MULTICOMPONENT RANDOM SYSTEMS

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CONTENTS

chapter 16

COMPLETE CLUSTER EXPANSIONS FOR WEAKLY COUPLED GIBBS RANDOM FIELDS

V. A. Malyshev

It is well known [3] that cluster expansions used to derive uniqueness, analyticity and exponential decay of correlations for weakly coupled Gibbs random field are called vacuum cluster expansions. The reason is that they give uniqueness of the vacuum and the lower mass gap.

Our terminology "complete cluster expansion" follows closely one of Glimm, Jaffe, and Spencer [1]. They deal with N-particle cluster expansions where the cluster expansion provides some information about the spectrum of the Hamiltonian in the N-particle region. The methods of [1] allow such expansions for $N \leq N_o(\beta)$, where β is a small coupling constant and $N_o(\beta) \to \infty$ when $\beta \to 0$.

Here, we treat the expansion which gives information about the N-particle region for all N uniformly in β such that $|\beta| < \beta_o$. In this article we consider only lattice Gibbs random fields with bounded interaction potential.

Our methods are quite different from [1]. We shall work in a special basis which was first used in [2].

The cluster expansion proofs are crucially based upon delicate estimates of semiinvariants together with rather complicated combinatorial lemmas. They seem to be of independent interest, and are essentially better than all known estimates.

This expansion seems to be the necessary step towards asymptotic completeness.

The main results are formulated in § I. The next paragraphs (§§ 2-4) contain detailed estimates for semiinvariants. The proof of the main result is given in § 5-7.

This article is a more general and refined version of [7]. It contains, in particular, the case of a gauge field.

I. THE COMPLETE CLUSTER EXPANSION

We consider the set $T = T_o \times \mathbb{Z}$, where $\mathbb{Z}$ is the one-dimensional lattice and T_o is an arbitrary denumerable set. Furthermore, we identify T_o with $T_o \times \{0\} \subset T$. Assume that for each $t \in T$ we are given the probability triple $(\Omega_t, \Sigma_t, \mu_t^{(o)})$. We define cartesian products

$$\Omega = \underset{t\in T}{\times} \Omega_t ; \qquad \Sigma_A = \underset{t\in A}{\times} \Sigma_t , \qquad A \subset T,$$

$$\mu^{(o)} = \underset{t\in T}{\times} \mu_t^{(o)}, \quad \Sigma = \Sigma_T , \qquad \Sigma_o = \Sigma_{T_o}$$

The function F on Ω is said to be <u>local</u> iff it is measurable w. r. t. Σ_A for some finite $A \subset T$. The minimal such A is the <u>support</u> of F.

Also, assume that we are given a system $\mathfrak{A}$ of finite sets $A_i \subset T$ such that

$$|A_i| \leq d_1 < \infty \tag{1. 1}$$

$$\sup_{t\in T} |\{i : t \in A_i\}| < d_2 < \infty \tag{1. 2}$$

for all i and all

$$t_1 = (t_1^{(o)}, \tau_1) \in A_i \tag{1.3}$$

and

$$t_2 = (t_2^{(o)}, \tau_2) \in A_i \quad \text{with} \quad t_i^{(o)} \in T_o$$

$$\tau_i \in \mathbb{Z} \quad \text{such that} \quad |\tau_1 - \tau_2| \leq 1$$

For the proof of our main result we must also assume that T_o is a lattice in $\mathbb{R}^\nu$, i. e. a denumerable set in $\mathbb{R}^\nu$ which is invariant w. r. t. translations by ν linearly independent vectors. We note that this assumption is used only in the proof of Lemmas 6. 1 and 1. 1. In this case we also assume that for all i

$$\text{diam } A_i \leq d_3 < \infty \tag{1.4}$$

For arbitrary $A_i \in \mathfrak{A}$, let Φ_{A_i} be a function with support A_i. Such a system $\Phi = (\Phi_{A_i})$ is called a potential. We will always assume that Φ_{A_i} is uniformly bounded, i. e.

$$|\Phi_{A_i}| \leq C_\Phi < \infty \tag{1.5}$$

For construction of a transfer matrix we will also need invariance w. r. t. translations

$$u_\tau (t^{(o)}, \tau^1) = u_\tau (t^{(o)}, \tau^1 + \tau)$$

and reflections

$$\theta t = \theta (t^{(o)}, \tau) = \theta (-t^{(o)}, \tau)$$

We assume that u_τ identifies Ω_t with $\Omega_{u_\tau t}$, etc.

In a standard way, one defines translations U_τ and reflections θ of random variables on Ω. We assume that if $\Phi_{A_i} \in \Phi$ and $\theta(\Phi_{A_i}) \in \Phi$, then $U_\tau(\Phi_{A_i}) \in \Phi$ (1.6)

Let μ be the limit Gibbs measure which is a limit in the usual sense of measures μ_V with densities

$$\frac{d\mu_V}{d\mu^{(o)}} = Z_V^{-1} \exp\Big(-\beta \sum_{\substack{A \subset V \\ A \in \mathfrak{A}}} \Phi_A\Big) \tag{1.7}$$

The existence and properties of such limits for small β easily follow from considerations given below. We shall not dwell any further on the matter. We denote $\mathcal{H} = L_2(\Omega, \Sigma_o, \mu)$, and let $P_{\mathcal{H}}$ be the orthogonal projection onto $\mathcal{H}$ in $L_2(\Omega, \Sigma, \mu)$. The transfer matrix $\mathcal{T}$ is defined as the following operator in $\mathcal{H}$:

$$\mathcal{T} = P_{\mathcal{H}} U_1 P_{\mathcal{H}} .$$

It has matrix elements

$$(\xi_1, \mathcal{T}\xi_2)_{\mathcal{H}} = \langle \xi_1 (U_1 \xi_2) \rangle \tag{1.8}$$

where $\langle \cdot \rangle$ is the expectation w. r. t. Gibbs measure μ.

We now define the special basis in $\mathcal{H}$. Let

$$g_x \in L_2(\Omega, \Sigma_x, \mu), \qquad x \in T_o$$

Let T_o be well-ordered in some manner by the relation $<$ and let

$$T_{o,x} = \{y : y \in T_o,\ y < x\}$$

Furthermore, let P_x be the orthogonal projection of $\mathcal{H}$ onto $L_2(\Omega, \Sigma_{T_{o,x}}, \mu)$, and put $\hat{g}_x = g_x - P_x g_x$.

For each finite $I \subset T_o$ and for an arbitrary array $G_I = \{g_x\}_{x \in I}$, we put $\hat{G}_I = \prod_{x \in I} \hat{g}_x$. It is clear that $\hat{G}_I$ and $\hat{G}_{I'}$ are orthogonal if $I \neq I'$.

Lemma 1.1. Constants (i.e. G_ϕ) and $\hat{G}_I$ (which belong to $\mathcal{H}$) span all $\mathcal{H}$.

One can often see that a more detailed assumption about Φ_A is valid

$$\Phi_A = \sum_{\alpha=1}^{d} \prod_{t \in A} \varphi_t^{(\alpha)} \tag{1.9}$$

where $\varphi_t^{(\alpha)}$ are Σ_t-measurable and bounded uniformly in α and t. (We do not use it here.)

We note that in some cases one can prove that (see [4])

$$\mathcal{F} \geq 0 \tag{1.10}$$

We note that (1.9) takes place for gauge fields and (1.10) also but in radiation gauge [5].

We proceed now to our main result. Let us be given two arrays

$$G_I = \{g_x\}_{x \in I}, \quad G_{I'} = \{g'_x\}_{x \in I'}, \quad \text{with} \quad \|g_x\|, \ \|g'_x\| \leq 1 \tag{1.11}$$

From the definition of the semiinvariants $\omega(J)$, we know that

$$(\hat{G}_I, \mathcal{F}\hat{G}_{I'})_{\mathcal{H}} \equiv \langle \hat{G}_I (U_1 \hat{G}_{I'}) \rangle = \Sigma\, \omega(J_1) \ldots \omega(J_k) \tag{1.12}$$

where the summation is through all partitions $J_1 \cup \ldots \cup J_k$ of the set $I \cup u_1 I' \subset T_o \cup T_1$ $T_1 = u_1 T_o$. $\omega(J_i)$ is the semiinvariant of $|J_i|$ random variables $\hat{g}_x$, $x \in I \cap J_i$ and $\hat{g}'_x$, $x \in (u_1 J_i) \cap I'$.

Due to the orthogonality of $\hat{g}_x$ for different x, the summation in (1.12) is in fact only made over partitions such that for all i,

$$J_i \cap I \neq \phi, \quad J_i \cap (u_I I') \neq \phi \tag{1.13}$$

Furthermore, we denote by C positive constants which differ from one proposition to another. All of them can be explicitly defined.

Theorem 1.2. There exist constants $C > 0$ and $\beta_0 > 0$ such that for all β, $|\beta| < \beta_0$, and all G_I, $G_{I'}$ which satisfy (1.11),

$$|\omega(J_i)| \leq (C\beta)^{d_{J_i}} \tag{1.14}$$

Here, d_J is the minimal d such that there exist $A_1, \ldots, A_d \in \mathfrak{A}$ such that the array $\Gamma(A_1, \ldots, A_d)$ is connected and $J_i \subset \bigcup_{j=1}^{d} A_j$. We emphasize that C does not depend on β, $|J_i$, G_I, $G_{I'}$.

Definition of connectedness. For this we construct a graph G_Γ with vertices $A_1, \ldots, A_d$. We connect A_i and A_j by a line iff $A_i \cap A_j \neq \phi$. Γ is called connected iff G_Γ is connected.

The proof of this theorem takes the larger part of the following text. We also obtain absolutely convergent series (in β) for $\omega(J)$ $\hat{g}_x$, etc. Let us consider the subspace $\mathcal{H}_N \subset \mathcal{H}$ spanned by all $\hat{G}_I$ with $|I| \leq N$; let P_N be the projection onto $\mathcal{H}_N$.

Corollary 1.3. There exist constants $C > 0$ and $\delta > 0$ which do not depend on β and N and such that

$$\|(1 - P_N)\mathcal{F}\| \leq (C\beta)^{\delta(N+1)} \tag{1.15}$$

Proof of this proposition as a corollary of Theorem 1.2 for the case of the Ising model (where $\delta \equiv 1$) is given in [10]. One can in fact find much more detailed estimates there. The same estimates can be easily obtained in our case by using the method of [10].

Corollary 1. 4. Let E_a be the spectral family for $\mathcal{T}$. Then for $a > (C\beta)^{\delta(N+1)}$

$$(1 - E_a)\mathcal{H}_N = (1 - E_a)\mathcal{H} \tag{1.16}$$

This is evident from (1. 15).

II. BOUNDS FOR SEMIINVARIANTS OF PARTIALLY DEPENDENT RANDOM VARIABLES

Let $\sigma_1, \ldots, \sigma_N$ be random variables defined on (Ω, Σ, μ) and assumed to have finite momenta. For each $A = \{i_1, \ldots, i_p\} \subset \{1, \ldots, N\}$, we define $\sigma_A = \Pi_{k=1}^{p} \sigma_{i_k}$ and $\langle \sigma_A \rangle^c = \langle \sigma_{i_1}, \ldots, \sigma_{i_p} \rangle$; the latter being the semiinvariant of random variables $\sigma_{i_1}, \ldots, \sigma_{i_p}$.

Let $\mathfrak{A}$ be the partially-ordered set of all partitions α of the set $\{1, \ldots, N\}$. We define $\alpha < \beta$ iff each block of partition α is contained in some block of partition β. e_0 is the minimal partition (each block consists of one point), e_1 is the maximal partition which consists of one block only. We define the following functions on $\mathfrak{A}$:

$$f(\alpha) = \prod_{i=1}^{k} \langle \sigma_{A_i} \rangle, \qquad g(\alpha) = \prod_{i=1}^{k} \langle \sigma_{A_i} \rangle^c$$

where $\alpha = (A_1, \ldots A_k)$ is the partition with blocks A_i.

Let us denote

$$C_\sigma = \max_{\alpha \in \mathfrak{A}} |f(\alpha)| \tag{2.1}$$

That is, if $|\sigma_i| \leq C_i$ with probability 1, then

$$C_\sigma \leq \prod_{i=1}^{N} C_i \tag{2.2}$$

One can also bound C_σ by using Hölder inequalities. The main result of this paragraph is

<u>Theorem 2.1.</u>

$$|<\sigma_1, \dots, \sigma_N>| \le C_\sigma \prod_{i=1}^{N} (3 v_i) \tag{2.3}$$

<u>where</u> v_i <u>is the number of indices</u> j, $j \neq i$ <u>such that random variables</u> σ_i <u>and</u> σ_j <u>are connected with a line in the graph</u> Γ <u>defined below.</u>

Before proceeding to the proof of this theorem, we note that the definition of semiinvariants gives

$$f(\alpha) = \sum_{\beta \le \alpha} g(\beta) \tag{2.4}$$

By the Möbius inversion formula [8],

$$g(\alpha) = \sum_{\beta \le \alpha} f(\beta)\mu(\beta, \alpha) \tag{2.5}$$

That is, we have [8]

$$g(e_1) = \sum_{\beta} f(\beta)(-1)^{|\beta|-1}(|\beta| - 1)! \tag{2.6}$$

where $|\beta|$ is the number of blocks in β. Now we want to take into consideration the fact that some pairs of random variables σ_i may be independent.

For this reason we shall define instead of $\mathfrak{A}$ the other partially ordered set $\mathfrak{A}_\Gamma$.

Assume that we are given a graph Γ with vertices labeled by $\sigma_1, \dots, \sigma_N$. Suppose that Γ enjoys the following remarkable property: $\sigma_A = \sigma_{A_1}\sigma_{A_2}$ if $A_1 \cup A_2 = A$, $A_1 \cap A_2 = \phi$, and if there are no lines in Γ between A_1 and A_2.

Let $\mathfrak{A}_\Gamma \subset \mathfrak{A}$ be the partially-ordered set of all partitions $\alpha = (A_1, \dots, A_k)$ of the set of vertices of Γ such that the subgraph Γ_{A_i} of Γ with vertices from A_i is connected for all i. (Two vertices in Γ_{A_i} are connected with a line iff the same condition is fulfilled in Γ.)

We denote by $\mu_\Gamma(\beta, \alpha)$ the Mobius function for $\mathfrak{A}_\Gamma$. Using

$$\langle \sigma_A \rangle^c = 0$$

if Γ_A is not connected, we obtain instead of (2. 4),

$$f(\alpha) = \sum_{\beta : \beta \leq \alpha, \ \beta \in \mathfrak{A}_\Gamma} g(\beta), \qquad \alpha \in \mathfrak{A}_\Gamma$$

The Modius inversion formula then gives

$$g(\alpha) = \sum_{\beta : \beta \leq \alpha, \beta \in \mathfrak{A}_\Gamma} f(\beta) \mu_\Gamma(\beta, \alpha) \tag{2. 7}$$

Our aim is to now calculate $\mu_\Gamma(\beta, \alpha)$. Let us define a circuit T as a set of lines of Γ such that every vertex of any line $\gamma \in T$ is a vertex of exactly one other line $\tilde{\gamma} \in T$.

Let us assume that all lines of Γ are well-ordered in some way: $\gamma_1 < \gamma_2 < \dots < \gamma_n$. If $T = \{\gamma_{i_1}, \dots \gamma_{i_k}\}$, $\gamma_{i_1} < \dots < \gamma_{i_k}$, is a circuit then we call $T' = \{\gamma_{i_1}, \dots \gamma_{i_{k-1}}\}$ a broken circuit.

<u>Lemma 2. 2.</u>

$$\mu_\Gamma(e_o, e_1) = (-1)^{N-1} m_1 \tag{2. 8}$$

where m_1 *is the number of subsets* G *of the set of lines of* Γ *such that* $|G| = N - 1$ *and such that* G *does not contain any broken circuit.*

<u>Proof:</u> We note that $\mathfrak{A}_\Gamma$ is a geometric lattice [8] or M-lattice (see [9], VII, §5, problem 9). Therefore, the lemma is a slightly revised formulation of proposition 1, p. 358, of [8].

Corollary 2. 3.

$$|\mu_\Gamma(e_o, e_1)| \leq \prod_{i=1}^{N} v_i \tag{2.9}$$

where v_i is the number of lines incident with vertex σ_i.

Proof: We bound the number $\tilde{m}_1$ of subsets $\tilde{G}$ of lines of Γ such that $|\tilde{G}| = N-1$ and $\tilde{G}$ does not contain any circuit. Clearly $m_1 \leq \tilde{m}_1$. For this reason, we define for each $\tilde{G}$ a one-to-one map $\varphi_{\tilde{G}}$ of $\tilde{G}$ into the set V_Γ of vertices of Γ.

We first note that $\tilde{G}$ is a connected tree: it is a tree since $\tilde{G}$ does not contain circuits, and it is connected since $|\tilde{G}| = N-1$.

We consider an arbitrary line $\gamma_1 \in \tilde{G}$ and define $\varphi_{\tilde{G}}(\gamma_1)$ as an arbitrary vertex incident with γ_1. Let us assume inductively that $\gamma_1, \ldots, \gamma_m$ and $\varphi_{\tilde{G}}(\gamma_1), \ldots \varphi_{\tilde{G}}(\gamma_m)$ are already defined. Let V_m be the set of vertices incident with at least one of $\gamma_1, \ldots \gamma_m$. We choose γ_{m+1} in such a manner that one of its vertices belongs to V_m. Both vertices of γ_{m+1} cannot belong to V_m, as we obtain a circuit in that case. Let us define $\varphi_{\tilde{G}}(\gamma_{m+1})$ as that of the two vertices of γ_{m+1} which belongs to V_m if it does not coincide with one of $\varphi_{\tilde{G}}(\gamma_1), \ldots, \varphi_{\tilde{G}}(\gamma_m)$, and the other if it does.

We then see that $\varphi_{\tilde{G}}^{-1}$ are different maps (since $\tilde{G}$ are different of V_Γ into the set of lines of Γ, and that for each $\tilde{G}$ and each $v \in V_\Gamma$, $\varphi_{\tilde{G}}^{-1}(v)$ is incident with v. That is why the number of such $\varphi_{\tilde{G}}^{-1}$ (which is the number of $\tilde{G}$) does not exceed Πv_i.

Q. E. D.

Lemma 2. 4.

$$\sum_{\beta \in \mathfrak{A}_\Gamma} |\mu_\Gamma(\beta, e_1)| \leq \prod_{i=1}^{N} (3 v_i) \tag{2.10}$$

Proof: For arbitrary $\beta \in \mathfrak{A}_\Gamma$, we define the new graph Γ_β which identifies vertices of Γ which belong to the same block of β, i. e. vertices of Γ_β are blocks of β. Between two new vertices b_1 and b_2 there is a line iff there is a line in Γ between some vertices i $i_1 \in b_1$ and $i_2 \in b_2$. Note that

$$\mu_\Gamma(\beta_1 e_1) = \mu_{\Gamma_\beta}(e_o, e_1)$$

We shall label partitions $\beta \in \mathfrak{A}_\Gamma$ of Γ by vertices of some tree $\mathcal{G}$.

We shall construct vertices of our tree $\mathcal{G}$ inductively by the following algorithm

1. Vertices of $\mathcal{G}$ have order $1, \ldots, N$ and may be red or blue.

2. To each vertex of $\mathcal{G}$ will correspond some Γ_β and a subset D_β of vertices of Γ_β.

3. The vertex of order 1 is unique. It is red and corresponds to $\Gamma_\beta \equiv \beta$, $D_\beta = \phi$.

4. Over an arbitrary vertex of order k, there is only one vertex of order $k-1$, and there is only one blue vertex of order $k+1$ under it.

5. Assume that a red vertex t or order k has already been constructed and that Γ_β and D_β correspond to it. Let v_i be the number of lines of Γ_β incident with vertex i of Γ_β, and let

$$v_1 \leq \cdots \leq v_m$$

We construct under t some number of red vertices not exceeding v_1 and 1 blue vertex. To this blue vertex will correspond the same Γ_β, but to D_β we add vertex v_1.

From graph Γ_β, we can construct by not more than v_1 ways a new graph Γ_{β_r} by identifying vertex 1 of Γ_β with one of the v_1 other vertices of Γ_β which are connected with 1 by a line, and do not belong to D_β. Each Γ_{β_r} will correspond to a new vertex. All new new vertices have the same D_β. (This is correct since the vertices of D_β are not identified in construction of Γ_{β_r}.)

We note that

A. for arbitrary $\beta \in \mathfrak{A}_\Gamma$, there is a red vertex which corresponds to Γ_β. This follows easily from construction.

B. if we put $\gamma_t = \Pi_{r=1}^m v_r$, where v_r are assumed to correspond to t, then for any vertex t of order k, and for any red vertex t' of order $k+1$ under t,

$$\frac{\gamma_{t'}}{\gamma_t} \le \frac{v_1 + v_r - 1}{v_1 v_r} \le \begin{cases} \frac{2}{v_1} & v_1 \ge 2 \\ 1 \ , & v_1 = 1 \end{cases}$$

Furthermore,

$$\sum_{\beta \le 1} |\mu(\beta, 1)| \le \sum_{\text{red } t} \gamma_t$$

However, it follows from construction that

$$\sum_{\substack{\text{red } t \\ \text{of order } k+1}} \gamma_t \le 2 \sum_{\substack{\text{red } t \\ \text{of order } k}} \gamma_t + \sum_{\substack{\text{blue } t \\ \text{of order } k}} \gamma_t$$

$$\le 2 \sum_{\substack{\text{red } t \\ \text{or order } k}} \gamma_t + \sum_{\substack{\text{red } t \\ \text{of order } < k}} \gamma_t$$

The inequalities

$$a_k \le 2 a_{k-1} + \sum_{i=1}^{k-2} a_i , \qquad a_1 = 1$$

imply that $a_k \le 3^k$. Thus Lemma 2. 4 is proved.

Theorem 2. 1 clearly follows from (2. 7) if we use (2. 10).

III. BOUNDS ON SEMIINVARIANTS OF FUNCTIONALS OF INDEPENDENT RANDOM FIELDS

Let T be a denumerable set, $\mathcal{X}$ an arbitrary measurable space with σ-algebra Σ_t and probability measure μ_t defined for all $t \in T$. We put

$$\mathcal{X} = \underset{t\in T}{\times} \mathcal{X}_t, \quad \Sigma = \underset{t\in T}{\times} \Sigma_t, \quad \mu = \underset{t\in T}{\times} \mu_t, \quad \Sigma_T = \underset{t\in T}{\times} \Sigma_t$$

Lemma 3.1. *Let $\sigma_1, \ldots, \sigma_N$ be local and have supports $B_1, \ldots, B_N$ We denote*

$$I = \{1, \ldots n\}, \quad I' = \{n+1, \ldots, N\}$$

*for $i \in I$ $(i \in I')$. We define $\tilde{V}_i$ to be equal to the number of σ_j such that $j \in I (j \in I')$ and $B_i \cap B_j \neq \phi$. Then,**

$$|\langle\sigma_1, \ldots \sigma_N\rangle| \leq C_\sigma \prod_{i=1}^{N} (3\tilde{V}_i 4^{|B_i|}) \tag{3.1}$$

Proof: This lemma will clearly follow from Theorem 2.1 if we can prove that**

$$\prod_{j\in I'} v_j \leq \prod_{i\in I} 4^{|B_i|} \prod_{j\in I'} 3V_j \tag{3.2}$$

where v_j is the number of $i \neq j$, $i = 1, \ldots N$ such that $B_i \cap B_j \neq \phi$.

For $j \in I'$, $v_j^{(1)} \equiv v_j^{(1)}, v_j^{(2)}$ is the number of $i \neq j$, $i = 2, 3, \ldots, N$, such that $B_i \cap B_j \neq \phi$. We have $v_j^{(1)} = v_j^{(2)} + \delta_j$, where $\delta_j = 1$ if $B_j \cap B_1 \neq \phi$, and is otherwise equal to 0.

---

* The factor $C^{|B_i|}$ was omitted in Lemma 3.1 and Corollary 3.3 of [7]. This does not affect the remaining results of [7].

** We define the graph Γ with vertices $\sigma_1, \ldots, \sigma_N$ connecting σ_i and σ_j by a line iff $B_i \cap B_j = \phi$.

For each $t \in B_1$, we choose an arbitrary (perhaps empty) subset $\beta(t) \in I'$ satisfying the following assumptions:

1. $t \in \beta_i$ for each $i \in \beta(t)$,
2. $\beta(t) \cap \beta(t') = \phi$ if $t \neq t'$,
3. if $j \in I'$ is such that $B_j \cap B_1 \neq \phi$, then j belongs to some $\beta(t)$.

It follows from these assumptions that

$$\prod_{j\in I'} v_j^{(2)} = \prod_{j\in I'} (v_j^{(1)} - \delta_j) \geq \prod_{j\in I'} - v_j^{(1)} [\prod_{t\in B_1} \prod_{j\in\beta(t)} (1 - \frac{\delta_j}{v_j^{(1)}})]$$

$$\geq \prod_{j\in I'} v_j^{(1)} \prod_{t\in B_1} (1 - \frac{1}{\alpha(t)})^{|\beta(t)|}$$

(where $\alpha(t) \geq 2$, $\alpha(t) \geq |\beta(t)|$)

$$\geq \prod_{j\in I'} v_j^{(2)} \prod_{t\in B_1} \frac{1}{4} = \prod_{j\in I'} v_j^{(1)} 4^{|B_1|}$$

From this we obtain

$$\prod_{j\in I'} v_j^{(1)} \leq 4^{|B_i|} \prod_{j\in I'} v_j^{(2)}$$

Inductively, we obtain the same inequalities for similarly defined $v_j^{(3)}, \ldots v_j^{(n)}$. From this (3. 2), a symmetric inequality, the the next Lemma follow.

Let us now be given a system $\mathfrak{A}$ of finite subsets $A_i \in T$ satisfying the following assumptions:

1. $|A_i| \leq d_1 < \infty$,
2. $\sup_{t\in T} |\{i : t \in A_i\}| \leq d_2 < \infty$.

where

$$b_n = \frac{(-1)^n}{n!} \sum_{A_1} \cdots \sum_{A_n} \langle F_1, \ldots, F_k, \Phi_{A_1}, \ldots, \Phi_{A_n} \rangle_o \tag{4.3}$$

One can easily see that for fixed n in the sum (4.3) only a finite number of terms are different from zero.

We denote by $d = d(B_1, \ldots, B_k)$ the minimal integer $d \geq 0$ such that there exist $A_1, \ldots, A_d \in \mathfrak{A}$ for which the array $\Gamma = (B_1, \ldots, B_k, A_1, \ldots, A_d)$ is connected.

Theorem 4.1. The series (4.2) is absolutely convergent for $|\beta| < \beta_o$, $\beta_o > 0$ being sufficiently small, and

$$|\langle F_1, \ldots, F_k \rangle^c| \leq C_\sigma \; 2[\prod_{i=1}^{k} 3 v_i \, 8^{|B_i|}] \, (2^{d_1 d_2} C\beta)^d \tag{4.4}$$

(All notation is the same as in Lemma 3.2.)

Proof: For a given ordered array $\Gamma = (A_1, \ldots, A_N)$, we denote by $\hat{\Gamma} = (A_{i_1}, \ldots, A_{i_p})$ the minimal unordered array such that:

1. $A_{i_1}, \ldots, A_{i_p}$ are all different,
2. Any $A_i \in \Gamma$ is equal to one of $A_{i_1}, \ldots, A_{i_p}$.

We note that all b_n, $n < d$, are equal to zero. Using (3.3), we obtain for b_n, $n \geq d$, the following bound

$$|b_n| \leq 2C_\sigma \prod_{i=1}^{k} (3v_i 4^{|B_i|}) \sum_{\hat{\Gamma} : |\hat{\Gamma}| \geq d} (C\beta)^{|\hat{\Gamma}|} \tag{4.5}$$

Let R_N be the number of unordered arrays $\hat{\Gamma}$ such that $|\hat{\Gamma}| = N$, and the array $(B_1, \ldots, B_k) \cup \hat{\Gamma}$ is connected. We shall prove that

$$R^N \leq 2^{|B_1| + \ldots + |B_k|} R'_N \tag{4.6}$$

Let Φ_{A_i} be given with supports $A_i \in \mathfrak{A}$. We consider local $F_1, \ldots F_k$ with supports $B_1, \ldots, B_k$.

We denote by γ_σ the constant defined in (2. 1) for the system $F_1, \ldots, F_k, \Phi_{A_1}, \ldots, \Phi_{A_n}$ of random variables.

We consider the partition of $\{A_1, \ldots, A_n\}$ into m groups such that A_i and A_j are identical if they belong to the same group and are otherwise different. Let $n_1, \ldots, n_m$ denote the number of elements in these groups.

Lemma 3. 2.

$$|\langle F_1, \ldots, F_k, \Phi_{A_1}, \ldots \Phi_{A_n} \rangle| \leq C_\sigma [\prod_{i=1}^{k} 3v_i 4^{|B_i|}] n_1! \ldots n_m! C^n \tag{3. 3}$$

where v_i *is the number of* j, $j = 1, \ldots, k$ *such that* $B_i \cap B_j \neq \phi$ and

$$C = 3 \cdot 4^{d_1] \;\; [\log_2 (d_2 + 1)] \;\; + 1 [} \tag{3. 4}$$

where $]a[= a$ *is* a *is an integer and* $[a] + 1$ *otherwise.*

Proof: We use (3. 1) first in the case where

$$\sigma_1 = F_1, \ldots, \sigma_n = F_k, \sigma_{n+1} = \Phi_{A_1}, \ldots \sigma_N = \Phi_{A_n}$$

To bound

$$\prod_{j \in I'} V_j$$

we inductively apply (3. 2) with I' being partitioned on the 2, 4, 8,... group by the dichotomic rule. For this we need the following

Lemma 3. 3. There exists a partition of I' *on* $d_2 + 1$ *subsets* $I_1, \ldots, I_{d_2+1}$ *such that* $I_i \cap I_j = \phi$ *for* $i \neq j$, $\bigcup I_i = I'$ *and if* i *and* j *belong to different subsets of this partition then* $A_i \cap A_j = \phi$.

Proof: We first define the graph G with I' as the set of its vertices.

Vertices i and j are connected with a line iff $A_i \cap A_j \neq \phi$. We set the length of each line equal to 1. We set the distance $\rho(i,j)$ between two vertices equal to the length of the minimal path from i to j. (If there is no such path we put $\rho(i, j)$ equal to ∞.) We choose a subset I' of vertices of G such that:

1. the distance between two arbitrary vertices of I' is at least 2,

2. each vertex not belonging to I' is at distance 1 from I'.

We delete all vertices I' from G, and also all lines incident with them. In the remaining graph we choose I_2 in a similar way, etc. The procedure will be complete after $d_2 + 1$ steps. In fact, we can consider arbitrary vertex i. In the first step either i belongs to I_1 (and then everything is proved), or one of at most d_2 vertices j, with $\rho(i, j) = 1$, belongs to I'. We repeat this reasoning in the next step, and continue in the same manner. The Lemma is thus proved. The theorem follows immediately.

IV. BOUNDS ON SEMIINVARIANTS OF THE GIBBS RANDOM FIELD IN A HIGH TEMPERATURE REGION

We will change the notation of the preceding paragraph slightly and will denote by μ_0 the measure μ on $(\mathcal{X}, \Sigma)$. $\langle\cdot\rangle_0$ will denote the expectation of μ_0.

We define the new measure μ_V with density

$$\frac{d\mu_V}{d\mu_0} = Z_V^{-1} \exp\left(-\beta \sum_{A \subset V} \Phi_A\right) \tag{4.1}$$

where $\langle\cdot\rangle_V$ denotes expectation w. r. t μ_V.

If $V \to \mathbb{Z}^\nu$ in the usual sense, we obtain the formal Taylor expansion

$$\langle F_1, \ldots, F_k\rangle^c \equiv \lim_V \langle F_1, \ldots, F_k\rangle_V^c = \sum_{n=0}^{\infty} b_n \beta^n \tag{4.2}$$

where the number

$$R'_N \le 2^{d_1 d_2 N} \tag{4.7}$$

is the number of arrays $\hat{\Gamma} = (A_1, \ldots, A_N)$, $|\hat{\Gamma}| = N$, such that some fixed point t belongs to $\tilde{\Gamma} = \bigcup_{i=1}^{N} A_i$.

We shall prove (4.6) first. Let all points of T be enumerated in some fixed manner. $\hat{\Gamma}$ is the union of r maximal connected subarrays, $\hat{\Gamma}_1, \ldots, \hat{\Gamma}_r$. One can easily see that $r \le |B_1| + \ldots + |B_k|$. In each $\hat{\Gamma}_i$, we take the first (in this fixed order) point

$$t_i \in \tilde{\Gamma}_i \cap \left(\bigcup_{j=1}^{k} B_j\right), \quad \tilde{\Gamma}_i = \bigcup_{A_j \in \hat{\Gamma}_i} A_j .$$

The number of sequences of the form $(t_1, \ldots, t_r)$ does not exceed $2^{|B_1| + \ldots + |B_k|}$. This proves (4.6).

<u>Lemma 4.2.</u>

$$R'_N \le 2^{d_1 d_2 N}$$

<u>Proof:</u> Let the set of all A_i be well-ordered in some way. We shall give the algorithm for construction of an arbitrary array $\hat{\Gamma}$, with $t \in \tilde{\Gamma}$ (t being fixed), with simultaneous ordering of each $\hat{\Gamma}$. We first order the elements of $\hat{\Gamma}$ containing t:

$$A_1 \le \ldots \le A_m, \quad 1 \le m \le d_2 .$$

This order is induced by the order in question of all systems of A_i. The number of such sequences $(A_1, \ldots, A_m)$ is not more than $2^{d_2} - 1$. We then take all elements of $\hat{\Gamma}$ intersecting A_1 (the number of such elements is not more than $2^{d_1 d_2}$) and order them in a similar manner after A_m. Repeating this procedure with A_2, etc., we demonstrate the Lemma.

V. THE SERIES FOR CONDITIONAL EXPECTATION

Let us consider the Gibbs measure μ defined in §4, and let P_Λ be the orthogonal projection onto $L_2(\Omega,\Sigma_\Lambda,\mu)$ in $L_2(\Omega,\Sigma,\mu)$. (We return now to the notations of §1.)

Our aim here is to explicitly find the absolutely convergent series for $P_\Lambda \; F_t$, $t \in \Lambda$, $F \in L_2(\Omega,\Sigma_t,\mu)$.

Let us consider finite Λ first. Let $<\cdot>_o^{(\Lambda)}$ be the expectation of the measure $\mu_{o,\Lambda} = \underset{t\in I\setminus\Lambda}{\times} \mu_t^{(o)}$, and let $<\cdot>^{(\Lambda)}$ be the expectation of the Gibbs measure $\lim_V <\cdot>_V^{(\Lambda)}$, where for $\Lambda \subset V$, we put

$$<F>_V^{(\Lambda)} = \frac{<F \exp(-\beta U_{V\setminus\Lambda})>_o^{(\Lambda)}}{<\exp(-\beta U_{V\setminus\Lambda})>_o^{(\Lambda)}} ,$$

where

$$U_{V\setminus\Lambda} = \sum_{A\subset V\setminus\Lambda} \Phi_A$$

Then

$$P_\Lambda F_t = M(F_t|\Sigma_\Lambda) = \lim_V \frac{<F_t \exp(-\beta \sum_{A\in G_\Lambda} \Phi_A)>_V^{(\Lambda)}}{<\exp(-\beta \sum_{A\in G_\Lambda} \Phi_A)>_V^{(\Lambda)}}$$

$$= \frac{<F_t \exp(-\beta \sum_{A\in G_\Lambda} \Phi_A)>^{(\Lambda)}}{<\exp(-\beta \sum_{A\in G_\Lambda} \Phi_A)>^{(\Lambda)}}$$

where G_Λ is the set of all $A \in \mathfrak{A}$ such that $A \cap \Lambda \neq \phi$, and

$A \cap (T \setminus \Lambda) \neq \phi$. From the last formula one can obtain

$$P_\Lambda F_t = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{|I'| = n} \eta_{I'} \tag{5. 1}$$

where the summation is over all ordered arrays $I' = (A_1, \ldots, A_n)$, $A_i \in G_\Lambda$, and where

$$\eta_{I'} = (-\beta)^n < F_t, \Phi_{A_1}, \ldots, \Phi_{A_n} >$$

We can rewrite (5. 1) as the sum over all unordered arrays $I = (A_1, \ldots, A_n)$, $A_i \in G_\Lambda$:

$$P_\Lambda F_t = \sum_{n=0}^{\infty} \sum_{|I| = n} \eta_I \tag{5. 2}$$

where

$$\eta_I = (-\beta)^n \frac{1}{n_1! \ldots n_m!} < F_t, \Phi_{A_1}, \ldots, \Phi_{A_n} >^{(\Lambda)}$$

and where $n_1, \ldots, n_m$ are defined as in Lemma 3. 2. Using (4. 4), we have for some $C > 0$,

$$\| \eta_I \| \leq \| F_t \| (C\beta)^{n + d_{I,t}} \tag{5. 3}$$

where $d_{I,t}$ is the minimal p such that there exist $\tilde{A}_1, \ldots, \tilde{A}_p$, $\tilde{A}_i \subset T \setminus \Lambda$, $\tilde{A}_i \in \mathcal{A}$, such that the array $(t, A_1 \cap \Lambda, \ldots, A_n \cap \Lambda, \tilde{A}_1, \ldots, \tilde{A}_p)$ is connected. We note that C does not depend on t, n, I.

Theorem 5. 1. If Λ is infinite, then (5. 2) also holds, η_I satisfy the bound (5. 3), and the series

$$\sum_{n=0}^{\infty} \sum_{|I| = n} \| \eta_I \|_n \tag{5. 4}$$

is convergent for sufficiently small β.

We omit the rather straightforward proof of this statement.

VI. PROOF OF THE MAIN THEOREM

We want to prove the estimate (see §1)

$$|\omega(J)| \leq (C\beta)^{d_J} \tag{1.14}$$

For I as in (5.2), we write

$$I = (A_1 \setminus \Lambda) \cup \ldots \cup (A_n \setminus \Lambda)$$

and

$$\alpha_B = \sum_{I:\tilde{I} = B} \eta_I \tag{6.1}$$

From (5.3), it easily follows that

$$\|\alpha_B\| \leq \|F_t\| (C\beta)^{n_B + d_{B,t}} \tag{6.2}$$

with a different constant $c > 0$. Here, $d_{B,t} \equiv d_{I,t}$ for $\tilde{I} = B$, and n_B is the minimal n such that there exists I with $\tilde{I} = B$ and $|I| = n$.

To prove (1.14), we insert into $\omega(J)$

$$\hat{g}_x = g_x - P_x g_x \equiv \alpha_x - \Sigma_B \alpha_B$$

where $\alpha_x \equiv g_x$. (We put $F_t = g_x$.) Then, $\omega(J)$ will be expanded into the sum of terms

$$\langle \alpha_{B_1}, \ldots, \alpha_{B_m} \rangle$$

where $m = |J|$, $J = (x_1, \ldots, x_m)$. From this we obtain

$$|\omega(J)| \leq \sum |<\alpha_{B_1}, \ldots, \alpha_{B_m}>|$$

$$\leq \sum_{B_1} \cdots \sum_{B_m} [\prod_{i=1}^{m} \|\alpha_{B_i}\| v_i C^{|B_i|}] \cdot (C\beta)^{d(\hat{B}_1, \ldots, \hat{B}_m)} \tag{6. 3}$$

where v_i is the number of j such that $B_j \cap B_i \neq \phi$, $j = 1, \ldots, m$. Here, we have used (4. 4). C is some constant which does not depend on β and B_i and $\hat{B}_i = B_i \cup \{x_i\}$.

Lemma 6. 1. Assume that m (pairwise different) subsets $\hat{B}_1, \ldots, \hat{B}_m \subset T$ are given. Let v_i be the number of j such that $\hat{B}_i \cap \hat{B}_j \neq \phi$, $j = 1, \ldots, m$. Then, there exists a $C > 0$ such that for all m and all $\hat{B}_1, \ldots, \hat{B}_m$,

$$\sum_{i=1}^{m} d_{\hat{B}_i} \geq C \sum_{i=1}^{m} \ln v_i \tag{6. 4}$$

We note that C depends only on T and $\mathfrak{A}$. The rather complicated proof of this Lemma for the case $T = \mathbb{Z}^{\nu}$ is given in [7]. (See Lemma A. 4.) In the general case, the proof is quite similar. We note that

$$\sum d_{B_i, x_i} \geq \sum d_{\hat{B}_i} \tag{6. 5}$$

Using (6. 2), (6. 4), and (6. 5), we now bound $|\omega(J)|$. We insert (6. 2) and (6. 4) into (6. 3), and obtain

$$|\omega(J)| \leq \sum_{B_1} \cdots \sum_{B_m} \prod_{i=1}^{m} C^{n_{B_i} + d_{B_i, x_i}} \cdot (C\beta)^{n_{B_i} + d_{B_i, x_i} + d(\hat{B}_1, \ldots, \hat{B}_m)}$$

with some (new) constant $C > 0$. Changing the constant and notation, we rewrite the last inequality as

$$|\omega(J)| < \sum_{R} (C\beta)^{\delta_R} \tag{6. 6}$$

where the summation is over all $R = (\hat{B}_1, \ldots, \hat{B}_m)$, and where

$$\delta_R = d(\hat{B}_1, \ldots, \hat{B}_m) + \sum_{i=1}^{m} d_{B_i, x_i}$$

It remains to prove that

$$\sum_R (C\beta)^{\delta_R} \leq (C_1\beta)^{d_J} \tag{6.7}$$

for some $C_1 > 0$. But,

$$\sum_R (C\beta)^{\delta_R} = \sum_{D=1} (C\beta)^D \sum_R{}' (C\beta)^{d_{B_1, x_1} + \ldots + d_{B_m, x_m}} \tag{6.8}$$

where in $\sum{}'$ the summation is over all R satisfying

$$d(\hat{B}_1, \ldots, \hat{B}_m) = D \tag{6.9}$$

We shall prove the auxiliary

<u>Lemma 6. 2.</u> <u>Let</u> $\mathcal{R}$ <u>be the set of all arrays</u> $R = (\hat{B}_1, \ldots, \hat{B}_m)$. <u>Then for each subset</u> $\mathcal{R}' \subset \mathcal{R}$,

$$\sum_{R \in \mathcal{R}'} (C\beta)^{d_{B_1, x_1} + \ldots + d_{B_m, x_m}} \leq (C_1\beta)^d$$

<u>where</u>

$$d = \min_{R \in \mathcal{R}'} (d_{B_1, x_1} + \ldots + d_{B_m, x_m})$$

<u>Proof:</u> The problem evidently reduces to the case when $\mathcal{R}'$ is the set of all arrays R such that

$$d_{B_1, x_1} + \ldots + d_{B_m, x_m} \geq d$$

Let us fix an ordered array $(r_1, \ldots, r_m)$ of integers with

$$0 \leq r_i \leq d, \quad \sum r_i = d$$

The number of such arrays is at most 4^d. If we can prove that

$$\sum_{B_1 : d_{B_1, x_1} \geq r} (C\beta)^{d_{B_1, x_1}} \geq (C_2\beta)^r \tag{6.10}$$

then our result with $C_1 = 4C_2$ will follow. But the proof of (6.10) is quite standard.

To prove (6.7), we take $\mathcal{R}' = \mathcal{R}_D$ as the set of all R satisfying (6.9). Then

$$|\omega(J)| \leq \sum_{D=1}^{\infty} (C\beta)^{D + d(D)} \tag{6.11}$$

where

$$d(D) = \min_{R \in \mathcal{R}_D} (d_{B_1, x_1} + \dots + d_{B_m, x_m})$$

But

$$D + d(D) \geq d_J \tag{6.12}$$

which follows from the definitions.

From (6.12) and (6.11), the proof of the main result (1.14) is straightforward.

VII. PROOF OF LEMMA 1.1

We choose some $x_o \in T_o$ and choose finite subsets $R = \{x_1, \dots x_k\} \subset T_o$ satisfying the following conditions:

1. $x_o > x_1 > \dots > x_k$, where $>$ is defined by lexicographic order on T_o,

2. $R = \{x : x < x_o, \rho(x, x_o) \leq d\}$, where ρ is the euclidean distance in $T_o \subset R^\nu$.

Let us first assume that $\{x : x_i < x < x_{i+1}\}$ is empty for each $i = 0, 1, \ldots, k-1$. We want to approximate bounded g_{x_o} within arbitrary degree of accuracy by products of $\hat{g}_{x_o}, \hat{g}_{x_1}, \ldots, \hat{g}_{x_k}$. It is easy to prove (we omit it) that $g_{x_o} - P_{x_k} g_{x_o}$ can be approximated with arbitrary accuracy by such products. Using the exponential bound

$$\| P_{x_k} g_{x_o} \| \leq \| g_{x_o} \| (C\beta)^{\delta d} \tag{7. 1}$$

where $\delta > 0$ and $C > 0$ do not depend on β and d, we obtain the proof. The bound (7. 1) can be proved in the following way. We apply

$$\| P_{x_k} g_{x_o} \|^2 = \langle M(g^2_{x_o} | \Sigma_{\{x < x_k\}}) \rangle \cdot M(g^2_{x_o} | \Sigma)$$

which can be calculated for any given "boundary conditions" on $\{x : x < x_k\}$ by using e. g. (4. 2), (4. 3) for $F_1 \equiv g_{x_o}$ and $k = 1$.

One easily observes from the estimates of §3 that the "boundary terms give a negligible contribution, and we obtain (7. 1).

In the general case (i. e. when $\{x : x_i < x < x_{i+1}\}$ are not empty), we use the following decomposition

$$x_o = x^{(o)} + x^{(1)} + \ldots + x^{(k+1)} \tag{7. 2}$$

where

$$x^{(o)} = x_o - P_{x_o} x_o, \quad x^{(1)} = P_{x_o} x_o - \hat{P}_{x_1} x_o,$$

$$x^{(2)} = \hat{P}_{x_1} x_o - \hat{P}_{x_2} x_o, \ldots,$$

$$x^{(k)} = \hat{P}_{x_{k-1}} x_o - \hat{P}_{x_k} x_o, \quad x^{(k+1} = \hat{P}_{x_k} x_o$$

and $\hat{P}_x$ is the projection onto $L_2(\Omega, \Sigma_{\{x', \quad x\}}, \mu)$. Furthermore, we proceed as earlier, using bounds (7. 1) for each $P\{x_i < x < x_{i+1}\}x_o$. Since K is bounded by d^ν, the exponential bound (7. 1) dominates. The proof is complete.

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