$, then we do not specify the protocol, since it has no influence on the ergodicity of the system.

MODEL 2

Let us consider a customer (computer, bus etc.) which has access to N data bases, but can use at time t at most two of them, $r_1(t)$ and $r_2(t)$, where $r_i(t) \in \{1, \dots, N\}$, $i = 1, 2$. This user has two buffers 1, 2 accumulating data coming from the data bases $r_1(t)$ and $r_2(t)$, respectively. Let λ_i be the intensity of arrivals from the data basis i, $i = 1, 2$. Let $\xi_i(t)$ be the number of units in buffer i at time t (necessarily issuing from the data basis $r_i(t)$).

Under the conditions $\xi_1(t) > 0$, $\xi_2(t) > 0$ and $r_i(t) = k_i$, $i = 1, 2$, the customer "serves" the buffers with the following intensities:

- $\mu^i_{k_i}$, when serving one unit from buffer i;
- $\mu_{k_1 k_2}$, when, simultaneously one unit from buffer 1 and one unit from buffer 2 are served.

When, e.g., $\xi_1(t) = 0$, $\xi_2(t) \neq 0$, the user switches to another data basis. The intensity of switching to data basis k is

$$\lambda_1(k; l, \xi_2(t)),$$

where l is the latter data basis used by buffer 1. In a similar way, we introduce

$$\lambda_2(k; \xi_1(t), l).$$

It is also assumed that the above switching is accompanied with the arrival of one unit from the data basis K to the corresponding buffer.

In model 1, a *regime* corresponds to a quarter-plane. In model 2, a single data basis corresponds to a one-dimensional complex and a pair of data bases corresponds to a quarter-plane. Both models can be treated with similar methods, since they induce random walks in 2-dimensional complexes. They do not completely fit our main theorems, as the probabilities of jumps from 1-dimensional faces depend on the last visited quarter-plane. We could solve this case also, by applying similar methods. Instead of this, we shall discuss in more detail a particular case of model 1.

ANALYSIS OF MODEL 1

Let us consider the set P of all $2N$ pairs (i, k), $i = 1, 2$ and $k = 1, \ldots, N$. Thus $(1, k)$ and $(2, k)$ can be considered as the 1-dimensional faces of the kth quarter plane. We assume that P is subdivided into $M \leqslant 2N$ equivalence classes α, so that (i, k) and (j, l) belong to different classes if $i \neq j$. This is equivalent to a *gluing* of the corresponding 1-dimensional faces. Let $\alpha(i, k)$ be the class to which (i, k) belongs.

So, we assume that $p^i_{kl} = 0$, if (i, k) and (i, l) belong to different classes and that p^i_{kl} depends only on $\alpha = \alpha(i, k)$. We shall write *ad libitum* $p_{\alpha,l}$ instead of p^i_{kl} if $(i, k) \in \alpha$. Our process is a triple $(\xi_1(t), \xi_2(t), r(t))$ and forms a countable Markov chain.

CORRESPONDENCE WITH THE RANDOM WALK

Let us consider the N quarter-planes $(\mathbb{Z}^2_+)_i$, $i = 1, \ldots, N$, glued together and $r(t) = 1, \ldots, N$. The one-dimensional faces correspond to the equivalence classes which have been introduced in the above gluing. This defines in fact a complex

T. It will be convenient to consider an imbedded discrete-time Markov chain instead of the original chain which is in continuous time. To that end, let us choose $c > 0$ sufficiently large (ergodicity conditions do not depend on the choice of this constant) and we define the following transition probabilities on T:

For $k, l \geq 1$, $(k, l)_i \in (\mathbb{Z}_+^2)_i$, we can write

$$cp((k, l)_i, (k', l')_i) = \begin{cases} \lambda_1, \; k' = k+1, \; l' = l; \\ \lambda_2, \; k' = k, \; l' = l+1; \\ \mu_{1i}, \; k' = k-1, \; l' = l; \\ \mu_{2i}, \; k' = k, \; l' = l-1. \end{cases}$$

On the boundaries

$$cp((0, l)_i, (0, l')_i) = \begin{cases} \lambda_2, \; l' = l+1; \\ \mu, \; l' = l-1; \end{cases}$$

$$cp((0, l)_\alpha, (1, l')_i) = \lambda_1 p_{\alpha i}, \; l' = l;$$

$$cp((k, 0)_j, (k', 0)_j) = \begin{cases} \lambda_1, \; k' = k+1; \\ \mu, \; k' = k-1; \end{cases}$$

$$cp((k, 0)_\alpha, (k', 1)_j) = \lambda_2 p_{\alpha j}, \; k' = k.$$

We write $(0, l)_\alpha = (0, l)_i$, if $(i, l) \in \alpha$ and similarly, $(k, 0)_\alpha = (k, 0)_j$ if $(j, k) \in \alpha$. The probabilities $p((k, l)_i, (k, l)_i)$ are defined by the normalization condition.

MEAN JUMPS

Up to the multiplicative constant C^{-1}, we have the following possibilities:

- If $\Lambda_i^{(2)}$ corresponds to the *regime* i, then

$$M_i \stackrel{\text{def}}{=} M_{\Lambda_i^{(2)}} = (\lambda_1 - \mu_{1i}, \lambda_2 - \mu_{2i}).$$

- If $\Lambda_\alpha^{(1)}$ corresponds to the equivalence class α, then

$$M_{\alpha\alpha} \stackrel{\text{def}}{=} Pr_{\Lambda_\alpha^{(1)}}\left[M_{\Lambda_\alpha^{(1)}, \Lambda_\alpha^{(2)}}\right] = \lambda_\alpha - \mu,$$

where $\lambda_\alpha = \lambda_1$ (resp. λ_2), if $\xi_2(t) = 0$ (resp. $\xi_1(t) = 0$);

$$M_{\alpha,l} \stackrel{\text{def}}{=} M_{\Lambda_\alpha^{(1)}, \Lambda_l^{(2)}} = \begin{cases} (0, \lambda_2 p_{\alpha l}), & \text{if } \alpha \text{ corresponds to } \xi_2(t) = 0; \\ (\lambda_1 p_{\alpha l}, 0), & \text{if } \alpha \text{ corresponds to } \xi_1(t) = 0. \end{cases}$$

(Recall that $\xi_i(t)$ is the content of buffer i, $i = 1, 2$ at time t.)

TRANSIENCE AND ERGODICITY

Would there exist only one regime, e.g. j, then necessary and sufficient conditions for the system to be ergodic for all $\mu > 0$ sufficiently large, should be

$$\lambda_i < \mu_{ij}, \quad i = 1, 2. \tag{9.1}$$

In fact, for several regimes, sufficient ergodicity conditions are: *inequalities (9.1) hold for all $j = 1, \ldots, N$ and μ sufficiently large.* In this case, the system is ergodic, independently of the p^i_{kl}'s. As to be expected, these conditions are not necessary and we get below the exact necessary and sufficient conditions, which connect the parameters p^i_{kl}, λ_i, μ_{ij} together. In particular, we show that if, for some i, j, we have

$$\lambda_i > \mu_{ij},$$

then, for some p^i_{kl}, the system can be ergodic. However, theorem 1.1 yields the following

COROLLARY 9.1

If there exists j such that

$$\lambda_1 > \mu_{1j}, \quad \lambda_2 > \mu_{2j}, \tag{9.2}$$

then the system is transient. □

Let us note that a 1-dimensional face $\Lambda^{(1)}_\alpha$ (for simply the equivalence class α) such that $(1, k) \in \alpha$ is ergodic if and only if $\lambda_2 < \mu_{2k}$. In this case, we have (see section 1)

$$sgn V_{\Lambda^{(1)}_\alpha} = sgn\left(\lambda_2 - \mu + \sum_{k;(1,k)\in\alpha} \left[(\lambda_1 - \mu_{1k})\frac{\lambda_2 p_{\alpha k}}{\lambda_2 - \mu_{2k}}\right]\right).$$

One could write similar formulas for ergodic faces $\Lambda^{(1)}_\alpha$, such that $(2, j) \in \alpha$. This can be summarized in the following. Let us assume that for all ergodic α

$$\lambda_\alpha - \mu + \sum_{ki:(i,k)\in\alpha} \frac{\lambda_i - \mu_{ik}}{\lambda_j - \mu_{jk}} \neq 0.$$

Then the following proposition holds.

COROLLARY 9.2

For the Markov chain $\mathscr{L}_T$ to be ergodic, it is necessary that, for all ergodic α,

$$\lambda_\alpha - \mu + \sum_{k:(i,k)\in\alpha} \frac{\lambda_i - \mu_{ik}}{\lambda_j - \mu_{jk}} \lambda_j p_{\alpha k} < 0. \tag{9.3}$$

Theorem 3.1 gives the necessary and sufficient conditions for ergodicity in the cases which are not covered by corollaries (9.1) and (9.2). To show even more explicit computations let us consider a system admitting only two classes, α_1 and α_2. In the usual language of queueing theory, this means that the quantities

$$p_{kl}^1 = p_l^1, \quad p_{kl}^2 = p_l^2, \tag{9.4}$$

do not depend on k. Let us classify the regimes in the latter case. The regime j is said to be *1-outgoing* if

$$\lambda_2 > \mu_{2j}. \tag{9.5}$$

We have seen in this case that, for the system to be ergodic, it is necessary to have

$$\lambda_1 < \mu_{1j}.$$

The set of all *1-outgoing* regimes will be denoted by A_1. Similarly, we define *2-outgoing* regimes A_2, when

$$\lambda_2 < \mu_{2j}, \quad \lambda_1 > \mu_{1j}. \tag{9.6}$$

Clearly, for all other regimes (belonging to $\{1, \dots, n\} \setminus (A_1 \cup A_2)$), we have

$$\lambda_1 < \mu_{ij}, \quad \lambda_2 < \mu_{2j}.$$

Let us note that the stationary probabilities of the associated Markov chain (irreducible and aperiodic) are given by

$$\pi_i \overset{\text{def}}{=} \pi\left(\Lambda_i^{(2)}\right) = \tfrac{1}{2} p_{sc}\left(\Lambda^{(1)}, \Lambda_i^{(2)}\right), \tag{9.7}$$

where $\Lambda^{(1)}$ is α_1 for *1-outgoing* $\Lambda^{(2)}$ and α_2 for *2-outgoing* $\Lambda_i^{(2)}$. Using (2.2) we have, e.g. for *2-outgoing* $\Lambda^{(2)}$,

$$p_{sc}\left(\Lambda^{(1)}, \Lambda^{(2)}\right) = \frac{q_{01}^{\Lambda^{(1)}} \gamma_{\Lambda^{(2)}}}{\sum\limits_{\Lambda^{(2)}} q_{01}^{\Lambda^{(2)}} \gamma_{\Lambda^{(2)}}},$$

where $\gamma = \gamma_{\Lambda_i^{(2)}}$ is defined from eq. (2.4), which reduces to

$$\lambda_i(1-\gamma) + \mu_{1i}(1-\gamma)^{-1} - (\lambda_i + \mu_{1i}) = 0,$$

whence

$$\gamma = 1 - \frac{\mu_{1i}}{\lambda_i}.$$

Let us define the following real number

$$M_1 = \frac{\sum\limits_{j \in A_1} \log\left[\dfrac{|\lambda_2 - \mu_{2j}|}{|\lambda_1 - \mu_{1j}|}\right] \lambda_2 p_j^2 \left(1 - \dfrac{\mu_{2j}}{\lambda_2}\right)}{\sum\limits_{j \in A_1} \lambda_2 p_j^2 \left(1 - \dfrac{\mu_{2j}}{\lambda_2}\right)}.$$

In a similar way, we can define M_2. Our final result, for the example under consideration, is contained in the next

THEOREM 9.3

Assume in model 1 that (9.4) holds and both A_1 and A_2 are not void. Then the system is recurrent if

$$M_1 + M_2 < 0$$

and transient if

$$M_1 + M_2 > 0.$$

Let us consider a system with only two classes α_1 and α_2 and assume that there is only one 1-outgoing regime $j = 1$

$$\lambda_2 > \mu_{21}, \quad \lambda_1 < \mu_{11};$$

there are only two 2-outgoing regimes $j = 2, 3$

$$\lambda_2 < \mu_{22}, \quad \lambda_1 > \mu_{12}$$

and

$$\lambda_2 < \mu_{23}, \quad \lambda_1 > \mu_{13}.$$

For the other regimes $j = 4, \dots, N$, we have

$$\lambda_2 < \mu_{2j} \quad \text{and} \quad \lambda_1 < \mu_{1j}.$$

Let us note that the matrix (3.5) in our case is given by

$$A_{11} = A_{22} = A_{33} = A_{23} = A_{32} = 0,$$

and

$$A_{21} = \sqrt{\mathrm{tg}\ \phi_{\Lambda_2^{(2)}}\ \mathrm{tg}\ \phi_{\Lambda_1^{(2)}}},$$

$$A_{31} = \sqrt{\mathrm{tg}\ \phi_{\Lambda_3^{(2)}}\ \mathrm{tg}\ \phi^{\Lambda_1^{(2)}}},$$

$$A_{12} = \sqrt{\mathrm{tg}\ \phi_{\Lambda_1^{(2)}}\ \mathrm{tg}\ \phi_{\Lambda_2^{(2)}}} \cdot p_2^2,$$

$$A_{13} = \sqrt{\mathrm{tg}\ \phi_{\Lambda_1^{(2)}}\ \mathrm{tg}\ \phi_{\Lambda_3^{(2)}}} \cdot p_3^2.$$

One can show that the maximal eigenvalue of this matrix is given by

$$\lambda_1^2 = p_2^2\ \mathrm{tg}\ \phi_{\Lambda_1^{(2)}}\ \mathrm{tg}\ \phi_{\Lambda_2^{(2)}} + p_3^2\ \mathrm{tg}\ \phi_{\Lambda_1^{(2)}}\ \mathrm{tg}\ \phi_{\Lambda_3^{(2)}}.$$

And so we have

THEOREM 9.4

Assume in model 1 that (9.4) holds and $A_1 = \{1\}$, $A_2 = \{2, 3\}$. Then the system is ergodic if

$$L = \tfrac{1}{2}\log\left[p_2^2\ \mathrm{tg}\ \phi_{\Lambda_1^{(2)}}\ \mathrm{tg}\ \phi_{\Lambda_2^{(2)}} + p_3^2\ \mathrm{tg}\ \phi_{\Lambda_1^{(2)}}\ \mathrm{tg}\ \phi_{\Lambda_3^{(2)}} \right] < 0$$

and non-ergodic if

$L > 0.$

Let us note that under the conditions of theorem 9.4 it is not necessary to calculate the maximal eigenvalue of the matrix (3.5). It is sufficient to note that for any $n \in \mathbb{Z}_+$ in (3.3) we have

$$E\left(\prod_{j=1}^{2n} \operatorname{tg} \phi_{\Lambda_j} \mid \Lambda_0 = \Lambda_1^{(2)}\right) = \left[E\left(\operatorname{tg} \phi_{\Lambda_1} \operatorname{tg} \phi_{\Lambda_2} \mid \Lambda_0 = \Lambda_1^{(2)}\right)\right]^n.$$

10. Remarks and problems

Remark 1

In our case, we could have defined an associated Markov chain with a different set of states: $\{\Lambda^{(1)}\}$ instead of $\{\Lambda^{(2)}\}$ with transition probabilities

$$p(\Lambda^{(1)}, \Lambda_1^{(1)}) = p(\Lambda^{(2)}, \Lambda_1^{(2)}),$$

for any $\Lambda^{(2)} \in S_+(\Lambda^{(1)})$ and the unique $\Lambda_1^{(2)} \in S_+(\Lambda_1^{(1)}) \cap S_-(\Lambda^{(1)})$, if such $\Lambda_1^{(2)}$ exists and 0 otherwise. In the more general case when condition (1a) of boundedness of jumps does not take place, the situation becomes more complicated: the scattering probabilities depend on the ingoing bristle of the hedgehog. Thus, the probabilities $p(\Lambda^{(2)}, \Lambda_1^{(2)})$ and $p(\Lambda_2^{(2)}, \Lambda_1^{(2)})$ can be different for different $\Lambda^{(2)}$, $\Lambda_2^{(2)} \in S_+(\Lambda^{(1)})$. But this case can be treated as well by the same methods.

Remark 2

It is of interest to generalize our results to the case when the simplexes of our complex are not $\mathbb{Z}_+^2$, but angles in $\mathbb{R}_+^2$ or in $\mathbb{Z}_+^{(2)}$. Of special interest is the situation when these angles in $\mathbb{Z}_+^2$ are not commensurable with π.

Remark 3

Our methods allow to get the classification of random walks in $\mathbb{Z}_+^N$ under the same "non-zero" and homogeneity assumptions, when all vectors of mean jumps inside all *faces* Λ with $dim\Lambda > 3$ have their coordinates negative. Then M_Λ, with $dim\Lambda = 1$ or 2, will become vectors derived from the corresponding induced chains.

Remark 4

We want to show now that all our assumptions have Lebesgue measure 1 in the parameter space. 0_1, 0_3 are fulfilled when all the $p_{\alpha\beta}$'s are positive. 0_2 is satisfied except for a finite number of hyperplanes. 0_4, 0_5 are not fulfilled only when $v_{\Lambda^{(1)}}$ in lemma 1.3 is equal to zero.

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