Mathematics for some classes of networks

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Network (as a general notion) is not a mathematical object - there is no even any definition. However, there is a lot of good rigorous mathematics for well-defined classes of networks. In sections 1-3 we give a short overview of classes of networks which interested the authors for some time. In section 4 we consider in detail a new class of networks, related to markets with many agents.

1 Random field dynamics on a fixed graph

The basic element of most networks is a graph G with the set $V = V(G)$ of vertices and the set $L = L(G)$ of links (lines, edges). Second basic element is a function $s = f(v) : V \to S$ with values in some space S The elements of S may be called marks, spins, field values, queues etc. The function f is subjected to random dynamics.

Simplest example is (an earlier stuff) random walks on graph, where $f = 0$ everywhere except one point where the particle is situated. This is related to electric networks, see for example [6].

In general there are two different situations. First one is a local continuous time Markov dynamics given by infinitesimal transitions. Classical reference is [9], mostly such processes model stochastic dynamics of particles or spins. The latter are related to Gibbs random fields (invariant measures for this dynamics) on graphs, see [10] and references therein.

Queuing, communication and transportation networks The simplest case is when the particles jump (from one node to another) freely without seeing each other, the only interaction is only through queues at the nodes, where they spend some time. There are two main theories concerning such class of networks:

1. Most popular - Jackson network (1963) and its generalizations (Gordon-Newel, BCMP). This theory gives explicit formulas for the stationary distribution and is the origin of many other analytical results. One of the applications is to describe jams and phase transitions in communication [11] and transportation [12] networks.

2. Stability theory (1968-1995) exhibits in many cases of large time qualitative behavior. If the walking clients are identical then it is described by random walks in orthants and strongly uses It uses Lyapounov functions, Euler scaling (fluid approximation), ergodic theory of dynamical systems and Lyapounov exponents, see [13]. If the walking clients can be of finite number of types then the corresponding theory [4] is the union of the one type case and the theory of random grammars (see below).

For more sophisticated restrictions - network protocols (TCP etc.) - there are many partial results but no comparable (deeply elaborated) mathematical theories.

Chemical kinetics - mean field Markov chain

Mean field network means that there is no specified local structure on the graph. Example of such theory is the chemical kinetics. It describes the following situation. Molecular types are indexed by V, n_v - number of molecules of type v

$$n_1 + ... + n_{|V|} = N$$

There are also reaction types $r = 1, 2, ..., R$, formally - multigraph defined by finite number of equations

$$\sum_v s_{vr} M_v = 0$$

where M_v - molecule of type v, s_{vr} - stoihiometric coefficients of molecule type v in reaction of type r , negative for substrates, positive for products. Reaction rates (continuous time Markov chain) are given by

$$\lambda_r = A_r \prod_{v: s_{vr} < 0} n^{-s_{vr}}$$

for the jump (transition)

$$n_v \to n_v + s_{vr}, v \in V$$

To get ODE of **classical chemical kinetics**

$$\frac{dc_v}{dt} = \sum_r Q_{vr}(c_1, ..., c_{|V|})$$

for some polynomials Q_r, in the limit $N \to \infty$

$$c_v(t) = \lim \frac{n_v^{(N)}(t)}{N}$$

one uses canonical scaling of reaction rates

$$A_r = a_r N^{s_r + 1}, s_r = \sum_{v: s_{vr} < 0} s_{vr}$$

To deduce **chemical thermodynamics** is more difficult [14], one should, together with molecular types v, introduce more degrees of freedom: kinetic energy $T_{v,i}$ and internal energy $K_{v.i}$ of i-th molecule of type v. Also, one should define more complicated mean field dynamics - introduce energy mechanism in reactions. As there is kinetic energy - there should be Newtonian movement, and the dynamics become mixed: **local + mean field**. Molecule move freely (as in ideal gas) but kinetic energies randomly interchange with internal energies.

Network homeostasis [15] Network is defined by

1. large graph G of compartments, this graph G has metrics and the boundary,
2. in any compartment chemical kinetics is defined, that is there are molecules with chemical reactions,
3. there is transport of molecules between compartments
4. there is input and output of molecules on the boundary

Under some conditions (the main is that reactions are unary) it is possible to prove that **far from the boundary there is equilibrium** - concentrations almost do not change with change of input.

2 Dynamics of graphs and of marked graph

Earlier the science of random graphs considered mainly the properties of graphs with fixed number of vertices and/or random number (for example Bernoulli) of edges, see for example [6, 8, 18, 19]. The simplest dynamics (appending edge by edge) appeared already in [7], see also [3]. What more general dynamics on graphs one should study? First of all, it is more reasonable to consider evolution of marked graphs. Most general dynamics of marked graphs (local random dynamics of a graph. jointly with a field on it) is called random graph grammars [1, 2, 5] and [21, 22]. It appears to be quite natural in connection with the emerging new physical theories [16, 17, 25] and social networks [20]. Namely, if eventually the local space-time appears to be discrete, then the most natural language for it is a graph with some physical fields on it. The dynamics of the space time is local. The example is the following.

Macrodimension of a graph - invariant of local dynamics We consider infinite (countable) graphs G. Let $O_n(v)$ be the neighborhood of vertex v of radius n. Put

$$D_n(v) = \frac{\ln|O_n(v)|}{\ln n}, \overline{D}(v) = \lim\sup_{n\to\infty} D_n(v), \underline{D}(v) = \lim\inf_{n\to\infty} D_n(v)$$

If for all v

$$\overline{D}(v) = \underline{D}(v) = D_S$$

then D_S is called **scaling macrodimension** of graph G. For example any homogeneous lattice in euclidean space R^d has scaling macrodimension $D_S = d$. Note that there are many other definitions of variants of macrodimension: connectivity, Hausdorf, entropy, inductive macrodimension.

Denote $\mathbf{G}_M$ the class of connected graphs where each vertex has degree $\leq M$. Let U - **any local dynamics** (graph grammar).

There is the following result [16]. If for some sufficiently large M U leaves the **class** $\mathbf{G}_M$ **invariant** and the corresponding Markov chain is **locally reversible** then the scaling macrodimension is an invariant.

Local reversibility means that Kolmogorov cycle criteria relations

$$a_{i_1i_2}...a_{i_Li_1} = 1, a_{ij} = \frac{\lambda_{ij}}{\lambda_{ji}}$$

follow from such relations of bounded length.

Random graph grammars Consider words $\alpha = x_1...x_N$ (ordered sequences of symbols), where x_N belongs to some finite alphabet A. Grammar is defined by the list *Sub* of productions (allowed substitution types)

$$S_j: \ \alpha_j \to \beta_j, j = 1, ..., S$$

Random grammar includes also positive numbers λ_j (rates). That is at time interval $(t, t+dt)$ in the word $\alpha(t)$ any subword α_j is independently replaced by β_j with probability $\lambda_j dt$ (continuous time Markov chain).

For **graph grammar** $\alpha(t)$ are marked graphs, α_j, β_j are (small) connected marked graphs. Thus, α_j is deleted from the graph and β_j is pasted instead (some restrictions needed of course). Note that ordinary grammar is a particular case, corresponding to linear marked graphs.

One of the problems - invariant measure and conserved characteristics with respect to given graph grammar dynamics was considered in [12, 21, 22].

3 Quantum Graph Grammar

What is quantum graph [26]. Consider Hilbert space $l_2(\{G\})$ with (orthonormal) basis e_G, enumerated by finite graphs G. Or by finite marked graphs if the set of marks is finite. First example is linear marked graphs - **quantum words**.

To define quantum dynamics assume that if $a_j = (\alpha_i \to \beta_i) \in Sub$ then also inverse substitution $a_j^* = (\beta_i \to \alpha_i) \in Sub$. Denote $S_j(k)$ the substitution S_j applied to subword of the word α starting on k-th symbol of the word α. Introduce the Hamiltonian

$$\sum_{j=1}^{|Sub|} \sum_{k=1}^{\infty} (\lambda_j a_j(k) + \lambda_j^* a_j^*(k))$$

The first simple result is: this Hamiltonian is selfadjoint in $l_2(\{G\})$, that is the quantum evolution is well-defined.

Gibbs and Quantum Spaces $\mathbf{G}$ - class of finite graphs with a function $f : V \to R$, called spin graphs (G, f), $\mathbf{G}_N$ - class of such spin graphs of radius $\leq D$. Potential is defined as some function $\Phi : \mathbf{G}_D \to R$. Hamiltonian $H : \mathbf{G} \to R$ is

$$H((G, f)) = \sum \Phi(\gamma)$$

where the sum over all sub spin subgraphs of (G, f). Partition function

$$Z_N = \sum_{(G,f) \in G_N} \exp(-\beta H(G, f))$$

Gibbs measure on $\mathbf{G}_{f,N}$

$$\mu_N(G, f) = Z_N^{-1} \exp(-\beta H(G, f))$$

There are many results-examples (by physicists and mathematicians) related to “quantum gravity”, see for example [17, 25] and references therein.

4 Trading network as Boltzmann mechanics of communicating vessels

Standard financial mathematics considers games of one or small number of players against the chance (random market). Recently, a new approach (called multi-agent models) appeared which considers the games of many players against each other. This theory is at the starting point and its models are mainly **mean-field** models.

In this section some **local** models are considered where there are many players and many financial or trading instruments. Our model develops simpler models of ([23, 24]). The model resembles communication and transportation networks - the main difference is that the nodes have special dynamical values (moving boundaries, or real prices). The clients have also their own subjective prices and their interaction (transaction) with the nodes depend on these prices. This model does not describe any real situation (and any other existing multi-agent model as well) but we hope that some features of this model will be useful for future more realistic models.

Free one-phase Boltzmann dynamics Consider the phase space $\mathbf{S} = I \times I_0$, where $I \subset R$ is an infinite interval and $I_0 = [-V_0, V_0]$, $0 < V_0 < \infty$. On $\mathbf{S}$ at any time $t \geq 0$ a random locally finite configuration $\{(x_i(t), v_i(t))\}$ of particles is given with coordinates $x_i \in I$ and velocities $v_i \in I_0$. Assume that this configuration at any time t has distribution P_t with one-particle correlation function $f(x, v, t)$ defined so that for any subset $A \subset \mathbf{S}$ of the phase space

$$E\#\{i : (x_i, v_i) \in A\} = \int_A f(x, v, t) dx dv$$

One can have in mind Poisson measure P_0 at time $t = 0$. Any particle moves always with its initial velocity, independently of other particles. Also there is Poisson income flow of particles from exterior with rate $\lambda(x,v,t)$, that is during time interval $[t,t+dt]$ the mean number of incoming particles to the cell $[x,x+dx]\times[v.v+dv]$ of the phase space is $\lambda(x,v,t)dxdvdt$. Assume moreover that each particle can die (disappear) with exponential distribution having rate $\mu(x,v,t)$. This means that during time dt $\mu(x,v,t)dxdvdt$ particles leave the cell $dxdv$.

Remind that we assume boundedness of velocities, that is

$$f(x,v,t) = \lambda(x,v,t) = \mu(x,v,t) = 0, |v| \geq V_0$$

Lemma 1 *For any $x \in I$ and $t < \frac{d(x,\partial I)}{V_0}$, where $d(x,\partial I)$ is the distance of the point x from the boundary of I, the standard linear Boltzmann equation holds*

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = -\mu(x,v,t)f(x,v,t) + \lambda(x,v,t) \tag{1}$$

This is trivial for $\mu = \lambda = 0$. In fact, for small $\delta > 0$ we have

$$f(x,v,t+\delta) = f(x - v\delta, v,t) \tag{2}$$

if x is not on the boundary of I and δ is sufficiently small. Subtracting $f(x,v,t)$ from both parts of this equality, dividing by δ and taking the limit $\delta \to 0$, we have

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0 \tag{3}$$

The unique solution of the Cauchy problem for (3) is

$$f(x,v,t) = f(x - vt, v, 0)$$

If there $\lambda \neq 0, \mu = \mu(x,v) \neq 0$ then it is also easy to see that the equation (1) holds. Note that if $\lambda = 0$ and μ does not depend on t, there is also explicit solution, see section XI.12 in [27]

$$f(x,v,t) = f(x - vt, v, 0)\exp\left(\int_0^t \mu(x - vs, v)ds\right)$$

Two phases — particle dynamics We shall define two types of dynamics — particle dynamics and continuum media dynamics.

In the particle dynamics $(\pm)$-phases consist of $(\pm)$-particles so that each $(-)$-particle is to the left of any $(+)$-particle. Denote $b(t) \in R$ (boundary between phases) the coordinate of the leftmost $(+)$-particle. Then for $x \geq b(t)$ there is $(+)$-phase and for $x < b(t)$ there is $(-)$-phase. Particles move, as above, with their own velocities until a $(-)$-minus particle reaches the point $b(t)$, then it disappears together with the $(+)$-particle at $b(t)$ and the point $b(t)$ jumps to the coordinate of the new leftmost $(+)$-particle. After this, the process proceeds similarly.

Random configurations of particles are defined by the correlation functions $f_\pm(x,v,t)$ correspondingly. Assume that also the functions $\lambda_\pm(r,t), \mu_\pm(r,t), r \geq 0$, are defined, smooth on R_+ and zero if $r \geq R_0$ for some $0 < R_0 < \infty$.

The dynamics of one point correlation functions $f_\pm(x,v,t)$ for $x \neq b(t)$, that is on $(b(t),\infty)$ and $(-\infty, b(t))$ correspondingly, is given by the equations (already non-linear as $b(t)$ is unknown)

$$\frac{\partial f_\pm}{\partial t} + v\frac{\partial f_\pm}{\partial x} = -\mu_\pm(x - b(t),v,t)f_\pm(x,v,t) + \lambda_\pm(x - b(t),v,t) \tag{4}$$

This means that we assume that arrivals and departures depend only on the distance $r = |x-b(t)|$.

Thus two phases add reactions between particles of different phases. The following interpretation is useful. We consider one instrument (**stocks, futures, houses or other real estate** etc.). There are two types of traders - $(+)$-particles correspond to **sellers** and $(-)$-particles to **buyers,** x_i are subjective prices comfortable for the trader i. Collision between particles corresponds to transaction, after this both leave the market. In more general cases it will be possible that they do nor leave the market (see below).

We consider here a particular case when for some constant velocities $v_{\pm}$ and for any t

$$f_{\pm}(x,v,t) = \rho_{\pm}(x,t)\delta(v-v_{\pm})$$

For this to hold at any time t it is sufficient to demand that this holds for $t=0$. *Initial conditions* are defined by the initial densities $\rho_{\pm}(r,0)$. The velocities $v_{\pm}$ can be interpreted as averaged velocities for sellers and buyers correspondingly.

Two phases — fluid dynamics It can occur that under some scaling the defined particle dynamics tends to some kind of **continuous (fluid) picture,** see [23], but we shall not pursue this way here. Instead, we consider continuous densities of $(+)$-masses and $(-)$-masses and shall define their dynamics directly. We assume that at each time t there exists point $b(t)$ - boundary between phases. There are two phases with initial densities $\rho_{+}(r,0), \rho_{-}(r,0)$ where

$$r = r(t) = |x-b(t)| = \pm(x-b(t))$$

correspondingly. Phases move with velocities $v_{\pm}$correspondingly. Collision of plus and minus masses (at the point $b(t)$) leads to their cancellation in equal amount. There is more realistic possibility - to make the cancellation proportional to the current price, but we do not consider this possibility here.

We obtain equations for the triple $(b(t), \rho_{+}(r,t), \rho_{-}(r,t))$ similarly to the way how the equations of continuum mechanuics are derived in the textbooks, that is using conservation laws. Here there is only one - mass conservation law.

First of all, obtain the equation for the boundary. Assume $b(t)$ smooth and put $\beta = \frac{db(t)}{dt}$. Then for time dt the amount of positive mass, reaching the boundary will be

$$M_{+}(\beta,t)dt = \int_{v-\beta<0} \int_{r<(-v+\beta)dt} f_{+}(r,v,t)dv + o(dt) =$$
$$dt \int_{v-\beta<0} f_{+}(0,v,t)(-v+\beta)dv + o(dt)$$

In fact, income and outcome give the contribution $o(dt)$. Similarly for negative mass

$$M_{-}(\beta,t)dt = \int_{v-\beta>0} \int_{r<(v-\beta)dt} f_{-}(r,v,t)dv + o(dt) =$$
$$dt \int_{v-\beta>0} f_{-}(0,v,t)(v-\beta)dv + o(dt)$$

Lemma 2 *For any t there exists unique $\beta = \beta(t)$ such that*

$$M_{+}(\beta,t) = M_{-}(\beta,t) \tag{5}$$

In fact, consider the equation with respect to β

$$\int_{v-\beta<0} f_+(0,v,t)(-v+\beta)dv = \int_{v-\beta>0} f_-(0,v,t)(v-\beta)dv$$

Then if β increases, then the right-hand side increases and the left-hand side decreases.

We can rewrite the equation (5) in our case

$$\rho_+(0,t)(-v_+ + \beta(t)) = \rho_-(0,t)(v_- - \beta(t)) \tag{6}$$

from where we can get $\beta(t)$

$$\beta(t) = \frac{\rho_+(0,t)v_+ + \rho_-(0,t)v_-}{\rho_+(0,t)+\rho_-(0,t)} \tag{7}$$

Now we should write the equations for the densities. For $\rho_+(r,t)$ we get

$$\rho_+(r,t+\Delta t) = \rho_+(r-(v_+ - \beta(t))\Delta t, t) - \mu_+(r,t)\rho_+(r,t)\Delta t + \lambda_+(r,t)\Delta t + o(\Delta t) =$$

$$= \rho_+(r,t) - (v_+ - \beta(t))\frac{\partial \rho_+(r,t)}{\partial r}\Delta t - \mu_+(r,t)\rho_+(r,t)\Delta t + \lambda_+(r,t)\Delta t + o(\Delta t)$$

In the limit $\Delta t \to 0$

$$\frac{\partial \rho_+(r,t)}{\partial t} = -(v_+ - \beta(t))\frac{\partial \rho_+(r,t)}{\partial r} - \mu_+(r,t)\rho_+(r,t) + \lambda_+(r,t) \tag{8}$$

Similarly $\rho_-(r,t)$:

$$\rho_-(r,t+\Delta t) = \rho_-(r+(v_- - \beta(t))\Delta t, t) - \mu_-(r,t)\rho_+(r,t)\Delta t + \lambda_-(r,t)\Delta t + o(\Delta t)$$

$$\frac{\partial \rho_-(r,t)}{\partial t} = (v_- - \beta(t))\frac{\partial \rho_-(r,t)}{\partial r} - \mu_-(r,t)\rho_-(r,t) + \lambda_-(r,t) \tag{9}$$

It would be nice to prove accurately that the solution of equations (6,8,9) exists for any $t \geq 0$ and is unique, but we did not try to do this.

Fixed points and stationary points Assume that the functions $\lambda_\pm(r) = \lambda_\pm(r,t)$ and $\mu_\pm(r) = \mu_\pm(r,t)$ do not depend on t (remind that they were assumed to have compact support). Denote

$$\gamma_{cr}^{(+)} = -v_+^{-1}\int_0^\infty \lambda_+(x)\exp\left(\frac{1}{v_+}\int_0^x \mu_+(y)dy\right)dx,\ \gamma_{cr}^{(-)} =$$
$$= v_-^{-1}\int_0^\infty \lambda_-(x)\exp\left(-\frac{1}{v_-}\int_0^x \mu_-(y)dy\right)dx$$

and

$$\gamma_{cr} = \max\left(\gamma_{cr}^{(+)}, \frac{v_-\gamma_{cr}^{(-)}}{-v_+}\right)$$

We define the fixed point of our dynamics by the conditions: $\beta(t) = 0$ and $\rho_\pm(r,t)$ do not depend on time. Alternatively the fixed points are defined as any solutions of the stationary version

$$\rho_+(0)v_+ + \rho_-(0)v_- = 0 \tag{10}$$

$$-v_+ \frac{\partial \rho_+(r)}{\partial r} - \mu_+(r)\rho_+(r) + \lambda_+(r) = 0 \quad (11)$$

$$v_- \frac{\partial \rho_-(r)}{\partial r} - \mu_-(r)\rho_-(r) + \lambda_-(r) = 0 \quad (12)$$

of the system (6,8,9). We will prove that there exists a family of fixed points depending on a real parameter.

Similarly, we call stationary point any solution of the system of equations (6,8,9), where $\beta = \beta(t)$ and the densities do not depend on t. We shall prove that there is a family of stationary points depending on two real parameters.

We say that a fixed (or stationary) point has finite mass if

$$\int_0^\infty \rho_\pm(r)dr < \infty$$

Theorem 3 *Let the parameters $\lambda_\pm(r), \mu_\pm(r)$ and $v_\pm$ be fixed. Then*

1. *For any value of the parameter $\gamma_+ = \rho_+(0)$ there is at most one fixed point. For $\gamma_+ < \gamma_{cr}$ there is no any fixed point. For $\gamma_+ \geq \gamma_{cr}$ there exists exactly one fixed point defined by*

$$\rho_+(r) = e^{-\frac{1}{v_+}\int_0^r \mu_+(x)dx} \left(\rho_+(0) + v_+^{-1} \int_0^r \lambda_+(x) e^{\frac{1}{v_+}\int_0^x \mu_+(y)dy} dx \right) \quad (13)$$

$$\rho_-(r) = v_-^{-1} e^{\frac{1}{v_-}\int_0^r \mu_-(x)dx} \left(-v_+\rho_+(0) - \int_0^r \lambda_-(x) e^{-\frac{1}{v_-}\int_0^x \mu_-(y)dy} dx \right) \quad (14)$$

2. *The fixed point has finite mass if $\gamma_{cr}^{(+)} = \gamma_{cr}^{(-)}$*

3. *For any γ_+, γ_- such that*

$$\gamma_+ = \rho_+(0) \geq \gamma_{cr}^{(+)},\ \gamma_- = \rho_-(0) \geq \gamma_{cr}^{(-)}$$

there is exactly one fixed point. Then the densities are defined by formulas (13,14) and the boundary velocity is

$$\beta = \frac{\rho_+(0)v_+ + \rho_-(0)v_-}{\rho_+(0) + \rho_-(0)}$$

4. *Stationary point has finite mass iff $\gamma_+ = \gamma_{cr}^{(+)}$, $\gamma_- = \gamma_{cr}^{(-)}$.*

Proof. Solving equations (8,9) we get for any $r > 0$ equations (13) and (14). Note that, by equations (14) and (13), densities $\rho_-(r)$, $\rho_+(r)$ are positive iff $\gamma_+ \geq \gamma_{cr}^{(+)}$, $\gamma_- \geq \gamma_{cr}^{(-)}$. Taking into account equation (10) we get the first asserion of the theorem.

For the stationary points the densities are again defined by equations (13) and (14). We have two conditions for them to be non-negative. Then the boundary will move with constant velocity defined from equation (7).

More complicated one market model Note that collision of masses of two phases create total annihilation flow

$$\nu(t) = (v_- - \beta(t))\rho_-(0,t) = -(v_+ - \beta(t))\rho_+(0,t)$$

of the disappearing ($\pm$)-particles. Here we assume that a part of annihilating particles does not disappear but can transform to particles of the other phase jumping from the collision point 0 to some point r. On the language of continuous media this means that there are output flows of mass $\nu(+,-,r,t)$ and $\nu(-,+,r,t)$ such that

$$\int_0^\infty p(+,-,r,t)dr \le 1, \int_0^\infty p(-,+,r,t)dr \le 1$$

where

$$p(+,-,r,t) = \frac{\nu(+,-,r,t)}{\nu(t)}, p(-,+,r,t) = \frac{\nu(-,+,r,t)}{\nu(t)}$$

For such model we have the system of three equations

$$\begin{aligned} \beta(t) =& \frac{v_-\rho_-(0,t)+v_+\rho_+(0,t)}{\rho_-(0,t)+\rho_+(0,t)} \\ \frac{\partial\rho_+(r,t)}{\partial t} =& -(v_+-\beta(t))\frac{\partial\rho_+(r,t)}{\partial r} - \mu_+(r,t)\rho_+(r,t) + \lambda_+(r,t) + \\ &+ (v_- - \beta(t))\rho_-(0,t)p(-,+,r,t) \\ \frac{\partial\rho_-(r,t)}{\partial t} =& (v_- - \beta(t))\frac{\partial\rho_-(r,t)}{\partial r} - \mu_-(r,t)\rho_-(r,t) + \lambda_-(r,t) - \\ &- (v_+ - \beta(t))\rho_+(0,t)p(+,-,r,t) \end{aligned} \tag{15}$$

We again assume that the functions $\mu_\pm(r,t), \lambda_\pm(r,t), p(-,+,r,t), p(+,-,r,t)$ do not depend on t and have compact support. Introduce the functions

$$F_+(x) = -\frac{1}{v_+}\int_0^x \mu_+(y)dy,\ F_-(x) = \frac{1}{v_-}\int_0^x \mu_-(y)dy$$

Denote

$$\alpha_{-+} = \int_0^\infty p(-,+,x)\exp\left(-F_+(x)\right)dx,\ \alpha_{+-} = \int_0^\infty p(+,-,x)\exp\left(-F_-(x)\right)dx$$

and assume that α_{-+}, $\alpha_{+-} < 1$. Define

$$\hat{\gamma}_{cr} = \max\left(\frac{\gamma_{cr}^{(+)}}{1-\alpha_{-+}}, \frac{v_-\gamma_{cr}^{(-)}}{-v_+(1-\alpha_{+-})}\right)$$

Theorem 4 *Let the parameters $\lambda_\pm(r), \mu_\pm(r)$, $p(-,+,r,t), p(+,-,r,t)$ and $v_\pm$be given. Then*

1. *For any value of the parameter $\gamma_+ = \rho_+(0)$ there is at most one fixed point. For $\gamma_+ < \hat{\gamma}_{cr}$ there is no any fixed point. For $\gamma_+ \ge \hat{\gamma}_{cr}$ there exists exactly one fixed point. It is*

$$\rho_+(r) = e^{F_+(r)}\left(\rho_+(0) + v_+^{-1}\int_0^r \left(\lambda_+(x) - v_+\rho_+(0)p(-,+,x)\right)e^{-F_+(x)}dx\right)$$

$$\rho_-(r) = v_-^{-1}e^{F_-(r)}\left(-v_+\rho_+(0) - \int_0^r \left(\lambda_-(x) - v_+\rho_+(0)p(+,-,x)\right)e^{-F_-(x)}dx\right)$$

2. *There is a unique stationary point with finite mass. It is*

$$\rho_+(r)=e^{F_+(r)}\left(\rho_+(0)+v_+^{-1}\int_0^r(\lambda_+(x)+(v_--\beta)\rho_-(0)p(-,+,x))\,e^{-F_+(x)}dx\right)$$

$$\rho_-(r)=e^{F_-(r)}\left(\rho_-(0)-v_-^{-1}\int_0^r(\lambda_-(x)-(v_+-\beta)\rho_+(0)p(+,-,x))\,e^{-F_-(x)}dx\right)$$

and we denote

$$\rho_+(0)=\frac{-v_+\gamma_{cr}^{(+)}}{-v_+(1-\alpha_{-+})-\beta\alpha_{-+}}$$

$$\rho_-(0)=\frac{v_-\gamma_{cr}^{(-)}}{v_-(1-\alpha_{+-})+\beta\alpha_{+-}}$$

where β is a root (belonging to the interval (v_+,v_-)) of quadratic equation (23). It exists and is unique.

Proof. 1. Similarly to the first part of theorem 3.
2. As follows from system (15) the equations for the stationary points are

$$\begin{aligned}0&=(v_--\beta)\rho_-(0)+(v_+-\beta)\rho_+(0)\\0&=-v_+\frac{\partial\rho_+(r)}{\partial r}-\mu_+(r)\rho_+(r)+\lambda_+(r)+(v_--\beta)\rho_-(0)p(-,+,r)\\0&=v_-\frac{\partial\rho_-(r)}{\partial r}-\mu_-(r)\rho_-(r)+\lambda_-(r)-(v_+-\beta)\rho_+(0)p(+,-,r)\end{aligned}\tag{16}$$

Solving these linear first order equations we get

$$\rho_+(r)=e^{F_+(r)}\left(\rho_+(0)+v_+^{-1}\int_0^r(\lambda_+(x)+(v_--\beta)\rho_-(0)p(-,+,x))\,e^{-F_+(x)}dx\right)\tag{17}$$

$$\rho_-(r)=e^{F_-(r)}\left(\rho_-(0)-v_-^{-1}\int_0^r(\lambda_-(x)-(v_+-\beta)\rho_+(0)p(+,-,x))\,e^{-F_-(x)}dx\right)\tag{18}$$

We are looking for a stationary point with finite mass such that

$$\int_0^\infty\rho_\pm(r)dr<\infty\tag{19}$$

Then by (17), (18), (19), (16), a stationary point is uniquely defined by three parameters $\gamma_\pm=\rho_\pm(0)$, β which satisfy the following equations

$$\begin{aligned}-v_+\gamma_+&=-v_+\gamma_{cr}^{(+)}+(v_--\beta)\gamma_-\alpha_{-+}\\v_-\gamma_-&=v_-\gamma_{cr}^{(-)}-(v_+-\beta)\gamma_+\alpha_{+-}\\(v_--\beta)\gamma_-&=-(v_+-\beta)\gamma_+\end{aligned}\tag{20}$$

where $\beta\in(v_+,v_-)$. We show that this system has a unique solution. Using the third equation of the system, we get from the first two

$$\gamma_+=\frac{-v_+\gamma_{cr}^{(+)}}{-v_+(1-\alpha_{-+})-\beta\alpha_{-+}}\tag{21}$$

$$\gamma_- = \frac{v_- \gamma_{cr}^{(-)}}{v_-(1-\alpha_{+-}) + \beta\alpha_{+-}} \tag{22}$$

Substituting these expressions to the third one we come to the quadratic equation with respect to β:

$$(\sigma_+\alpha_{+-} - \sigma_-\alpha_{-+})(-v_+ + \beta)(v_- - \beta) + (\sigma_- v_+ - \sigma_+ v_-)\beta - v_+ v_-(\sigma_- - \sigma_+) = 0 \tag{23}$$

where, for shortness, we denote $\sigma_+ = -v_+\gamma_{cr}^{(+)}$, $\sigma_- = v_-\gamma_{cr}^{(-)}$.

Consider first the case when $\sigma_+\alpha_{+-} - \sigma_-\alpha_{-+} \neq 0$. Note that the boundary velocity should satisfy $v_+ < \beta < v_-$. One can show easily that there is always one root of the equation in the interval $v_+ < \beta < v_-$. Now one should verify that γ_+, γ_-, defined by (21) and (22) are non-negative. By (21) (22) one of the values γ_+, γ_- is always positive. Then by the third equation of the system (20) also the other value is positive as $v_- - \beta, -v_+ + \beta > 0$. Thus there exists the unique fixed point satisfying (17), (22), (21) (22).

Is $\sigma_+\alpha_{+-} - \sigma_-\alpha_{-+} = 0$, we have a linear equation with respect to β, we gives

$$\beta = \frac{\sigma_- - \sigma_+}{\sigma_- v_-^{-1} - \sigma_+ v_+^{-1}}$$

and from (21) (22) we get $\gamma_+ = \sigma_+ v_+^{-1}$ and $\gamma_- = \sigma_- v_-^{-1}$. In this case also a stationary point exists and is unique.

Networks with many markets Let us call the previous model an elementary market. A network is a set V of elementary markets with similar parameters and variables indexed by $m \in V$

$$v_{\pm,m}, \lambda_{\pm,m}(r,t), \mu_{\pm,m}(r,t), \rho_{\pm,m}(r,t), b_m(t), \beta_m(t)$$

There are also other parameters interconnecting the markets. Denote $\nu_{+,m}(t)$ ($\nu_{-,m}(t)$) the total annihilation flow of $(\pm)$-particles from the market m. As they are equal we denote $\nu_m(t) = \nu_{+,m}(t) = \nu_{-,m}(t)$. Let

$$\nu_{k,m}(+,+,r,t), \nu_{k,m}(+,-,r,t), \nu_{k,m}(-,+,r,t), \nu_{k,m}(-,+,r,t)$$

be the parts of these annihilation flows of $(\pm)$-particles, that after the transaction on the market m, become $(\mp)$-particles on the market k with the coordinate r. Denote

$$p_{km}(\pm,\pm,r,t) = \frac{\nu_{k,m}(\pm,\pm,r,t)}{\nu_m(t)}$$

We mean that $p_{km}(+,+,r,t) = p_{km}(-,-,r,t) \equiv 0$. Then for any k and t the conditions

$$\sum_{m\in V}\int_0^\infty p_{km}(+,-,r,t)dr \leq 1, \sum_{m\in V}\int_0^\infty (p_{km}(-,+,r,t)dr \leq 1$$

should hold. Denote by $|V|$ the cardinality of the set V. We have then the following system of $3|V|$ equations:

$$\begin{aligned}
\beta_m(t) =& \frac{v_{-,m}\rho_{-,m}(0,t) + v_{+,m}\rho_{+,m}(0,t)}{\rho_{+,m}(0,t) + \rho_{+,m}(0,t)} \\
\frac{\partial \rho_{+,m}(r,t)}{\partial t} =& -(v_{+,m} - \beta_m(t))\frac{\partial \rho_{+,m}(r,t)}{\partial r} - \mu_{+,m}(r,t)\rho_{+,m}(r,t) + \lambda_{+,m}(r,t) \\
&+ \sum_{k\in V}(v_{-,k} - \beta_k(t))\rho_{-,k}(0,t)p_{km}(-,+,r,t) \\
\frac{\partial \rho_{-,m}(r,t)}{\partial t} =& (v_{-,m} - \beta_m(t))\frac{\partial \rho_{-,m}(r,t)}{\partial r} - \mu_{-,m}(r,t)\rho_{-,m}(r,t) + \lambda_{-,m}(r,t) \\
&- \sum_{k\in V}(v_{+,k} - \beta_k(t))\rho_{+,k}(0,t)p_{km}(+,-,r,t)
\end{aligned}$$

Fixed points Again we assume $\lambda_{\pm,m}(r,t), \mu_{\pm,m}(r,t), p_{km}(\pm,\pm,r,t)$ do not depend on t and have a compact support. Put

$$F_+^{(m)}(x) = -v_{+,m}^{-1}\int_0^x \mu_{+,m}(y)dy,\ F_-^{(m)}(x) = v_{-,m}^{-1}\int_0^x \mu_{-,m}(y)dy$$

$$\hat{\lambda}_{+,m} = \int_0^\infty \lambda_{+,m}(x)e^{-F_+^{(m)}(x)}dx,\ \hat{\lambda}_{-,m} = \int_0^\infty \lambda_{-,m}(x)e^{-F_-^{(m)}(x)}dx \tag{24}$$

$$\alpha_{km}(-,+) = \int_0^\infty p_{km}(-,+,x)e^{-F_+^{(m)}(x)}dx,\ \alpha_{km}(+,-) = \int_0^\infty p_{km}(+,-,x)e^{-F_-^{(m)}(x)}dx$$

for $k,m \in V$.

Define matrices A_{-+}, A_{+-} with elements $\alpha_{km}(-,+)$ $\alpha_{km}(+,-)$, where $k,m \in V$, and assume, that they have the following property:

$$\forall k \sum_{m\in V}\alpha_{km}(\pm,\pm) \le 1,\ \exists k_0 \sum_{m\in V}\alpha_{km}(\pm,\pm) < 1 \tag{25}$$

For two vectors $a = (a_i)$ and $b = (b_i)$ we shall write $a \ge b (a > b)$ if $a_i \ge b_i$ $(a_i > b_i)$ for all coordinates. Consider the following system of inequalities with respect $\bar{s}$

$$\bar{s}(E - A_{-+}) \ge \overline{\lambda}_+,\ \bar{s}(E - A_{+-}) \ge \overline{\lambda}_- \tag{26}$$

where E is the identity matrix and $\overline{\lambda}_\pm$ are vectors with coordinates $\hat{\lambda}_{\pm,m}$ defined by (24). We say that this system has a positive solution if there is vector $\bar{s}$ with positive coordinates satisfying both inequalities in (26). Generally, this system may not have a positive solution. If one of the matrices A_{-+}, A_{+-} is diagonal or zero the set of positive solutions is nonempty.

Theorem 5 *Each solution $\bar{s} = (s_m,\ m \in V) > 0$ of the system (26) uniquely defines the fixed point as follows:*

$$\rho_{+,m}(r) = -v_{+,m}^{-1}e^{F_+^{(m)}(r)}\left(s_m - \int_0^r \left(\lambda_{+,m}(x) + \sum_{k\in V}s_k p_{km}(-,+,x)\right)e^{-F_+^{(m)}(x)}dx\right)$$

$$\rho_{-,m}(r) = v_{-,m}^{-1}e^{F_-^{(m)}(r)}\left(s_m - \int_0^r \left(\lambda_{-,m}(x) + \sum_{k\in V}s_k p_{km}(+,-,x)\right)e^{-F_-^{(m)}(x)}dx\right)$$

If the set of positive solutions of system (26) is empty there is no any fixed point.

Proof. The fixed points satisfy the system consisting of $3|V|$ equation:

$$\begin{aligned}
0 &= v_{-,m}\rho_{-,m}(0,t) + v_{+,m}\rho_{+,m}(0,t) \\
0 &= -v_{+,m}\frac{\partial \rho_{+,m}(r)}{\partial r} - \mu_{+,m}(r)\rho_{+,m}(r) + \lambda_{+,m}(r) + \sum_{k\in V} v_{-,k}\rho_{-,k}(0)p_{km}(-,+,r) \qquad (27)\\
0 &= v_{-,m}\frac{\partial \rho_{-,m}(r)}{\partial r} - \mu_{-,m}(r)\rho_{-,m}(r) + \lambda_{-,m}(r) - \sum_{k\in V} v_{+,k}\rho_{+,k}(0)p_{km}(+,-,r)
\end{aligned}$$

Solving first order linear differential equations we get

$$\rho_{+,m}(r) = e^{F_+^{(m)}(r)}\left(\rho_{+,m}(0) + v_{+,m}^{-1}\int_0^r \left(\lambda_{+,m}(x)+\sum_{k\in V} v_{-,k}\rho_{-,k}(0)p_{km}(-,+,x)\right)e^{-F_+^{(m)}(x)}dx\right) \qquad (28)$$

$$\rho_{-,m}(r) = e^{F_-^{(m)}(r)}\left(\rho_{-,m}(0) - v_{-,m}^{-1}\int_0^r \left(\lambda_{-,m}(x)-\sum_{k\in V} v_{+,k}\rho_{+,k}(0)p_{km}(+,-,x)\right)e^{-F_-^{(m)}(x)}dx\right) \qquad (29)$$

for $m \in V$.

Using equations $0 = v_{-,m}\rho_{-,m}(0) + v_{+,m}\rho_{+,m}(0)$, we conclude that solutions (28), (29) are uniquely defined by parameters $s_m = -v_{+,m}\rho_{+,m}(0)$, $m \in V$, and one can write

$$\rho_{+,m}(r) = -v_{+,m}^{-1}e^{F_+^{(m)}(r)}\left(s_m - \int_0^r \left(\lambda_{+,m}(x) + \sum_{k\in V} s_k p_{km}(-,+,x)\right)e^{-F_+^{(m)}(x)}dx\right) \qquad (30)$$

$$\rho_{-,m}(r) = v_{-,m}^{-1}e^{F_-^{(m)}(r)}\left(s_m - \int_0^r \left(\lambda_{-,m}(x) + \sum_{k\in V} s_k p_{km}(+,-,x)\right)e^{-F_-^{(m)}(x)}dx\right) \qquad (31)$$

Whereas the densities (30), (31) are nonnegative for all $r \geq 0$ the following conditions must be satisfied

$$s_m \geq \hat{\lambda}_{+,m} + \sum_{k\in V} s_k \alpha_{km}(-,+) \qquad (32)$$

$$s_m \geq \hat{\lambda}_{-,m} + \sum_{k\in V} s_k \alpha_{km}(+,-) \qquad (33)$$

for all $m \in V$. These inequalities are equvalent to system (26).

So the fixed points exist iff there exist positive solutions of system (26).

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