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Markov Processes and Related Fields

Static Charged Clusters

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Abstract. Shortly speaking, static charged cluster is a rigid body with charge distributed over it, For any system of such clusters the potential energy is the Coulomb potential energy. Here we give examples of system of clusters for which there exists stable configuration, that is minimum of potential energy.

KEYWORDS: Coulomb force, charged solids, clusters, stability

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1. Introduction

Celestial Mechanics, the famous science, explains configuration and dynamics of planets in the Universe via gravity forces. It is believed that molecular dynamics (for example, in biological organisms) exists mainly due to electromagnetic forces, where electrostatic (Coulomb) force is the most important. But why then the science, that could be called Coulomb mechanics, had never been even started. In despite of the fact that it has the same inverse square law (however, with not only attraction forces but also repulsive). One of the main reasons why it still does not exist – it seems to be more difficult.

First of all, it is well-known that static system of point charges is never stable by Earnshaw theorem. We give short review of corresponding classical theorems below. Moreover, in classical equilibrium statistical physics were widely studied large random systems (Gibbs fields) of charged particles, see for example [10–16]. For systems of point particles with different signs even additional external fields do not provide desired stability. Structured reviews of recent results in all these directions could be very useful for our project.

To get stable objects, instead of point particles one should find natural models for atoms and molecules. Now these are considered mainly in nonrelativistic quantum mechanics. However, one can try to come back to classical physics and try to study classical models for stable objects of this kind. Here point particles should move (due to mutual Coulomb interaction forces) inside some domain, then such system is called dynamical Coulomb cluster. Some results and techniques of Celestial mechanics can be used in the future Coulomb mechanics, for example to explain the existence of atoms, where centrifugal (that is, the rotation with velocity dependent of the radius) force is necessary. Example is two particle system with different charges, when one of the particle rotates around another. This is a classical basic result from Celestial mechanics. But completely new techniques is necessary to prove existence of stable long molecules without centrifugal forces and to prove existence of solid state using only Coulomb forces.

Simpler approach is to add other forces between charges in the cluster. If such system is without internal movement, then we call it static (or solid) Coulomb cluster (see below exact definition). Thus, we assume that other (unknown) forces make them solid. Moreover, Coulomb clusters may consist not only of point charges but also continuous charge distribution, resembling the electron wave function in quantum mechanics around positive nucleus.

The Model Static charged (or Coulomb) cluster in R^d is a pair (μ, ϑ) of finite measures on R^d with the same bounded support O of diameter $\varepsilon > 0$ (often we consider ε to be small). The measure μ is non-negative and corresponds to mass distribution (or even mass density), and ϑ can be alternating, and corresponds to the charge distribution, or defined by charge density $q(x), x \in O$. Below for $d = 1$ we consider arbitrary clusters but for $d \geq 2$ we consider only rotationally symmetric clusters, that is when O is a ball with the center at some point X and (μ, ϑ) are invariant w.r.t. rotation group around X.

The system of several, $i = 1, \ldots, N$, such clusters – $(\mu_i, \vartheta_i), O_i, \varepsilon_i$, with non intersecting supports O_i is called classical system of clusters. We will call $Q_i = \vartheta_i(O_i)$ the (full) charge of the cluster i, and the potential (Coulomb interaction) energy of such system of clusters is defined as

$$U = U(\mu_i, \vartheta_i, G) = \sum_{(i,j)\in I} \int_{O_i} \int_{O_j} \frac{1}{|x_i - x_j|} d\vartheta_i d\vartheta_j,$$

where I is a system of pairs $(i, j), i < j$, that is the set of edges of some graph G with the set of vertices $\{1, \ldots, N\}$ and the set of edges I.

Remark 1.1. We want also to emphasize that we use inverse square law for electrostatic forces in any dimension, that is "dimension 3" force. In dimension 1 it is an approximation to the 3-dimensional thin tube, in dimension 2 – to the thin layer between two parallel planes. Another reason is that it can be derived in any dimension not from Maxwell – Lorentz equations, and even without Newton law, but only using Galileo axioms, see [1].

We see that U does not depend on the measures μ_i. On the other hand, each measure μ_i defines the center of mass of the cluster i

$$X_i = M^{-1} \int x_i \mu_i(dx_i), \ \ M = \int \mu(dx_i),$$

that is the center of O_i. If for any i, j the distances

$$d_{ij} = \inf_{x \in O_i, y \in O_j} |x - y| > 0$$

between O_i and O_j are not zero, then the potential energy U is finite.

The potential energy $U(X_1, \ldots, X_N)$, if we assume rotation invariance of each cluster, is a function of N vectors X_i. And by translation invariance, the function of only $N-1$ vectors $R_i = X_{i+1} - X_i, = 1, 2, \ldots, N-1$. But in fact it depend only on the vector D of distances d_{ij} such that $(i, j) \in I$.

There are two ways how one could define dynamics of such clusters: 1) use equations of rigid body physics; 2) consider regular continuum particle systems (see [17]) and use approximation of such systems by N particle systems with asymptotically (as $N \to \infty$) infinite forces which conserve distances between any pair of particles (see examples in [18]). Then due to rotation symmetry of clusters only forces, for example between clusters i and j, will be only in the direction of the vector between mass centers.

We consider here the following definition of stability: assuming that $U(R_1, ... R_{N-1})$ is a smooth function, we say that the point $(R_1, \ldots, R_{N-1})$ is stable if U has strict minimum at this point. In other words, if two conditions hold: 1) all "forces" $-\nabla_{R_i} U$ are zero, 2) the corresponding Hessian is positive at this point. Intuitively, then X_i can move permanently in their own potential wells. Here we give many examples of such stability for two and three clusters.

Useful and more general definition kind of stability could be the following. Dynamic stability: 1) the system is called bounded for some class of initial conditions, if all distances d_{ij} rest bounded for any time $0 \leq t < \infty$, 2) clusters never collide, that is all distances d_{ij} are larger than some $\delta > 0$ for any $0 \leq t < \infty$.

In dimension 3 we will also consider the so called quasi-quantum systems of clusters, that is when O_i may intersect. This is inspired by analogy with quantum mechanics where wave function of a particle may have even unbounded support.

2. One-dimensional case

A cluster is called k-point cluster if the set O is a k-point set.

2-point cluster and point particle We consider two following clusters: 1) 2-point cluster, that is two point measure in the points $x_1 < x_2 = x_1 + \varepsilon, \varepsilon > 0$, with charges correspondingly: positive $q_+ > 0$ at x_1 and negative, which is convenient to denote $-q_-$ with $q_- > 0$, at x_2; 2) there is also 1-point cluster, that is a point particle with negative charge $q < 0$ at the point x_3 so that $r = x_3 - x_2 > 0$.

Proposition 2.1. *Let the parameters ε, q_+, q_-, q be fixed. If $Q_1Q_2 < 0$, that is $q_+ - q_- > 0$, that is full charges have different signs, then the function $U(r)$ has exactly one minimum on the interval $(0, \infty)$.*

Proof. One can show existence without any calculations, and the same proof will be used below in more general cases. Note first that $U(r) \to +\infty$, if $r \to 0$. Moreover, the repulsive force (for any value of q_+) also tends to $+\infty$, and it follows that the derivative $\frac{dU}{dr}$ tends to $-\infty$. At the same time, when $r \to +\infty$ the function $U(r) \to 0$. And by condition $q_+ > q_-$ the force on the point particle (with negative charge) will be negative for sufficiently large r. Then the derivative $\frac{dU}{dr}$ should be positive for sufficiently large r. This means that $U(r)$ increases for sufficiently large r and tends to zero "from below". It follows, as $U(r)$ is smooth, that it should have global minimum. This will be point minimum as $U(r)$ is a rational function. And then at the point minimum its second derivative is positive.

Remark 2.1. In fact, for existence of minimum, it is sufficient that $\frac{dU}{dr} > 0$ for all sufficiently large r, and at least one of the following two conditions hold:

1) $U(r) > 0$ for some $r > 0$;

2) $\dfrac{dU}{dr} < 0$ for some $r > 0$.

However, in this case one can do more exact statements with all calculations done explicitly. Potential energy is

$$U(d) = \frac{q_+q}{r+\varepsilon} - \frac{q_-q}{r}.$$

The condition $\frac{\partial U}{\partial r} = 0$ gives the equation

$$\frac{q_+}{(r+\varepsilon)^2} = \frac{q_-}{r^2} \Longrightarrow \frac{q_+}{q_-} - 1 = \frac{\varepsilon}{r}\Big(2 + \frac{\varepsilon}{r}\Big).$$

We see immediately that for existence of the solution it should be $q_+ > q_-$. Moreover, the solution for the minimum is unique and is given by

$$\frac{\varepsilon}{r} = \sqrt{\frac{q_+}{q_-}} - 1 \Longrightarrow r = \frac{\varepsilon}{\sqrt{q_+/q_-} - 1}.$$

Non-point cluster and point particle Now we consider again cluster on the interval $x_1 < x_2 = x_1 + \varepsilon, \varepsilon > 0$, with point charge $q_+ > 0$ at x_1, but negative charge $-q_-$ is assumed to be uniformly distributed on the interval (x_1, x_2). The point particle with negative charge $q < 0$ is again at the point x_3, so that $r = x_3 - x_2 > 0$.

Proposition 2.2. *Let the parameters be fixed. If again $Q_1 Q_2 < 0$, then the function $U(r)$ has unique minimum on the interval $(0, \infty)$.*

Proof. To prove existence of the minimum similarly to the previous proposition, it is sufficient to show that $U \to +\infty$ as $r \to 0$.

The charge density ρ_- on the interval (x_1, x_2) is defined from

$$q_- = \int_0^\varepsilon \rho_- dx = \varepsilon \rho_-$$

and the potential energy is:

$$\begin{aligned} U(r) &= \frac{q_+ q}{r+\varepsilon} - \int_0^\varepsilon \frac{\rho_- q dx}{r+x} = \frac{q_+ q}{r+\varepsilon} - \frac{q_- q}{\varepsilon} \int_0^\varepsilon \frac{dx}{r+x} \\ &= \frac{q_+ q}{r+\varepsilon} - \frac{q_- q}{\varepsilon} \left(\ln(r+\varepsilon) - \ln r \right). \end{aligned}$$

It follows that as $r \to 0$ we have $U(r) \to +\infty$. Note that even the solution of the equation $\frac{\partial U}{\partial r} = 0$, or

$$\begin{aligned} 0 = \frac{q_+}{(r+\varepsilon)^2} - \frac{q_-}{\varepsilon} \left(\frac{1}{r} - \frac{1}{r+\varepsilon} \right) &= \frac{q_+}{(r+\varepsilon)^2} - \frac{q_-}{r(r+\varepsilon)} = \\ = \frac{q_+ r - q_-(r+\varepsilon)}{r(r+\varepsilon)^2} &= \frac{r(q_+ - q_-) - \varepsilon q_-}{r(r+\varepsilon)^2}, \end{aligned}$$

is unique and is equal to

$$r = \varepsilon \frac{q_-}{q_+ - q_-} = \frac{\varepsilon}{q_+/q_- - 1}.$$

Then, as minimum exists, it is exactly this unique solution.

Two 2-point clusters Consider two clusters: 1) the first one at the points $x_1 < x_2 = x_1 + \varepsilon$; 2) the second – at the points $x_3 = x_2 + r, x_4 = x_3 + \varepsilon$. The charges are $q_1 > 0, q_2 < 0, q_3 < 0, q_4 > 0$. Assume also that $q_1 + q_2 > 0, q_3 + q_4 < 0$.

Proposition 2.3. *Potential energy has minimum on the interval $(0, \infty)$. Moreover, it is unique.*

It can be proved similarly to the proof of Proposition 2.1 that at least one minimum of $U(r)$ exists for $r > 0$. Now we will prove its uniqueness. We have

$$U(r) = \frac{q_2 q_3}{r} + \frac{q_1 q_3 + q_2 q_4}{r+\varepsilon} + \frac{q_1 q_4}{r+2\varepsilon} =$$

$$= \frac{(q_1+q_2)(q_3+q_4)r^2 + b_1 r + b_0}{r(r+\varepsilon)(r+2\varepsilon)},$$

where b_1, b_0 are not interesting for us. Note that $U \to +\infty$ as $r \to 0$, and $U \to -0$ as $r \to +\infty$. Then the graph of the function $U(r)$ can intersect real axis only odd number of times. But as the polynomial in the numerator is or order 2, only once. That is, $U(r)$ has exactly one zero in $(0, \infty)$.

Extrema of the function $U(r)$ in the set $(0, +\infty)$ are zeros of the following equation:

$$0 = -\frac{dU(r)}{dr} = \frac{q_2 q_3}{r^2} + \frac{q_1 q_3 + q_2 q_4}{(r+\varepsilon)^2} + \frac{q_1 q_4}{(r+2\varepsilon)^2} =$$

$$= \frac{P(x)}{(x-\varepsilon)^2 x^2 (x+\varepsilon)^2}$$

where we denoted $x = r + \varepsilon$, and

$$P(x) = (q_1+q_2)(q_3+q_4)x^4 + 2(q_2 q_3 - q_1 q_4)x^3\varepsilon +$$

$$+(q_1 q_4 + q_2 q_3 - 2q_1 q_3 - 2q_2 q_4)x^2\varepsilon^2 + (q_1 q_3 + q_2 q_4).$$

Again change the variable $y = x/\varepsilon$:

$$0 = -\frac{dU(r)}{dr} = (q_1+q_2)(q_3+q_4)\frac{y^4 + a_3 y^3 + a_2 y^2 + a_0}{\varepsilon^2 (y-1)^2 y^2 (y+1)^2},$$

where

$$a_0 = \frac{(q_1 q_3 + q_2 q_4)}{(q_1+q_2)(q_3+q_4)}.$$

The set $r \in (0, +\infty)$ becomes the set $y \in (1, +\infty)$. The transformed function we shall denote also by $U(y)$. We saw that $U(y)$ has only one root on $(1, +\infty)$. As the numerator of the derivative of $U'(y)$ is a polynomial of 4-th degree, the number of its roots is not more than 4. As $U \to +\infty$ if $y \to 1$, and $U \to -0$ if $y \to +\infty$, then the derivative has odd number of roots on $(1, +\infty)$, that is 1 or 3.

Let us show that it cannot be 3. Assume they are 3. It follows, that as the polynomial in the numerator of the derivative has degree 4, then all roots of it are real. Note that their product is

$$y_1 y_2 y_3 y_4 = a_0 = \frac{(q_1 q_3 + q_2 q_4)}{(q_1+q_2)(q_3+q_4)} > 0,$$

as in the numerator both terms are negative, and the denominator is negative by our assumption. Then, as we assumed that 3 roots are positive, then also the fourth one. But then

$$a_1 = a_0\Big(\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} + \frac{1}{y_4}\Big) = 0$$

which is impossible if all roots are positive. So, the function $U(r)$ has only one extremum in $(0, +\infty)$, which is the desired minimum.

Two non-point clusters Consider two clusters: 1) the first one at the point x_1 with charge $q_1 > 0$, and on the interval $(x_1, x_2 = x_1 + \varepsilon)$ with constant charge density of total charge $q_2 < 0$; 2) the second on the interval $(x_3 = x_2 + r, x_3 + \varepsilon = x_4)$ with constant charge density with total charge $q_3 < 0$, and positive charge q_4 at the point x_4. Assume also that $q_1 + q_2 > 0, q_3 + q_4 < 0$.

Proposition 2.4. *Potential energy has at least one minimum on the interval* $(0, \infty)$.

Define charge densities on the intervals:

$$q_2 = \int_0^{\varepsilon} \rho_2 dx \Rightarrow \rho_2 = \frac{q_2}{\varepsilon}, \quad \rho_3 = \frac{q_3}{\varepsilon}.$$

Then potential energy is

$$U(r) = \frac{q_1 q_4}{r + 2\varepsilon} + (q_1\rho_3 + q_4\rho_2)\int_0^{\varepsilon} \frac{dx}{r + \varepsilon + x} + \rho_2\rho_3 \int_0^{\varepsilon}\int_0^{\varepsilon} \frac{dxdy}{r + x + y} =$$

$$= \frac{q_1 q_4}{r + 2\varepsilon} + (q_1\rho_3 + q_4\rho_2) I_1(r) + \rho_2\rho_3 I_2(r).$$

We want to show that as $r \to 0$

$$0 < \lim_{r\to 0} U(r) < \infty, \quad -\frac{dU}{dr} \to \infty.$$

This is clear from the following calculations:

$$I_1(r) = \int_0^{\varepsilon} \frac{dx}{r + \varepsilon + x} = \int_0^{\varepsilon} \frac{d(x + r + \varepsilon)}{r + \varepsilon + x} = \ln(x)|_{r+\varepsilon}^{r+2\varepsilon} = \ln(r + 2\varepsilon) - \ln(r + \varepsilon),$$

$$-\frac{dI_1(r)}{dr} = \frac{1}{r + \varepsilon} - \frac{1}{r + 2\varepsilon} = \frac{\varepsilon}{(r + \varepsilon)(r + 2\varepsilon)},$$

$$I_2(r) = \int\limits_0^{\varepsilon} \int\limits_0^{\varepsilon} \frac{dxdy}{r+x+y} =$$

$$= \int\limits_0^{\varepsilon} (\ln(r+y+\varepsilon) - \ln(r+y))dy = \int\limits_{r+\varepsilon}^{r+2\varepsilon} \ln(z)dz - \int\limits_{r}^{r+\varepsilon} \ln(z)dz =$$

$$= (z\ln(z) - z)\,|_{r+\varepsilon}^{r+2\varepsilon} - (z\ln(z) - z)\,|_{r}^{r+\varepsilon} = (r+2\varepsilon)\ln(r+2\varepsilon) +$$

$$+ r\ln(r) - 2(r+\varepsilon)\ln(r+\varepsilon) = r\ln\left(\frac{r(r+2\varepsilon)}{(r+\varepsilon)^2}\right) + 2\varepsilon\ln\left(\frac{r+2\varepsilon}{r+\varepsilon}\right),$$

$$-\frac{dI_2(r)}{dr} = \int\limits_0^{\varepsilon} \left(\frac{1}{r+y} - \frac{1}{r+y+\varepsilon}\right) dy =$$

$$= 2\ln(r+\varepsilon) - \ln(r) - \ln(r+2\varepsilon) = \ln\left(\frac{(r+\varepsilon)^2}{r(r+2\varepsilon)}\right),$$

$$-\frac{dU(r)}{dr} = \frac{q_1 q_4}{(r+2\varepsilon)^2} + \frac{\varepsilon(q_1\rho_3 + q_4\rho_2)}{(r+\varepsilon)(r+2\varepsilon)} + \rho_2\rho_3 \ln\left(\frac{(r+\varepsilon)^2}{r(r+2\varepsilon)}\right) =$$

$$= \frac{q_1 q_4 (r+\varepsilon) + (q_1 q_3 + q_4 q_2)(r+2\varepsilon)}{(r+\varepsilon)(r+2\varepsilon)^2} + \frac{q_2 q_3}{\varepsilon^2} \ln\left(\frac{(r+\varepsilon)^2}{r(r+2\varepsilon)}\right).$$

Now the argument similar to one in the proof of Proposition 2.1 gives that there exists at least one minimum.

A chain of N clusters with nearest-neighbor interaction Consider N one-dimensional clusters $i = 1, \ldots, N$, where the measures (μ_i, ϑ_i) are three-point measures at the points $x_i - \varepsilon_{i,1}, x_i, x_i + \varepsilon_{i,2}$ with charges $q_{i1} < 0, q_{i2} > 0, q_{i3} < 0$ and masses $\mu_{i1}, \mu_{i2}, \mu_{i3}$ correspondingly. We assume that cluster i is symmetric , that is $\mu_{i1} = \mu_{i3}.\varepsilon_{i,1} = \varepsilon_{i,2}, q_{i1} = q_{i3}$.

Moreover, it is always assumed that $x_1 < \ldots < x_N$ and $r_i = (x_{i+1} - \varepsilon_{i+1}) - (x_i + \varepsilon_i) > 0$ for any $i = 1, \ldots, N-1$. Then $U = U(r_1, \ldots, r_{N-1})$ is the function of all $r_i, i = 1, \ldots, N-1$, and $Q_i = q_{i1} + q_{i2} + q_{i3}$ is the full charge of the cluster i. Moreover, assume that I consists of $N-1$ edges $(i, i+1), i = 1, \ldots, N-1$.

Proposition 2.5. *If $Q_i < 0$ for odd i, $Q_i > 0$ for even i, then the function $U(r_1, \ldots, r_{N-1})$ has at least one minimum in R_+^{N-1}.*

To see this, it sufficient to note that the potential energy is the sum $U = \sum_{i=1}^{N-1} U_i(r_i)$. It follows that all mixed derivatives in the Hessian are zero. The Hessian is diagonal, and as each U_i has minimum at some point $r_{i,0}$, U also has minimum at $(r_{1,0}, \ldots, r_{N-1,0})$.

Complete 3-cluster interaction Note first that it is easy to prove that the system of 3 particles may have fixed configuration but only not stable.

Now consider the same model as above and let $N = 3$. Assume that all 3 clusters are symmetric, remind that also $Q_1 < 0$, $Q_3 < 0$, $Q_2 > 0$. But now assume that any pair of clusters interact, that is the potential energy is

$$\begin{aligned} U &= U_{1,2}(r_1) + U_{2,3}(r_2) + U_{1,3}(r_1 + r_2 + 2\varepsilon) \\ &= V(r_1) + V(r_2) + U_{1,3}(r_1 + r_2 + 2\varepsilon), \end{aligned}$$

where functions $U_{1,2}$ and $U_{2,3}$ are assumed to be the same functions which we denoted by $V(r)$. Let $r_{min} > 0$ be minimum of $V(r)$.

Denote $V^{(k)}(r) = kV(r)$ the same function but where all charges of the cluster 2 are multiplied by k (then cluster 2 we denote as $2(k)$). It has minimum at the same point r_{min}. Then the potential energy $U_k(r_1, r_2)$ of the system of clusters $1, 2(k), 3$ is

$$U^{(k)}(r_1, r_2) = kV(r_1) + kV(r_2) + U_{1,3}(r_1 + r_2 + 2\varepsilon).$$

Proposition 2.6. *There exists $k_0 > 0$ such that for any $k > k_0$ the potential energy $U^{(k)}(r_1, r_2)$ of the system of clusters $1, 2(k), 3$, has minimum at the point (r_o, r_o) for some $r_0 = r_0(k)$. Moreover, $r_0(k) \to r_{min}$ as $k \to \infty$.*

Proof. Instead of $U^{(k)}(r_1, r_2)$ we can consider the function

$$\frac{1}{k}U^{(k)}(r_1, r_2) = V(r_1) + V(r_2) + \frac{1}{k}U_{1,3}(r_1 + r_2 + 2\varepsilon).$$

Note that all these functions are analytic for $r_1 > 0, r_2 > 0$. Then we can consider $\frac{1}{k}U_{1,3}(r_1+r_2+2\varepsilon)$ as a small perturbation of the function $V(r_1)+V(r_2)$. Due to symmetry we start with the function $\frac{1}{k}U^{(k)}(r, r)$. We know from section 3.1 that the graph of the function $V(r)$ intersects real axis at some point $x_0 > 0$. Then the graph of the function $\frac{1}{k}U^{(k)}(r, r)$ intersects real axis at the point $x_0 + \delta$, where $\delta = \delta(k) \to 0$ as $k \to \infty$. Also, the critical point of the function $\frac{1}{k}U^{(k)}(r, r)$ of one variable r tends to the minimum of $V(r)$ as $k \to \infty$. Moreover, at the point $r_1 = r_2 = r$

$$\frac{\partial}{\partial r_1}\frac{1}{k}U^{(k)}(r_1, r_2) = \frac{1}{2}\frac{\partial}{\partial r}\frac{1}{k}U^{(k)}(r, r).$$

Thus, $\nabla U^{(k)}(r_0, r_0) = 0$.

Now, to show that it is minimum of the function $U^{(k)}(r_1, r_2)$ of two variables r_1, r_2, it remains to note that, by analyticity, two Minors of the Hessian

$$\frac{\partial^2 U}{\partial r_{1,2}^2}(r_o, r_o) = \frac{\partial^2 U_{1,2}}{\partial r_{1,2}^2}(r_o) + \frac{\partial^2 U_{1,3}}{\partial r_{1,2}^2}(r_o, r_o),$$

$$\frac{\partial^2 U}{\partial r_1^2}(r_o, r_o)\frac{\partial^2 U}{\partial r_2^2}(r_o, r_o) - \left(\frac{\partial^2 U}{\partial r_1 \partial r_2}(r_o, r_o)\right)^2$$

stay positive at some neighborhood of the point (r_{min}, r_{min}).

3. Two-dimensional case

Circle-type cluster and point particle Consider the following two clusters: 1) at point $z = (x_1, 0) \in R^2$ there is charge $q_+ > 0$, and the charge $-q_-, q_- > 0$, is uniformly distributed on the circle $\{y : |z - y| = \varepsilon\}$ of radius $\varepsilon > 0$ with center at z; 2) there is also point particle with negative charge $q < 0$ at the point $(x_2, 0\}$, such that $r = x_2 - (x_1 + \varepsilon) > 0$.

Proposition 3.1. *If* $Q_1 Q_2 < 0$, *then* $U(r)$ *has minimum on the interval* $(0, \infty)$.

Proof. To prove the existence of the minimum, similarly to the Proposition 2.1, it is sufficient to show that $U \to +\infty$ as $r \to 0$.

The charge density on the circle is defined as follows:

$$q_- = \int_0^{2\pi} \rho_- d\varphi = 2\pi\rho_-$$

Then the potential energy is

$$U(r) = \frac{q_+ q}{r+\varepsilon} - \int_0^{2\pi} \frac{\rho_- q d\varphi}{\sqrt{(r+\varepsilon - \varepsilon\cos(\varphi))^2 + \varepsilon\sin(\varphi)^2}} =$$

$$= \frac{q_+ q}{r+\varepsilon} - \int_0^{2\pi} \frac{q_- q d\varphi}{2\pi\sqrt{(r+\varepsilon)^2 + \varepsilon^2 - 2(r+\varepsilon)\varepsilon\cos(\varphi)}} =$$

$$= \frac{q_+ q}{r+\varepsilon} - \frac{q_- q}{2\pi\sqrt{2(r+\varepsilon)\varepsilon}} \int_0^{2\pi} \left(\sqrt{\frac{(r+\varepsilon)^2 + \varepsilon^2}{2(r+\varepsilon)\varepsilon}} - \cos(\varphi)\right)^{-1/2} d\varphi.$$

It is easy to see that for $r = 0$ this integral diverges in the neighborhood of $\varphi = 0$.

Remark 3.1. In fact, in this case we can get explicit formula for $U(r)$. Consider the last integral. Denote

$$\frac{(r+\varepsilon)^2 + \varepsilon^2}{2(r+\varepsilon)\varepsilon} = a,$$

and as $r > 0$, we get that $a > 1$. Then

$$\int_0^{2\pi} \frac{d\varphi}{\sqrt{a+\cos(\varphi)}} = 2\int_0^{\pi} \frac{d\varphi}{\sqrt{a+\cos(\varphi)}} = 2\int_0^{\pi} \frac{d\varphi}{\sqrt{a+1-2\sin^2(\varphi/2)}} =$$

$$= 2\int_0^{\pi} \frac{d\varphi}{\sqrt{a+1-2\sin^2(\varphi/2)}} = \frac{4}{\sqrt{a+1}}\int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1-2\sin^2(\psi)/(a+1)}} =$$

$$= \frac{4}{\sqrt{a+1}} K\left(\sqrt{\frac{2}{a+1}}\right),$$

where $K(k)$ is the complete elliptic integral of the first kind. It is known that $K(1) = \infty$, which gives that $U \to +\infty$ as $d \to 0$.

Remark 3.2. Denote r_{min} an existing minimum. Then we get configuration with interesting form of stability. The point particle can stand still (with zero tangential velocity $v - v(0) = 0$) at any point on the distance r_{min} from the circle. Also it can rotate stationary along the same circle with any fixed tangential velocity $v \neq 0$. Note that this is impossible in Celestial mechanics, where in the famous 2-body problem the radius uniquely defines the tangential velocity.

If we choose the initial coordinate $(x_2(0), 0)$ with $x_2(0) = x_1(0) + \varepsilon + r_{min} + \delta$ and initial tangential velocity $v(0) = 0$, for sufficiently small $\delta > 0$, then the point particle will oscillate on a small interval of R^1 around $x_1(0) + \varepsilon + r_{min}$.

Two circle-type clusters Define two clusters on the plane $R^2 = \{(x, y)\}$. The first one consists of the particle at the point $x_1 = (x_1, 0)$ with charge $q_{1+} > 0$, and the measure on the circle of radius $\varepsilon > 0$ with center at x_1. This measure has full charge $-q_{1-}, q_{1-} > 0$, uniformly distributed on this circle. The second consists of the point charge $q_{2+} > 0$ at the point $x_2 = (x_2, 0)$, and the measure with full charge $-q_{2-}, q_{2-} > 0$, uniformly distributed on the circle of radius $\varepsilon > 0$ with center at the point x_2. Moreover, we assume that $r = x_2 - x_1 - 2\varepsilon > 0$.

The charge densities on both circles are (we introduce angles φ on the left circle and φ on the right

$$q_{1-} = \int_0^{2\pi} \rho_{1-} d\varphi = 2\pi\rho_{1-}, \quad q_{2-} = \int_0^{2\pi} \rho_{2-} d\varphi = 2\pi\rho_{2-}$$

and the potential energy is

$$U(r) = \frac{q_{1+}q_{2+}}{r+2\varepsilon} - \left(\frac{q_{1+}q_{2-} + q_{1-}q_{2+}}{2\pi}\right) I_1(r) + \frac{q_{1-}q_{2-}}{4\pi^2} I_2(r),$$

where

$$I_1(r) = \frac{1}{\varepsilon}\int_0^{2\pi} \frac{d\varphi}{\sqrt{(\frac{r}{\varepsilon} + 2\varepsilon - \varepsilon\cos(\varphi))^2 + \varepsilon^2\sin^2(\varphi)}},$$

$$I_2(r) = \frac{1}{\varepsilon}\int_0^{2\pi}\int_0^{2\pi} \frac{d\varphi d\psi}{\rho(\varphi,\psi)},$$

where $\rho(\varphi,\psi)$ is the distance between points on the circles corresponding to angles φ and ψ.

Proposition 3.2. *$U(r) < \infty$ for $r \in (0,\infty)$, but the repulsive force $-\frac{dU(r)}{dr}$ tends to $+\infty$ when $r \to \infty$.*

Note first that for sufficiently small $\varepsilon > 0$ the function $U(r) > 0$ for not too large r, and this fact is sufficient for existence of the minimum.

Proof. It is easy to see that, for small ε, $I_1(r)$ is uniformly bounded in $r \in [0,\infty)$. Now we want to prove that $0 < I_2(0) < \infty$.

For this consider on the plane $R^2 = \{(x,y)\}$ two circles of radius ε with centers at the points $(-\varepsilon, 0)$ and $(\varepsilon, 0)$ correspondingly. They intersect only at $(0,0)$. To get the upper bound for I_2 it is sufficient to consider the interaction of parts of these circles correspondingly in the two upper quarter planes $\{(x_1,y) : 0 < x_1 < a, 0 < y < a\}$ and $\{(-x_2,y) : -a < -x_2 < 0, 0 < y < a\}$ for small $0 < a < \varepsilon$. Choose two points of these arcs L_1, L_2:

$$\xi_1 = (x_1, y_1 = \sqrt{\varepsilon^2 - (x_1-\varepsilon)^2}), \quad \xi_2 = (-x_2, y_2).$$

The distance between them is

$$\rho(\xi_1,\xi_2) = \sqrt{(x_1+x_2)^2 + (y_1-y_2)^2} > \sqrt{x_1^2 + x_2^2}.$$

Then the potential energy between two negative uniform charge distributions is, for some constant $C > 0$ (note that charge density is the function of x_1, x_2, bounded from below and from above),

$$U_{12} = \int_{L_1}\int_{L_2} \frac{d\xi_1 d\xi_2}{\rho(\xi_1,\xi_2)} = C\int_0^a\int_0^a \frac{dx_1dx_2}{\rho(\xi_1,\xi_2)} < C\int_0^a\int_0^a \frac{dx_1dx_2}{\sqrt{x_1^2+x_2^2}}. \tag{3.1}$$

It can be considered as the integral over domain $Q = (0,a)\times(0,a)$ of the quarter-plane $R_+^2 = \{(x_1,x_2)\}$. Then, in the polar coordinate system $(R.\varphi)$ on the plane (x_1,x_2) we have that the last integral is less than

$$\int_Q \frac{Rdrd\varphi}{R} < 2\pi a^2.$$

The second assertion of the Proposition can be proved quite similarly. Namely, instead of arcs take two segments: $y = x_1, 0 < x_1 < a$, on the right quarter-plane and $y = x_2, -a < -x_2 < 0$ on the left quarter-plane. The distance on R^2 between points $\xi_1 = (x_1, x_1)$ and $\xi_2 = (-x_2, x_2)$ is

$$\rho(\xi_1, \xi_2) = \sqrt{(x_1 + x_2)^2 + (x_1 - x_2)^2} = \sqrt{2}\sqrt{x_1^2 + x_2^2}.$$

Then the modulus of the force between these two points is $F = \rho^{-2}(\xi_1, \xi_2) = \frac{1}{2}(x_1^2 + x_2^2)^{-1}$. And its projection F_x onto x-axis is

$$\frac{1}{\sqrt{2}} F^{-1}(\xi_1, \xi_2) \leq F_x^{-1}(\xi_1, \xi_2) \leq F^{-1}(\xi_1, \xi_2)$$

and, as the integral

$$\int_0^a \frac{dx_1 dx_2}{x_1^2 + x_2^2} = \infty,$$

this give the result.

4. Three-dimensional case

4.1. Classical Theorems

Mean-value Theorem Remind that C^2-function $u = u(x)$ in open domain $O \subset R^3$ is called harmonic if it satisfies the Laplace equation $\Delta u = 0$ in this domain. Let $B(x, r)$ be the a ball with center at some $x \in O$, with radius $r > 0$, and $\partial B(x, r)$ be its boundary. The mean-value theorem [9] says that for any harmonic u and any x, r such that $B(x, r) \subset O$

$$u(x) = \int_{B(x,r)} u(y) dy = \int_{\partial B(x,r)} u d\sigma. \tag{4.1}$$

Earnshaw theorem This theorem says that harmonic function $u(x)$ can have neither point maximum nor minimum in O. For example, let x be such a minimum, then the mean-value theorem gives us contradiction if we choose r sufficiently small. We do not know the history of this theorem, just Earnshaw is the most often name mentioned in the literature, see the related papers in [2–8], in particular chapter 7.5 in [4].

Newton's Shell Theorem Newton's Shell theorem consists of two parts:

1) a point charge q at point $x \in R^3$ and the charge uniformly distributed on either the sphere $\partial B(x, r)$ or on the whole ball $B(x, r)$, for any r, produce the same scalar electrostatic field at any point y outside the ball, that is

$$\varphi(y) = \frac{q}{|x - y|} = \frac{1}{4\pi r^2} \int_{\partial B(x,r)} \frac{q}{|z - y|} d\sigma(z) = \frac{3}{4\pi r^3} \int_{B(x,r)} \frac{q}{|z - y|} dz.$$

To deduce it from (4.1) take some point z outside $B(x,r)$ and consider harmonic function $u(x) = \frac{1}{|x-z|}$. And note that the interaction energy of charge 1 at point z with charge q at the point z, and with and distributed charge on $B(x,r)$ or $\partial B(x,r)$ are given by the same integrals, and thus they are all equal. And these integrals can be considered also as the scalar field at the point z, created by any of these 3 charged clusters – point x, ball $B(x,r)$ and $\partial B(x,r)$.

2) $\varphi(y)$ is constant inside the closed ball $B(x,r)$ if the only charge is uniformly distributed on the boundary $\partial B(x,r)$. That is such distributed charge produces zero force on any particle inside the ball. For this assertion there is a simple geometric proof.

4.2. Examples of stability for quasi-quantum systems of clusters

Consider two clusters:

1) A ball $B(x,1)$ has the charge $q_+ > 0$, uniformly distributed on its boundary $\partial B(x,1)$ and the charge $-q_- < 0$, uniformly distributed inside this ball.

2) A ball $B(y,\varepsilon)$ has the charge $q < 0$ uniformly distributed inside $B(y,\varepsilon)$.

The distance between two clusters is $r = |x-y| - 1-\varepsilon$. Then $U(r)$ depends only on r. We assume that $q_+ > q_-$. Then for r sufficiently large U is negative and tends to zero. For any $r > 0$ $U(r)$ is also negative and equals $(q_+ - q_-)q/(1+\varepsilon+r)$. When $|x-y|$ becomes less than $1-\varepsilon$, the function $U(r)$ becomes positive as positive charge q_+ does not interact anymore with cluster 2 (by second Shell Theorem) as this cluster is inside the ball $B(x,1)$. It follows that the force on y will be repulsive. And thus U will have unique minimum inside the interval $(-\varepsilon, 0)$.

One could of course give more examples: 1) cluster 2 has the same structure as cluster 1 but the positive charge of the sphere is less than the negative charge inside the ball; 2) cluster 2 has a point charge $0 < q_1 < |q|$ at the center. And many others.

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