Russian Math. Surveys **54**:4 685–728 ©1999 RAS(DoM) and LMS
Uspekhi Mat. Nauk **54**:4 3–46 UDC 519.2

Probability related to quantum gravity. Planar gravity

V. A. Malyshev

Abstract. The stochastic dynamics preserving an equilibrium distribution of quantum gravity is considered. This is the first detailed theoretical investigation of this dynamics (earlier it was used for Monte-Carlo simulation). The main result is related to the existence and certain properties of local correlation functions in the thermodynamic limit. At the same time, the paper can serve as a mathematical introduction to quantum gravity because we give a rigorous exposition of quantum gravity in the case of planar pure gravity. We mainly use the combinatorial approach instead of matrix models, which are more popular in physics, and the central point is the famous exponent $\alpha = -\frac{7}{2}$.

Contents

§1. Introduction

On the history. A few words should be said about the history of quantum gravity. The classical gravity deals with a smooth (not necessarily four-dimensional) manifold M, a pseudo-metric tensor g_{ij} on it, and the classical Einstein-Hilbert action

$$S = \int_M (\lambda R(x) + L(x) + \mu)\sqrt{\det g}\, dx$$

whose stationary points are studied. Here $R(x)$ stands for the intrinsic curvature at the point x, L is a functional of the matter fields $\phi(x)$, and μ is the cosmological constant. In the case of pure gravity (without matter) we have $L = 0$, and we treat

AMS 1991 *Mathematics Subject Classification.* Primary 83C45, 81T40; Secondary 60K35, 83C27, 05C30, 57M20, 60J15, 60J05, 82B10, 68Q50, 65C05.

this case only. Quantum gravity takes into account not only stationary points but also all other configurations, with some weights, that is, with a formal density

$$Z^{-1}\exp(-\mu S)$$

(which becomes positive under passage to the Euclidean metric) on some configuration space Ω of matter fields and metric tensors. All earlier attempts to eliminate the divergences led to the conclusion that Ω must also include the smooth structures on M and even the manifold M itself, that is, both the space and its topology must be random. Nowadays, the only reasonable way to pursue this program is to start from the consideration of discrete objects only and then to perform some scaling limits. In other words, the space becomes a finite complex, a smooth structure becomes a piecewise-linear structure, the metric and the curvature are coded by the one- and two-dimensional skeletons of the complex, and the matter fields pass into spins that live on cells of the complex. It turns out that such a quantization (discretization) is equally applicable to quite different physical systems such as relativistic particles, strings, and so on, but with a different interpretation. For instance, a string quantized in this way becomes a two-dimensional complex, and the spin at a particular vertex is a vector in $\mathbb{R}^d$ and thus provides a map of the vertices of the complex into $\mathbb{R}^d$ that approximates the classical string.

A discretization of classical gravity was considered first by Regge [1] who gave rigorous definitions in the discrete case for finite space-time, its curvature, and the Einstein-Hilbert action. His theory already appeared in the fundamental monograph [2], but in the 1970s it was still regarded as a deviation from the mainstream in physics, and few papers were devoted to it. However, among these papers was a well-known paper by Hawking [3] in which discrete quantum gravity was discussed.

In the 1980s more than 100 papers concerning discrete quantum gravity appeared.

In the 1990s, the number of papers has exceeded 1000 and is still increasing. This is due in particular to numerous relationships of discrete gravity both with other contemporary mathematical theories and with physical theories, including string theory, random matrices, and so on. Moreover, recent papers in theoretical physics often contain maxims of the following type: "Two-dimensional random geometry is now placed at the heart of many models of modern physics, from string theory and two-dimensional quantum gravity in an attempt to describe fundamental interactions, to membranes and interface fluctuations in various problems of condensed matter physics" (see [4]).

For a probabilist, quantum gravity is a source of inspiration and also of new mathematics and a new philosophy of probability. This paper can serve as a mathematical introduction to discrete quantum gravity for probabilists. However, it should be noted that the subject of quantum gravity is much broader; for various aspects of it, see the recent survey [5].

Dynamics contra equilibrium. We mainly use the combinatorial approach instead of that based on random matrix models (RMM), which is more popular in physics. However, the central point of our paper is the famous exponent $\alpha = -\frac{7}{2}$. Another objective of the paper is the study of the stochastic dynamics that preserves the equilibrium distribution treated in quantum gravity. Here we start a theoretical study of this dynamics (earlier it was used only for Monte-Carlo simulation;

the study of this dynamics can be regarded as the third part of the author's series of papers [6] and [7], but knowledge of these papers is not needed to read the text below). The point is that similar processes possess a universal character in probability, cover most examples of Markov processes, and have many analogues in computer science and biology.

By probability we mean here classical probability. The theory of quantum gravity now amounts to a large number of papers that have filled well-known physical journals during the last ten years. Discrete quantum gravity is regarded as a promising direction towards unifying the largest and smallest space-time scales in the present picture of nature. We restrict ourselves to a part of this field which uses probabilistic intuition, but in which it is difficult to find even rigorous formulations, not to mention proofs that could be satisfactory for a mathematician: even when the "probabilities" are (we hope) positive, they are not normalized. It seems that this is so not because of the authors' disdain for rigour but for more substantial reasons.

In the physical literature the techniques of algebraic geometry dominate the subject. Therefore, it is worth discussing also the simplest probabilistic aspects of quantum gravity: even the attempt gives rise to many natural questions.

Two approaches to discrete quantum gravity are known, namely, the quantum Regge calculus (where the lengths of edges are random variables) and dynamic triangulations (where the lengths of edges are constant). The term *dynamic* in the second approach can lead to confusion because in fact it has no dynamics, and the main apparatus uses the Gibbs equilibrium distributions on the set of large random matrices. For this reason, such an approach is called here an equilibrium approach.

The dynamics in quantum gravity arose in Monte Carlo simulations. The new point in this paper is in our analytic and probabilistic study of the relevant Markov processes (no recent rigorous results in this direction are known to the author). How can this study be useful? There are many reasons, and we present here an incomplete list.

- A well-known difficulty in averaging over all topologies is that, in dimension four, this generates algorithmically insoluble problems. The dynamics replaces such a problem by another one, namely, instead of averaging we seek a process (with arbitrary initial state) that generates a class of topologies. This process must have some symmetry, but at the same time it must be legitimate from the viewpoint of probability theory, and already this remark leads to interesting observations.
- It is desirable to have a Markov process that preserves a Gibbs measure, as is the case for stochastic quantization in ordinary quantum field theory and for the Glauber dynamics in statistical mechanics. This dynamics is useful (but not fundamental) in quantum field theory, which, however, has the Whiteman axioms, and in statistical mechanics, where the original deterministic Newton dynamics is regarded as a more fundamental source than the Gibbs measure itself. In quantum gravity, where both these factors are absent, one can unexpectedly take the viewpoint that the process itself is more fundamental than the Gibbs measure.
- The dynamics enables one to consider the domain under the critical point, in which the equilibrium distribution is meaningless. However, this domain becomes no less natural from the probabilistic viewpoint, like the expanding

universe (in "computer time", a term which the author learned from a paper by Migdal). Moreover, the dynamics gives meaning to quantities at the critical point without performing scaling limits. A true physical interpretation of this fact is not known to the author, but it is clear that this phenomenon is natural from the probabilistic viewpoint.

- We have no physical arguments for (or against) dynamic models, and only the probabilistic intuition can be a guiding thread. The main problems are as follows. What is universality, and what is the generic situation? It was recently suggested [8] that computer science can play a role in future physical theories. The dynamic approach enables one to interpret the evolution of the universe as a random grammar describing a random evolution of the corresponding language.
- Mathematical investigations of the thermodynamic passage in statistical mechanics and quantum theory have enriched probability theory with many new ideas. However, attempts to construct a similar theory for growing random complexes have yeilded additional and unexpected phenomena (see [6] and [7]), which we hope have physical meaning. One of these new effects is that one cannot choose an origin in the infinite universe without using the Zermelo axiom, and any constructive introduction of a local observer into the system modifies the structure of space-time in a neighbourhood of the observer.
- It is a difficult problem to find the critical exponents by means of Monte-Carlo computer simulation because an exponential factor dominates the asymptotic behaviour, and this factor is not universal and strongly depends on details of the model. A Markov process modifying the number of cells would be more useful in this situation than simulation of the uniform distribution on the set of triangulations. It turns out that for the desired exponent $\alpha = -\frac{7}{2}$ it is more natural to use a special transformation of measures, a so-called non-linear Markov process in terms of probability theory, instead of an ordinary Markov process.

Contents of the paper. In § 2, the one-dimensional case is treated, which is useful as a bridge between classical probability theory, for instance, queuing theory, and two-dimensional quantum gravity.

In § 3 we give a minimal list of necessary definitions concerning complexes and curvature in the two-dimensional case.

§ 4 contains the introductory definitions, the main problems, and some known results. In § 4.2 we give a brief exposition of the RMM approach to pure planar gravity, and the sole object of this exposition is to stress some problems related to the combinatorial approach which is basic in the present paper.

In § 5 we study a simple dynamic model in which cells are randomly appended to the boundary of the disc. This model is explicitly soluble (via random walks), and we calculate the main quantities. The exponent for this model is -2, and thus it belongs to a universal class which differs from that accepted in physics. However, the continuum limit in this model is well defined, and gives a space of constant curvature.

§ 6 is central. Here we construct and study non-linear Markov processes that preserve a Gibbs measure. Here changes can again occur on the boundary only.

We use the Tutte method of functional equations to obtain the desired exponent $-\frac{7}{2}$. On the other hand, we develop a new (combinatorial) approach to the study of local correlation functions.

In § 7 we consider Markov dynamics in which changes can occur at any cell of the complex.

I thank L. Pastur for elucidating some points of the theory of random matrices and also S. Shlosman and R. Minlos for reading the paper and for numerous useful comments.

§2. One-dimensional gravity

For the physical interpretation and beautiful calculations in one-dimensional gravity, see Chapter 2 of the Ambjørn lectures [9]. Our aim here is the probabilistic investigation of gravity and the discussion of unexpected relationships. There is no topology in dimension one, and a one-dimensional cell complex is a linear graph.

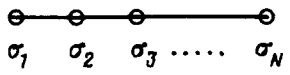

Figure 1. Linear spin graph

A chain of symbols drawn from some alphabet can be regarded as a function on the set of vertices of such a graph (see Fig. 1).

2.1. Equilibrium distribution. Let us give the basic definition in a more abstract form than that in [9], without preliminary embedding in Euclidean space, which gives better agreement with Polyakov string quantization. Let us consider probability distributions on the set of finite linear spin graphs (sometimes we use the terminology of [6] and [7], but such deviations can mostly be skipped). That is, we consider distributions on the set Ω of strings (or chains) $\sigma = s_1 \dots s_N$, where $N = 0, 1, 2, \dots$, $s_i \in S$, and S is an alphabet (the spin space). For example, S can be the unit sphere in $\mathbb{Z}^d$ or in $\mathbb{R}^d$. The case $N = 0$ corresponds to the empty string, which has no S-value. Let us define a non-negative measure on Ω by the relation

$$Q(\sigma) = \exp\Bigl(-\mu N - \beta \sum_{i=0}^{N-1} \bigl(f_1(s_{i+1}) + f_2(s_i, s_{i+1})\bigr)\Bigr) \tag{1}$$

for some functions f_1 and f_2. It is convenient to assume that some element s_0 is chosen. The simplest example is obtained for $s \in \mathbb{Z}^d$, $f_2 = 0$, and $f_1(s) = \infty$ for all s except for a finite set. This measure can be normalized if

$$Z = \sum_{N,\sigma} Q(N, \sigma) < \infty.$$

Intuitively, a sequence of values s_0, s_0+s_1, $\dots$, $s_0+s_1+\dots+s_N$ can be regarded as a "random walk." However, it differs from classical random walks. In what follows we find a relationship of the above sequence with certain computer science problems,

and then connect this formal object with Euclidean space; in physics, a similar procedure is now common, namely, an abstract object (a random triangulation, an intrinsic metric, and so on) is tied to physical space-time.

Simplest examples. In the first example we set $S = \mathbb{Z}^d$, $f_2 \equiv 0$, and $f_1(s) = 1$ for $|s| = 1$ and $f_1(s) = \infty$ otherwise. Let $s_0 = 0$. In other words, we consider the non-normalized measures $\exp(-l(s))$ on all possible finite paths $r = (0, s_1, s_1 + s_2, \ldots, s_1 + s_2 + \cdots + s_N)$ in $\mathbb{Z}^d$ that start from 0, where $N = l(r)$ is the length (the number of steps) of the path r. These measures can be infinite. There is a number μ_{cr}, $0 < \mu_{\mathrm{cr}} < \infty$, such that the series

$$Z = \sum_r \exp\bigl(-\mu l(r)\bigr)$$

for the partition function is convergent for $\mu > \mu_{\mathrm{cr}}$ and divergent for $\mu \leqslant \mu_{\mathrm{cr}}$. In our case, $\mu_{\mathrm{cr}} = \ln 2d$.

For the second example we take $S = \mathbb{R}^d$, $e^{-f_1(s)} = \delta(|s| - 1)$, and $f_2(s, s') = \phi((s, s'))$ for some bounded function ϕ of the angle between two vectors. For the third example we set $S = \mathbb{R}^d$, $f_2 \equiv 0$, and $f_1(s) = s^2 = (s^1)^2 + \cdots + (s^d)^2$.

These examples, which are highly simple and have nothing special from the probabilistic viewpoint, correspond to one-dimensional analogues of rather famous actions: that for a free relativistic point particle in d-dimensional space-time,

$$S(L) = \mu \int_L dl,$$

where l stands for the Euclidean length and $L = L(0, x)$ is a path from 0 to x; the Hilbert–Einstein action

$$S(L) = \mu \int_L dl + \lambda \int_L |\kappa|\, dl,$$

where κ stands for the curvature of the curve embedded in Euclidean space; the action for a boson string $x(l)$,

$$S(L) = \frac{1}{\alpha'} \int_L \sqrt{g(l)}\, dl \left(g(l) \frac{d^2 x}{dl^2} + \mu \right),$$

where g is a metric on the interval of variation of the parameter. One can regard the above distribution as a quantization of the corresponding classical action, and it does coincide with the discretization. The length of any link of the discrete path is assumed to be 1, and thus the length of a path is the number of links.

We discuss the first example only; the other two can be treated in a similar way [9]. For $\mu > \mu_{\mathrm{cr}}$ we can define a probability distribution on the set of all finite paths starting from the origin 0,

$$P(r) = Z^{-1} \exp\bigl(-\mu l(r)\bigr).$$

The Green functions are defined as measures on $\mathbb{Z}^d$,

$$G(x) = \sum_{s:\, 0 \to x} \exp\bigl(-\mu l(r)\bigr) = \sum_N C(N; x) \exp(-\mu N) = \sum_N P_{RW}^{(N)}(x) e^{(-\mu + \mu_{\mathrm{cr}})N},$$

where $C(N;x)$ stands for the number of paths from 0 to x of length N, and $P_{RW}^{(N)}(x)$ are the probabilities of an N-step transition from 0 to x for a classical simple random walk in $\mathbb{Z}^d$. By the local limit theorem, the number of paths is

$$C(N;x) \sim (2\pi)^{-d/2}(2d)^N N^{-d/2}$$

if $x = O(\sqrt{N})$. The term "Green functions" originates not only from the functions in physics but also from the Green functions for Markov processes. Indeed, for a random walk we have

$$G_{RW}(0,x) = \sum_n p_{0x}^{(n)} = \sum_{L:\,0\to x} p(L),$$

where L ranges over all paths from 0 to x.

The following observations are of importance.

- The number μ_{cr} and the exponent $-d/2$ do not depend on the choice of x.
- The value μ_{cr} is not universal; it depends on the dimension and on the lattice. Other values of μ_{cr} can also be obtained when considering piecewise-linear paths in $\mathbb{R}^d$ with sides of chosen length [9].
- At the same time, the exponent $-d/2$ does not depend on the lattice, as follows immediately from the local limit theorem.
- For $\mu = \mu_{\mathrm{cr}}$, the series is convergent if and only if $d > 2$.

Let us study the scaling limit $\mu \to \mu_{\mathrm{cr}} + 0$. We write $x = s_1 + \cdots + s_N$. In the scaling limit, one studies the following exponents: the mass exponent ν (the inverse correlation length), the susceptibility exponent γ, the anomalous dimension η, and the Hausdorff dimension d_H. They are determined by the leading terms in the asymptotic behaviour for small $\mu - \mu_{\mathrm{cr}}$,

$$m(\mu) \approx (\mu - \mu_{\mathrm{cr}})^{\nu}, \qquad \sum_x G(x) \approx (\mu - \mu_{\mathrm{cr}})^{-\gamma},$$

$$G(x) \approx |x|^{-d+2-\eta}, \qquad \sum NC(N;x)\exp(-\mu N) \approx x^{d_H}.$$

Let us give a detailed calculation. The susceptibility

$$\chi(\mu) = \sum_x G(x) = \sum_N e^{(-\mu+\mu_{\mathrm{cr}})N} = \frac{1}{1-\exp(-\mu+\mu_{\mathrm{cr}})}$$

does not depend on the dimension of the spin s and remains the same for more general spins and interactions (however, not always). Thus, $\gamma = 1$.

If $(\mu - \mu_{\mathrm{cr}})x \ll 1$, then $G(x) \approx G_{RW}(x)$, where $G_{RW}(x)$ are the Green functions for a simple random walk. Therefore, $\eta = 0$.

Let $P_{RW}^{(N)}(x)$ be the transition probabilities for a simple random walk on $\mathbb{Z}^d$ and let $f(p,z) = \sum_N \sum_x P_{RW}^{(N)}(x)e^{ipx}z^N$, $z \in \mathbb{C}$, be their generating function. Then

$$G(p) = f(p, e^{\mu_{\mathrm{cr}}-\mu}) = \sum_N e^{(-\mu+\mu_{\mathrm{cr}})N}\left(\frac{1}{d}\sum \cos p_i\right)^N = \frac{1}{1 - e^{-\mu+\mu_{\mathrm{cr}}}\frac{1}{d}\sum \cos p_i}$$

$$\approx \frac{1}{\mu - \mu_{\mathrm{cr}} + \frac{1}{d}\sum \cos p_i},$$

which is the classical propagator for a free quantum relativistic particle with mass $\sqrt{\mu - \mu_{\rm cr}}$. Thus, $\nu = \frac{1}{2}$.

We can readily prove that

$$d_H = \nu^{-1}, \qquad \gamma = \nu(2 - \eta).$$

In the second example, different exponents can be obtained as $\lambda \to \infty$ and $\mu \to \mu_{\rm cr}$ [9].

2.2. Gravity as a queuing model. Let us start from the first example introduced above. One can construct a reversible dynamics preserving the measure $Q(\sigma)$. This is given by the following Markov chain with continuous time. In our case, the set S consists of $2d$ unit vectors $e_1, -e_1, \dots, e_d, -e_d$ in $\mathbb{Z}^d$. The state

$$\sigma = s_1 \dots s_N$$

is interpreted as a queue, where N is the length of the queue, S is the set of customer types, and $s_1 + \dots + s_N \in \mathbb{R}^d$ is the generalized length of the queue. This is a LIFO-type queue (last in, first out), and the transitions consist in appending and/or deleting the rightmost link (which is similar to the arriving and service of customers in queuing theory). More exactly,

$$s_1 \dots s_N \to s_1 \dots s_{N-1}$$

for all $\sigma = s_1 \dots s_N$ with rate (intensity) ν, and

$$s_1 \dots s_N \to s_1 \dots s_N s_{N+1}$$

with rate λ, where all values of s_{N+1} are equiprobable. The transitions $\emptyset \to s_1$ from the empty queue have the rate λ with equiprobable values of s_1.

Lemma 1. *The process is ergodic if and only if $\nu > \lambda$ and the distribution* (1) *is invariant with respect to this dynamics.*

For the proof we note that the length $N_t = N(\sigma(t))$ (of the queue $\sigma(t)$ related to the instant t) is also a Markov chain, a birth-and-death process, and its stationary probabilities are

$$Z^{-1} \exp(-\mu N - N \ln 2), \qquad \mu = \ln \frac{\nu}{\lambda}.$$

We also note that a system of two queues corresponds to two interacting particles, and so on, but we do not go into the details of this problem.

Supercritical case. Although there is no equilibrium distribution in the case $\nu \leqslant \lambda$, this case is not worse than that of $\nu > \lambda$ from the viewpoint of dynamics, namely, at any instant there is a distribution $Q(t)$, and the behaviour of its local correlation functions as $t \to \infty$ can be studied. By $N(\sigma(t))$ we again denote the length of $\sigma(t)$ at the instant t.

Theorem 1. *If $\lambda > \nu$, then $N(\sigma(t)) \to \infty$ almost surely. Moreover, there are limiting local correlation functions (not very near the endpoints of the string) that define a translation-invariant Gibbs measure on $\mathbb{Z}$. For instance, a one-particle correlation function is defined as follows:*

$$\lim_{k\to\infty}\lim_{t\to\infty} P(s_k(t) = i) \to p_i, \qquad i \in S.$$

In fact, in the first example, this Gibbs field is a Bernoulli measure in each of the three regions $\nu > \lambda$, $\nu = \lambda$, and $\nu < \lambda$; for details, see [10].

Critical case and the scaling limit. There are two ways to treat the critical case. First, we can consider the dynamics for critical values of the parameters (in the first example, this is the value $\frac{\lambda}{\mu} = 1$). The critical exponents can be obtained immediately from the dynamics. There are results for rather general transitions. For instance, let $\nu(s)$ and $\lambda(s, s')$ be two positive functions, and let us define the transition rates by the formulae

$$\nu(s_1 \dots s_N \to s_1 \dots s_{N-1}) = \nu(s_N), \qquad \lambda(s_1 \dots s_N \to s_1 \dots s_{N+1}) = \lambda(s_N, s_{N+1}).$$

Thus, the rates depend on the rightmost symbols. We assume that the functions $\nu(s)$ and $\lambda(s, s')$ are such that the Markov chain is null-recurrent; for the conditions ensuring this property, see [11]. Let S be finite, $S = \{s_1, \dots, s_k\}$. Let $n(t) = (n_1(t), \dots, n_k(t))$, where $n_i(t)$ is the number of symbols s_i in the string $\sigma(t)$. Then the following assertion holds.

Theorem 2. *The central limit theorem holds for the random vector $n(t)$, that is, the limit*

$$\frac{n(t)}{\sqrt{t}} \to |w|\, c$$

exists in probability as $t \to \infty$, where w has the standard Gaussian distribution and c is a constant vector.

This gives the same canonical exponents as under the equilibrium approach. We note that, in the reversible case, the proof reduces to the case of a reflected random walk. The proof for irreversible dynamics is more involved: for finite S, see [11]. For compact S, there is no proof, but it seemingly must be similar. It would be of interest to find examples for the non-compact case with non-Gaussian limiting distribution.

The other approach corresponds to the scaling limit in the equilibrium case. The scaling of the parameters of dynamics is performed simultaneously with the scaling of time t so that the difference $\nu - \lambda$ tends to zero and is scaled as $t^{-\frac{1}{2}}$. In the dynamics, this scaling limit corresponds to the diffusion approximation in queuing theory. Moreover, under the scaling

$$t = \tau N, \quad x = r\sqrt{N}, \quad \nu - \lambda = N^{-\frac{1}{2}}, \quad N \to \infty$$

(with some constant τ and r) we obtain a Brownian motion with drift. The drift defines the mass gap in the spectrum of the infinitesimal generator of the corresponding diffusion process.

Random grammars. We have considered a dynamics that is called a right-linear grammar (not necessarily context-free, see [6]) in computer-science terminology. Let us now consider a more general dynamics in which the transitions can occur at any place of the string (not only at its right endpoint as above).

For the first example, one can construct the following reversible Markov chain that preserves our measure; this chain is a context-free random grammar (for the definition, see [6]). Each symbol of the string is deleted with rate ν and, for any $i = 0, 1, \dots, n$, we insert a new symbol between the symbols s_i and s_{i+1} (for $i = 0$, we put it before s_1 and, for $i = n$ after s_n) in the string $s_1 \dots s_n$ with rate λ. The appended symbol takes one of the $2d$ values of the coordinate vector e with probability $1/2d$. To prove that the measure is invariant, we note that the path lengths also form a Markov chain (on $\mathbb{Z}_+$). This is a birth-and-death process on $\mathbb{Z}_+$ with jump rates $q_{i,i+1} = \lambda i$, $q_{i+1,i} = \nu i$, and $q_{0,1} = \lambda$. Then its stationary probabilities (for $\lambda < \nu$) are

$$\pi_k = \frac{q_{0,1} q_{1,2} \cdots q_{k-1,k}}{q_{1,0} \cdots q_{k,k-1}} = C \exp(-\mu k), \qquad \mu = \ln \frac{\nu}{\lambda}.$$

For the other two examples, the dynamics (which is not context-free) can also be constructed, and we shall treat this in another paper in the general case.

§3. Spin complexes

3.1. Cell structures. Here we present a minimal list of basic definitions concerning cell structures.

A complex is obtained by gluing together its elementary components, or cells, like matter consisting of molecules. First of all, one must be very careful in defining the rules of gluing and the probability distributions that arise. On the other hand, it seems doubtful that some types of cellular structures have some *a priori* advantages over the others, because there are no physical reasons to prefer one cell structure or gluing rule to another. Thus, various possibilities must be considered to see that some laws are universal. In this paper we encounter two universal classes, one of which is popular in physics. Moreover, having flexibility in choosing a cell structure, we can obtain models that are simpler and even soluble.

3.1.1. Abstract complexes. A (labelled) complex Γ is a set of elements, the so-called cells, for which a function $A \mapsto \dim A$ on Γ is defined, the so-called dimension of a cell A, that takes values in the set $\{0, 1, 2, \dots\}$. By definition, the dimension of Γ is $\dim \Gamma = \sup_A \dim A$. Let $\Gamma_d \subset \Gamma$ be the set of cells of dimension d. For each cell $A \in \Gamma_d$, $d > 0$, a subset $\partial A \subset \cup_{i=0}^{d-1} \Gamma_i$ is defined which is called the boundary of A. A subcomplex Γ' of Γ is a subset of Γ such that, if $A \in \Gamma'$, then $\partial A \subset \Gamma'$.

An isomorphism of two complexes is a one-to-one map preserving the dimension and the boundaries. The equivalence classes of complexes with respect to the isomorphism are called the unlabelled complexes.

We note that a complex Γ can be regarded as a special case of a spin graph (G, s); see [7]. The correspondence can be constructed in different ways. For example, let the vertices i of a graph G correspond to cells of C and let the function $s(i)$ be the dimension of the corresponding cell. The links (edges) are defined by the incidence

matrix as follows: two vertices A and A' of G are connected by a link if and only if $A' \in \partial A$.

A labelled spin complex is a pair (Γ, s), where Γ is a complex and $s\colon C(\Gamma) \to S$ is a function on the set $C(\Gamma)$ of cells of Γ with values in some spin space S. The isomorphism of two spin complexes is an isomorphism of the complexes respecting the spins. The equivalence classes are called (unlabelled) spin complexes. Unless otherwise stated, we consider only functions s defined on the cells of maximal dimension; sometimes, if a dual complex exists, we may assume that the function s is defined on the vertices of the dual complex.

3.1.2. Topological complexes. There are many topological incarnations of abstract complexes. In each of them, a cell is represented by an open disk. A CW-complex is a topological space K that is a union of its d-skeletons K_d, which are defined by induction as follows. Let $K_0 = \Gamma_0$ be a disjoint union of points (vertices), which are the cells of dimension 0. In general, K_d is obtained from K_{d-1} by joining some cells A by the following rule. Each such cell A of dimension d is identified with an open d-dimensional disc D_A, and a continuous (pasting) map $\phi_A\colon \partial D_A \to K_{d-1}$ is chosen. Then K_d is the quotient space of the union of K_{d-1} and $\cup_{A:\dim A=d} D_A$ via identifications of the points $x \in \partial D_A$ and $\phi_A(x)$.

For example, K_1 is a graph whose vertices are the zero-dimensional cells and whose links (edges) are the one-dimensional cells. A link A is called a *loop* if the boundary of A is mapped to one vertex. Some restrictions are often imposed on the attaching maps. Here we restrict ourselves to the case in which $\dim \Gamma = 2$ and, for all A with $\dim A = 2$, the boundary ∂D_A is a union of some cells B with $\dim B = 0, 1$ (in some books, this condition is included in the definition of a CW-complex; cf. [12]). With a CW-complex of this type one can associate an abstract complex with $\partial A = \{B \in K_0 \cup K_1 : B \subset \phi_A(D_A)\}$.

We obtain a class of simplicial complexes (where the cells are called simplices) if the boundary of any two-cell contains precisely three one-dimensional cells and, for any cell A, the set $\partial A \cap K_0$ of vertices determines the cell A uniquely. In particular, each graph without multiple edges and without loops is a simplicial complex of dimension one.

3.1.3. Cell surfaces. In the paper we consider various classes of two-dimensional complexes. A class can be defined either by some restrictions on the above (maximal) class of complexes or by a constructive procedure by which we can obtain all complexes of this class. In other words, a class is defined as a language obtained by means of a grammar on the set of graphs; see [6] and [7].

The following restrictions hold for all complexes in this paper: a complex is a (closed compact) surface. A (closed) pseudosurface is a topological space isomorphic to a finite two-dimensional simplicial complex with the following additional property: each link belongs to the boundary of exactly two two-dimensional cells. A surface has an additional property that each vertex has a neighbourhood homeomorphic to a disc.

Here is the list of all compact closed two-dimensional surfaces (without holes). The orientable surfaces are the spheres with ρ handles, S_ρ, $\rho = 0, 1, 2, \ldots$. The non-orientable surfaces are P_1 (the projective plane), P_2 (the Klein bottle),

and the surfaces P_k, $k = 3, \ldots$, where P_k is the sphere in which k holes are cut out and a Möbius band (a crosscap) is attached to each of them along its boundary.

If a complex is isomorphic to such a space, then K_1 is a graph homeomorphically embedded in the corresponding surface S. Such complexes are studied in topological graph theory [13] and in combinatorics, where they are called *charts* (or *maps*). A surface with holes is obtained from a closed surface by cutting out finitely many discs with disjoint boundaries. The boundary of any surface with holes belongs to K_1.

An isomorphism of charts is an isomorphism of abstract complexes. In other words, two charts are called isomorphic if there is a homeomorphism of S that maps the vertices onto the vertices, the edges onto the edges, and the (two-dimensional) cells onto the cells. We consider here only orientation-preserving maps of this kind.

A chart B is a *subdivision* of a chart A if the graph $K_1(A)$ of A is a subgraph of the graph $K_1(B)$ of B. As is known, two two-dimensional (and even three-dimensional) topological complexes are homeomorphic as topological spaces if they have subdivisions that are isomorphic as abstract complexes.

The Euler characteristics of a complex (for a compact closed surface) is defined by the formula

$$\chi = V - L + F,$$

where F is the number of faces, V is the number of vertices, and L is the number of links. It depends on the surface itself rather than on the complex; for the orientable surfaces we have $\chi = 2 - 2\rho$, where ρ is the genus (the number of handles), and for non-orientable surfaces we obtain $\chi = 2 - n$, where n is the number of crosscaps.

In what follows we are mainly interested in the following five classes of complexes (charts).

Arbitrary charts. This is the class we have just defined. No further restrictions are imposed. The simplest examples are a vertex inside the sphere (a vertex chart), and an edge with two vertices inside the sphere (an edge chart).

Smooth cell surfaces. A smooth cell surface [14] is a compact connected smooth two-dimensional manifold M with finitely many closed subsets (cells) F_i such that

1) $\cup F_i = M$,
2) for each i, there is a one-to-one smooth map f_i of the cell F_i onto a polygon with n_i edges, $n_i \geqslant 3$,
3) for $i \neq j$ either we have $F_i \cap F_j = \emptyset$ or $f_i(F_i \cap F_j)$ is an edge or a vertex of the corresponding polygon.

Triangulations. These are smooth cell surfaces such that $n_i = 3$ for all i.

A set V of vertices is said to be *separating* if there are two subgraphs G_1 and G_2 such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = V$. For a triangulation, the absence of separating sets is equivalent to the absence of loops in K_1.

A disc triangulation is defined as a smooth cell surface homeomorphic to a disc and having no separating singletons. In this case, it can be regarded as a triangulation of the sphere for which a face (corresponding to the exterior of the disc; this face is said to be *exterior*) is singled out and the number of edges n_i is equal to three for all other faces.

Simplicial complexes. These are triangulations without multiple edges; moreover, every three edges define at most one cell. Note that a triangle (a cycle of length 3) having at least one vertex inside and outside is not regarded as a cell.

Convex polyhedra. A convex polyhedron is a special case of a chart with $\rho = 0$. There is a purely combinatorial characterization of charts corresponding to convex polyhedra. If a triangulation has neither loops nor multiple edges, then for $L \geqslant 4$ it corresponds to a convex polyhedron (by the Steinitz–Rademacher theorem).

3.1.4. Local observer (root). A complex is said to be *labelled* if the set of vertices is enumerated. An important point is that often one does not speak of the enumeration, with the understanding that the set of vertices is defined in some way. However, the labels are useful to choose a coordinate system in the space but are superfluous for geometry and topology. At the same time, equivalence classes of complexes are not useful for other reasons. There is another way of avoiding superfluous coordinatization while keeping the group of automorphisms trivial. In essence, this is the same coordinatization by means of which we can enumerate all vertices of a given complex by means of some algorithm. A root (or a local observer) in a labelled complex is a triple (f, l, v), where f is a two-dimensional cell, l is one of its edges, and v is a vertex of l. An isomorphism of two complexes with roots is an isomorphism of complexes that respects the roots but does not necessarily preserve the enumeration. A rooted chart (a rooted complex, a complex with a local observer) of class **A** is an equivalence class of complexes with a root in the class **A**. We always assume that a rooted edge is oriented and directed away from the rooted vertex. For disc triangulations, we agree that the exterior face i is always rooted, and in this case it is possible that $n_i \neq 3$.

Lemma 2. *The automorphism group of any rooted chart is trivial.*

This can readily be proved by induction on the number of cells by successively extending an automorphism from the rooted face to its neighbours.

3.1.5. Moves. The grammars on graphs, or graph grammars, corresponding to transformations of complexes are less studied. Here the corresponding substitutions are called moves. In the next section, § 4, we study *Tutte moves* (see Fig. 5), which consist in appending an edge between two vertices of a cell or in joining two complexes by identifying two of their vertices. However, partitions have always played an important role in the topology. In [15] and [16] some moves are studied in detail.

Let A be a commutative associative algebra over Z_2 (the algebra of simplicial chains over Z_2) generated by the symbols from some (countable) alphabet L with the commutation relations $s_i^2 = 0$ and $s_i s_j = s_j s_i$. Thus, this is the linear span of the strings (simplices) of the form $\alpha = s_1 \dots s_n$, where all s_i's are different. Let us define the boundary operator as a linear operator $\partial\colon A \to A$ such that

$$\partial\alpha = \sum \beta,$$

where the sum is taken over all submonomials $\beta = s_{i_1} \dots s_{i_{n-1}}$, $1 \leqslant i_1 < \dots < i_{n-1} \leqslant n$, of the monomial α. We treat here two-dimensional complexes only, that is, $n \leqslant 2$.

The other linear operators in this algebra are called *Alexander moves* and denoted by $S_{i,j;x}$, $i, j, x \in L$. They are defined by the formula

$$S_{i,j;x}\alpha = x(i+j)\beta$$

for $\alpha = ij\beta$; we set $S_{i,j;x}\alpha = 0$ otherwise.

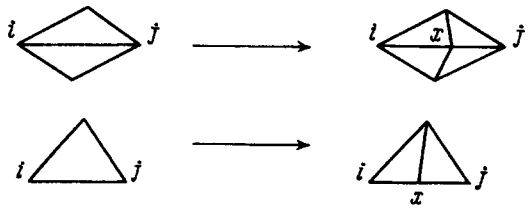

Figure 2. Alexander moves

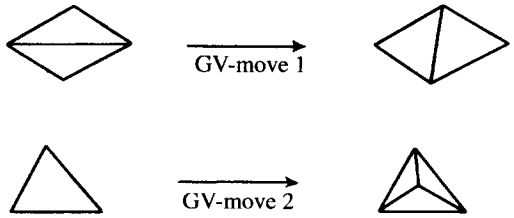

Figure 3. Gross–Varsted moves

The next example is given by the *Gross–Varsted moves* defined in Fig. 3. It is proved in [16] that each of the Alexander moves can be represented as a composite of Gross–Varsted moves, and conversely. A set R of moves is said to be *irreducible* in a class $\mathbf{A}$ of complexes if, for each pair T_1, T_2 of complexes in $\mathbf{A}$, there is a sequence of moves in R that lead to T_2 starting from T_1 (in the physical literature, the term *ergodic* is used in this case, but we use the standard probabilistic terminology).

Theorem 3. *In the class* $\mathbf{A}$ *of simplicial complexes, both the set of Alexander moves and the set of Gross–Varsted moves are irreducible.*

For the proof, see [15] and [16].

3.1.6. Automorphism groups. Let $\mathbf{A}$ be one of the five classes of complexes introduced above.

Theorem 4. *For almost any complex with* N *two-dimensional cells that belongs to the class* $\mathbf{A}$, *the automorphism group is trivial, that is,* $\dfrac{\mathbf{A}_{\text{non-trivial}}(N)}{\mathbf{A(N)}} \to 0$ *as* $N \to \infty$, *where* $\mathbf{A}(N)$ ($\mathbf{A}_{\text{non-trivial}}(N)$) *is the set of all complexes with* N *two-dimensional cells that belong to* $\mathbf{A}$ (*the subset of* $\mathbf{A}(N)$ *formed by the complexes with non-trivial automorphism groups, respectively*).

Tutte noticed earlier that this assertion is intuitively clear. Since then, many rigorous results appeared, see [17], [18], [19]. For the proof in the case of disc-triangulations, see [20].

3.2. Metric and curvature. A metric structure is defined if it is given on each of the closed cells so that the metrics on the boundaries of the cells coincide. There are two basic approaches to the definition of a metric, namely, the dynamic triangulations for which all edges are of unit length, and the quantum Regge calculus in which these lengths are random. Let us use the first approach. In this case, all cells with equal number of edges are identical, and the metric on them is standard.

We can use another approach. Let us consider first the graph K_1 embedded in the plane as a system of smooth arcs. Let us first define the metric on K_1 so that all edge lengths are equal to a constant and the area of any face (if this face is a triangle) is equal to one. Inside a cell with n edges, we define a metric by a smooth one-to-one map of an equilateral polygon Q_n with n edges onto this cell and such that the smoothness holds in a neighbourhood of the edges. In this case, the curvature is zero inside the cells. The same holds on the faces away from the vertices, as is shown in the figure in the piecewise-linear case.

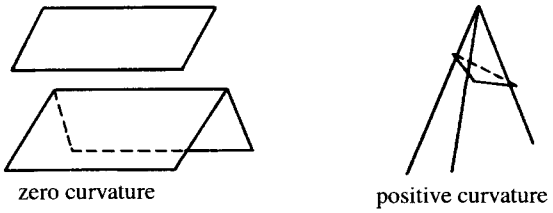

Figure 4. Curvature

Let us define the curvature R_v at a vertex v. As usual, the curvature is measured by the parallel translation (with respect to the Levi-Civita connection) of a vector (lying in the plane of a face in the piecewise-linear case) along a closed path as follows: along the internal part of a triangle, this is done as on the Euclidean plane, and when passing through an edge, by developing the two half-planes separated by this edge. Thus, only paths around vertices can give non-zero curvature. Around a vertex v, the angle between the initial and the translated vector is equal to $\varepsilon_v = 2\pi - \sum_f \varphi_{fv}$, where φ_{fv} is the angle of the simplex f at the vertex v. We note that

$$2\pi V - \sum_v \varepsilon_v = \sum_v \sum_f \varphi_{fv} = \sum_f \sum_v \varphi_{fv} = \pi F.$$

This, together with the Euler formula $\chi = V - L + F$, yields the Gauss–Bonnet formula

$$\sum_v \varepsilon_v = 2\pi\chi$$

for the triangulations, because $L = 3F/2$.

The classical examples are as follows: positive curvature corresponds to elliptic geometry (sphere, projective plane), zero curvature to Euclidean geometry (plane, torus, Klein bottle), and negative curvature to hyperbolic geometry (all other cases).

We claim that the curvature R_i at a vertex i is defined by the number q_i of edges incident to i. The Einstein–Hilbert action on a smooth manifold is

$$\int (c_1 R + c_2)\sqrt{g}\, dx,$$

where R stands for the Gaussian curvature and g is the metric. By the Gauss-Bonnet formula, $\int R\, d\sigma = \int R\sqrt{g}\, dx = 4\pi\chi$ for smooth surfaces. The connection of this formula with the discrete case for a partition of the unit sphere into polygons with n_i edges and areas A_i (the area A of a triangle with angles α_1, α_2, and α_3 is $A = \alpha_1 + \alpha_2 + \alpha_3 - \pi$) is given by

$$\sum k_i A_i = \sum_i \left[\sum_j \alpha_{ij} - (n_i - 2)\pi\right] = \sum_{ij} \alpha_{ij} - \pi \sum n_i + 2\pi F = 2\pi V - 2\pi E + 2\pi F,$$

where the α_{ij} are the angles of the cells. Thus, the discrete action must be $\lambda\rho + \mu N$ (up to a constant), where ρ stands for the genus and N is the number of triangles. Let us rewrite this action in terms of curvature by summing over the vertices instead of summing over the triangles. All triangles are equilateral; we assume that they have unit areas. Then each vertex obtains the area $\frac{1}{3}$ from any incident triangle, which gives the total $q_i/3$. Therefore,

$$\int R\, d\sigma \simeq \sum_i \frac{q_i}{3} R_i,$$

and the formula

$$\sum_i \frac{q_i}{3} R_i = 4\pi\chi$$

holds only if we set $R_i = 2\pi \dfrac{6 - q_i}{q_i}$.

§4. Equilibrium planar pure gravity

There are two kinds of technique in use in two-dimensional quantum gravity. Historically, the first of them is the combinatorial approach, which was initiated by Tutte and continued (without any mention of physics) by many researchers, and their papers are published in journals of combinatorics. The other is based on random matrix models (RMM's), which originated in physics itself. The calculations in the RMM approach are very persuasive, but the arguments are not completely rigorous. As far as the author knows, no mathematically rigorous connections between these approaches have been established. We use the first approach and give a short survey of the other.

4.1. Definitions and the combinatorial approach. Let $\mathbf{A}$ be a class (for example, defined in the previous section) of complexes T homeomorphic to the sphere, let $|T| = F(T)$ be the number of two-dimensional cells in T, and let $C(N) = \natural\{T \in \mathbf{A} : |T| = N\}$. The main example is the class of all triangulations of the sphere.

The grand canonical ensemble is defined by the relations

$$P(T) = Z^{-1}\exp\bigl(-\mu F(T)\bigr),$$
$$Z = \sum_{T\in\mathbf{A}} \exp\bigl(-\mu F(T)\bigr) = \sum_{N} C(N)\exp(-\mu N). \tag{2}$$

In particular for fixed N the conditional distribution of $T \in \mathbf{A}$ is uniform. There is a general and often useful method for the rough estimation of the number $C(N)$.

Lemma 3 (exponential a priori bounds). $c_1\gamma_l^N < C(N) < c_2\gamma_u^N$, *where* $1 < \gamma_l < \gamma_u < \infty$ *and* $c_i > 0$.

Proof. The idea for proving the lower bound is as follows: the sphere partitioned into N triangles is represented as a stretched tube of length of order N and with cross-section of order n, which is a constant (the metric is defined here by the condition that the lengths of all edges are equal to unity). We take two non-isomorphic complexes homeomorphic to an annulus in the plane, with the same number of edges on both sides and with n cells. The first complex is made up of alternating triangles standing on an edge and on a vertex, and the other of alternating pairs of triangles (two on an edge and two on a vertex, and so on). Let us glue these triangles in succession in an arbitrary order. We thus obtain the lower bound $2^{N/n}$.

To prove the upper bound, we use the following approach that works even in a somewhat more general situation. One can suggest an algorithm for constructing all possible complexes with N cells of dimension two. Let us start with one cell. We enumerate its edges by 1, 2, 3. At each step we add at most one cell to the boundary and enumerate the new edges in the order of their appearance. Let us describe this inductive construction. We take the edge with index 1 and make one of the following four decisions: 1) to add no more triangles to this edge, 2) to add to this edge a triangle with exactly two new edges, 3) to add a triangle to this edge and to the next edge on the boundary (clockwise), 4) the same for the anticlockwise direction. For each of the 4^k decision sequences ω, let $f(k,\omega)$ be the number of edges after k steps; we have $f(k,\omega) \leqslant 3 + 2k$ because at most two edges are added at each step. Moreover, if there are n triangles, then at most $3n$ decisions of type 1 are possible.

However, an exact asymptotic expression is needed. All known examples show the following asymptotic behaviour:

$$C(N) \sim c_1 N^{\alpha} c^N. \tag{3}$$

The next assertion follows from (3).

Corollary. *There is a number* μ_{cr}, $0 < \mu_{\mathrm{cr}} < \infty$, *such that the series* (2) *is convergent for* $\mu > \mu_{\mathrm{cr}}$ *and divergent for* $\mu < \mu_{\mathrm{cr}}$. *If* $\mu = \mu_{\mathrm{cr}}$, *then* $Z < \infty$ *if and only if* $\alpha < -1$.

Thus, for values $\mu < \mu_{\mathrm{cr}}$ of the parameter the distribution does not exist. However, the dynamics introduced below enables us to interpret these values of μ as well.

Neither of the constants $c_1 > 0$ and $c > 1$ is universal but for all known examples α is (it has the same value for all examples). This property of α is not intuitively clear. For example, predictions based on physical (non-rigorous) arguments (see, for instance, [21]) failed to give the correct numerical value $\alpha = -\frac{7}{2}$ even in the planar case.

Theorem 5. *The asymptotics* (3) *holds for all five classes defined in* § 3.1.3. *Moreover,* $\alpha = -\frac{7}{2}$ *in all these cases.*

Proof. We prove this for triangulations only; for the other cases, see [17]–[20] and [22]–[26]. We define a distribution on the class $\mathbf{A}_0$ of rooted complexes (that is, triangulations with a distinguished "root") in a similar way,

$$P_0(T) = Z_0^{-1} \exp\bigl(-\mu F(T)\bigr), \qquad Z_0 = \sum_{T \in \mathbf{A}_0} \exp\bigl(-\mu F(T)\bigr) = \sum_N C_0(N) \exp(-\mu N),$$

where the zero subscript means that only the rooted complexes of class $\mathbf{A}$ are considered.

Lemma 4. *For triangulations,*

$$C(N) \sim (3N)^{-1} C_0(N).$$

This follows from the fact that most complexes have the trivial automorphism group. Indeed, a non-rooted complex can be made a rooted one by choosing first a cell of dimension two, which gives the factor N if the automorphism group of the complex is trivial. Similarly, the choice of one of its three edges gives the factor 3.

We denote by $C(N, m)$ the number of disc triangulations for which the exterior face (that contains the north pole of the sphere) has m edges, and by $C_0(N, m)$ the number of these triangulations for which the exterior face is a rooted face (in other words, one of boundary edges is singled out and oriented anticlockwise). The following result is similar to the previous one but can be proved in a simpler way.

Lemma 5. $C(N, m) \sim m^{-1} C_0(N, m)$ *for large* N *and for any chosen* m.

Proof. Let us consider the automorphisms of the disc. We enumerate the edges of the boundary in a cyclic order, $1, 2, \ldots, m$. An automorphism ϕ of the disc is uniquely defined if $j = \phi(1)$ is given, that is, if we know the edge to which the edge with index 1 is mapped. Indeed, in this case, the triangle adjacent to the edge with index 1 is mapped by ϕ to the triangle adjacent to the edge j, and so on according to the connectivity.

We claim that for a given j almost all complexes (for N large enough) admit no automorphism such that $j = \phi(1)$. To this end, let us show that, for each complex A with a non-trivial automorphism ϕ, we can decompose the complex into two subcomplexes, $A_1 \cup A_2 = A$, in such a way that each face belongs to only one of these subcomplexes and $\phi A_1 = A_2$. This can be done by induction as follows. We take a boundary edge, and put a triangle T with this edge in A_1. Then we let $\phi T \in A_2$. Each step of the induction consists of taking another triangle that has a common edge with the part of A_1 already constructed. We can now modify A_2 internally, in a number $u(N)$ of ways bounded below by some function $u(N)$ with

$u(N) \to \infty$ uniformly with respect to A as $N \to \infty$. This can be done by choosing $u(N)$ triangles in A_2 that are not too close to each other and by independently modifying some neighbourhood of each of them, while preserving the boundary of the neighbourhood and the number n of cells in it. This is indeed possible because $C(n, b) > 1$, where b is the number of edges on the boundary. Thus, for given A_1, the proportion of complexes with $\phi A_1 = A_2$ is small. We have thus proved that only very few complexes have an automorphism ϕ such that $j = \phi(1)$. Since m is fixed, this holds for all j.

By the above lemmas, to prove the theorem it remains to establish the relation $C_0(N) \sim c_2 N^{-\frac{5}{2}} c^N$.

The hope that formula (3) has a universal nature is strongly supported by the fact (proved below for triangulations) that for all such examples the first positive singularity of the generating function $\sum_N C_0(N) z^N$ is algebraic, and this yields the asymptotics (3).

Remark 1. Let us assume that an algebraic function $y(x)$ is analytic at the origin and has its minimal positive singularity at $a > 0$. We say that its leading exponent is b if there are non-integer rational numbers b_i, $i = 1, 2, \ldots, d$, and functions $g_i(x)$, $i = 0, 1, \ldots, d$, such that $b = \max b_i$, the functions $g_i(x)$ are analytic at the point $x = a$, and

$$y(x) = \sum_{i=1}^{n} g_i(x) \left(1 - \frac{x}{a}\right)^{b_i} + g_0(x).$$

Then we have the following expansion (known as Darboux' theorem):

$$y(x) = \sum_n c_n x^n, \qquad c_n \sim \frac{g_1(a)}{\Gamma(-b)} n^{-b-1} a^{-n}.$$

In our case (for $C_0(N)$) we have $b = \frac{3}{2}$. Tauberian theorems can also be applied in this situation.

We give some examples for which all constants in the asymptotics are known; see [22] and its bibliography. The first example is the above class of all triangulations. Here $C_0(N) \sim \gamma_2 N^{-\frac{5}{2}} c^N$ and $c = 3\sqrt{\frac{3}{2}}$. For convex polyhedra we obtain $C_0(N) \sim \gamma_3 N^{-\frac{5}{2}} c_1^N$ and $c_1 = \frac{16}{3\sqrt{3}}$. For simplicial triangulations $C_0(N) \sim \gamma_4 N^{-\frac{5}{2}} c^N$ and $c = \frac{3\sqrt{3}}{2}$. Many other examples could be given; however, we only want to understand the general underlying mechanism.

Tutte [23]–[25] began to study the asymptotics for $C(N, m)$ and developed a powerful "quadratic" method. Afterwards, many authors developed this method and obtained the asymptotics for various classes **A** (see the review [26] and the recent paper [20]).

The main idea of Tutte is to use the following recurrent equations for $C(N, m)$, $N = 0, 1, \ldots$, $m = 2, 3, \ldots$:

$$C(N, m) = C(N-1, m+1) + \sum_{\substack{N_1+N_2=N-1,\\ m_1+m_2=m+1}} C(N_1, m_1) C(N_2, m_2), \quad m \geqslant 3, \ N \geqslant 1,$$

$$C(0, 2) = 1, \qquad C(0, m) = 0, \quad m > 2.$$

These equations can readily be derived from Fig. 5, where the orientation of the rooted edge is marked by an arrow, the rooted vertex is the beginning of the arrow, and the rooted face is to the right of the arrow (it contains the north pole of the sphere). These pictures define two types of moves, called Tutte moves. Let us take any rooted chart with parameters $(N-1, m+1)$, make a Tutte move of type 1, take any ordered pair of rooted charts (N_1, m_1), (N_2, m_2), and make a Tutte move of type 2. Any rooted chart (N, m) can be obtained uniquely in this way. The pair $(0, 2)$ corresponds to the so-called edge chart, or edge complex, with a single edge that is counted twice.

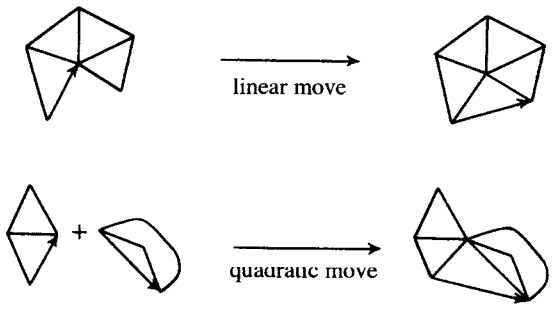

Figure 5. Linear and quadratic moves

Introducing the generating function

$$U(x,y) = \sum_{N=0}^{\infty} \sum_{m=2}^{\infty} C(N,m) x^N y^{m-2},$$

we obtain the following functional equation:

$$U(x,y) = U(x,y)xy^{-1} + U^2(x,y)xy + 1 - xy^{-1}U(x,0).$$

Let us use this equation to prove that $U(x,0)$ is algebraic, and let us find its principal singularity in a somewhat more general setting.

Green functions. We consider some class $\mathbf{A}$ of complexes. Let $\mathbf{A}(m_1, \ldots, m_k)$ be the class of complexes defined by the same restrictions as $\mathbf{A}$ that are homeomorphic to the sphere with k holes and m_i edges, $i = 1, \ldots, k$, on the boundaries of these holes. We also assume that the boundaries are disjoint. The Green functions are defined as follows:

$$\begin{aligned} Z(m_1, \ldots, m_k) &= \sum_{T \in \mathbf{A}(m_1, \ldots, m_k)} \exp\bigl(-\mu F(T)\bigr) \\ &= \sum_{N} C(N, m_1, \ldots, m_k) \exp(-\mu N), \end{aligned} \tag{4}$$

where Z corresponds to the case $k = 0$. The rooted Green functions are defined similarly,

$$Z_0(m_1, \ldots, m_k) = \sum_{T \in \mathbf{A}_0(m_1, \ldots, m_k)} \exp\bigl(-\mu F(T)\bigr) = \sum_{N} C_0(N, m_1, \ldots, m_k) \exp(-\mu N),$$

where the subscript 0 means that we consider complexes with a distinguished edge on the first boundary with m_1 edges, the so-called local observer in the terminology of [6] and [7]. It is desirable to have an expression for Green functions in terms of the basic distribution (as for Markov chains).

The Green functions are associated with the derivatives $\dfrac{d^n Z}{d\mu^n} = -\chi^{(n)}(\mu)$, that is, with the factorial moments of N.

Lemma 6. *The partition function and its first two derivatives are finite for* $\mu = \mu_{\mathrm{cr}}$, *and for* $n > 2$

$$\frac{d^n Z(\mu)}{d\mu^n} \sim c(n)(\mu - \mu_{\mathrm{cr}})^{-\alpha-1-n}$$

as $\mu \to \mu_{\mathrm{cr}} + 0$.

Proof. We shall see below that Z is an algebraic function of $z = e^{-\mu}$ and has the principal singularity $C(1 - \frac{z}{z_0})^{-\alpha-1}$ at the point $z_0 = e^{-\mu_{\mathrm{cr}}}$. In a neighbourhood of z_0 we have

$$1 - \frac{z}{z_0} = 1 - e^{-(\mu-\mu_{\mathrm{cr}})} \sim \mu - \mu_{\mathrm{cr}}.$$

This agrees well with the following simple intuitive argument.

Lemma 7. *For fixed* k, $m_1, \ldots, m_k$, *there are constants* c_1 *and* c_2, $0 < c_1 < c_2 < \infty$, *such that*

$$c_1 N^{k-1} C_0(N) < C_0(N, m_1, \ldots, m_k) < c_2 N^{k-1} C_0(N).$$

Proof. We first take $k = 1$ and prove the upper bound. Let us take some complex $A \in \mathbf{A}_0(m)$ with N faces and glue up the hole using some complex $B \in \mathbf{A}_0(m)$ with r faces, where r depends only on m. We obtain a complex $C = C(A) \in \mathbf{A}_0$ with $N + r$ faces; the number of the complexes thus obtainable is at most $C_0(r, m)$. Therefore, $C_0(N, m) < C_0(r, m) C_0(N + r) < c_2 C_0(N)$. The lower bound and the induction on k can be treated similarly. However, another method is more convenient, namely, we can take a complex in $\mathbf{A}_0$ and construct from it complexes with k holes. To this end, we must arbitrarily choose $k - 1$ faces in the above complex (this gives the factor N^{k-1}) that are adjacent to the boundaries with $m_2, \ldots, m_k$ edges, respectively. Then one should arbitrarily choose closed paths of lengths $m_2, \ldots, m_k$, respectively, starting from each of the selected edges.

4.1.1. Uniform asymptotics. Two questions arise. What is the asymptotics of $C(N, m)$ as both N and m tend to infinity, and what is the asymptotics of other global variables, for instance, of the number of vertices, and so on? We shall see that these two questions are related.

The quantities V, L, and F are well-defined random variables in the grand canonical ensemble, and we may speak of their joint distribution. In general, only two of them are independent because of Euler's formula, $V - L + F = 2$. For triangulations, where each face has three incident edges, we have only one independent variable because $L = 3F/2$. For the class of all rooted charts, where there are two independent variables, we have the following lemma.

Lemma 8. *Let* $\mathsf{E}(V \mid F = N)$ *be the conditional mean number of vertices when the number of faces is* N. *Then*

$$\mathsf{E}(V \mid F = N) \sim cN$$

for some $c > 0$.

As follows from the formula on p. 157 of [22], the number of rooted charts with $N + 1$ faces and $m + 1$ vertices is

$$c(N, m) = \frac{1}{2N - 1}\binom{2N + m - 2}{m}\frac{1}{2m - 1}\binom{2m + N - 2}{N}.$$

Thus,

$$\mathsf{E}(V \mid N) = \frac{\sum_m mc(N, m)}{\sum_m c(N, m)} \sim \alpha_{\max} N,$$

where $\alpha_{\max}$ is defined as the maximum of the function $c(N, m = \alpha N)$ with respect to α (related to the asymptotics of large deviations).

Let us now consider one-particle Green functions.

Lemma 9. *The series*

$$Z(\nu, \mu) = \sum_{N,m} e^{-\mu N - \nu m} C(N, m)$$

is convergent for $\nu > \nu(\mu)$, *where* $\nu(\mu)$ *is a non-increasing function that can be infinite in a neighbourhood of the origin* (*see Fig.* 6).

Proof. The proof is obvious because the coefficients of the series are positive.

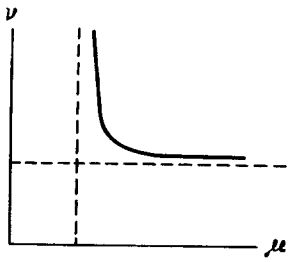

Figure 6. Critical curve

For the family of distributions defined by the partition function $Z(\nu, \mu)$, it is of interest to study the asymptotics and the exponents for the case in which $\mu = a\nu$, where the constant a is fixed, $0 \leqslant a \leqslant \infty$.

This can readily be done by means of the explicit formula (see [22]) for the number $C_0(N, m)$ of triangulations with a distinguished edge on the boundary (rooted triangulations):

$$C_0(N, m) = \frac{2^{j+2}(2m + 3j - 1)!\,(2m - 3)!}{(j + 1)!\,(2m + 2j)!\,((m - 2)!)^2}, \tag{5}$$

where $N=m+2j$ is the number of cells that are not exterior. We have $C_0(N,m)=0$ if $N-m$ is odd.

Hence, as $j,m\to\infty$ we obtain

$$C_0(N,m)\sim \text{const}\,\frac{\sqrt{m}}{j(2m+3j)}\,2^{j+2m}\binom{2m+3j}{j};$$

in particular, for $j\sim\beta m$ with $\beta>0$ we have

$$C_0(N,m)\sim \text{const}\,\frac{1}{m^2}\,2^{j+2m-\beta m\log\frac{\beta}{2+3\beta}-(2+2\beta)m\log\frac{2+2\beta}{2+3\beta}}.$$

Thus, the exponent is $\alpha=-2$ for all β such that $0<\beta<\infty$. For a chosen m, the exponent does not depend on m and is equal to $\alpha=-\frac{5}{2}$. Moreover,

$$C_0(N,m)\sim\phi(m)N^{-\frac{5}{2}}c^N.$$

Furthermore, $\phi(m)\sim Cm^{\frac{1}{2}}c_1^m$ as $m\to\infty$.

4.2. RMM approach. A random matrix model is a probability distribution μ on the set of self-adjoint $n\times n$ matrices $\phi=(\phi_{ij})$ with density

$$\frac{d\mu}{d\nu}=Z^{-1}\exp\Big(-\operatorname{tr}\Big(\frac{\phi^2}{2h}\Big)-\operatorname{tr}(V)\Big),$$

where $V=\sum a_k\phi^k$ is a polynomial in ϕ that is bounded below, and ν is Lebesgue measure on the real n^2-dimensional space of vectors $(\phi_{ii},\operatorname{Re}\phi_{ij},\operatorname{Im}\phi_{ij},i<j)$. This distribution can be rewritten in the form of the density

$$\frac{d\mu}{d\mu_0}=Z_0^{-1}\exp\big(-\operatorname{tr}(V)\big),$$

with respect to the Gaussian measure μ_0. We can readily see that the covariances of the measure μ_0 are

$$\langle\phi_{ij},\phi_{kl}^*\rangle=\langle\phi_{ij},\phi_{lk}\rangle=h\delta_{ik}\delta_{jl}.$$

We note that for the mere existence of the probability measure μ it is necessary that the leading coefficient a_p of V be positive and p even. In this case we can use the well-developed theory of RMM models, which we do not treat here; see [27].

The fundamental relationship (due to t'Hooft) between RMM's and two-dimensional quantum gravity is given by the formal series in semi-invariants or in diagrams (see [28]),

$$\log Z_0=\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\langle\operatorname{tr}(V),\dots,\operatorname{tr}(V)\rangle=\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\sum_{D_k}I(D_k),$$

where $\sum_{D_k}$ is the sum over all connected diagrams with k vertices. For instance, let $V=a_4\phi^4$. Then each diagram has labelled vertices $1,\dots,k$ and $L=2k$ edges,

and each vertex has labelled legs $1, 2, 3, 4$ corresponding to the factors of the product $\phi_{ij}\phi_{jk}\phi_{kl}\phi_{li}$. Each leg, say the one corresponding to ϕ_{ij}, can be represented as a two-sided strip whose sides are marked with the matrix indices i and j. We assume that the coupling of the legs-strips is such that coupled sides have the same indices. Since each index occurs twice at each of the vertices, it follows that, starting from some side with index i, there is a unique path in the diagram along edges with index i. These paths are closed; they are called index loops. Summing over the indices, we obtain the factor n^N, where $N = N(D_k)$ is the number of index loops. As a result we have

$$\sum \frac{(-a_4)^k}{k!} \sum_{D_k} h^{2k} n^{N(D_k)}.$$

For each graph D, there is a minimal cell embedding $f(D)$ of this graph in a compact orientable surface S_ρ of genus ρ [13]. The term "cell embedding" means that each index loop bounds an open domain of the surface $S_\rho \setminus f(D)$ homeomorphic to a disk, and the minimality means that ρ is the minimal genus with this property. The chart $f(D)$ has k vertices, $2k$ edges, and N faces. Setting $a = -a_4$ and using the Euler formula $k = N + 2\rho - 2$, we obtain

$$(-a_4)^k h^{L(D_k)} n^{N(D_k)} = a^k h^{2k} n^N = (ah^2 n)^N (ah^2)^{2\rho-2} = (ha^2)^{-1} \exp(-\mu N - \nu\rho)$$

with $\mu = -\ln(ah^2 n)$ and $\nu = -2\ln(ah^2)$.

The calculations in RMM are usually performed as $n \to \infty$ only, and therefore to obtain finite values of μ one must use a scaling of the form $ah^2 = b/n$ and $b = e^{-\mu}$. Passing to the limit as $n \to \infty$, we obtain $\nu \to \infty$, and only the terms with $\rho = 0$ have minimal order. The limit $\lim_{\mu \to \mu_{cr}} \lim_{n\to\infty}$ is called the simple scaling limit. It was proved in [29] that $\alpha = -\frac{7}{2}$ in this case, which again shows the universal property of this exponent.

Let us stress some important points of this approach.

- For $V = a_4\phi^4$, the order of all vertices is four. Therefore, the dual complex D^* (whose vertices are inside the faces of D) has two-dimensional faces with four edges. The main question is whether this list contains all complexes with quadrangle cells (homeomorphic to closed surfaces) exactly once.
- This question is related to the consideration of the combinatorial factors. Dividing by 4^k, we eliminate the enumeration of the four legs; however, we leave them cyclically ordered. The automorphism group of our diagram can be factorized into two factors. The first one, $C_v(D)$, is related to the permutation of vertices, and the other, $C_l(D)$, is related to the permutation of legs at each vertex. For almost all diagrams the first factor is trivial; however, for some of them we have $C_v(D) > 1$. We can sum over the non-labelled diagrams (which is customary), in which case each diagram acquires the factor

$$\frac{C_l(D)}{C_v D)}.$$

  This means that this counting under consideration does not coincide with the natural counting used above in the combinatorial approach. There are no rigorous proofs of the fact that there is no difference asymptotically.

- It is impossible to have positive coefficients simultaneously for the RMM and on the set of complexes thus obtained. For the RMM, a probability distribution can be obtained for $a < 0$ only, and a probability distribution on the diagrams is achieved only if $a > 0$. Thus, we must justify the possibility of analytic continuation from $a < 0$ to $a > 0$. Therefore, although the free energy for the above scaling can be calculated rigorously, a complete proof that this scaling singles out exactly the planar complexes remains to be found.
- There are other pure gravity models that can be treated by this approach; in these models, the numbers $n(q, T)$ of vertices v with $q_v = q$ are explicitly used,

$$Z = \sum_{T \in \mathbf{A}} \prod_{q>2} t_q^{n(q,T)},$$

where the t_q are parameters (see [4]).

§5. Linear boundary dynamics

The probability distribution

$$P(T) = Z^{-1} \exp\bigl(-\mu F(T)\bigr)$$

on some class $\mathbf{A}$ of complexes is invariant with respect to a simple Markov process $T(t)$ with range in $\mathbf{A}$. The process $T(t)$ is defined by the following infinitesimal transition rates. At the instant t we decompose T into separate cells with some rate $\lambda_+(N)$, where $N = |T|$, add another cell, and randomly glue all cells together, that is, if $N = F(T(t))$, then we choose $T(t + 0)$ according to the uniform distribution on the complexes of class $\mathbf{A}$ with $N + 1$ cells. We also make a random choice of a complex with $N - 1$ cells with rate $\lambda_-(N)$.

If $\lambda_+(T) = bf(N)$ and $\lambda_-(T) = df(N - 1)$ for some positive function $f(N)$ and positive constants b and d, then the probability distribution $P(T)$ is invariant with respect to this process. The proof consists of the remark that the process induced on the numbers N is a reversible Markov chain, namely, a birth-and-death process on $\mathbb{Z}_+$ with jump rates $q_{i,i+1} = bf(i)$ and $q_{i,i-1} = df(i - 1)$ and with $q_{0,1} = 1$.

The simplest method of Monte-Carlo simulation is to choose N as large as possible and simulate the uniform distribution on complexes with N cells. A dynamics in which N is varying in the course of time can be more convenient. In the remaining part of the paper we study the local dynamics. We start with a simplest local dynamics, which does not preserve our measure, but still it is of interest to study its invariant distribution and calculate the exponent. We shall see that we are thus in another universal class (that is, the exponent takes another value).

5.1. Local growth. We consider the class of smooth cell surfaces and assume that the cells are triangles. We begin with one triangle, and each step consists in attaching a new triangle to the boundary. There are two kinds of attachment (see Fig. 7), either to one edge or to two edges with a common vertex. To any edge on the boundary we attach a triangle with rate λ_1. To any pair of neighbouring edges on the boundary we attach a triangle with rate λ_2. At any instant, the complex is homeomorphic to a closed two-dimensional disc, and its boundary to a circle.

We assume that the initial state of the process is a single triangle and that, if the number $m(t)$ of edges on the boundary is equal to three at an instant t, then only λ_1-transitions are possible. We can consider the states with $m = 3$ as triangulations of the sphere for which the exterior of the disc is identified with the face of the sphere that contains the north pole. The instants at which we are in these states can be interpreted as the closing of a hole in the sphere. At the other instants, the state is a triangulation of the disc.

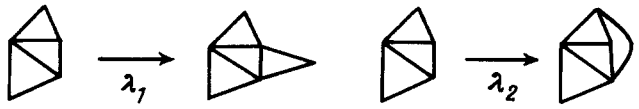

Figure 7. Dynamics on the boundary

Remark 2. It is important to note that there are two modifications of this dynamics. In the first modification we regard the set of equivalence classes of cell surfaces as the set of states. In this case, the transition rates are $\lambda_i m'$, $i = 1, 2$, instead of $\lambda_i m$, where m' is equal to m divided by the order of the automorphism group of the disk triangulation. In the other modification, we single out a (rooted) edge on the boundary, and transitions can occur only if they touch this edge.

There are three different types of behaviour of this Markov process: subcritical or ergodic, critical or null-recurrent, and supercritical or non-recurrent. For all these cases, we study the behaviour of local correlation functions and of the following global variables at time t,

$$F(t) = F(T(t)), \quad V(t) = V(T(t)), \quad L(t) = L(T(t)), \quad m(t) = m(T(t)),$$

where L is the total number of edges and m is the number of edges on the boundary.

Subcritical case. By definition, in this case we have $\lambda_2 > \lambda_1$.

Theorem 6. *If* $\lambda_2 > \lambda_1$, *then*

$$\pi(m = N) = \lim_{t\to\infty} \mathsf{P}\big(m(t) = N\big) \sim_{N\to\infty} CN^{-1} \exp(-\nu N), \qquad \nu = \ln \frac{\lambda_2}{\lambda_1}.$$

Let τ *be the random number of jumps until the first return to the state with* $m = 3$, *and let* $P_0(T) = \mathsf{P}(T(\tau) = T)$ *for any triangulation* T *of the disc. Then*

$$P_0\big(F(T) = N\big) \sim C \left(\frac{\lambda_2 - \lambda_1}{\lambda_1 + \lambda_2}\right)^N.$$

Proof. We first note that the length $m(t)$ of the boundary is itself a Markov process, and the evolution of the boundary can be regarded as a random grammar with the alphabet consisting of a single symbol 1 (representing the only edge) and with the substitutions

$$\lambda_1 \colon 1 \to 11, \qquad \lambda_2 \colon 11 \to 1.$$

This process is obviously reduces to a branching process with one particle type in which λ_1 is the birth rate and λ_2 is the death rate. The Markov chain $m(t)$ (with continuous time) thus obtained whose states are the points of the lattice interval $[3,\infty)$ has jumps ± 1, and the corresponding rates are $\lambda_1 k$ and $\lambda_2 k$ from a point $k \in [3,\infty)$. The initial state is $m(0) = 3$. The stationary measure for the process $m(t)$ is

$$\pi(m=k) = \frac{\prod_{i=3}^{k-1} \lambda_1 i}{\prod_{i=4}^{k} \lambda_2 i} \sim Ck^{-1}\exp(-\nu k), \qquad \nu = -\ln\frac{\lambda_1}{\lambda_2}.$$

Remark 3. We note that, for the dynamics with a local observer, the same considerations show that

$$\pi(m=k) = \frac{\prod_{i=3}^{k-1} \lambda_1}{\prod_{i=4}^{k} \lambda_2} \sim C\exp(-\nu k).$$

As was expected, the exponents differ by one.

Let us consider the (discrete-time) jump process $z_n = m(t_n)$ for the process $m(t)$, where $t_1 < t_2 < \cdots$ are the instants of the jumps. Let $f(s) = \dfrac{1}{\lambda_1+\lambda_2}(\lambda_2 + \lambda_1 s^2)$ be the generating function for the transitions of the jump process and let $f_n(s)$ be its nth iterate. Let $n(\omega)$ be the first instant at which $m(t_n) = 3$. As is known [30],

$$P(n(\omega) = N) = f_N(0) - f_{N-1}(0) \sim Cm^N, \qquad m = \frac{\lambda_2-\lambda_1}{\lambda_1+\lambda_2}.$$

This completes the proof.

Supercritical case. If $\lambda_1 > \lambda_2$, then the boundary has exponential growth. The array (V, L, F, m) behaves like a degenerate branching process with four types of particles and with the following rates:

$$\lambda_1 m\colon V \to V+1, L \to L+2, F \to F+1, m \to m+1,$$
$$\lambda_2 m\colon V \to V, L \to L+2, F \to F+1, m \to m-1.$$

Therefore, for some random variable ξ we have

$$m(t) \sim \xi\exp(\lambda_1-\lambda_2)t, \qquad F(t) \sim c_1\xi\exp(\lambda_1-\lambda_2)t,$$
$$V(t) \sim c_2\xi\exp(\lambda_1-\lambda_2)t, \qquad L(t) \sim c_3\xi\exp(\lambda_1-\lambda_2)t;$$

the proof repeats the analogous argument in [7] with regard to the Euler formula $V - L + F = 1 + m$ and the relation $L = \frac{3F}{2}$.

Critical case and the exponents. If $\lambda_2 = \lambda_1$, then the process $m(t)$ is null-recurrent. One cannot speak of its stationary probabilities, but there is an infinite positive stationary measure. It has the asymptotics $\pi_N = const N^\alpha c^N$ with the exponent $\alpha = -1$.

We can introduce another analogue of stationary probabilities: let us consider $P(T(\tau) = N)$, where $T(\tau)$ is the number of cells at the terminal instant τ.

Lemma 10. *Let* $\lambda_1 = \lambda_2 = \lambda$. *Then*

$$P(T(\tau) = N) \sim \frac{C}{N^2}.$$

This is well known; see [30].

Remark 4. The exponent here is slightly different from the equilibrium case, $\alpha = -2 \neq -\frac{5}{2}$. Moreover, the following phenomenon occurs. The exponents in the ergodic case and in the null-recurrent case are different: $\alpha = 0$ and $\alpha = -2$, respectively. That is, the partition function for the critical case is convergent, but the limit of the partition functions as $\mu \to \mu_{\mathrm{cr}}$ is infinite.

Limit correlation functions. Even if the exponents differ from those of the physical theory, it is of interest to study local correlation functions, which define the fluctuations of the curvature.

We denote by i the vertex that appeared at the ith step. We set $q_i(t) = 0$ if i did not appear before the instant t and we set $q_i(t)$ equal to the number of triangles (or edges) incident to i otherwise. Let $\tau(i)$ be the first instant at which the vertex i appears.

Theorem 7. *There are numbers* χ_k, $\sum \chi_k = 1$, *such that*

$$\lim_{s\to\infty} \lim_{i\to\infty} P\big(q_i(\tau(i)+s) = k\big) \to \chi_k$$

for $\lambda_1 > \lambda_2$. *For any two vertices* $i < j$, *let* $\rho(i,j)$ *be the initial distance between them (that is, the distance at the instant* $\tau(j)$*). Then there are constants* $a, b > 0$ *such that*

$$\Big|\lim_{s\to\infty} P\big(q_i(\tau(j)+s) = k,\, q_j(\tau(j)+s) = l \,|\, \rho(i,j) = d\big) - \chi_k\chi_l\Big| < b\exp(-ad)$$

for any $i < j$. *If* $\lambda_1 \geqslant \lambda_2$, *then*

$$\lim_{s\to\infty} \lim_{m\to\infty} P\big(q_i(\tau(i)+s) = k \,|\, m(\tau(i)) = m\big) \to \chi_k.$$

Moreover, all local correlation functions χ_k *are analytic functions of* λ_1 *and* λ_2 *for all values* $\lambda_1, \lambda_2 > 0$.

Proof. For the vertex i we set $\xi_i(s) = q_i(\tau(i)+s)$. We note that $\xi_i(0) = 2$, and the number q_i increases until both adjacent links to this vertex become covered by a new triangle. Let us consider some vertex i on the boundary and take two edges, $(i-2, i-1)$ and $(i-1, i)$, to the left and two edges, $(i, i+1)$ and $(i+1, i+2)$, to the right of this vertex. In the limit of an infinite boundary we have five independent Poisson processes ξ_{-2}, ξ_{-1}, ξ_0, ξ_1, and ξ_2. The processes ξ_{-2} and ξ_2 describe the simultaneous covering of the pair $(i-2, i-1)$, $(i-1, i)$ of edges and the pair $(i, i+1)$, $(i+1, i+2)$, respectively. The processes ξ_{-1} and ξ_1 describe the covering of the edge $(i-1, i)$ and the edge $(i, i+1)$, respectively. The process ξ_0 is the appending of a triangle to the edges $(i-1, i)$ and $(i, i+1)$. Thus, χ_k, $k \geqslant 2$, is equal to the probability that a Poisson process with rate $2\lambda_1 + 2\lambda_2$ has exactly $k-2$ jumps

before the first jump of another Poisson process that has rate λ_2 and is independent of the first process.

The other assertion of the theorem can be proved quite similarly if we note that the processes $q_i(t)$ and $q_j(t)$ are independent until the distance between i and j becomes less than four. For the distance to become less than four, many λ_2-events must occur "between" i and j, which has exponentially small probability.

We note that, for $\lambda_2 \geqslant \lambda_1$, a similar property of exponential correlation decay holds. Let us define the mean curvature

$$k = \mathsf{E}R = \sum 2\pi \frac{6-q}{q} \chi_q,$$

where the sum is taken over all q. It follows from the exponential decay that the central limit theorem holds for the scaled curvature.

Theorem 8. *For $\lambda_2 < \lambda_1$ the random variables*

$$\frac{\sum_i R_i - kV(t)}{\sqrt{V(t)}}$$

are convergent to the Gaussian distribution as $t \to \infty$.

5.2. Reversible boundary processes. We did not obtain the desired exponent for the dynamics introduced above for rather delicate reasons. Let us give an intuitive probabilistic explanation. The invariant measure under consideration is quite simple: for each N, all complexes are equiprobable. At the same time, one can expect that the invariant measure is rather simple if the process is reversible. The above process is not reversible and, to obtain a reversible process we must add the possibility of deleting faces. For instance, let $\lambda_1 = \lambda_2 = \lambda$; we assume that each boundary triangle can be deleted with rate μ. The following argument shows that this Markov process is reversible. Let α_i be complexes. We consider closed paths of the form $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_1$, where each α_{i+1} is obtained from α_i by appending or deleting a triangle on the boundary. The number of appendings in this closed path $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_1$ must be equal to the number of deletions, and thus it is equal to $n/2$. We denote by λ_{ab} the rate of transitions from the complex α to the complex β. Then

$$\lambda_{\alpha_1\alpha_2} \cdots \lambda_{\alpha_n\alpha_1} = \lambda_{\alpha_1\alpha_n} \cdots \lambda_{\alpha_2\alpha_1} = (\lambda\mu)^{n/2}.$$

However, the deletions can move the modified complex outside chosen class. For example, deleting some triangles, we can obtain a disconnected complex. Thus, one should assume that the deletion is impossible if the resulting complex does not belong to our class **A** of complexes. This "boundary" set of states has a rather complicated structure, and it is difficult to use the standard procedure of obtaining the stationary probabilities via the balance equations. However, by admitting disconnected complexes but preserving the irreducibility of the process, we obtain a (non-local) operation of joining connected components. It is exactly the last operation that leads us to non-linear Markov processes, which we study in the next section.

§6. Non-linear boundary dynamics

In the preceding section we considered a Markov dynamics whose states were complexes. The dynamics was local and homogeneous on the boundary. This setting did not provide the needed exponent. Here we construct a dynamics giving exactly the exponent $\alpha = -\frac{7}{2}$. However, this dynamics is not a Markov process. It can be represented as a system of independent particles (or universes) interacting with one another, but this is not like a Markov infinite system of particles. The simplest pairwise interaction (gluing) of two complexes gives a quadratic functional equation and leads to the desired exponent.

6.1. Quadratic quasi-processes. Let Δ be the set of probability measures on some space S. Let us consider a class of transformations $M\colon \Delta \to \Delta$ that generalize the Markov chains. These transformations are not generated by random maps $S \to S$ and are non-linear on Δ.

Let S be a countable set and let a Markov chain be defined on the state space S with transition probabilities $p_{\alpha\beta}$ from α to β for which $\sum_{\beta:\beta\neq\alpha} p_{\alpha\beta} = 1$. This determines a linear transformation L on Δ by the formula

$$q = \{q_\beta\} \to Lq = \left\{\sum_\alpha q_\alpha p_{\alpha\beta}\right\}.$$

We also assume that a probability kernel is given,

$$P((\alpha,\gamma)\to\beta)\colon S\times S\to S, \qquad \sum_\beta P((\alpha,\gamma)\to\beta) = 1.$$

It is assumed to be symmetric,

$$P((\alpha,\gamma)\to\beta) = P((\gamma,\alpha)\to\beta) \quad \text{and} \quad P((\alpha,\gamma)\to\gamma) = 0.$$

It can even be deterministic. This kernel determines a quadratic transformation Q on Δ by the rule

$$q = \{q_\alpha\} \to Qq = \left\{\sum_{\beta,\gamma} q_\beta q_\gamma P\big((\beta,\gamma)\to\alpha\big)\right\}.$$

Taking a convex combination, we obtain the following transformations (formally, $c_0 + c_1 L + c_2 Q$) on the class of measures on S:

$$q(\alpha,t+1) = r_1 \sum_\beta q(\beta,t)p_{\beta\alpha} + r_2 \sum_{\beta,\gamma} q(\beta,t)q(\gamma,t)P\big((\beta,\gamma)\to\alpha\big) + (1-r_1-r_2)c_0(\alpha), \quad (6)$$

where $c_0(\alpha)$ is a probability measure and r_1 and r_2 are non-negative real numbers such that $0 \leqslant r_1 + r_2 \leqslant 1$. We see that the total mass $\sum_\alpha q(\alpha) = 1$ is preserved.

This can be interpreted as follows. Let us consider countably many particles on S and denote by q_α the mean number of particles at a point α. With probability r_1, each particle (independently of the others) makes a jump according to the probabilities $p_{\alpha\beta}$. This defines a linear transformation. With probability r_2, the particles combine into pairs, and the mean number of pairs (α, β) is the product of the means; moreover, each pair (α, γ), independently of the others, gives birth to a single particle at the point β with probability $P(\alpha, \gamma \to \beta)$. Furthermore, with probability $1 - r_1 - r_2$, immigration of particles to the points α occurs with mean values $c_0(\alpha)$. We stress that there is no stochastic process in the ordinary sense, and we have a transformation of measures only. This seems to be related to field theory for strings (second quantization of strings), but this physical theory still has no mathematical status.

A little general theory. There is no general theory of such quadratic quasi-processes, but we need at least a little of it. Let us represent a quadratic process in the form

$$W = p_1 W_1 + p_2 W_2 + (1 - p_1 - p_2) W_0.$$

Let k_1 be the contraction coefficient for W_1, that is, a number satisfying the relation $\|W_1(\mu_1 - \mu_2)\| \leqslant k_1 \|\mu_1 - \mu_2\|$ for any probability measures μ_1 and μ_2. Let $k_2(x)$ be the contraction coefficient for the stochastic matrix $P_{yz}(x) = P((x, y) \to z)$, and let $k_2 = \sup_x k_2(x)$.

Theorem 9. *Suppose that $p_1 k_1 + 2p_2 k_2 < 1$. If the number of states is finite, then there is exactly one fixed point of W, and the rate of convergence to this point is exponential. If the number of states is denumerable, then the same assertion holds under the assumption that there is a (Lyapunov) function $f(x)$ such that $\sum_x f(x) < \infty$ and $(W\mu)(x) \leqslant f(x)$ for $\mu(x) \leqslant f(x)$.*

Proof. Let us take two probability measures ν and $\mu = \nu + \varepsilon$. We obtain the following contraction property for $\rho(\nu, \mu) = \|\varepsilon\|$:

$$\begin{aligned}
&\rho(W\mu, W\nu) \\
&\quad = \left\| p_1 W_1 \varepsilon + p_2 \sum \mu(x)\varepsilon(y) P\big((x, y) \to z\big) + p_2 \sum \nu(x)\varepsilon(y) P\big((x, y) \to z\big) \right\| \\
&\quad \leqslant (p_1 k_1 + 2p_2 k_2) \|\varepsilon\| .
\end{aligned}$$

This implies the first assertion of the theorem. To prove the other assertion, we note that, because of compactness, there is a fixed point ν in $A = \{\mu : \mu(x) \leqslant f(x)\}$, and for any μ_0 the sequence $W^n \mu_0$ is convergent to ν.

Quasi-processes with continuous time are defined similarly. Instead of the probabilities p, we introduce the rates λ; with some rate λ_1 we perform a linear transformation and with some rate λ_2 we make a quadratic transformation, and immigration occurs with some rate λ_0. The equations for the stationary measure are as follows:

$$(\lambda_1 + \lambda_2 + \lambda_0)\pi(\alpha) = \lambda_1 \sum_\beta \pi(\beta, t) p_{\beta\alpha} + \lambda_2 \sum_{\beta,\gamma} \pi(\beta, t)\pi(\gamma, t) P\big((\beta, \gamma) \to \alpha\big) + \lambda_0 c_0(\alpha).$$

The time evolution is governed by the equation

$$\frac{dq(\alpha,t)}{dt} = \lambda_1 \sum_{\beta} \big(q(\beta,t) - q(\alpha,t)\big) p_{\beta\alpha} + \lambda_2 \sum_{\beta,\gamma} \big(q(\beta,t)q(\gamma,t) - q(\alpha,t)\big) P\big((\beta,\gamma) \to \alpha\big) + \lambda_0 \big(c_0(\alpha) - q(\alpha,t)\big).$$

6.2. Generating functions. For the set S we now take the set of all rooted disc triangulations (with root on the boundary). We also take some probability measure μ_0 on S for the initial measure and assume that the dynamics is given by the Tutte moves shown in Fig. 5. For the measure $c_0(\alpha)$ we take the unit measure on the complex with $N = 0$ and $m = 2$ that consists of a single edge and two vertices. It is convenient to assume that this complex belongs to the class **A** under consideration. All the randomness is included in the probabilities p_0, p_1, and p_2, and the "linear" jumps and the stochastic kernel are assumed to be deterministic,

$$p_{(N,m),(N+1,m-1)} = 1,$$
$$P\big(((N_1,m_1),(N_2,m_2)) \to (N_1+N_2+1, m_1+m_2-1)\big) = 1, \qquad c_0((0,2)) = 1.$$

If the initial measure μ_0 is such that the value of the measure $\mu_0(T)$ on any complex K depends on $N = N(K)$ only, $m = m(K)$, then all iterations $T^n\mu_0$ of μ_0 have this property. We always assume below that this property holds; we claim that the limit of the iterates $T^n\mu_0$ then exists and is unique. In fact, it can be shown that this statement remains valid for any initial probability measure μ_0 because the dynamics "sweeps out" the initial measure from any finite subset of S.

Under our assumption we can consider the projection of this dynamics on $\mathbb{Z}_+^2$, where the points of $\mathbb{Z}_+^2$ are denoted by $\alpha = (N,m)$, N stands for the number of faces, and m is the number of boundary edges. That is, the measure of a point (N,m) is the sum of measures of the corresponding complexes. It turns out that this projection is a quadratic quasi-process of the above type. Let us introduce the generating function

$$U(x,y) = \sum_{N,m=0}^{\infty} q(N,m) x^N y^m.$$

We note that the following homogeneity condition holds: for any α, β, α_1, β_1, and γ such that all points α, β, $\alpha+\gamma$, $\beta+\gamma$, α_1, β_1, $\alpha_1+\beta_1+\gamma$ belong to the quarter plane we have

$$p_{\alpha,\alpha+\gamma} = a_\gamma, \qquad P\big((\alpha,\beta) \to \alpha+\beta+\gamma\big) = P\big((\alpha_1,\beta_1) \to \alpha_1+\beta_1+\gamma\big) = b_\gamma.$$

Thus, the invariant measure $q(N,m)$ of the dynamics is specified by the recurrence equations

$$q(N,m) = r_1 q(N-1,m+1) + r_2 \sum_{N_1+N_2=N-1, m_1+m_2=m+1} q(N_1,m_1)q(N_2,m_2) + r_0 \delta_{(N,m),(0,2)}$$

and the boundary conditions

$$C(N,0) = C(N,1) = 0, \qquad C(0,2) = 1, \qquad C(0,m) = 0, \quad m \neq 2.$$

Multiplying by $x^N y^m$ and adding, we obtain the corresponding equation for the generating functions,

$$U = r_1 U x y^{-1} + r_2 U^2 x y^{-1} + (1 - r_1 - r_2) y^2 - x y^{-1} r_1 \sum_N q(N,2) x^N y^2.$$

One can carry out the scaling

$$x \to x r_1^{-1}, \qquad q(N,m) \to a q(N,m)$$

in the main equation. By setting $a r_2 = 1$ and $\beta = \dfrac{(1 - r_1 - r_2) r_2}{r_1}$, we obtain

$$U = U x y^{-1} + U^2 x y^{-1} + \beta y^2 - x y^{-1} \sum_N q(N,2) x^N y^2.$$

This equation is said to be *canonical*. We note that the case $\beta = 1$, or in other terms $r_1 = \dfrac{r_2(1 - r_2)}{1 + r_2}$, corresponds to the counting problem for complexes and was completely solved by Tutte. In fact, we only reproduce his analysis in a more general setting.

If there is a solution U of the canonical equation with non-negative coefficients $q(N,m)$, then the invariant measure $\tilde{q}(N,m)$ for this dynamics is of the form $\tilde{q}(N,m) = a^{-1} r_1^N q(N,m)$.

We restrict ourselves to the class E of invariant measures satisfying the following conditions:

1) (exponential bounds) $q(N,m) < C^{N+m}$ for some $C > 0$,
2) $q(N,m)$ are non-negative.

The main result is the following theorem.

Theorem 10. *For any r_1 and r_2, a positive invariant measure exists and is unique. It is finite if and only if* $\dfrac{2(1 - r_1 - r_2) r_1 r_2}{27} < 1$.

The theorem is a consequence of the next lemma with regard to the above scaling.

Lemma 11. *For any β there is a solution of the canonical equation with positive coefficients in the class of measures under consideration, and this solution is unique. The series $\sum_{N,m} q(N,m)$ is convergent if and only if* $\sqrt{\frac{2}{27\beta}} > 1$.

6.2.1. Analysis of the functional equation. We prove here the main lemma. By setting $U = y^2 W$ and defining $S_m(x)$ by the relation

$$W(x,y) = \sum_{m=2}^{\infty} S_{m-2}(x) y^{m-2},$$

we can rewrite the functional equation as follows:

$$W = \beta + xyW^2 + xy^{-1}(W - S), \qquad S(x) = S_0(x) = W(x, 0). \tag{7}$$

If the expression $S_0(x)$ is known as a formal series, then all the S_m are determined recursively from it by

$$yS_0 = \beta y + yxS_1, \qquad y^2S_1 = y^2xS_0^2 + y^2xS_2, \dots .$$

We follow in part the discussion in [22]. Let us represent the above functional equation (7) in the form

$$(2xU + x - y)^2 = 4x^2y^2S + (x - y)^2 - 4\beta xy^3 \tag{8}$$

and denote its left-hand side by D. Using the left-hand side, we introduce the analytic set $\{(x, y) : 2xU + x - y = 0\}$ in a small neighbourhood of the point $x = y = 0$. We note that this set is not empty, since it contains the point $(0, 0)$, and it defines a function $y(x) = x + O(x^2)$ in a neighbourhood of $x = 0$. Let us prove that the functions $y(x)$ and $S(x)$ are algebraic. We have two equations that hold at the points of this set:

$$D = 0, \qquad \frac{\partial D}{\partial y} = 0,$$

or

$$\begin{aligned} &4x^2y^2S(x) + (x - y)^2 - 4\beta xy^3 = 0, \\ &8x^2yS(x) - 2(x - y) - 12\beta xy^2 = 0, \end{aligned} \tag{9}$$

from which we can find both $y(x)$ and $S(x)$:

$$x = y(1 - 2\beta y^2), \qquad S = \frac{\beta(1 - 3\beta y^2)}{(1 - 2\beta y^2)^2}.$$

The algebraic function $y(x)$ satisfies the equation $y^3 + py + q = 0$ with

$$p = -\frac{1}{2\beta}, \qquad q = \frac{x}{2\beta}.$$

Its discriminant

$$\Delta = -4p^3 - 27q^2 = p^2\left(\frac{2}{\beta} - 27x^2\right)$$

is not a square in the field of rational functions. Therefore [31], the Galois group is S_3, and the ramification points are $x = \pm\sqrt{\frac{2}{27\beta}}$. Let us (formally) write out the Cardano solution of the cubic equation in the form

$$y = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

We note that

$$\left|\frac{q}{2}\right| \neq \left|\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right|,$$

and thus there is no singularity inside the circle of radius $x_1 = \sqrt{\frac{2}{27\beta}}$. We need a branch with $y(0) = 0$. Since $y(x_1) > 0$, it follows that, if the discriminant vanishes, then $y(x_1) = \sqrt[3]{x_1/(2\beta)}$. Iterating the equation

$$y = \frac{x}{1-2\beta y^2},$$

we see that all coefficients of the expansion of $y(x)$ at the point $x = 0$ are positive. The leading singularity of $y(x)$ is $(x-x_1)^{\frac{1}{2}}$. The function S is algebraic, and it is analytic for $|x| < \sqrt{\frac{2}{27\beta}}$. Indeed, S can have a pole at a point x with $|x| < \sqrt{\frac{2}{27\beta}}$ only if $1 - 2\beta y^2(x) = 0$, which yields $x = 0$, and this is impossible. We write $y_0(x) = y(x)$ and make the substitution

$$S = \frac{\beta(1-3\beta y_0^2(x))}{(1-2\beta y_0^2(x))^2}, \qquad x = y_0(x)(1-2\beta y_0^2(x))$$

in (8). In the expansion of S, two terms with opposite sign give a cancellation of the lowest-order singularity $(x-x_1)^{1/2}$ of the function $y_0(x)$, and therefore the function S has leading singularity of the form $(x-x_1)^{3/2}$. To represent the expansion of U explicitly, we write

$$\begin{aligned} D &= 4y^2y_0^2(1-3\beta y_0^2) + \left(y_0(x)(1-2\beta y_0^2(x)) - y\right)^2 - 4\beta y^3 y_0(x)(1-2\beta y_0^2(x)) \\ &= (y-y_0)^2(a(x)+b(x)y), \end{aligned}$$

because $y = y_0(x)$ is a double root of the main equation, and

$$a(x) = (1-2\beta y_0^2(x))^2, \qquad b(x) = -4\beta y_0(x)(1-2\beta y_0^2(x)).$$

Choosing the minus sign, we obtain

$$U(x,y) = \frac{y-x}{2x} - \frac{\sqrt{D}}{2x} = \frac{-x+y_0(x)}{2x} - (y-y_0(x))\frac{\sqrt{a(x)+b(x)y}-1}{2x},$$

which gives a legitimate expansion.

For a given x, the radius of convergence of U regarded as a function of y is determined by the zeros of $\sqrt{a(x)+b(x)y}$, or $\sqrt{1-\frac{4\beta y_0^2}{x}y}$. Since $\frac{y_0^2}{x}$ increases on the interval $[0,x_1]$, it follows that the radius of convergence for $x = x_1$ is $R = \frac{x_1}{4\beta y_0^2(x_1)}$.

6.3. Correlation functions. Using the representation below of complexes by trees, we shall prove that for most vertices the conditional distributions of the random variables q_v are convergent to a common limit as $N \to \infty$. To this end, we use a special method of singling out a vertex v in a recursively constructed complex. We discussed this problem in [6] and [7] in a more general situation. The proof is combinatorial and has no analytic nature; it is rather sophisticated, and we present it in an informal way.

6.3.1. Representation of complexes by trees. The process of constructing complexes by the successive use of Tutte moves (or "Tutte substitutions") can be represented as a planar tree. This gives a one-to-one correspondence between the set of complexes and a certain class of planar trees. A planar tree has a rooted vertex and grows upwards (this is shown in Fig. 8 by the directions of the arrows). Let $\mathbf{A}_0$ be the class of rooted disc triangulations. We denote the vertices of trees by the symbol v and the vertices of complexes by w.

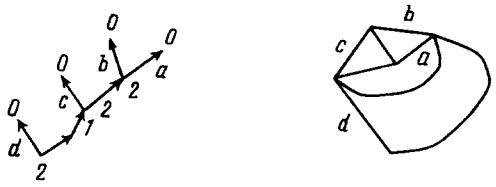

Figure 8. Planar trees and charts

Let us introduce a class $\mathbf{T}_0$ of planar trees defined as follows. Each tree of this class can have three types of vertices: 0, 1, 2, according to the number of edges going upwards from a given vertex. The vertices of type 0 are also called end vertices. Let n_i be the number of vertices of type i. Among 2-vertices (that is, vertices of type 2) there are vertices incident (that is, connected by an edge) to a single 0-vertex; we denote the number of such vertices by n_{20}, and the number of 2-vertices that have two incident 0-vertices by n_{00}. For any vertex v we denote by T_v the tree formed by v and all vertices above v. We denote by $n_i(v)$ and so on the corresponding numbers for the tree T_v and note that $n_0(v) = n_2(v) + 1$. In these terms the class $\mathbf{T}_0$ is characterized by the following restriction: $n_0(v) - n_1(v) - 1 \geqslant 0$ for all vertices v of type 1.

Lemma 12. *The construction below defines a one-to-one correspondence between* $\mathbf{A}_0$ *and* $\mathbf{T}_0$.

Let us show first that each complex generates a planar tree belonging to $\mathbf{T}_0$, that is, there is a function f from the set of complexes to the set of planar trees. We prove this by induction on the number of faces. The complex itself is represented by the rooted vertex of the tree. We reduce the complex by inverse Tutte substitutions in which we draw upwards one edge in the case of a substitution of type 1 and two edges in the case of a substitution of type 2 that divides the complex into two complexes. These two complexes we associate with the right and left vertices, according to the orientation of the rooted edge of the complex, by the following rule. The rooted edges of both new complexes have the same orientation, and one of them succeeds the other. To the left vertex we associate the complex with the preceding rooted edge and to the right vertex that with the subsequent rooted edge. At each step, the number of faces is reduced by one, and we proceed by induction. At a step at which we obtain a complex formed by a single edge (an edge complex), the corresponding vertex becomes an end vertex of the tree.

Let us note that $N = n_1 + n_2$, $L = n_0 + n_1 + n_2$, $V = n_0 + 1$, and $m = n_0 - n_1 + 1$. Here N, L, m, and V are related to the complex, and the numbers n_i, n_{00}, and so

on to the tree. To prove the last of the above relations, we note that each 1-vertex and each 2-vertex (except for the vertices that are incident to 0-vertices) reduce the number m by one, each of the n_{20} vertices increases m by one, and each of n_{00} vertices increases m by three. Thus, $m = n_{20} + 3n_{00} - n_1 - (n_2 - n_{00} - n_{20}) = n_0 - n_1 + 1$. Since $m \geqslant 2$ for the complexes under consideration, we obtain the restriction introduced above.

Let us show now that each planar tree of class $\mathbf{T}_0$ generates a complex, that is, f is one-to-one. Let us take a tree and proceed by induction from upper vertices to lower ones. Any end vertex is declared to be an edge complex. Let us take V edge complexes and enumerate them from left to right as the end vertices of the tree, $v = 1, 2, \ldots, V$. These edges give $V + 1$ vertices of the complex. Let us mark all these end vertices. At each step of the induction we take a non-marked vertex v such that all vertices above it are marked. If v is of type i, then at the given step of the induction we use a Tutte substitution of type i. After this step we mark the vertex v. Conversely, a complex constructed in this way is mapped by the function f to the tree from which we started. All complexes in the course of induction are legitimate because $m \geqslant 2$. This proves the lemma.

The contribution of the tree $G(T)$ is defined as the product $r_0^{n_0} r_1^{n_1} r_2^{n_2}$, where $n_i = n_i(T)$ is the number of vertices of type i, which is certainly equal to the probability of the corresponding rooted complex.

Remark 5. The planar trees are in one-to-one correspondence with the systems of parentheses that can be arranged in a product $a_1 \cdots a_n$ in a non-associative and non-commutative algebra, see Fig. 9. However, the above restriction on $\mathbf{T}_0$ makes the situation more complicated.

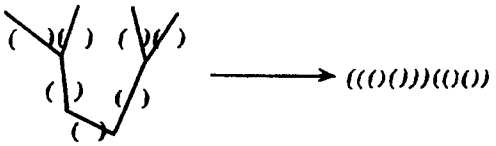

Figure 9. Planar trees and parentheses

6.3.2. Local curvature. Let us take an end vertex v and set $l(v) = \max(v, V - v)$.

Theorem 11. *As* $l(v) \to \infty$ *(in which case* $V \to \infty$ *and* $N \to \infty$*), the equilibrium distribution* P *has the limit*

$$\lim P(q_v = k) = p(k), \qquad \sum_{k=2}^{\infty} p(k) = 1.$$

Proof. Corresponding to any end vertex is a complex consisting of a single edge. The number of these complexes is V. Let us imagine that these edges are placed horizontally along a horizontal line and are directed from left to right. Let us take an end vertex v and the left (that is, the rooted) vertex $w = w(v)$ of the corresponding edge complex. We choose the orientation of the boundary to be anticlockwise. Let us define a history of each vertex $w(v)$. This history can be described both in

terms of the complex and in terms of the tree, and it is convenient to use both descriptions. By definition, a history consists of several parts, the so-called stages of the history, which are formed by a unique path from v to the root of the tree T downwards. From the viewpoint of complexes, the history terminates either at the instant the vertex $w(v)$ disappears from the boundary of the complex (this means that q_w stops changing) or at the root of the tree if w remains on the boundary of the complex. The first part of the history continues until the vertex is covered by an edge, and q_w can change only on such pieces of the history. From the viewpoint of trees, the first part of the history starts from the end vertex v, goes downwards from vertex to vertex, and continues until it meets the first right 2-vertex. A 2-vertex is said to be a right vertex if we (that is, the vertex w) arrived at this vertex from the right, that is, the vertex $w(v)$ belongs to the right complex. The next part of the history starts at an instant $w(v)$ arrives at the rooted edge of the complex, and it terminates exactly as the first part does.

If a vertex a of the tree is below the vertex $v = w(v)$, then w belongs to the corresponding complex $f^{-1}(a)$. We denote by $a(w)$ the least a such that $a < v$ and the vertex w belongs to the boundary of $f^{-1}(a)$.

Let us denote by $b(w)$ the least vertex b such that $b < v$ and the vertex w is a rooted vertex of the corresponding complex for any b_i belonging to the interval $v > b_1 > b_2 > \cdots > b$ in the tree. The length $L(v(w))$ of this interval (path) in the tree is exactly equal to the difference $q_w(j) - q_w(k)$, where j is the first complex of the path (that is, an edge complex) and k is the last one. We claim that $P_V(L(v(w)) = n) < C\exp(-\gamma n)$ for some $\gamma, C > 0$ and note that, for the proof, it suffices to consider an arbitrary invariant measure because for each of these measures the conditional probabilities (for chosen N and m) are the same (we are mainly interested only in the case $m = 3$).

If one deletes all 1-vertices from the tree, then the resulting tree (without 1-vertices) is said to be bare. It defines an equivalence class of dressed trees each of which can be obtained by appending several 1-vertices to it. Each non-negative measure on the set of trees induces a measure on the equivalence classes, that is, on the bare trees. We begin with these.

The case of bare trees. If there are no 1-vertices at all, then $m = n_0 + 1 = N$, and the number of faces is also equal to m. In this case, all vertices belong to the boundary of the complex and, after a vertex is covered by an edge, it stops participating in the process, that is, its history consists of only one part. Moreover, this vertex can be covered by an edge only when something is joined from the left to the complex containing w, and until this happens, there can be, say, k adjunctions from the right, which give k extra edges to q_w. Under any right adjunction we add to w some complex that corresponds to a tree covering an interval to the right of v.

For example, let us estimate the probability $P(k; j_1, \ldots, j_k)$ that there are exactly k adjunctions of intervals of the lengths $j_1, \ldots, j_k$ that cover the interval $I_r = \{v + 1, \ldots, v + R\}$ and occur before the left adjunction that covers the interval $I_l = \{v - L, \ldots, v - 1\}$ for some L. We make use of the Markov property that can be described as follows. Let I be an interval. A vertex $v' = v(I)$ is said to be I-separating if it exactly covers the interval I. Then the number of trees with separating vertex v' is equal to $t(I)t(V - I + v')$, where $t(I)$ is the number of trees

on the interval I, because $t(V - I + v')$ is also the number of quotient trees with respect to the set of subtrees on I. Thus, the probability that v' is I-separating can be estimated as

$$\frac{t(I)t(V - I + v')}{t(V)} < \min(b, cI^{-3/2}),$$

where $t(n+1) = \dfrac{1}{n+1}\binom{2n}{n} \sim cn^{-\frac{3}{2}}2^n$ are the well-known Catalan numbers and $b < 1$. In particular, according to this relation, the probability P_v that a right adjunction with a vertex v occurs before a left adjunction does not exceed some constant $P_v < d < 1$.

Therefore, by the Markov property, we see by induction that

$$P(q_v \geqslant r) < a^r$$

for large values of r and for some $a < 1$. Thus, we have obtained exponential estimates. The existence of the thermodynamic limit follows from the fact that the influence of the boundaries manifests itself only with probabilities less than $a^{l(v)}$.

The case $n_1 > 0$. We choose here a bare tree and consider an auxiliary problem concerning the distribution of 1-vertices on it. We obtain estimates that are uniform for bare trees.

We consider first the probability that the vertex w acquires a large value of q_w due to 1-vertices before it is covered for the first time. We find such estimates for each part of the history separately, and consider in detail only the first part of the history. Let v be a vertex at which the first part of the history is terminated. Let T_v be the tree over this vertex (with root at v) and let $f^{-1}(v)$ be the corresponding complex.

We consider first the case in which T_v has only one 00-vertex. In this case, 0-vertices are joined in succession to trees that already exist. This means that the possible ways of arranging 1-vertices can be characterized as the number of all possible ordered tuples of non-negative integers, $a_1, \ldots, a_n$, such that $\sum_{i=1}^{k} a_i \leqslant k - 2$ for all $k = 1, 2, \ldots, n$.

We can pose the following abstract urn problem. Suppose that we have n urns and m balls placed in these urns. Let a_i be the number of balls in the ith urn. Let $c(n, m)$ be the number of arrangements $a_1, \ldots, a_n$ of balls satisfying the conditions $\sum_{i=1}^{k} a_i \leqslant k$ and $\sum_{i=1}^{m} a_i = m \leqslant n$. The following recurrence relation holds:

$$c(n, m) = \sum_{k=0}^{m} c(n-1, m-k),$$

or

$$c(n, m) = c(n, m-1) + c(n-1, m-1),$$

which readily implies the asymptotics for the number $c(n, m; i) = c(n-1, m-i)$ of arrangements among the above ones (whose number is $c(n, m)$) such that there

are exactly i balls in the last urn. We can easily obtain an explicit formula for the generating function,

$$f(x,y) = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} c(n,m) x^n y^m = \frac{y}{1-y(1+x)} \sum_{k=1}^{\infty} k x^k,$$

whose coefficients coincide with ours for $n \geqslant m$. We claim that $\dfrac{c(n,m)}{c(n,m-1)} < b$ for some $b < 1$, and we want to find a method (different from that of generating functions) that might work in a more general situation. To this end, we first rewrite $c(n,m)$ in terms of the number of paths from the line $m = 1$ to the point (n,m), where a step of the path can be either $(0,1)$ or $(1,1)$. We have

$$c(n,m) = \sum_{k=m+1}^{n} kL(k;n,m),$$

where $L(k;n,m)$ is the number of paths from the point $(k,1)$ to the point (n,m) (because $c(k,1) = k$). Since $L(k;n,m-1) = L(k-1;n-1,m-1)$, we see that $\dfrac{c(n,m-1)}{c(n-1,m-1)} = 1 + O(1/m)$. This proves our result.

Let us pass now to the general case. Instead of the urn problem on the interval $[1,n]$, we have an urn problem on an arbitrary planar tree T satisfying the conditions

$$\sum_{i \in T_v} a_v \leqslant V(T_v), \qquad \sum_{v \in T} a_v = m,$$

where a_v is the number of balls in the urn (at the vertex) v of the tree and $V(T_v)$ is the number of vertices of the tree T_v. Let $c(T_v,m)$ be the number of such arrangements on the tree T_v with m balls. For example, if there are two edges going upwards from the vertex v to vertices $v(1)$ and $v(2)$, then

$$c(T_v,m) = \sum_{i=0}^{m} c(T_v,m;i), \qquad c(T_v,m;i) = \sum_{m_1+m_2=m-i} c(T_{v(1)},m_1) c(T_{v(2)},m_2).$$

The rest of the argument is similar to the above. Let us note that $c(T_v,1) = V(T_v)$. We want to compare $c(T_v,m;i)$ and $c(T_v,m;i+1)$; to this end, we iterate the last recurrence relation for $c(T_v,m;i+1)$ to the very end, that is, we obtain the sum of terms of the form B_s^{i+1} in each of which all factors are of the form $c(T_{v'},1)$ for some v'. To the iteration process for $c(T_v,m;i+1)$ there corresponds a similar process for $c(T_v,m;i)$, and thus to each summand B_s^{i+1} there corresponds a summand B_s^i in the expansion for $c(T_v,m;i)$. In this summand, one of the factors is $c(T_{v'},2)$ instead of the factor $c(T_{v'},1)$ in the summand B_s^{i+1}. Thus, we have $\dfrac{c(T_v,m)}{c(T_v,m-1)} \gtrsim 2$ as above. This yields estimates that are uniform with respect to the bare trees. The influence of the boundary is again exponentially small.

Repeating this argument for two-particle functions, we can prove the following theorem.

Theorem 12. *As* $V \to \infty$,

$$|\langle q_{v_1} q_{v_2} \rangle - \langle q_{v_1} \rangle \langle q_{v_2} \rangle| < c \exp(-\alpha |v_1 - v_2|)$$

for any two vertices v_1 *and* v_2 *such that* $l(v_i) \to \infty$.

§7. Internal dynamics

In the previous sections we considered the dynamics for which changes during one step are possible either only in a neighbourhood of the rooted edge or only on the boundary of a disc triangulation. Here we consider a dynamics for which changes are possible at any place of the complex.

Thermodynamic limit of local processes. For a given stochastic dynamics, by a local process we mean the process $q_v(t)$ given by the number of edges incident to a given vertex v at an instant t. It is assumed that this vertex, which was singled out at the instant $t = 0$, cannot disappear. Let us present two examples of interesting behaviour of such local processes.

The first example is related to Gross–Varsted moves. We note that these can be introduced not only for simplicial complexes but also for the other classes above of complexes with triangle cells. Let v be a vertex. By the star $\mathrm{St}(v)$ of v we mean the edges and triangles incident to it, together with their boundaries.

Let us consider GV-moves of types 1 and 2 and the move inverse to a GV-move of type 2, and a Markov chain with state space the set of complexes of class **A** in which all cells are triangles. The jumps are defined by the jump rates λ_i, $i = 1, 2$, and μ of these moves, for each pair of adjacent triangles in the case of a move of type 1, for each triangle in the case of a move of type 2, and for the star of each vertex of order three in the case of the move inverse to a GV-move of type 2, respectively.

If $\lambda_2 = \mu = 0$, then V, L, and N are invariants. Let $A' \subset A(N, L)$ be an irreducible component of the set of complexes with given N and L. We set $C(A') = |A'|$. We state the next lemma without proof.

Lemma 13. *If $\lambda_2 = \mu = 0$ and $\lambda_1 = \lambda$, then the Markov chain on each component A' is reversible with respect to the uniform measure. On the class of simplicial complexes, the unique irreducible component coincides with the entire class.*

We merely note that the first assertion of the lemma is verified by the condition $\pi_\alpha \lambda_{\alpha\beta} = \dfrac{\lambda}{C(A')} = \pi_\beta \lambda_{\beta\alpha}$ for $\lambda_{\beta\alpha} = \lambda_{\alpha\beta} = \lambda$.

At the instant 0 we take a complex with N vertices, choose a vertex v in it, and consider the random variables $q_v^{(N)}(t)$ which are the numbers of edges incident to v at time t. The following example shows that the large-time limits and the large-N limits can fail to commute, that is,

$$\lim_{t\to\infty} \lim_{N\to\infty} \neq \lim_{N\to\infty} \lim_{t\to\infty}$$

for local quantities. Thus, Monte-Carlo simulation can be misleading in this case.

Lemma 14. *As $N \to \infty$, the limit process exists and is a random walk on $[3, \infty)$ with transition rates $\lambda_{i,i+1} = \lambda_{i,i-1} = \lambda i$. Moreover,*

$$\lim_{t\to\infty} \lim_{N\to\infty} P(q_v^{(N)}(t) = k) = 0.$$

At the same time,

$$\lim_{N\to\infty} \lim_{t\to\infty} P(q_v^{(N)}(t) = k) \neq 0.$$

Proof. We have $L=3N/2$, $V=(N+4)/2$, and $\lim_{N\to\infty}\lim_{t\to\infty} q_v^{(N)}(t)=L/V=3$. We also note that for a chosen N the process $q_v^{(N)}(t)$ has the Markov property, with state space $3,\dots,N$ and rates $\lambda_{i,i+1}=\lambda_{i,i-1}=\lambda i$. In fact, each edge incident to v can be changed to a transversal one; moreover, for each triangle containing v, its edge not containing v can be deleted by a GV-move, which gives an additional edge incident to v. The limiting random walk is null-recurrent. This proves the lemma.

Note that we have made essential use of the fact that the star of a vertex v looks like an umbrella with centre at v.

Let us consider now a Markov chain in which the only transitions are A-moves (Alexander moves). Each vertex i competes for the right to make an A-move; in other words, two Poisson processes go out from this vertex, with rates λ and μ, respectively, and the first event occurring at one of the vertices accomplishes an A-move. If the event corresponds to the first process, then we randomly choose (with probability q_i^{-1}) one of the edges on the boundary of $\mathrm{St}(i)$ and make an A-move corresponding to this edge. If the event corresponds to the other process, then we randomly choose one of the vertices v of degree four on the boundary $\partial\,\mathrm{St}(i)$ and one of the two possible inverse A-moves, and precisely this move is performed. If there are no vertices of degree four on $\partial\,\mathrm{St}(i)$, then no moves are made. After this, the process starts from the beginning.

We note that a vertex i that has appeared in this way can later disappear. Let $t(i)$ be the instant at which the vertex i appears.

Theorem 13. *If $\lambda>\mu$, then $q_i(t)\to\infty$ with positive probability. If $\lambda<\mu$, then the vertex disappears almost surely, and $\mathsf{E}q_i(t)$ is uniformly bounded.*

Proof. For each vertex v, let $a(t)=a_v(t)$ be the number of vertices j on $\partial\,\mathrm{St}(v)$ with $q_j=4$, and let $b(t)=q_v(t)-a(t)$. Let us choose a vertex i. If $v\in\partial\,\mathrm{St}(i)$, then we denote by $v+1$ the next vertex on $\partial\,\mathrm{St}(i)$ in the clockwise direction.

Let us consider the process $(a_i(t),b_i(t))$ and, for a fixed configuration outside $\mathrm{St}(i)$, write out its infinitesimal jumps in $\mathbb{Z}_+^2$. For a direct and inverse A-move, there are only three possibilities.

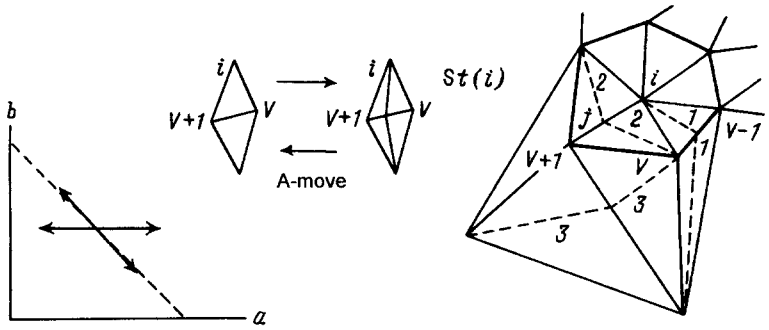

Figure 10. Proof of the theorem

1. Two edges (marked by 1 in Fig. 10) appear on some edge (this is the edge $(v-1,v)$ in Fig. 10). Thus, in this case, the transition is $a,b\to a+1,b$

with rate $2\lambda\frac{q}{q} = 2\lambda$. Here and below, the factor 2 occurs because the same move can also be produced by the opposite vertex. The inverse move $a, b \to a, b+1$ has the rate $2\mu\frac{a}{a} = 2\mu$.

2. This move is produced by the vertex $v \in \partial \mathrm{St}(i)$ (the dotted edges 2 in Fig. 10), and a new vertex appears on the edge $(i, v+1)$. This produces a change in the vector (a, b) only if $q_{v+1} \neq 4$. Thus, here we have the transition $a, b \to a+1, b-1$ with rate $2\lambda a\frac{b}{q_v}$. The inverse move gives the jump $a, b \to a-1, b$ with rate $2\alpha\mu\frac{a}{a_v}$.
3. The next move is also produced by the vertex v (the edges marked by 3 in Fig. 10). For q_v, the transitions $4 \to 3$ and $5 \to 4$ can occur.

In fact, we do not need the rates for Cases 2 and 3: we note only that these jumps preserve $q_i = a + b$. We assume first that $\mu > \lambda$. In this case the embedded process $f(n) = q_i(t_n) = a(t_n) + b(t_n)$, where t_n are the jump instants, satisfies the inequality

$$\mathsf{E}(f(n+1) \mid f(n)) < f(n) - \varepsilon$$

for some $\varepsilon > 0$. The proof can be given by means of the apparatus of submartingales (see, for example, [32]). For $\mu < \lambda$ we have

$$\mathsf{E}(f(n+1) \mid f(n)) > f(n) + \varepsilon,$$

and we use a similar method.

It seems plausible that if $\lambda < \mu$, then, for all sequences $t(i)$ tending to infinity, the process $q_i(t)$ tends to some distribution if $s = t - t(i)$ is fixed. For $\lambda \ll \mu$, this can be rigorously proved. On the other hand, for the critical case $\mu = \lambda$, the random variables $q_i(t)$ fluctuate as in Brownian motion. Comparison with the results of the previous sections gives an argument that we do not obtain the desired invariant measure under this dynamics.

Bibliography

[1] T. Regge, "General relativity without coordinates", *Nuovo Cimento* **19** (1961), 558–571.

[2] C. Misner, K. Thorne, and J. Wheeler, *Gravitation*, Freeman, San Francisco 1973.

[3] S. Hawking, "Space-time foam", *Nuclear Phys.* B **144** (1978), 349–361.

[4] V. Kazakov, M. Staudacher, and T. Wynter, "Exact solution of discrete two-dimensional R^2 gravity", *Nuclear Phys.* B **471** (1996), 309–333.

[5] C. Rovelli, "Strings, loops and others: A critical survey of approaches to quantum gravity", LANL e-print gr-qc/9803024 (1998).

[6] V. A. Malyshev, "Random grammars", *Uspekhi Mat. Nauk* **53**:2 (1998), 107–134; English transl., *Russian Math. Surveys* **53** (1998), 345–370.

[7] V. A. Malyshev, "Stochastic evolution via graph grammars" Rapport de Recherche INRIA, no. 3380, March 1998; see also *Diskret. Mat.* **10**:2 (1998), 30–44; English transl., *Discrete Math. Appl.* **8** (1998), 247–262.

[8] R. Penrose, *The emperor's new mind. Concerning computers, minds and the laws of physics*, Oxford Univ. Press, Oxford 1990.

[9] J. Ambjørn, *Quantization of geometry*, Les Houches Lectures 1994, North-Holland, Amsterdam 1996.

[10] V. A. Malyshev, "Interacting strings of symbols", *Uspekhi Mat. Nauk* **52**:2 (1997), 59–86; English transl., *Russian Math. Surveys* **52** (1997), 299–326.
[11] A. Gairat, R. Iasnogorodski, and V. Malyshev, "Null recurrent random string", *Markov Process. Related Fields* **2** (1996), 427–460.
[12] A. T. Fomenko and D. B. Fuks, *A course in homotopic topology*, Nauka, Moscow 1989. (Russian)
[13] J. Gross and T. Tucker, *Topological graph theory*, Wiley, New York 1987.
[14] S. Sternberg, *Lectures on differential geometry*, Prentice-Hall, Englewood Cliffs, New Jersey 1964.
[15] J. Alexander, "The combinatorial theory of complexes", *Ann. of Math.* (2) **31** (1930), 292–320.
[16] M. Gross and S. Varsted, "Elementary moves and ergodicity in D-dimensional simplicial quantum gravity", *Nuclear Phys.* B **378** (1992), 367–380.
[17] W. Tutte, "On the enumeration of convex polyhedra", *J. Combin. Theory Ser.* B **28** (1980), 105–126.
[18] N. Wormald, "On the number of planar maps", *Canad. J. Math.* **33** (1981), 1–11.
[19] N. Wormald, "Counting unrooted planar maps", *Discrete Math.* **36** (1981), 205–225.
[20] L. Richmond and N. Wormald, "Almost all maps are asymmetric", *J. Combin. Theory Ser.* B **63** (1995), 1–7.
[21] A. B. Zamolodchikov, "On the entropy of random surfaces", *Phys. Lett.* B **117** (1982), 87–90.
[22] I. Goulden and D. Jackson, *Combinatorial enumeration*, Wiley, New York 1983.
[23] W. Tutte, "The enumerative theory of planar maps", in: *A survey of combinatorial theory* (Proc. Internat. Sympos., Colorado State Univ., Fort Collins, Colo., 1971) (J. Srivastava et al., eds), North-Holland, Amsterdam 1973, pp. 437–448.
[24] W. Tutte, "A census of planar maps", *Canad. J. Math.* **15** (1963), 249–271.
[25] W. Tutte, "What is a map?", in: *New directions in the theory of graphs* (Proc. Third Ann Arbor Conf., Univ. Michigan, Ann Arbor, Mich., 1971) (F. Harary, ed.), Academic Press, New York 1973, pp. 309–325.
[26] E. Bender and L. Richmond, "A survey of the asymptotic behaviour of maps", *J. Combin. Theory Ser.* B **40** (1986), 297–329.
[27] L. A. Pastur, "Spectral and probabilistic aspects of random matrix models", in: *Algebraic and geometric methods in mathematical physics* (A. Boutet de Monvel and V. A. Marchenko, eds.), Kluwer, Dordrecht 1996, pp. 205–242.
[28] V. A Malyshev and R. A Minlos, *Gibbs random fields*, Nauka, Moscow 1985; English transl., Kluwer, Dordrecht 1991.
[29] D. Bessis, C. Itzykson, and J. Zuber, "Quantum field theory techniques in graphical enumeration", *Adv. in Appl. Math.* **1** (1980), 109–157.
[30] K. Athreya and P. Ney, *Branching processes*, Springer-Verlag, New York-Heidelberg 1972.
[31] S. Lang, *Algebra*, Addison-Wesley, Reading, MA 1965.
[32] G. Fayolle, V. Malyshev, and M. Menshikov, *Topics in the constructive theory of countable Markov chains*, Cambridge Univ. Press, Cambridge 1995.

Moscow State University

Received 18/SEP/98

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX