COMBINATORICS AND PROBABILITY OF MAPS

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Abstract. We give an introductory pedagogical review of rigorous mathematical results concerning combinatorial enumeration and probability distributions for maps on compact orientable surfaces, with an emphasize on applications to two-dimensional quantum gravity.

1. Introduction

We give an introductory pedagogical review of rigorous mathematical results concerning combinatorial enumeration and probability distributions for maps on compact orientable surfaces. The main goal of this paper is to present the main ideas concerning central unsolved mathematical problems in the field related to quantum gravity and strings. I could not include, for example, billions of results concerning various classes of plane maps, there are very nice results among them but most of them just confirm that those classes of maps belong to the same universality class. The reader will find such results in more special reviews.

We give also a rigorous description of some terminology, related to discrete quantum gravity. Bibliographical reviews on quantum gravity see [2–4]. Earlier mathematical ideas see in [1, 16], some fresh mathematical exposition of some parts of discrete quantum gravity see in [34]. We do not touch an enormous number of relations of maps with other fields of mathematics like moduli of curves [8], Galois theory see [5, 6], topology [28].

After the definition of map itself it is important to know main examples of subclasses maps, they are defined via some topological restrictions on maps. Next we give an introduction to Tutte's enumeration theory, using partly different techniques. After this we present elements of dynamical triangulation calculus. A short exposition of random matrix approach is given as well.

Gibbs families are defined as a far going generalization of Gibbs distributions. For easiest subclass of maps, called planar Lorentzian models, we present more or less complete results. The last section is devoted to the most difficult problem, maps with matter fields on them. Rigorous results can only be obtained now with cluster expansion techniques.

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2. Maps are defined

Consider a compact closed orientable smooth surface S_ρ of genus ρ, that is a sphere with ρ handles. It is useful but not necessary to have a differentiable structure (which is known to be unique) on it. Denote also $S_{\rho,k}$ the surface S_ρ with k holes (that is k disks deleted), thus $S_{\rho,0} = S_\rho$. The boundary of $S_{\rho,k}$ consists of k smooth circles. A smooth map on $S_{\rho,k}$ is a triple $(S_{\rho,k}, G, \phi)$, where G is a connected graph, considered as a one-dimensional complex. G may have loops and multiple edges. Let $V(G), L(G)$ are the sets of vertices and (open) edges of G. ϕ is an embedding of G into $S_{\rho,k}$ such that the following conditions are satisfied:

1. for each edge $l \in L(G)$ $\phi(l)$ is a smooth curve on $S_{\rho,k}$;
2. the connected components (called cells or faces) of the complement $S_{\rho,k} \setminus \phi(G)$ are homeomorphic to an (open) disk. It follows that if $k > 0$ then the boundary is contained in $\phi(L(G) \cup V(G))$.

Two smooth maps $(S_{\rho,k}, G, \phi), (S'_{\rho,k}, G', \phi')$ are called equivalent if there is a one-to-one homeomorphism $f : S_{\rho,k} \to S'_{\rho,k}$, respecting orientation and such that $f : \phi(V(G)) \to \phi'(V(G'))$ and $f : \phi(L(G)) \to \phi'(L(G'))$ are one-to-one. Combinatorial map is an equivalence class of smooth maps. Further on we consider only combinatorial maps and call them maps for shortness.

Combinatorial definition of maps
There is a pure combinatorial definition of maps, that does not use surfaces at all. It is based on the notion of a ribbon (or ordered, or fat) graph: graph with a cyclic order of edge-ends at each vertex. Edge-end (or leg) is a pair (v, l) where v is a vertex and l is one of its incident "half-edges".

The combinatorial definition starts with a triple (E, ω, P), where E is a finite set with even number of elements and two permutations ω and P. It is assumed that all cycles of ω have length 2 and that the group generated by ω, P acts transitively on E. For each map $(S_{\rho,k}, G, \phi)$ one can canonically define a triple (E, ω, P) so that E is the set of edge ends of G, ω interchanges two edge-ends of each edge, and for each vertex v P is a cyclic permutation of the set of all edges incident to v corresponding to the clockwise order for a given orientation of $S_{\rho,k}$. Inverse construction is given by the following theorem. In combinatorics community it is refered to Edmonds [7], but algebraists indicate that it goes back to Hamilton.

Theorem 1. *For any triple (E, ω, P) there is a unique map T where the factor set E/ω is the set of edges, the factor set E/P is the set of vertices (with a clockwise order of incident edges), the factor set $E/(\omega P)$ is the set of faces, denoted by $F(T)$.*

The construction of the map proceeds by induction: on each step one face from $E/(\omega P)$ is appended by identifying corresponding edges.

Fat (ribbon) graph can be obtained from an ordinary graph by replacing each edge with fat edge, thus an edge will have two distinct sides. If vertex v has $l(v)$

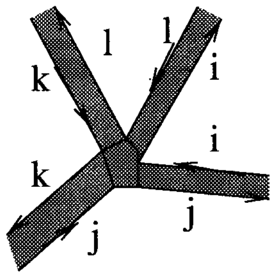

Figure 1. Vertex of a ribbon graph

incident edges then $2l(v)$ sides will also have cyclic order, and in a neighbourhood of v they will look as in the Figure 1. This combinatorial definition was used also in algebraic problems, for example, concerning moduli space of curves, see [8].

Classes of maps
Denote $\mathcal{A}$ the class of all maps. There are many subclasses of $\mathcal{A}$, defined by some restrictions on maps. We give a few number of important examples.

1. Maps of genus ρ, that is all maps with fixed ρ and any k. In particular planar maps have $\rho = 0$. See a long list of subclasses of planar maps in [9, 10].

2. We define triangulations (called quasi-triangulations in [11]) by the following restrictions: the boundary of each cell consists exactly of three edges and the map is nonseparable, thus multiple edges are allowed but no loops. For example, let $\mathcal{A}_{0,1}(N,m)$ be the class of all triangulations of $S_{0,1}$, that is of the disk, with N triangles and m edges on the boundary of the disk. Note that all combinatorial triangulations can be obtained by taking N copies of a triangle and identifying some pairs of their edges, that is why their number is finite for fixed N. Note also that the class of simplicial complexes is a subclass of triangulations in this terminology.

3. Rooted triangulations $\mathcal{A}^0_{0,1}(N,m)$ of $S_{0,1}$ that is in each triangulation one boundary edge is distinguished with one of its vertices, this edge is called the root. Equivalence relation is modified correspondingly so that the mappings f respect the roots.

4. One face maps on S_ρ, that is all $T \in \mathcal{A}$ with $|F(T)| = 1$.

5. Slice-triangulations. We define slice-triangulations (in physical literature they are called Lorentzian models) of the 2-dim cylinder $S_{0,2}$. Consider triangulations T of the cylinder $S^1 \times [M,N]$, where S^1 is a circle, $M < N$ are integers. Assume the following properties of T: each triangle belongs to some strip $S^1 \times [j,j+1], j = M,\ldots,N-1$, and has all vertices and exactly one edge on the boundary $(S^1 \times \{j\}) \cup (S^1 \times \{j+1\})$ of the strip $S^1 \times [j,j+1]$. Let

$k_j = k_j(T)$ be the number of edges on $S^1 \times \{j\}$. We assume $k_j \geq 1$. Then the number of triangles $F = F(T)$ of T is equal to

$$F = 2 \sum_{j=M+1}^{N-1} k_j + k_M + k_N. \tag{1}$$

3. Enumeration for fixed genus

3.1. GENUS ZERO

Starting from 1962 Tutte publishes a series of papers where he solves the enumeration (censoring) problem for various classes of planar maps. He invents a new method of solving such problems which was developped in hundreds of papers. The heart of this method consists of two parts: deleting of an edge to get recurrent equation and a method to solve the resulting quadratic functional equation, containing two unknown generating functions: from two and one variables correspondingly. I use this occasion to note that some years later quite independently I developped (in connection with the Riemann–Hilbert problem for two complex variables) a new method for solving linear functional equation with three unknown functions: one function of two variables and two functions of one variable. These methods are quite different but have one common point—projection of the equation on some algebraic curve.

Let $C_0(N,m)$ be the number of triangulations in $\mathcal{A}^0_{0,1}(N,m)$, that is with N triangles and m edges on the boundary. It is not difficult (see [13]) to prove apriori bounds: there exist $0 < C_1 < C_2 < \infty$ such that for all N and m

$$C_1^N < C_0(N,m) < C_2^N$$

There are more general results concerning such exponential apriori bounds for the number of triangulations of a manifold for dimensions more than 2, see [17].

More difficult is to find exact asymptotics. One of the theorems by Tutte can be formulated as follows

Theorem 2. *If $N \to \infty$ and m is fixed then*

$$C_0(N,m) \sim cN^{\alpha}\gamma^N$$

where $\gamma = \sqrt{\frac{27}{2}}, \alpha = -\frac{5}{2}, c = c(m)$.

Proof. The following recurrent equations

$$C_0(N,m) = C_0(N-1,m+1) + \sum_{N_1+N_2=N-1, m_1+m_2=m+1} C_0(N_1,m_1)C_0(N_2,m_2)$$

can be obtained by deleting the rooted edge. After deletion of the rooted edge there is a rule to define a new rooted edge for each of the resulting maps. This is

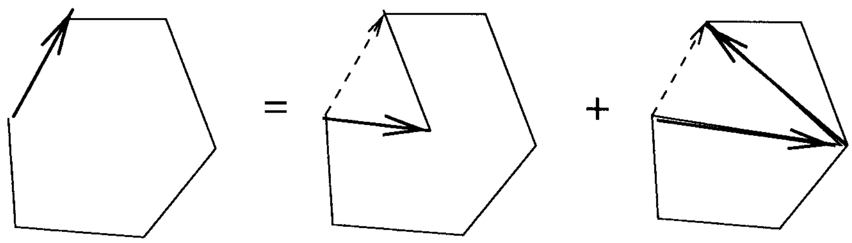

Figure 2. Recurrent equation

shown on Figure 2. There are two possibilities which correspond to the linear and quadratic terms correspondingly.

It is very convenient to assume the following conditions which will be the boundary conditions for the systems of equations below

$$C_0(N,0) = C_0(N,1) = 0, C_0(0,m) = \delta_{m,2}, C_0(1,m) = \delta_{m,3}$$

Only the case $N = 0$, $m = 2$ needs comment: this corresponds to a degenerate disk, an edge with two vertices.

Multiplying (7) on $x^N y^m$ and summing $\sum_{N=0}^{\infty}\sum_{m=2}^{\infty}$ we get the following equation in a small neighbourhood $\Omega \in C^2$ of $x = y = 0$

$$F(x,y) = F(x,y)xy^{-1} + F^2(x,y)xy^{-1} + y^2 - xyF_2(x) \tag{2}$$

where we introduced the generating functions

$$F(x,y) = \sum_{N=0}^{\infty}\sum_{m=2}^{\infty} C_0(N,m)x^N y^m, \quad F_m(x) = \sum_{N=0}^{\infty} C_0(N,m)x^N.$$

F and F_2 are unknown functions.

Here we solve the functional equation (2). We rewrite it in the following form

$$(2xF(x,y) + x - y)^2 = 4x^2y^2F_2(x) + (x-y)^2 - 4xy^3 \tag{3}$$

and denote D its righthand side. Consider the analytic set $\{(x,y) : 2xF + x - y = 0\}$ in a small neighbourhood of $x = y = 0$. Note that it is not empty, $(0,0)$ belongs to this set and it defines a function $y(x) = x + O(x^2)$ in a neighbourhood of $x = 0$. In particular, it will be shown that $y(x)$ and $F_2(x)$ are algebraic functions. Because of the square in the righthand side of (3) we have two equations valid at the points of this analytic set

$$D = 0, \quad \frac{\partial D}{\partial y} = 0$$

or

$$4x^2y^2F_2(x) + (x-y)^2 - 4xy^3 = 0 \tag{4}$$

$$8x^2yF_2(x) - 2(x-y) - 12xy^2 = 0$$

from where one can exclude the function $F_2(x)$ by multiplying second equation (4) on $\frac{y}{2}$ and subtracting it from the first equation. Then

$$y = x + 2y^3 \tag{5}$$

or

$$y = \frac{x}{1 - 2y^2} \tag{6}$$

By the theorem on implicit functions this equation gives the unique function $y(x)$, analytic for small x with $y(0) = 0$. It is evident from (6) that the convergence radius of $y(x)$ is finite. Note that $y(x)$ is odd and $F_2(x)$ is even, because for any triangulation $N - m$ is even.

We continue to rededuce here Tutte results in a different way. $y(x)$ is an algebraic function satisfying the equation $y^3 + py + q = 0$ with $p = -\frac{1}{2}, q = \frac{x}{2}$. The polynomial $f(y) = y^3 + py + q$ can have multiple roots only when $f = f'_y = 0$, which gives $x_{\pm} = \pm\sqrt{\frac{2}{27}}$. These roots are double roots because $f''_y \neq 0$ at these points. For $x_+ = \sqrt{\frac{2}{27}}$ we have $y_+ = y(x_+) = \frac{1}{\sqrt{6}}$, that can be seen from $f'_y = 3y^2 - \frac{1}{2} = 0$ and $f = 0$. From (6) it also follows that $x(-y) = -x(y)$ and thus $y(x)$ is odd. It follows that $y(x)$ has both $x_{\pm} = \pm\sqrt{\frac{2}{27}}$ as its singular points.

From (4) we know $F_2(x)$ explicitely, after that $F(x, y)$ is explicit from equation (3). The unique branch $y(x)$, defined by equation (6), is related to the unique branch of $F_2(x)$ by the equation

$$F_2 = \frac{(1 - 3y^2(x))}{(1 - 2y^2(x))^2} = x^{-2}y^2(1 - 3y^2)$$

that is obtained by substituting $x = y - 2y^3$ to the first equation (6).

We know that $F_2(x)$ has positive coefficients, that is why $x = \sqrt{\frac{2}{27}}$ should be among its first singularities. Then $x = -\sqrt{\frac{2}{27}}$ should also be a singularity of both $y(x)$ and $F_2(x)$.

The principal part of the singularity at the double root x_+ is $y(x) = A(x - x_+)^{d+\frac{1}{2}}$ for some integer d. As $y_+ = y(x_+)$ is finite then $d \geq 0$. At the same time $y'(x) = \frac{1}{1 - 6y^2(x)}$ that is ∞ for $x = x_+$. It follows that $d = 0$. For F_2 we have the same type of singularity $A(x - x_+)^{d+\frac{1}{2}}$ but here $d = 1$ as $F_2(x_+)$ and $F'_2(x_+)$ are finite but $F''_2(x_+)$ is infinite.

The theorem follows from this. We proved also that the generating functions are algebraic. One can check also that

$$F(x, y) = \frac{y - x}{2x} - \frac{(y - y(x))\sqrt{\frac{x^2}{y^2(x)} - 4xy}}{2x}.$$

□

Automorphism groups. How from the results for rooted triangulations to get results for unrooted? Everything here is based on the following principle. Let

$\mathcal{A}_{01}(N,m)$ be the class of unrooted triangulations of the disk with N triangles and m edges on the boundary, let $|\mathcal{A}_{01}(N,m)| = C(N,m)$ and let $C^{nontr}(N,m)$ be the number of such maps with nontrivial automorphism group.

Theorem 3. *For almost all maps in the classes $\mathcal{A}_{01}(N)$ and $\mathcal{A}_0(N)$ the automorphism group is trivial, more exactly, for fixed m*

$$\frac{C^{nontr}(N,m)}{C(N,m)} \underset{N\to\infty}{\longrightarrow} 0$$

There are many proofs of this theorem for many classes of maps, see [9]. This is convenient to do in two steps.

Lemma 4. *If m is fixed and $N \to \infty$ then*

$$C(N,m) \sim \frac{1}{m} C_0(N,m)$$

Proof. Let us enumerate the edges of the boundary $1, 2, \dots, m$ in the cyclic order, starting from the root edge. Automorphism ϕ of the triangulation of the disk is uniquely defined if the image $j = \phi(1)$ of the edge 1 is fixed. In fact, then the triangle adjacent to the edge 1 has to be mapped by ϕ onto the triangle adjacent to j, and so on by connectedness.

Consider the strip of width 1, adjacent to the boundary, that is the set of triangles of three types: those which have a common edge with the boundary (type 1), two common adjacent edges (type 2) and those, having with the boundary only a common vertex (type 0). Thus the strip, that is the sequence of triangles can be identified with the word $\alpha = x_1 \dots x_n$, $n > m$, where $x_i = 0, 1, 2$ represent the triangle types. Consider the set $W(m, n_0, n_1, n_2)$ of words with given m and numbers n_i of symbols $i = 0, 1, 2$. An automorphism of a disk gives a cyclic automorphism of the word α. The sets $W(m, n_0, n_1, n_2)$ are invariant however. Note that

$$m = n_1 + 2n_2, \quad n_0 \geq n_1 + n_2$$

and the length of other boundary of the strip equals $m' = n_0$. Thus, if there is no nontrivial cyclic automorphism of the word, then there are no automorphisms of the whole triangulation of the disk. It is easy to see, that for a given sequence $n_0(m)$ and as $m \to \infty$ the set of words from $\cup_{n_1,n_2} W(m, n_0, n_1, n_2)$, having nontrivial cyclic automorphisms, is asymptotically zero compared with the number of words in $\cup_{n_1,n_2} W(m, n_0, n_1, n_2)$. □

Let now $C(N)$ be the number of triangulation of the sphere with N triangles.

Lemma 5. *If $N \to \infty$ then*

$$C(N) \sim \frac{1}{3N} C_0(N,3).$$

Proof of this intiotively clear assertion can be obtained along similar lines.

3.2. ARBITRARY GENUS

Let $C(N, \rho, 0)$ be the number of (unrooted) triangulations of S_ρ, and $C_0(N, \rho, k+1; m, m_1, \ldots, m_k)$, $m, m_1, \ldots, m_k \geq 2$, be the number of triangulations of $S_{\rho,k+1}$, $k = 0, 1, 2, \ldots$, where one boundary (with m edges) has a rooted edge, and where the remaining k boundaries have $m_1, \ldots, m_k$ edges correspondingly.

Theorem 6. *Then for fixed $\rho, k, m_1, \ldots, m_k$ as $N \to \infty$*

$$C(N, \rho, 0) \sim f(\rho) N^{a\rho+b} \gamma^N, \quad a = \frac{5}{2}, \; b = -\frac{5}{2} - 1, \; \gamma = \sqrt{\frac{27}{2}}$$
$$C_0(N, \rho, k; m, m_1, \ldots, m_k) \sim f(\rho, k, m, m_1, \ldots, m_k) N^{a\rho+b+1+k} \gamma^N.$$

There are many similar results on enumerating such maps but not in terms of the number of triangles, see [14, 15]. This theorem can be proved by using Tutte's idea of deleting the rooted edge. We shall see what are the resulting recurrent equations. Start with some triangulation with parameters $N, \rho, k+1, m, m_1, \ldots, m_k$.

The operation of deleting the rooted edge can produce more possibilities than in case $\rho = 0, k = 1$:

1. neither ρ nor k change, only $m \to m+1$, $N \to N-1$;
2. the hole merges with another hole, thus ρ is not changed, k becomes less by 1;
3. the hole cuts a handle, thus $\rho \to \rho - 1$, $k \to k+1$;
4. the hole cuts the surface itself thus producing two surfaces with parameters ρ_i, k_i, $i = 1, 2$, such that $\rho_1 + \rho_2 = \rho$, $k_1 + k_2 = k+1$.

For example, we get a closed system of equations for $\rho = 0$, ρ cannot increase. Moreover, for $\rho = 0$ the parameter k cannot increase, moreover equations for $C_0(N, 0, k; m, m_1, \ldots, m_k)$ are linear assuming we know $C_0(N, 0, j; m, m_1, \ldots, m_j)$ for $0 \leq j < k$. Thus, equations for the generating functions are nonlinear only on the first step with $\rho = 0$, $k = 0$. Afterwards (for $\rho = 0$, $k > 1$ and for $\rho > 0$) the equations for the generating functions become linear, however very bulky. One can find treatment of similar problems in [14, 15] and references therein.

Universality classes. We considered only the case of triangulations. There are many results (see [9]) showing that γ (and also c) strongly depends on the class of maps, on the contrary α (called the critical exponent) does not. However, γ does not seems not to depend on $\rho, k, m, m_1, \ldots, m_k$. An example of another universality class is the class of slice-triangulations, see below.

4. What is two-dimensional gravity

Metric structure on the surface related to a given triangulation is defined once it is defined for each closed cell so that the lengths of the edges are compatible. There are two basic approaches for defining such metric structure. In the Dynamical Triangulation approach all triangles are identical, lengths of edges are equal a and the metrics inside triangles is the standard euclidean metrics. In the Quantum Regge Calculus they are random.

We will use further on the Dynamical Triangulation approach. One can show then that the differentiable structure exists such that the metrics is sufficiently smooth everywhere except for the vertices, where the number q_v of triangles adjacent to the vertex v is different from 6. One can prove this by induction putting pairs of adjacent triangles on the plane.

We can show that the curvature is zero everywhere except for the vertices v with $q_v \neq 6$, in those vertices the curvature becomes discontinuous. In fact, the curvature is measured using the parallel transport (Levi–Civita connection). That is the curvature should be proportional to the difference of the angles for initial and transported vectors of a vector (lying in the plane of the triangle) along a small closed path. If the path lies inside a triangle then the angle is zero as on the euclidean plane, if the path encircles a point on some edge then it is zero (by unfolding the two half planes separated by this edge). Only paths around vertices may give nonzero difference. Around the vertex v the angle between the initial and the transported vector is $\varepsilon_v = 2\pi - \sum_f \varphi_{fv} = \frac{\pi}{3}(6 - q_v)$, where φ_{fv} is the angle of the simplex f at vertex v. Note that

$$2\pi V - \sum_v \varepsilon_v = \sum_v \sum_f \varphi_{fv} = \sum_f \sum_v \varphi_{fv} = \pi F$$

Using the Euler formula

$$\chi = 2 - 2\rho = V - L + F$$

one can get from this the Gauss–Bonnet formula

$$2 \sum_v \varepsilon_v = 4\pi\chi$$

for triangulations, using $L = \frac{3F}{2}$.

Einstein–Hilbert action on the smooth manifold for the so called pure gravity (that is without matter fields) is

$$\int (c_1 R + c_2)\sqrt{g}\, dx$$

where R is the gaussian curvature, g-metrics. By Gauss–Bonnet formula for smooth surfaces $\int R\sqrt{g}\, dx = 4\pi\chi$. Thus the discrete action should be (up to a constant) $\lambda\rho + \mu N$, where ρ is the genus and N is the number of triangles.

When the genus is fixed, that is the surface itself is fixed (but not its metrics), the first term becomes superfluous, and pure gravity (without matter fields) is defined by the following grand canonical partition function

$$Z = \sum_T \exp(-\mu N(T)) = \sum_N C(N) \exp(-\mu N)$$

with the corresponding canonical partition function

$$Z_N = \sum_{T:|F(T)|=N} \exp(-\mu N) = C(N) \exp(-\mu N)$$

There is a critical point $\mu_{cr} = \log \gamma$. If $\mu > \mu_{cr}$ then $Z < \infty$, and if $\mu < \mu_{cr}$ then $Z = \infty$. In the critical point the terms of the series have power decrease (or increase), defined by the critical exponent α. This critical point does not depend on ρ.

However, if the surface has a boundary then the situation is different, there are some extra degrees of freedom (see more about this in the section 7.2). If the surface has a boundary then Euler formula is

$$\chi = V - L + F + k$$

where k is the number of components of the boundary. Gauss–Bonnet formula has the form

$$2 \sum_v \varepsilon_v = 4\pi(\chi - k + \partial L)$$

where ∂L is the number of edges (or vertices) on the boundary, if for the vertices v of the boundary we define ε_v by $\pi - \varepsilon_v = \frac{\pi}{6} q_v$.

4.1. SOME CENTRAL PROBLEMS

The situations when ρ is fixed and when ρ is random are physically quite different. The first one corresponds in physics to string diagrams, the second—to the so called spin foam, when there are local fluctuations of the topology. In fact, in this case ρ is of the order N.

Spin foam. If the genus is not fixed then the grand canonical partition function would be

$$Z = \sum_T \exp(-\mu N(T) - \lambda \rho(T))$$

where T runs over all triangulations. However one can easily show that this series is divergent for any μ, λ. In fact, the number of maps with N triangles irregardless to genus has a factorial growth, that cannot be compensated by exponential factors. Thus it is natural to consider the canonical partition function

$$Z_N = \sum_{T:|F(T)|=N} \exp(-\lambda \rho(T))$$

instead of the grand canonical. Unfortunately, there are no results concerning this canonical partition function—this is one of the central unsolved problems. However, there are another interesting possibilities to choose a canonical ensemble. For example,

$$Z_N = \sum_{T:L(T)=N} \exp(-\lambda\rho(T) - \mu F(T)).$$

For this case we shall get some results in the next section.

Strings. Assume now that to each triangle f of $F(T)$ there corresponds a spin σ_f, taking its values in the space S with some positive measure μ_0 on it. For some symmetric function $\Phi : S \times S \to R$ define the following partition function

$$Z_N = \sum_{T:N(T)=N} \prod_{f\in|F(T)|} \int d\mu_0(\sigma_f) \exp(-\lambda\rho(T) - \beta \sum_{<f,f'>} \Phi(\sigma_f, \sigma_{f'})).$$

The case $S = R^d$ and $\Phi(\sigma, \sigma') = (\sigma - \sigma')^2$ corresponds to Polyakov action for strings with the target space-time of dimension d. There are no results (even on the physical level) concerning calculating this partition function even for $\rho = 0$, random matrix models do not work here. The latter circumstance is quite surprising because another physical approache (Hamiltonian quantization) to free quantum boson strings work quite well.

Z_N gives an example of probability distributions which we call Gibbs families below. We shall say some word about the general theory of such probability distributions.

Continuum limit. It is reasonable to assume that under some scaling limit random discrete spaces converge in the Hausdorf–Gromov distance to a smooth manifold. Even for simple examples it is a formidable problem to prove such a convergence.

Conventional string theory approach. An alternative approach to two-dimensional gravity, see [29], is similar to the conventional continuum path integral approach, and its more refined BRST procedure. String theory approach to treat the continuum analog of Z_N consists heuristically in the following remark. Due to invariance of the action with respect to the diffeomorphism group G_1 and the group G_2 of Weyl transformations, the path integral can be factored on the integrals over G_1, G_2 and some residual factor. The residual factor can be reduced to the integral over moduli of complex algebraic curves. It seems difficult however to make this illuminating staff well-defined. Relation between the two approaches were discussed in [8].

5. Random genus.

Consider the class $\mathcal{A}^0$ of all rooted maps of S_ρ irregardless to ρ. Let $F_{b,p}(\rho)$ be the number of rooted (one edge is specified together with its direction) maps with

$p+1$ vertex and $b+p$ edges of the closed compact surface of genus ρ. We give a simple asymptotic formula for canonical partition function

$$Z_N(y) = \sum_{b+p=N} \sum_{\rho=0}^{\infty} F_{b,p}(\rho) y^p$$

with any complex y.

The operation of deleting a rooted edge gives the following recurrent equations, see [19], for the numbers $F_{b,p} = \sum_\rho F_{b,p}(\rho)$

$$F_{b,p} = \sum_{j=0}^{p-1} \sum_{k=0}^{b} F_{k,j} F_{b-k,p-j-1} + (2(b+p)-1) F_{b-1,p}, \quad b,p \geq 1. \tag{7}$$

In fact these equations hold also if either $b=0$ or $p=0$, except the case $b=p=0$, if we put

$$F_{0,0} = 1, \quad F_{-1,p} = F_{b,-1} = 0$$

We should say more about the cases $b=0$ and $p=0$. The case $b=p=0$ corresponds to an imbedding of one vertex into the sphere. The cases $p \geq 1$, $b=0$ correspond to trees, imbedded to the sphere, $F_{0,p} = \frac{(2p)!}{p!(p+1)!}$. Numbers $F_{b,p}$ with $p=0, b \geq 1$ are equal to $(2b-1)(2b-3)\ldots = \frac{(2b)!}{b! 2^b}$, that is to the number of partitions of $\{1,2,\ldots,2b\}$ onto pairs.

Put

$$b_k = b_k(y) = \sum_{b+p=k} F_{b,p} y^p, b_{-1} = 0, b_0 = 1$$

for $y \geq 0$. The main result is the following theorem.

Theorem 7. *For any complex $y > 0$ and as $k \to \infty$*

$$b_k(y) \sim f(y)\, k!\, 2^k k^{y-\frac{1}{2}}$$

where $c(y) > 0$ is a constant analytic in y.

This result has physical interpretation. Consider the probability measure on the set of general rooted maps with $b+p=N$ (canonical ensemble) with the following partition function

$$Z_N(y) = \sum_{b,p: b+p=N} F_{b,p} \exp(-\lambda b) = \sum_{b,p: b+p=N} F_{b,p} y^p$$

with $y = \exp(-\lambda)$. Then we have the following asymptotics for the partition function

$$Z_N(y) \sim c(y)\, N! \exp((\ln 2 - \lambda) N) N^{y-\frac{1}{2}}$$

One can, using $-2\rho - F - m - 2 = -b$, rewrite this in other terms up to a constant factor

$$Z_N(y) = \sum_{T: L(T)=N} \exp(-2\lambda\rho(T) - \mu F(T))$$

only for $\mu = \lambda$. It follows that the critical point $\lambda_{cr} = \ln 2$ and the critical exponent is $\exp(-\lambda) - \frac{1}{2}$.

6. Random matrix techniques

For mathematical initialization to this beautiful methods we refer the reader to [24], complete but more physical presentations one can find in many physical papers, see for example [21].

6.1. FIXED GENUS

One of the central models of random matrix theory is the following probability distribution μ on the set of selfadjoint $n \times n$-matrices $\phi = (\phi_{ij})$ with the density

$$\frac{d\mu}{d\nu} = Z^{-1} \exp(-tr(\frac{\phi^2}{2h}) - tr(V))$$

where $V = \sum a_k \phi^k$ is a polynomial of ϕ, bounded from below, ν is the Lebesgue measure on the real n^2-dimensional space of vectors $(\phi_{ii}, \operatorname{Re}\phi_{ij}, \operatorname{Im}\phi_{ij}, i < j)$. If $V = 0$, then the measure $\mu = \mu_0$ is gaussian with covariances $\langle \phi_{ij}, \phi^*_{kl} \rangle = \langle \phi_{ij}, \phi_{lk} \rangle = h\delta_{ik}\delta_{jl}$. The density of μ with respect to μ_0 is equal to

$$\frac{d\mu}{d\mu_0} = Z_0^{-1} \exp(-tr(V)).$$

For the existence of μ it is necessary that the degree p of the polynomial V were even and the coefficient a_p were positive. For this case there is a deep theory of such models, see [25].

Fundamental connection (originated by Hooft) between matrix models and counting of maps on surfaces is given by the formal series in semiinvariants or in diagrams (see for example [20]) (we neglect $\log Z_0$)

$$\log Z = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} <tr(V), \ldots, tr(V)> = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \sum_{D_k} I(D_k)$$

where $<tr(V), \ldots, tr(V)>$ is the semiinvariant of order k of the random variable $tr(V)$ with respect to the gaussian measure μ_0, $\sum_{D_k}$ is the sum over all connected diagrams D_k with k vertices and $L = L(D_k)$ edges.

Assume, for example, $V = a\phi^4$, $a > 0$. Then each diagram has enumerated vertices $1, \ldots, k$, to each vertex corresponds the random variable $tr(\phi^4)$. Moreover, each vertex has enumerated edge-ends (legs) $1, 2, 3, 4$, corresponding to all factors in $\phi_{ij}\phi_{jk}\phi_{kl}\phi_{li}$, one should further sum over all indices i, j, k, l. By coupling legs we $L = 2k$ edges. The leg, for example corresponding to ϕ_{ij}, can be imagined as a strip (ribbon) (defining thus a ribbon or fat graph), the sides of the leg have indices i and j correspondingly. In a neighborhood of the vertex the ribbons are placed on the surface as on the Figure 1. It is assumed that coupling of ribbon legs is such that the coupled sides have the same indices.

As any index appears even number of times in each vertex, then for a given edge l with index i, there is a unique closed connected path in the diagram consisting of sides with index i, and passing through l. These paths are called index loops. Summing over indices we will get then the factor n^N where $N = N(D_k)$ is the number of index loops in the diagram. Finally we get the following formal series

$$\sum_k \frac{(-a)^k}{k!} \sum_{D_k} h^{2k} n^{N(D_k)} = \sum_k (-4ah^2)^k \sum_{E_k} n^{N(E_k)} = \sum_{k,N} (-4ah^2)^k n^N M(k,N)$$

where in the second sum the summation is over all graphs with unordered set of vertices, and the set of legs of each vertex is only cyclically ordered. While changing from D to E we introduced the factor $\frac{k!4^k}{A(E)}$, where $A(E)$ is the power of the automorphism group of E. We omit $A(E)$, assuming that for "almost all" (as $k \to \infty$) graphs $A(E) = 1$.

In the third sum $M(k, N)$ is the number of maps with k vertices and N faces. Here a map means a map from the class of 4-regular maps, that is each vertex has degree 4. By Edmonds theorem (see section 2) for any graph E there exists a unique (up to combinatorial equivalence) embedding $f(E)$ of this graph to some oriented compact closed surface S_ρ for some genus ρ and such that each index loop bounds an open subdomain of S_ρ homeomorphic to a disk. This map has k vertices, $2k$ edges and N faces. By Euler formula $k = N + 2\rho - 2$ we get the following formal expansion, putting $h = 1$, $a = \frac{b}{4n}$

$$\log Z = \sum_{N,\rho} (-b)^{N+2\rho-2} n^{-2\rho+2} C_\rho(N)$$

where $C_\rho(N)$ is the number of maps with N faces and having genus ρ. It follows that for example for $\rho = 0$, the sum of terms corresponding to maps with N faces and having genus ρ can be formally obtained as

$$\lim_{n\to\infty} \frac{\log Z}{n^2} = \sum_N (-b)^{N-2} C_0(N).$$

There is a beautiful techniques to extract $C_0(N)$ from this, however I do not know any completely rigorous treatment.

There are generalization of the random matrix model with Q matrices M_q, where the action is

$$Tr(\sum_{q=1}^{Q} V(M_q) + \sum_{q=1}^{Q-1} M_q M_{q+1})$$

and its continuum analog, see review [23]. Such generalizations allow to do calculations for some models with spin.

One face maps. There is another direction in the combinatorics of maps, now more related to algebra. One considers triangulations with one (see [26]) or fixed finite number of cells (see [8, 28]). Maps with $F = 1$ for S_0 coincide with trees imbedded in S_0, that is with plane trees. Their number is given by Catalan numbers. Generally, their number was calculated in [26, 27]. This is related to string theory approach to two dimensional gravity, see section 4.1.

7. Gibbs Families

Here we introduce very shortly a unified general framework for discrete random spaces, a generalization of the now classical theory of Gibbs fields (see [30, 31]), which we call Gibbs families ("Gibbs fields on Gibbs graphs").

We define graph with a local structure. Let $\mathcal{F}_d$ be the set of all finite graphs with diameter d.

Definition 8. Let the function $\sigma(\gamma)$ on $\mathcal{F}_d$ be given with values in some set S. Local structure (of diameter d) on the graph G is given by the set of values $\{\sigma(\gamma)\}$, where γ runs all regular subgraphs γ of diameter d of the graph G.

Examples. – Graphs G with some function $\sigma(v)$ on the set $V(G)$ of vertices (spin graphs) correspond to a local structure with $d = 0$.

- Gauge fields on graphs: for each vertex and for each edge values from a group R are defined, in this case one can take $d = 1$.
- Simplicial complex is completely defined the following function on complete regular subgraphs (of its one-dimensional skeleton) of diameter 1: it takes value 1, iff this subgraph defines a simplex of the corresponding dimension, and 0 otherwise. Here one can also take $d = 1$.
- Penrose quantum networks [33, 32]: in a finite graph, each vertex of which has degree 3, to each edge l some integer $p_l = 2s_l$ is prescribed, where halfintegers s_l are interpreted as degrees of irreducible representations of $SU(2)$. Moreover, in each vertex the following condition is assumed satisfied: the sum of p_i, for all three incident edges, is even and any p_i does not exceed the sum of two other values. Then for the tensor product of three representations there exists a unique (up to a factor) invariant element, which is prescribed to this vertex.

We fix some function $s(\gamma)$, defining a local structure. Let $\mathcal{F}^{(s)}$ be the set of all finite graphs with the local structure. Potential is a function $\Phi : \mathcal{F}^{(s)} \to R \cup \{\infty\}$. The energy of a finite graph $\Gamma \in \mathcal{F}^{(s)}$ is defined as

$$H(\Gamma) = \sum_{\gamma \subset \Gamma} \Phi(\gamma)$$

where the sum is over all regular subgraphs $\gamma \in \mathcal{F}^{(s)}$ of Γ. For any subclass $\mathcal{F}_0^{(s)} \subset \mathcal{F}^{(s)}$ the Gibbs family with potential Φ is the following probability distribution on $\mathcal{F}_0^{(s)}$

$$\mu(\Gamma) = Z_{\mathcal{F}_0}^{-1} \exp(-\beta H(\Gamma)), \Gamma \in \mathcal{F}_0^{(s)}$$

$$Z_{\mathcal{F}_0^{(s)}} = \sum_{\Gamma \in \mathcal{F}_0^{(s)}} \exp(-\beta H(\Gamma))$$

it is should be assumed that $Z_{\mathcal{F}_0^{(s)}} \neq 0, \infty$. Another possibility is not to introduce a subclass $\mathcal{F}_0^{(s)} \subset \mathcal{F}^{(s)}$ by hand but to choose "hard core" (that is taking values ∞) potential which distinguishes exactly the subclass $\mathcal{F}_0^{(s)}$, see more details in [34].

For some increasing sequences $\mathcal{F}_1^{(s)} \subset \mathcal{F}_2^{(s)} \subset \cdots \subset \mathcal{F}_n^{(s)} \subset \cdots \subset \mathcal{F}^{(s)}$ of finite graphs with the local structure defined by $s(\cdot)$ one can naturally define weak limits of the Gibbs families with given Φ, β. See a detailed exposition of this theory in [34]: for example, analog of DLR-condition.

7.1. CORRELATION FUNCTIONS FOR MAPS

For maps without a local structure correlation functions define frequencies of "small" subgraphs in the map. We consider only the probabilities of graphs containing the rooted edge. Then using the analogs of the theorem 3 one can get rid of specifying rooted edge. Let us consider the class $\mathcal{A}_{01}^0(N, m)$ of rooted triangulations T of the disk with N triangles and m edges on the boundary. Then the potential $\Phi \equiv 0$ gives equal probabilties for all maps in $\mathcal{A}_{01}^0(N, m)$. Fix some map Γ of the disk and let $p^N(\Gamma)$ be the probability that a neighborhood of the rooted edge is isomorphic to the map Γ.

Theorem 9. *For any* Γ *there exists the limit* $\lim p^N(\Gamma) = \pi(\Gamma)$.

The *proof* is easy. Let Γ have $n(\Gamma)$ triangles, $m(\Gamma)$ edges on its own boundary, among them $m_{in}(\Gamma)$ are internal edges of T. Delete all triangles of Γ from T. This can produce one or more maps of the disk. Each of these maps has at least one boundary edge in common with some of $m_{in}(\Gamma)$ edges. The first one (in the clockwise order) we define to be rooted for that map. However, it can be proven using explicit formulae for $C_0(N, m)$ that the probability, that this deletion produces more than one map, tends to one as $N \to \infty$, this is a corollary of $N^{-\frac{5}{2}}$ factor. Then the theorem follows again from the explicit formulae for $C_0(N, m)$. $\square$

In the combinatorics papers there are many more refined results concerning the distribution of subgraphs, see [9].

7.2. PHASE TRANSITION FOR PLANAR TRIANGULATIONS

We consider triangulations $T \in \cup_m \mathcal{A}_{0,1}^0(N, m)$. The canonical distribution on this set of triangulations is defined by the probability $P_{0,N}(T)$ of triangulation T. One can introduce $P_{0,N}$ in four equivalent (easy to verify) ways:

- by the interaction proportional to the general number of edges, that is

$$P_{0,N}(T) = Z_N^{-1} \exp(-\mu_1 L(T))$$

  where $L(T)$ is the number of edges of T, including all boundary edges;
- by the interaction proportional to the sum of the degrees of vertices

$$P_{0,N}(T) = Z_{0,N}^{-1} \exp(-\frac{\mu_1}{2} \sum_{v \in V(T)} \deg v).$$

Thus
$$Z_N = \sum_m \sum_{T \in \mathcal{A}^0_{0,1}(N,m)} \exp(-\frac{\mu_1}{2} \sum_{v \in V(T)} \deg v).$$

Using a duscrete analog of Gauss–Bonnet theorem (see above), one can show that this is a discrete analog of Einstein–Hilbert action. Note that this is a particular case (with parameters $t_q = t$) of the interaction , considered in [22]

$$\prod_{q>2} t_q^{n(q,T)}$$

where $n(q,T)$ is the number of vertices of degree q.

- by the interaction proportional to the number of boundary edges.
- By the Gibbs family with fictitious spins σ_v in the vertices v of the triangulation, taking values in any compact set, and assume that the potential of the Gibbs family is
$$\Phi(\sigma_v, \sigma_{v'}) \equiv 1$$

for any two neighboring vertices v, v'.

We will observe phase transitions with respect to parameter μ_1. It has the critical point $\mu_{1,cr} = \log 12$. Let $\beta_0 = \beta_0(\mu_1)$ be such that

$$\frac{(1 + \frac{4\beta_0}{3(1-\beta_0)})}{(1 + \frac{2\beta_0}{1-\beta_0})^2} \exp(-\mu_1 + \log 12) = 1$$

Theorem 10. *The free energy* $\lim_N \frac{1}{N} \log Z_{0,N} = F$ *is equal to* $-\frac{3}{2}\mu_1 + c$, $c = 3\sqrt{\frac{3}{2}}$, *if* $\mu_1 > \mu_{1,cr}$, *and to*

$$-\frac{3}{2}\mu_1 + c + \beta_0(-\mu_1 + \log 12) + \int_0^{\beta_0} \log \frac{(1 + \frac{4\beta}{3(1-\beta)})}{(1 + \frac{2\beta}{1-\beta})^2} d\beta$$

if $\mu_1 < \mu_{1,cr}$.

Note that if $\mu_1 \to \mu_{1,cr}$ then $\beta_0 \to 0$.

Let $m(N)$ be the random length of the boundary when N is fixed. Its probability can be written as, using $|L(T)| = \frac{3N}{2} + \frac{m}{2}$,

$$P_{0,N}(m(N) = m) = \Theta_{0,N}^{-1} \exp(-\mu_1 \frac{m}{2}) C_0(N,m), \Theta_{0,N} = \sum_m \exp(-\mu_1 \frac{m}{2}) C_0(N,m)$$

Theorem 11. *There are 3 phases, where the distribution of* $m(N)$ *has quite different asymptotical behaviour:*

- *Subcritical region, that is* $12 \exp(-\mu_1) < 1$. *Here* $m(N) = O(1)$, *more exactly the distribution of* $m(N)$ *has a limit* $\lim_N P_N(m(N) = m) = p_m$ *for fixed* m *as* $N \to \infty$. *Thus the hole becomes neglectable with respect to* N.

– *Supercritical region (elongated phase), that is* $12\exp(-\mu_1) > 1$. *Here the boundary length is of order* $O(N)$. *More exactly there exists* $\varepsilon > 0$ *such that* $\lim P_{0,N}(\frac{m_N}{N} > \varepsilon) = 1$.
– *In the critical point, that is when* $12\exp(-\mu_1) = 1$, *the boundary length is of order* $\sqrt{N}$. *The exact statement is that the distribution of* $\frac{m_N}{\sqrt{N}}$ *converges in probability.*

Let us remove now the coordinate system from the boundary, that is we consider the class $\cup_m \mathcal{A}_{0,1}(N,m)$ of unrooted triangulations. The free energy remains the same. Only in the critical point the distribution of the length changes—stronger fluctuations appear.

Theorem 12. *In the critical point without coordinate system the boundary length is of order* N^α *for any* $0 < \alpha < \frac{1}{2}$. *The exact statement is that the distribution of* $\frac{\log m_N}{\log \sqrt{N}}$ *converges to the uniform distribution on the unit interval, that is* $P_{0,N}(\frac{\alpha}{2} \leq \frac{\log m_N}{\log \sqrt{N}} \leq \frac{\beta}{2}) \to \beta - \alpha$ *for all* $0 \leq \alpha < \beta \leq 1$.

7.3. SOME STOCHASTIC OPERATORS

One can construct probability distributions on $S_{\rho,k}$ generalizing the distribution of the previous section. For example, take $S_{0,2}$ and define a stochastic kernel on the set $\{2, 3, \ldots\}$

$$S(m \to m_1) = \lim_{N\to\infty} S_N(m \to m_1), \quad S_N(m \to m_1) = \frac{Z_N(m, m_1)}{\sum_m Z_N(m, m_1)}$$

if the limit exists, where the conditional (the lengths of the boundaries are fixed to m and m_1) partition function $Z_N(m, m_1)$ is defined as

$$Z_N(m, m_1) = C(N, m, m_1)\exp(-\mu_1(m + m_1))$$

where $C(N, m, m_1)$ is the number of unrooted triangulations T of $S_{0,2}$ with N triangles and boundary lengths m and m_1. We study these operators in a forthcoming paper.

Similar stochastic kernels can be defined for systems with spins. Consider, for example, triangulations of $S_{\rho,k}$ with spin σ_v in the vertices taking values in the finite set E. Assume that a function $f : E \times E \to R$ is given, which defines a nearest-neighbor potential Φ so that $\Phi(\sigma_v, \sigma_{v'}) = f(\sigma_v, \sigma_{v'})$ if at least one of two vertices is inside the triangulation and by $\Phi(\sigma_v, \sigma_{v'}) = \frac{1}{2} f(\sigma_v, \sigma_{v'})$, if both are on the boundary. One could say equivalently that a local structure is given such that it knows whether the 1-neighborhood of v is a disk or a "half-disk". Let $L(E)$ is the linear space having elements of E as a basis.

Such operators can have relation to topological field theory (TFT) and scattering matrices for string models. In TFT, for example, one considers an abstract set of such operators

$$S_{\rho,k}(L(E)^{\otimes m_1} \otimes \cdots \otimes L(E)^{\otimes m_p} \to L(E)^{\otimes m_{p+1}} \otimes \cdots \otimes L(E)^{\otimes m_k})$$

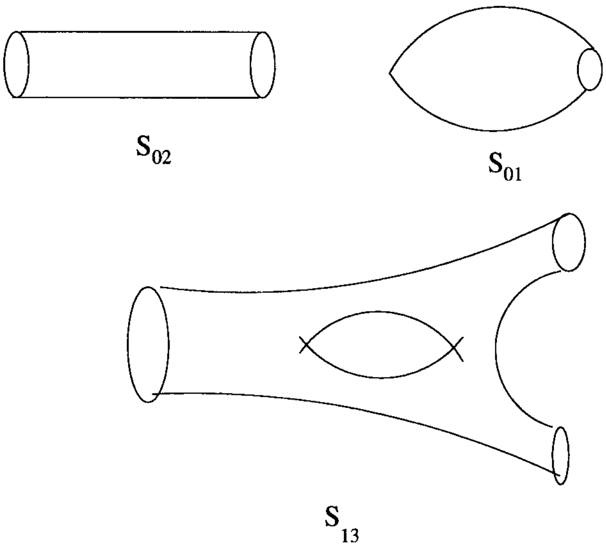

Figure 3. Surfaces with boundaries

satisfying some axioms concerning composition of these operators. These axioms can be given in different ways: that of Atiyah [35, 36] using functorial approach and the (not well-defined) approach using functional integrals. As far as I know now only simple examples of TFT, which are direct sums of the system without spin, see the last chapter of [18].

8. Lorentzian Models

Lorentzian models appeared in physical papers [37–39], where several approaches were suggested. Here we present rigorous formulations and results, see proofs in [40].

Above we defined slice-triangulations of the cylinder. They are easier to handle and complete control is possible.

Assume that $k_M = k$ and $k_N = l$ are fixed. Introduce Gibbs measure on the (countable) set $\mathcal{A}_{[M,N]}(k,l)$ of all such triangulations

$$\mu_{[M,N],k,l}(T) = Z^{-1}_{[M,N]} \exp(-\mu F(T)). \tag{8}$$

Define correlation functions for random variables $k_j, j \in [M+1, N-1]$, taking values from $\mathcal{N} = \{1, 2, \ldots\}$, and for finite subsets $J \subset [M+1, N-1]$

$$\mu_{[M,N],k,l}(k_j = n_j, j \in J). \tag{9}$$

The behaviour of the system depends on to which region μ belongs: define subcritical, critical or supercritical region if $2\exp(-\mu) < 1$, $2\exp(-\mu) = 1$ and $2\exp(-\mu) > 1$ correspondingly.

In the subcritical region we get a limiting probability measure on $\mathcal{N}^Z$.

Theorem 13. *If* $2\exp(-\mu) < 1$ *then the limiting correlation functions*

$$\lim_{N\to\infty} \mu_{[-N,N],k,l}(k_j = k_j^0, j \in J)$$

exist for any finite subset $J \subset Z$ *and any vector* $(k_j^0, j \in J)$*. The measure defined by these correlation functions is a stationary ergodic Markov chain on* $\mathcal{N}$ *with the following stationary probabilities*

$$\pi(n) = \left(1 - (\lambda_2(s))^2\right)^2 n\left(\lambda_2(s)\right)^{2(n-1)},$$

where

$$\lambda_2(s) = \frac{1 - \sqrt{1 - s^2}}{s}, \quad s = 2\exp(-\mu).$$

Theorem 14. *In the subcritical case the asymptotics of the partition function is*

$$Z_{[-N,N]}(k,l) \sim \left(1 - (\lambda_2(s))^2\right)^2 \left(\lambda_2(s)\right)^{k+l-2-4N}.$$

Theorem 15. *In the critical case the asymptotics of the partition function is*

$$Z_{[-N,N]}(k,l) \sim \frac{1}{4}N^{-2}, \quad \sum_{l=1}^{\infty} Z_{[-N,N]}(k,l) \sim \frac{1}{2}N^{-1}.$$

For any J *and any* k_j^0 *the correlation functions*

$$\lim_{N\to\infty} \mu_{[-N,N],k,l}(k_j = k_j^0, j \in J) = 0.$$

In the critical case one can study the continuous limit $N \to \infty$. Take the interval $[0, N]$. We take the length of a horizontal edge equal to 1. At time αN, $\alpha < 1$, let $k_{[\alpha N]} = k_{[\alpha N]}(k,l)$ be the volume (the number of edges) in our one-dimensional Universe. Define the macrolength as $\frac{k_{[\alpha N]}}{N}$ and the macrotime as α.

Theorem 16. *For any positive* k, l

$$\lim_{N\to\infty} \frac{Ek_{[\alpha N]}}{N} = 2\alpha(1-\alpha), \tag{10}$$

and limit does not depend on k *and* l*. If* l *is not fixed then*

$$\lim_{N\to\infty} \frac{Ek'_{[\alpha N]}}{N} = \alpha(2-\alpha).$$

This shows that in our scaling limit the continuous two-dimensional Universe looks like the surface of a paraboloid of revolution, with two (or one, if l is not fixed) singular points. This can be interpreted as the expansion of this Universe.

Proposition 17. *In the supercritical case the finite volume partition function* $Z_{[0,N]}$ *exists only if*

$$\mu > \ln\left(2\cos\frac{\pi}{N+1}\right).$$

As $N \to \infty$ *this region, where the partition function exists, becomes empty.*

9. Gravity with matter fields

Here we shortly describe central ideas of the new cluster expansion techniques, which allows to treat combinatorial and probability problems for maps with spins. In statistical physics cluster expansions proved to a powerful tools, sometimes a unique tool allowing complete control over the problem. Disadvantages of this method is the necessity of a small or large parameter in the interaction, and also sufficient difficulty. We are interested here the stability problem of the critical exponent in the asymptotics of the partition function, however other questions, as correlation functions, can be treated as well. We consider only planar triangulations with matter fields with compact set of values in high temperature region. We shall see that for Gibbs families, to which maps belong, cluster expansions have many new features unknown for statistical physics and quantum field theory. The main new feature is that the empty space has an entropy, to take into account this entropy Tutte's method should be incorporated into the expansion.

We again consider the class $\mathcal{A}^0_{0,1}(N,m)$ and denote T^* the dual graph of the triangulation T, its vertices $v \in V(T^*)$ correspond to triangles of T, edges $l \in L(T^*)$—to pairs of adjacent triangles. All vertices of T^* have degree 3 except vertices corresponding to the triangles (there are not more than $m = |B(T)|$ of such triangles), incident to at least one boundary edge, where $B(T)$ is the set of boundary edges of T.

In each triangle of T, or in each vertex v of the dual graph T^*, there is a spin σ_v with values in the set E, this set is assumed finite for simplicity.

Partition function for the canonical ensemble (with fixed number $N \geq 0$ of triangles and fixed number $m \geq 2$ of boundary edges) is defined as

$$Z(N,m) = Z_\beta(N,m) = \sum_{T:|F(T)|=N,|B(T)|=m} Z(T)$$

where the partition function $Z(T)$ for a given triangulation $T \in \mathcal{A}^0_{0,1}(N,m)$ is

$$Z(T) = |E|^{-N} \sum_{\{\sigma_v : v \in V(T^*)\}} \exp(-\beta \sum_{<v,v'>} \Phi(\sigma_v, \sigma_{v'})), N = |F(T)| = |V(T^*)|$$

where $<v,v'>$ means a pair of nearest neighbor vertices (that is of adjacent triangles) $v, v' \in V(T^*)$, $\Phi(s,s')$ is a symmetric real function on $S \times S$, $\beta > 0$—inverse temperature.

To be concrete we shall consider the ensemble with boundary conditions empty on the internal boundary, that is there are no spins on the triangles of $F(T)$ adjacent to the boundary of the disk, thus no interaction with these triangles.

We prove that in some cases the partition function has canonical asymptotics. This means, there is a constant $c = c(\Phi, \beta)$ such that for fixed m, β, Φ

$$Z(N,m) \sim \phi(m, \Phi, \beta) N^{-\frac{5}{2}} c^N.$$

The critical exponent $\alpha = -\frac{5}{2}$ is also called canonical. For example, we have the following result.

Theorem 18. *Let*

$$k = \sum_{\sigma,\sigma'} [\exp(-\beta\Phi(\sigma,\sigma')) - 1] < 0.$$

Then for β sufficiently small $Z(N,m)$ has canonical asymptotics.

Theorem 19. *If $\Phi \leq 0$ is not identically constant, then for β sufficiently small the asymptotics is not canonical.*

Example: scaling transformation
A simple example is the constant nearest-neighbor interaction $\Psi_\mu(\sigma,\sigma') \equiv \mu$. For non-rooted triangulations the term Ψ gives an overall factor $\exp(-\beta\mu L^*) = \exp(-\frac{3}{2}\beta\mu N)$. Appending Ψ_μ to some interaction Φ results in a scaling transformation of the generating functions (see below). Otherwise speaking, appending such interaction changes only the constant $c(\beta)$ in the asymptotics, and does not change the canonical exponent.

9.1. CLUSTER EXPANSION TECHNIQUES

To present the main ideas we consider a simpler model—maps with random impurities. A map T with impurities is a map where some triangles are colored.

Note the distance between triangles in dynamical triangulation models is the distance between the corresponding vertices in the dual graph, that is the length (number of edges) of the shortest path between them in T^*. A set Δ of triangles is called connected if between each pair of triangles $t, s \in \Delta$ there is a path, belonging to Δ, in which any pair of consecutive triangles are on the distance not greater than $d = 1$.

For each set Δ define the external boundary $\partial_e\Delta$ as the set of triangles on distance 1 from Δ. A cluster (more exactly, a T-cluster for a given triangulation T) of colored triangles is a maximal connected subset of the closure $cl(V^*_{col}(T)) = V^*_{col}(T) \cup \partial_e(V^*_{col}(T))$ of the set $V^*_{col}(T)$ of colored triangles.

We define now the hierarchy of T-clusters for a given triangulation T. For any set $V \subset F(T)$ the complement $F(T) \setminus V$ consists of the two parts: exterior part $Ext(F(T) \setminus V)$, consisting of all triangles of $F(T) \setminus V$, which can be connected with the boundary by connected path, belonging to $F(T) \setminus V$, and the interior part $Int(F(T) \setminus V)$, containing all other triangles.

Let V be one of the T-clusters. Then the interior part of its complement $F(T) \setminus V$ consists of some number r of connected components $V_1, \ldots, V_r$.

For given T a set $V \subset F(T) = V(T^*)$ of triangles is called simple if it is connected and its interior part is empty. We say that T-cluster has level 1 if it is simple.

We define clusters of level $n > 1$ by induction: T-cluster V has level n if n is the minimal number such that in its interior part there are only clusters of level less than n. Thus the T-clusters form a forest (a set of connected trees), where clusters are vertices of this forest. Two vertices of the tree are connected by an edge if one of the corresponding T-clusters is in the interior part of the other one, and their levels differ by 1.

A random cluster model on maps assumes the following properties for any T:

1. Cluster V has a weight $k(V) > 0$, so that the partition function of the random cluster model is
$$Z^{(1)}(N,m) = \sum_{T:F(T)=N} \prod_{V \subset F(T)} k(V)$$
where the product is over all T-clusters.

2. The cluster estimate holds
$$k(V) \le \beta^{|V|}.$$

3. Isomorphic clusters have equal weight.

4. All clusters are simple.

Nonempty T-cluster V is called complete if it contains all triangles of T. It is obviously simple and thus Σ consists only of this cluster. The complete T-cluster V is obviously unique and we define

$$k(T) = k(V).$$

Then the cluster generating function is defined as

$$W(x,y) = W_1(x,y) = \sum_{N=3}^{\infty} \sum_{m=2}^{\infty} W_{N,m}^{(1)} x^N y^m, W_{N,m}^{(1)} = \sum_{T:T \in \mathcal{A}_{01}^0(N,m)} K(T).$$

Then the equation for the generating functions

$$U_1(x,y) = \sum_{N=0}^{\infty} \sum_{m=2}^{\infty} Z^{(1)}(N,m) x^N y^m$$

is for $n = 1$

$$U_1(x,y) = U_1(x,y)xy^{-1} + U_1^2(x,y)xy^{-1} + y^2 + W_1(x,y) - xyS(x) \qquad (11)$$

with

$$S_1(x) = \sum_{N=0}^{\infty} Z^{(1)}(N,2)x^N.$$

Proof. We have recurrent equations, similar to the case $\beta = 0$

$$Z^{(1)}(N,m) = Z^{(1)}(N-1,m+1) + \delta_{N,0}\delta_{m,2} + W_{N,m}^{(1)} + \sum_{N_1+N_2=N-1, m_1+m_2=m+1} Z^{(1)}(N_1,m_1)Z^{(1)}(N_2,m_2) \qquad (12)$$

for $m \geq 2, N \geq 0$ and

$$Z^{(1)}(-1,m)) = Z^{(1)}(N,0)) = Z^{(1)}(N,1)) = 0.$$

Tutte's method is applicable to this equation, because W is analytic in a domain of radius which increases when β decreases.

If we do not assume the simplicity of clusters (property 4) then one uses an inductive procedure in n. For example, to get the partition function $Z^{(2)}(N,m)$, taking into account only maps having no clusters of level more than 2 but containing at least one cluster of level 2, we use already obtained level 1 generating function for $Z^{(1)}(N,m)$ as the function $W(x,y) = W_2(x,y)$ for the equation to find the generating function for $Z^{(2)}(N,m)$, etc. On each step the equations look like equation (11).

We omitted also the first step of the expansion, necessary to get the random cluster model from our high temperature model. On this step one uses a (standard, statistical physics type [42, 20]) cluster expansion for each fixed T, followed by the resummation. There are however some finer points, for example, the weights of clusters are not positive anymore.

See details in [41]. I hope that this method can be extended to more general situations: low temperature, unbounded spins etc. However, it demands additional efforts.

I also did not mention here two kind of physical results: methods of conformal field theory to get critical exponents and random matrix techniques to do calculations for Ising model on maps. There should be deep connections with cluster expansion. Isosystolic inequalities approach to genus independence of γ for models with matter fields see in [43].

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