

Stochastic micromodel of the Couette flow

V. A. Malyshev A. D. Manita*

Abstract

We study Markov exclusion process for a particle system with a local interaction in the integer strip. This process models the exchange of velocities and particle-hole exchange of the liquid molecules. It is shown that the mean velocity profile corresponds to the behaviour which is characteristic for incompressible viscous liquid. We prove the existence of phase transition between laminar and turbulent profiles.

Keywords: Couette flow, Markov processes with a local interaction, scaling, hydrodynamics

1 Introduction

From the mathematical point of view this paper considers a phase transition for a Markov multi-dimensional exclusion process. But as this problem originated from concrete physical models, we discuss these connections in detail.

One can study the liquid flow from macroscopic and from microscopic point of view. Although in the first case (that is in the continuum mechanics) most studies are based on the Navier-Stokes equations, there exist various generalizations introducing randomness in these equations (see the review of J. Marsden and other papers in [1], some history of the question one can find in the recent preprint [4]). It is precisely with Navier-Stokes equations the immense number of concrete theoretical and practical problems of the liquid mechanics, including the Couette flow, which is the simplest (but far from being trivial model, see [5], [6]) available for a mathematical study.

However, there are only very few microscopic models for concrete situations with nonzero viscosity, in despite of the abundance of lattice models with infinite number of particles (see books [10], [3], [9]). In theoretical physics the preferable direction is the derivation of the Navier-Stokes equations following the sequence: hamiltonian microdynamics

*Moscow State University, Faculty of Mechanics and Mathematics, Leninskie Gory 1, GSP-1, Moscow, 119991, Russia; e-mail: malyshev2@yahoo.com, e-mail: manita@mech.math.msu.su

→ BBGKY equations → Boltzmann equation → equation of hydrodynamics. However, there is also direct derivation of the Navier-Stokes equations from stochastic dynamics of particles on the lattice (see [11]).

Recently, there appeared several papers where some particle system were related to the Navier-Stokes equations so that for a certain scaling the dynamics of these particle system converges to these equations (see [3], [13], [14]). However, in these papers the particle dynamics is not local, it is close to the mean-field dynamics of McKean-Vlasov type. But most important is that it is derived not from proper molecular dynamics but from the computer simulation of Navier-Stokes equations, that is from the Navier-Stokes equations themselves.

The conclusion is that the derivation of Navier-Stokes equations from physically grounded molecular dynamics is still open. The main problem here is that it is far from clear how to describe the molecular dynamics. The model of hard balls, obeying the laws of classical mechanics, is simple in formulation but very difficult to study. Moreover, one cannot be sure that this model is adequate. The question is that the molecules in a liquid are very close to each other, and quantum effects seem to be important. If one will work in the framework of classical models, then one should take into account the interaction of rotating and oscillating degrees of freedom of the neighbor molecules, which presumably have a random distributions.

From the other side, before any attempt to derive the Navier-Stokes equations, it could be natural first to try particular cases. In the local model of the Couette flow considered below we use two physical processes: exchange of velocities between molecules and the exchange between molecules and holes (that is empty places). This corresponds to the intuitive picture often discussed in the physical literature. Moreover, one can hope that there is a kind of universality, that is independence of qualitative behaviour of the concretization of the model.

From probabilistic point of view we study a Markov system of locally interacting particles, the dynamics of which is a mixture of two well-known (see [8]) exclusion processes - symmetric and completely asymmetric. Introducing boundary conditions does not influence the moment closeness of the process [12], and this simplifies essentially the study. Precisely using moment closeness one can avoid such difficult techniques as Bethe [15] or matrix [16] ansatz.

It is worth notice that, due to different scaling, direct study of concrete micromodel is not equivalent to the derivation of the Navier-Stokes equations and their subsequent application to the same situation.

For our problem we get the phase transition for the velocity profile, corresponding to

the transition in the Couette flow [7], however there is a difference. The transition parameter in our paper is not the classical Reynolds number, which depends on the viscosity, cross-section and longitudinal velocity, but its analog, depending on the cross-exchange parameter (analog of viscosity), cross-section and random perturbation. The latter could be the consequence of fast vortex, resulting from deviations of velocity from longitudinal direction or from interaction of velocity with randomly distributed internal degrees of freedom. It appears interesting that the phase transition is sharp on the rougher scale and smooth on the finer scale.

2 Model and results

The Couette flow is the liquid flow in one (horizontal) direction in the space (which is discrete and two-dimensional, however the generalization to multi-dimensional case is straightforward) between two plates. One of the plates is fixed and the other moves with constant velocity, that draws the neighbor particles with it. Other particles are also involved due to viscosity, that is reflected microscopically in the exchange of velocities between neighbor particles.

We come now to exact definitions. The basic set for us is the discrete strip

$$L_S = \{0, 1, 2, \dots, S, S + 1\} \times \mathbf{Z}.$$

To each point $(s, x) \in L_S$ we assign random variable $v_{(s,x)}$, which can take one of the following three values: \emptyset , 0 or V . The value \emptyset means that there is no particle at the site (s, x) (“a hole”), $v_{(s,x)} = 0$ means the presence of one particle with zero velocity, and $v_{(s,x)} = V$ means the presence of one particle with velocity V . For notational convenience the “hole” will be called the particle with velocity \emptyset .

Dynamics of this particle system consists of jumps and is defined by the following transitions. On any small time interval $[t, t + dt]$ the following events can take place independently of one another:

(a) particles of each neighboring vertical pair (at points (s, x) and $(s + 1, x)$) exchange their velocities with probability $\lambda dt + o(dt)$ (*velocity exchange between vertical layers*);

(b) particles in each neighboring horizontal pair of the type $v_{(s,x)} = V$, $v_{(s,x+1)} = \emptyset$ (and only of this type) exchange their velocities with probability $\lambda_1 dt + o(dt)$ (*horizontal flow*). Of course, one could put $V = \lambda_1$, but in this paper it will be convenient to distinguish V and λ_1 .

(c) on the zero layer each particle that has velocity V acquires velocity 0 with proba-

bility $\beta dt + o(dt)$; on the top layer $S + 1$ each particle that has velocity 0 acquires velocity V with probability $\beta dt + o(dt)$ (*influence of the boundaries*);

(d) each node (s, x) with nonempty velocity ($v_{(s,x)} = 0$ or V) changes its velocity according to the rule $0 \leftrightarrow V$ with probability $\varepsilon dt + o(dt)$ (*random perturbation*).

In the sequel we assume that $\lambda > 0$, $\lambda_1 > 0$, $\beta > 0$ and $\varepsilon \geq 0$.

Thus we have just defined a continuous time *Markov* process $v(t) = \{v_{(s,x)}(t), (s, x) \in L_S\}$, $t \in \mathbf{R}_+$, with state space $\mathcal{W} = \{\emptyset, 0, V\}^{L_S}$. Its generator acts on the functions f that depend on velocities at finite number of nodes as follows

$$\begin{aligned}
\mathcal{L}f(v) &:= \lambda \sum_{x \in \mathbf{Z}} \sum_{k=0}^S (f(v^{(k,x) \leftrightarrow (k+1,x)}) - f(v)) \\
&+ \lambda_1 \sum_{x \in \mathbf{Z}} \sum_{k=0}^{S+1} (f(v^{(k,x) \leftrightarrow (k,x+1)}) - f(v)) \mathbf{1}(v_{(k,x)} = V, v_{(k,x+1)} = \emptyset) \\
&+ \beta \sum_{x \in \mathbf{Z}} (f(v^{(0,x)}) - f(v)) \mathbf{1}(v_{(0,x)} = V) \\
&+ \beta \sum_{x \in \mathbf{Z}} (f(v^{(S+1,x)}) - f(v)) \mathbf{1}(v_{(S+1,x)} = 0) \\
&+ \varepsilon \sum_{x \in \mathbf{Z}} \sum_{k=0}^{S+1} (f(v^{(k,x)}) - f(v)) \mathbf{1}(v_{(k,x)} \neq \emptyset), \tag{1}
\end{aligned}$$

where two configurations $w = v^{y \leftrightarrow z}$ and v from \mathcal{W} differ only by the velocities at the points y and z of L_S :

$$w_y = v_z, \quad w_z = v_y,$$

and $v^y \in \mathcal{W}$ is a configuration that can differ from v only in velocity value at node y due to the exchange $0 \leftrightarrow V$:

$$(v^y)_z = \begin{cases} v_z, & z \neq y, \\ \emptyset, & z = y, \quad v_y = \emptyset, \\ V \mathbf{1}(v_y = 0), & z = y, \quad v_y \neq \emptyset. \end{cases}$$

One can check that such definition of \mathcal{L} leads to a correct definition of the Markov process on \mathcal{W} (see [8, Ch. I, §3]).

Denote by η^t the distribution of the stochastic process $v(t)$ at time t .

Let $A = ((s_1, x_1), \dots, (s_m, x_m))$ be some ordered finite subset of L_S and $E = (e_1, \dots, e_m)$

be some ordered collection of $e_j \in \{\emptyset, 0, V\}$. Consider the following probabilities

$$q_t[A; E] := \mathbf{P} \{v_{(s_1, x_1)}(t) = e_1, \dots, v_{(s_m, x_m)} = e_m\}.$$

On the set L_S we define the action of the group $(T_a, a \in \mathbf{Z})$ of horizontal translations:

$$T_a(s, x) = (s, x + a).$$

In the present paper we restrict ourself to consideration of initial distributions η^0 of the process v that are *invariant* with respect to the action of the group T_a .

Since the Markov semigroup generated by (1) commutes with the translations T_a , for any time t the distribution η^t is translation invariant. In particular, for any choice of sets A and E

$$q_t[T_a A; E] = q_t[A; E] \quad (2)$$

for all $t > 0$ and $a \in \mathbf{Z}$. From physical point of view it means that we are interested only in homogeneous flows that are invariant with respect to shifts along the x -direction: $x \rightarrow x + a$.

As it is easy to see, the dynamics of the process v is such that the total number of particles, namely, the number of nodes with velocities 0 and V , is a conserved quantity. In particular, the mean (expected) number of nonempty nodes in a vertical section x is conserved in time and does not depend on x :

$$\sum_{k=0}^{S+1} (q_t[(k, x); 0] + q_t[(k, x); V]) \equiv M \quad \forall x, t. \quad (3)$$

Evidently, the constant $M = M(\eta^0)$ can be calculated in terms of the initial distribution η^0 . However, the mean number of nodes with velocity V is not conserved.

From now on we are interested only in distributions in a fixed vertical layer, therefore, via the homogeneity, the dependence on x will be omitted in the following notation

$$q_t[(s, x); e] = p_s^e(t). \quad (4)$$

Theorem 1 *Assume that the initial distribution η^0 is translation invariant. Then:*

1) (*existence of a stationary regime*) *the following limits exist for any s and e*

$$\mu_s^e = \lim_{t \rightarrow \infty} p_s^e(t),$$

we call μ_s^e stationary probabilities;

2) (uniformity of a particle density) the stationary probabilities of “holes” are the same for any layer $s = 0, 1, \dots, S + 1$:

$$\mu_s^\emptyset \equiv \text{const.}$$

First we consider the case when the random perturbation is absent, i.e., $\varepsilon = 0$.

Theorem 2 (linear velocity profile) *Assume that the initial distribution η^0 of the process is translation invariant and $\varepsilon = 0$. Then the mean velocities of particles form a linear profile, namely,*

$$\mu_k^V = \frac{\rho S}{S + 1 + 2\lambda\beta^{-1}} (k + \lambda\beta^{-1}), \quad k = 0, 1, \dots, S + 1,$$

where $\rho_S = M(\eta^0)/(S + 2)$.

The velocity profile of this theorem corresponds to the laminar flow in contrast to what is observed in the turbulent regime (see [7]). In the next theorem we consider nonzero random perturbations.

Theorem 3 *Assume that $S \rightarrow \infty$, the parameter $\varepsilon = \varepsilon_S$ depends on S and the sequence of initial distributions $\{\eta^{0,S}\}$ is such that the density of particles is fixed, i.e.,*

$$\frac{M(\eta^{0,S})}{S + 2} \rightarrow \rho \in [0, 1], \quad S \rightarrow \infty. \quad (5)$$

Then:

1) (rough scale picture of the transition) in scaling $s = uS$ the functions μ_s^0 and μ_s^V demonstrate a phase transition from laminar profile to turbulent one; namely, if $\varepsilon_S S^2 \rightarrow 0$, then the limiting profile of the function μ_{uS}^V , $u \in (0, 1)$, has a linear (laminar) form:

$$\mu_{uS}^V \rightarrow \rho u \quad (S \rightarrow \infty),$$

but, if $\varepsilon_S S^2 \rightarrow \infty$, then the limiting profile is constant (turbulent):

$$\mu_{uS}^V \rightarrow \frac{\rho}{2}, \quad (S \rightarrow \infty);$$

2) (finer scale picture of the transition) the case $\varepsilon_S S^2 = \text{const}$, $S \rightarrow \infty$, is intermediate between the situations described in point 1). Precisely, if we introduce a positive parameter

K putting $\varepsilon_S S^2 = \frac{1}{2}\lambda K^2$, then $\mu_{uS}^V \rightarrow g_K(u)$, where

$$g_K(u) := \frac{\rho}{2} \left(1 + \frac{\sinh(K(u - 1/2))}{\sinh(K/2)} \right).$$

Since $\lim_{K \rightarrow 0} g_K(u) = \rho u$ and $\lim_{K \rightarrow \infty} g_K(u) = \rho/2$, we can say that by changing the parameter $K > 0$ we observe a smooth transition between laminar and turbulent profiles.

Remark 1. The parameter K can be considered as analog of the Reynolds number Re . This can be easily explained by comparing these two numbers, that is by comparing the definition of K

$$K^2 = \frac{S^2 \varepsilon}{\lambda/2}$$

with the definition of the Reynolds number

$$\text{Re} = \frac{LV}{\nu},$$

where L is the cross-section of the flow, V is the mean velocity, ν is the viscosity.

The mentioned phase transition is based on the competition between processes which reinforce and the process which diminish the influence of the boundary. Thus, the factors L and S show that the greater the distance from the boundaries, the more random perturbation diminishes the influence of the boundaries. The factors V and ε show that if the velocity increases the particles are perturbed more often, and the influence of the boundaries is less visible. Finally, when λ increases, then the influence of the boundaries increases. One should note that, by definition of the model, the parameter V can be put proportional to λ_1 .

3 Proofs

3.1 Closed equations for one-particle functions

Our next task is to obtain a system of differential equations for the functions $p_k^e(t)$, $0 \leq k \leq S + 1$, $e \in \{\emptyset, 0, V\}$, defined in (4). First note that for any $k = 0, 1, \dots, S + 1$

$$p_k^\emptyset(t) + p_k^0(t) + p_k^V(t) = 1 \quad \forall t \geq 0, \tag{6}$$

hence, for example, the functions $p_k^0(t)$ can be easily expressed in terms of the functions $p_k^\emptyset(t)$ and $p_k^V(t)$.

Lemma 1 *Let the initial distribution η^0 be translation invariant. Then the functions $p_k^e(t)$ satisfy to a closed system of linear first-order differential equations having the following structure.*

1) *The functions $p_k^\varnothing(t)$ can be obtained from a subsystem*

$$\begin{aligned} \frac{d}{dt}p_k^\varnothing(t) &= \lambda(p_{k+1}^\varnothing(t) + p_{k-1}^\varnothing(t) - 2p_k^\varnothing(t)), & 1 \leq k \leq S, \\ \frac{d}{dt}p_0^\varnothing(t) &= \lambda(p_1^\varnothing(t) - p_0^\varnothing(t)), \\ \frac{d}{dt}p_{S+1}^\varnothing(t) &= \lambda(p_S^\varnothing(t) - p_{S+1}^\varnothing(t)). \end{aligned} \quad (7)$$

2) *The equations for $p_k^V(t)$ have the following form*

$$\begin{aligned} \frac{d}{dt}p_k^V(t) &= \lambda(p_{k+1}^V(t) + p_{k-1}^V(t) - 2p_k^V(t)) + \varepsilon(1 - p_k^\varnothing(t) - 2p_k^V(t)), \\ & 1 \leq k \leq S, \\ \frac{d}{dt}p_0^V(t) &= \lambda(p_1^V(t) - p_0^V(t)) + \varepsilon(1 - p_0^\varnothing(t) - 2p_0^V(t)) - \beta p_0^V(t), \\ \frac{d}{dt}p_{S+1}^V(t) &= \lambda(p_S^V(t) - p_{S+1}^V(t)) + \varepsilon(1 - p_{S+1}^\varnothing(t) - 2p_{S+1}^V(t)) \\ & + \beta(1 - p_{S+1}^\varnothing(t) - p_{S+1}^V(t)). \end{aligned} \quad (8)$$

Note that equations (7) and (8) do not depend on λ_1 .

Remark 2. It can easily be checked that the system (7) has the same form as the corresponding system for the simple symmetric exclusion process on the finite set $0, 1, \dots, S+1$ (with empty boundary conditions).

The remaining part of this subsection is devoted to the proof of Lemma 1. First we shall obtain equations for the marginal distributions $q_t[(k, x); e]$ *without assumption* of translation invariance of the initial distribution of velocities.

For inner layers $1 \leq k \leq S$ the marginal distributions satisfy the following equations.

$$\begin{aligned} \frac{d}{dt} q_t[(k, x); \emptyset] &= \lambda(q_t[(k+1, x); \emptyset] + q_t[(k-1, x); \emptyset] - 2q_t[(k, x); \emptyset]) \\ &\quad + \lambda_1(q_t[((k, x), (k, x+1)); (V, \emptyset)] - q_t[((k, x-1), (k, x)); (V, \emptyset)]), \end{aligned} \quad (9)$$

$$\begin{aligned} \frac{d}{dt} q_t[(k, x); 0] &= \lambda(q_t[(k+1, x); 0] + q_t[(k-1, x); 0] - 2q_t[(k, x); 0]) \\ &\quad + \varepsilon(q_t[(k, x); V] - q_t[(k, x); 0]), \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} q_t[(k, x); V] &= \lambda(q_t[(k+1, x); V] + q_t[(k-1, x); V] - 2q_t[(k, x); V]) \\ &\quad + \varepsilon(q_t[(k, x); 0] - q_t[(k, x); V]) + \lambda_1(-q_t[((k, x), (k, x+1)); (V, \emptyset)] \\ &\quad + q_t[((k, x-1), (k, x)); (V, \emptyset)]). \end{aligned} \quad (10)$$

On the bottom layer $k = 0$ the equations have the following form

$$\begin{aligned} \frac{d}{dt} q_t[(0, x); \emptyset] &= \lambda(q_t[(1, x); \emptyset] - q_t[(0, x); \emptyset]) \\ &\quad + \lambda_1(q_t[((0, x), (0, x+1)); (V, \emptyset)] - q_t[((0, x-1), (0, x)); (V, \emptyset)]), \end{aligned} \quad (11)$$

$$\begin{aligned} \frac{d}{dt} q_t[(0, x); 0] &= \lambda(q_t[(1, x); 0] - q_t[(0, x); 0]) + \beta q_t[(0, x); V] \\ &\quad + \varepsilon(q_t[(0, x); V] - q_t[(0, x); 0]), \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} q_t[(0, x); V] &= \lambda(q_t[(1, x); V] - q_t[(0, x); V]) - \beta q_t[(0, x); V] \\ &\quad + \varepsilon(-q_t[(0, x); V] + q_t[(0, x); 0]) + \lambda_1(-q_t[((0, x), (0, x+1)); (V, \emptyset)] \\ &\quad + q_t[((0, x-1), (0, x)); (V, \emptyset)]). \end{aligned} \quad (12)$$

On the top layer $k = S + 1$ we have the equations

$$\begin{aligned} \frac{d}{dt} q_t[(S + 1, x); \emptyset] &= \lambda(q_t[(S, x); \emptyset] - q_t[(S + 1, x); \emptyset]) \\ &\quad + \lambda_1(q_t[((S + 1, x), (S + 1, x + 1)); (V, \emptyset)] \\ &\quad \quad - q_t[((S + 1, x - 1), (S + 1, x)); (V, \emptyset)]), \end{aligned} \quad (13)$$

$$\begin{aligned} \frac{d}{dt} q_t[(S + 1, x); 0] &= \lambda(q_t[(S, x); 0] - q_t[(S + 1, x); 0]) + \beta q_t[(S + 1, x); V] \\ &\quad + \varepsilon(q_t[(S + 1, x); V] - q_t[(S + 1, x); 0]), \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} q_t[(S + 1, x); V] &= \lambda(q_t[(S, x); V] - q_t[(S + 1, x); V]) + \beta q_t[(S + 1, x); 0] \\ &\quad + \varepsilon(q_t[(S + 1, x); 0] - q_t[(S + 1, x); V]) \\ &\quad + \lambda_1(-q_t[((S + 1, x), (S + 1, x + 1)); (V, \emptyset)] \\ &\quad \quad + q_t[((S + 1, x - 1), (S + 1, x)); (V, \emptyset)]). \end{aligned} \quad (14)$$

We see that the equations for one-dimensional marginal distributions *are not closed*, since they contain two-dimensional distributions in the r.h.s.

Derivation of these equations is straightforward, but intermediate calculations are rather cumbersome. This derivation will be more transparent if we note that dynamics of the model consists of three processes. The first one is a symmetric exclusion process on vertical layers. The second component is a totally asymmetric exclusion process in the horizontal direction. And, finally, the Glauber spin-flip process at each point. This latter process enters the equations in a very simple manner, but it is very useful to consider the first two processes separately. We shall see that the equations for the first process are closed even with the boundary conditions, while the equations for the second process are not closed.

Model 1: symmetric exclusion process. First it is convenient to consider a simpler *auxiliary* model $\xi(t) = (\xi_s(t))$, which describes a single vertical layer $x = 0$, consisting of points $s = 0, 1, 2, \dots, S, S + 1$, under additional assumption $\lambda_1 = 0$, $\varepsilon = 0$ and $\beta = 0$. For notational convenience we assume that there are no ‘‘holes’’, i.e., $\xi_s(t) \in \{0, 1\}$, and put $V = 1$. Then the process $\xi(t)$, $t \geq 0$, is a well-known simple exclusion process.

For $1 \leq i < i + k - 1 \leq S$ and $e_j = 0, 1$ denote

$$p_t(i; e_1 \dots e_k) = \mathbf{P} \{ \xi_i(t) = e_1, \dots, \xi_{i+k-1}(t) = e_k \}.$$

One-particle functions $p_t(s; 1)$ satisfy closed equations. Indeed, using an explicit form of the transition functions of the Markov process ξ on a small time interval $[t, t + dt)$ and the

complete probability formula, for any $1 \leq i \leq S$ we have, up to terms of order $o(dt)$,

$$\begin{aligned}
 p_{t+dt}(i; 1) &= p_t(i-1; 111) + [p_t(i-1; 011) + p_t(i-1; 110)](1 - \lambda dt) \\
 &\quad + p_t(i-1; 010)(1 - 2\lambda dt) + [p_t(i-1; 100) \\
 &\quad + p_t(i-1; 001)]\lambda dt + p_t(i-1; 101) \cdot 2\lambda dt,
 \end{aligned} \tag{15}$$

hence

$$\begin{aligned}
 p_{t+dt}(i; 1) - p_t(i; 1) &= \lambda dt [-p_t(i-1; 01) - p_t(i; 10) + p_t(i-1; 10) + p_t(i; 01)] \\
 &= \lambda dt [p_t(i-1; 1) - 2p_t(i; 1) + p_t(i+1; 1)],
 \end{aligned} \tag{16}$$

and, finally,

$$\frac{d}{dt} p_t(i; 1) = \lambda [p_t(i-1; 1) - 2p_t(i; 1) + p_t(i+1; 1)]. \tag{17}$$

Model 2: asymmetric exclusion process. It is convenient also to consider an asymmetric exclusion process $\zeta(t) = (\zeta_x(t), x \in \mathbf{Z})$ with three values at the node: $\zeta_x \in \{\emptyset, 0, V\}$. We assume that the following transitions are only possible: each pair of closest neighbors of the type $V\emptyset$ become $\emptyset V$ with intensity λ_1 . In other words, on the time interval $[t, t + dt]$ each particle, having velocity V , independently of other particles jumps one unit to the right ($x \rightarrow x + 1$) with probability $\lambda_1 dt + o(dt)$ provided that the node $x + 1$ is empty. We need to do calculations similar to (15) and (16). Now $x \in \mathbf{Z}$. As before, let us consider all possible states in the neighboring nodes $x - 1$, x and $x + 1$ at time t and apply the complete probability formula:

$$\begin{aligned}
 p_{t+dt}(x; \emptyset) &= \sum_{e_i \in \{\emptyset, 0, V\}, i=1,2,3} \mathbf{P}(\zeta_x(t+dt) = \emptyset \mid (\zeta_{x-1}(t), \zeta_x(t), \zeta_{x+1}(t)) = (e_1, e_2, e_3)) \\
 &\quad \times \mathbf{P}\{(\zeta_{x-1}(t), \zeta_x(t), \zeta_{x+1}(t)) = (e_1, e_2, e_3)\}.
 \end{aligned}$$

Conditional probabilities under the sum sign can be easily found. The results are presented in the following table, where “*” means any of symbols $\emptyset, 0$ or V :

$x-1$	x	$x+1$	$\mathbf{P}(\zeta_x(t+dt) = \emptyset \mid (\zeta_{x-1}(t), \zeta_x(t), \zeta_{x+1}(t)) = (e_1, e_2, e_3))$
e_1	e_2	e_3	
\emptyset	\emptyset	*	$1 - o(dt)$
0	\emptyset	*	$1 - o(dt)$
V	\emptyset	*	$1 - \lambda_1 dt + o(dt)$
*	0	*	$o(dt)$
*	V	\emptyset	$\lambda_1 dt + o(dt)$
*	V	0	$o(dt)$
*	V	V	$o(dt)$

Up to terms of order $o(dt)$ we get

$$\begin{aligned} p_{t+dt}(x; \emptyset) &= p_t(x-1; \emptyset\emptyset) + p_t(x-1; 0\emptyset) + p_t(x-1; V\emptyset) \\ &\quad - \lambda_1 dt \cdot p_t(x-1; V\emptyset) + \lambda_1 dt \cdot p_t(x; V\emptyset) \\ &= p_t(x; \emptyset) + \lambda_1 dt \cdot [-p_t(x-1; V\emptyset) + p_t(x; V\emptyset)], \end{aligned}$$

and, hence,

$$\frac{d}{dt} p_t(x; \emptyset) = \lambda_1 [-p_t(x-1; V\emptyset) + p_t(x; V\emptyset)]. \quad (18)$$

In the same way, we obtain the equation for $p_t(x; V)$:

$$\frac{d}{dt} p_t(x; V) = \lambda_1 [p_t(x-1; V\emptyset) - p_t(x; V\emptyset)].$$

Similarly one can consider the components of the dynamics (1), corresponding to the random perturbations and to the behavior on the boundaries.

It appears that the equations for $q_t[(k, x); e]$ become closed if we consider only translation invariant distributions. Namely, to finish the proof of Lemma 1, from now on we assume that the process starts at time $t = 0$ from a *translation invariant* distribution. Recall that the dynamics conserves the translation invariance, i.e. at any time $t > 0$ the process has a translation invariant distribution. In particular, the property (2) holds. Therefore, in the right hand sides of the equations (9)–(14) all summands having the factor λ_1 disappear. Collecting together the equations with $\frac{d}{dt} q_t((k, x); \emptyset)$, $0 \leq k \leq S+1$, and using notation (4), we come to the statement 1) of Lemma 1. The statement 2) easily follows from the form of the equations (9)–(14) if we take into account (6).

This completes the proof of the lemma.

3.2 Convergence as $t \rightarrow \infty$

We prove here case 1) of Theorem 1, namely, we show the existence of the limits $\lim_{t \rightarrow \infty} p_k^e(t)$ for the functions $p_k^e(t)$, satisfying the equations (7)–(8). We use probabilistic arguments. On the *finite* state space $\mathcal{M} = \{\emptyset, 0, V\}^{\{0,1,\dots,S+1\}}$ let us define an auxiliary continuous time Markov process

$$\vartheta(t) = (\vartheta_k(t), k = 0, 1, \dots, S+1), \quad \vartheta_k(t) \in \{\emptyset, 0, V\},$$

with the following transitions. On a small time interval $[t, t+dt]$ the following events can occur independently of each other:

- any nodes k and $k + 1$ exchange their states with probability $\lambda dt + o(dt)$;
- any node of the zero layer $k = 0$, being in the state V , changes its state to 0 with probability $\beta dt + o(dt)$; any node of the top layer $k = S + 1$, being in the state 0 , with probability $\beta dt + o(dt)$ changes its state to V ;
- any node k in a state different from \emptyset changes its state according to the rule $0 \leftrightarrow V$ with probability $\varepsilon dt + o(dt)$ (*random perturbation*).

Note that the Markov process $\vartheta(t)$ is reducible, because its dynamics does not change the total number of nodes having the empty state (\emptyset). At the same time the process $\vartheta(t)$ considered on any set, invariant with respect to the dynamics

$$\mathcal{M}_n = \{\vartheta = (\vartheta_0, \dots, \vartheta_{S+1}) \in \mathcal{M} : \#\{j : \vartheta_j = \emptyset\} = n\}, \quad n = 0, 1, \dots, S + 1,$$

is irreducible and ergodic. Hence, for any n the following limits

$$\lim_{t \rightarrow \infty} \mathbf{P}(\vartheta(t) = \varkappa \mid \vartheta(0) = \varkappa_0) = g_n(\varkappa), \quad \varkappa_0, \varkappa \in \mathcal{M}_n,$$

exist and do not depend on the concrete choice of initial state \varkappa_0 . Thus for any initial distribution $\mu_{\vartheta(0)}$ there exists the limit $\lim_{t \rightarrow \infty} \mathbf{P}\{\vartheta(t) = \varkappa\}$, depending on $\mu_{\vartheta(0)}$.

If we permit, within the current subsection, the following notation

$$p_k^e(t) = \mathbf{P}\{\vartheta_k(t) = e\}, \quad e \in \{\emptyset, 0, V\}, \quad k = 0, 1, \dots, S + 1,$$

then it is easy to see that the functions $p_k^e(t)$ satisfy the system of equations (7)–(8). Proof of this fact is even simpler than the reasonings of subsection 3.1. Since

$$p_k^e(t) = \sum_{\varkappa \in \mathcal{M} : \varkappa_k = e} \mathbf{P}\{\vartheta(t) = \varkappa\},$$

the statement 1) of Theorem 1 easily follows.

3.3 Exact formulas for stationary solutions

Now we return to the process $v(t)$. Denote by μ_s^\emptyset , μ_s^0 and μ_s^V the stationary probabilities of the following three events: on the vertical layer s there is no particle, there is a particle with velocity 0 and there is a particle with velocity V correspondingly, that is $\lim_{t \rightarrow \infty} p_s^e(t) = \mu_s^e$. As it was explained in the previous subsection these limits depend on the initial distribution η^0 .

In particular,

$$\sum_{s=0}^{S+1} \mu_s^\varnothing = (S+2) - M(\eta^0), \quad (19)$$

where $M(\eta^0)$ is defined in (3).

Let us remark that the limiting probabilities μ_s^e satisfy the stationary versions of the equations (7), (8). It follows from (6) that

$$\mu_s^\varnothing + \mu_s^0 + \mu_s^V = 1 \quad \forall s.$$

Lemma 2 *The distribution of “holes” is uniform on the set of layers: $\mu_s^\varnothing = \mu_{s_1}^\varnothing$ for all $s, s_1 \in \{0, 1, \dots, S+1\}$.*

Proof. The system of stationary equations for μ_s^\varnothing , $s = 0, 1, \dots, S+1$, has the form:

$$\begin{aligned} 0 &= \mu_{s-1}^\varnothing + \mu_{s+1}^\varnothing - 2\mu_s^\varnothing, & 1 \leq s \leq S, \\ 0 &= \mu_1^\varnothing - \mu_0^\varnothing, \\ 0 &= \mu_S^\varnothing - \mu_{S+1}^\varnothing. \end{aligned}$$

It is evident that the set of all solutions of this system is the one-dimensional subspace $\mu_0^\varnothing = \dots = \mu_{S+1}^\varnothing$. From the probabilistic nature of our model we have

$$\mu_0^\varnothing = \dots = \mu_{S+1}^\varnothing = 1 - \rho_S \in [0, 1].$$

This proves the lemma and the statement 2) of Theorem 1.

The number $\rho_S \in [0, 1]$ will be called particle density. It is readily seen from (19) that the density depends on the initial distribution: $\rho_S = M(\eta^0)/(S+2)$.

Since $\mu_k^0 + \mu_k^V \equiv \rho_S$, we need only to find probabilities (μ_k^V , $k = 0, 1, \dots, S+1$). On the inner layers they satisfy the following equations:

$$0 = \lambda(\mu_{k+1}^V + \mu_{k-1}^V - 2\mu_k^V) + \varepsilon(\rho_S - 2\mu_k^V), \quad 1 \leq k \leq S, \quad (20)$$

with the following boundary conditions

$$0 = \lambda(\mu_1^V - \mu_0^V) + \varepsilon(\rho_S - 2\mu_0^V) - \beta\mu_0^V, \quad (21)$$

$$0 = \lambda(\mu_S^V - \mu_{S+1}^V) + \varepsilon(\rho_S - 2\mu_{S+1}^V) + \beta(\rho_S - \mu_{S+1}^V). \quad (22)$$

Case 1: $\varepsilon = 0$. Equations (20) take the following form $\mu_{k+1}^V + \mu_{k-1}^V - 2\mu_k^V = 0$.

Characteristic equation corresponding to this difference equation

$$z^2 + 1 - 2z = 0$$

has a root $z = 1$ of order 2. This is the *resonant* case, therefore a general solution is $\mu_k^V = D_0 + D_1 k$. The coefficients D_0 and D_1 can be found from the boundary conditions (21) and (22):

$$0 = \lambda D_1 - \beta D_0, \quad 0 = -\lambda D_1 + \beta(\rho_S - D_0 - D_1(S+1)).$$

Solving the latter system, we get explicit solution

$$\mu_k^V = \frac{\rho_S}{S+1+2\lambda\beta^{-1}} (k + \lambda\beta^{-1}), \quad k = 0, 1, \dots, S+1,$$

which is a linear function. Now the statement of Theorem 2 easily follows.

Case 2: $\varepsilon > 0$. It can be shown in a standard way that general solution of the inhomogeneous difference equation (20) has the form

$$\mu_k = \frac{\rho_S}{2} + C_1 z_1^k + C_2 z_2^k, \quad (23)$$

where z_1 and z_2 are the roots of the characteristic equation

$$\lambda z^2 + \lambda - 2(\lambda + \varepsilon)z = 0, \quad (24)$$

C_1 and C_2 are unknown coefficients. Immediately, by solving equation (24), we get

$$z_{1,2} = 1 + \frac{\varepsilon}{\lambda} \pm \sqrt{\left(1 + \frac{\varepsilon}{\lambda}\right)^2 - 1}. \quad (25)$$

Notice that if $\varepsilon > 0$ then the equation (24) has two *different* real roots: $z_1 > 1$ and $z_2 < 1$, moreover, $z_1 z_2 = 1$. Substituting (23) to the boundary conditions (21)–(22), after some algebra we obtain the next lemma.

Lemma 3 *The solution $(\mu_s^V, s = 0, 1, \dots, S+1)$ has the following explicit form:*

$$\mu_k^V = \frac{\rho_S}{2} - \rho_S \frac{(b(z_1)z_1^{S+1-k} - b(z_2)z_2^{S+1-k}) - (b(z_1)z_1^k - b(z_2)z_2^k)}{2(b^2(z_1)z_1^{S+1} - b^2(z_2)z_2^{S+1})},$$

$k = 0, 1, \dots, S+1$, where $b(z) := 1 + (2\varepsilon + \lambda(1 - z^{-1}))/\beta$.

3.4 Asymptotics and phase transition

Here we consider λ and β as fixed parameters and assume that $\varepsilon \rightarrow 0$, $S \rightarrow \infty$. We see that for small ε

$$z_1 = 1 + \sqrt{\frac{2\varepsilon}{\lambda}} + O\left(\frac{\varepsilon}{\lambda}\right), \quad z_2 = 1 - \sqrt{\frac{2\varepsilon}{\lambda}} + O\left(\frac{\varepsilon}{\lambda}\right).$$

Taking into account the assumptions of Theorem 3, for simplicity we put $\rho_S \equiv \rho$.

We are interested in the behaviour of $(\mu_k^V, 0 \leq k \leq S+1)$ on the space scale $k = [uS]$, $u \in [0, 1]$. To get a meaningful asymptotics in the limit $S \rightarrow \infty$, we assume that the parameter ε is a function of S :

$$\varepsilon_S = \frac{\lambda}{2} \psi^2(S) S^{-2},$$

where $\psi(S) > 0$ is such that $\psi(S)S^{-1} \rightarrow 0$ as $S \rightarrow \infty$. Hence, $\sqrt{2\varepsilon_S/\lambda} = \psi(S)S^{-1}$ and

$$\begin{aligned} z_{1,2} &= 1 \pm \psi(S)S^{-1} + O(\psi^2(S)S^{-2}), \\ b(z_{1,2}) &= 1 \pm \frac{\sqrt{\lambda}}{\beta} \psi(S)S^{-1} + \beta^{-1} O(\psi^2(S)S^{-2}). \end{aligned}$$

The calculations presented below show that the asymptotics of $\mu_{[uS]}^V$ strongly depends on the properties of the function ψ .

In case of *small* random perturbations $\psi(S) \rightarrow 0$

$$\begin{aligned} \mu_{[uS]}^V &\sim \frac{\rho}{2} - \rho \frac{2(1-u)\psi(S) - 2u\psi(S)}{2 \cdot 2\psi(S)} + \beta^{-1} O(S^{-1}) \\ &\sim \frac{\rho}{2} + \rho \frac{2u-1}{2} = u\rho, \quad u \in (0, 1). \end{aligned}$$

and the profile has a laminar character.

In case of *strong* random perturbations $\psi(S) \rightarrow \infty$

$$\mu_{[uS]}^V \sim \frac{\rho}{2} - \rho \frac{z_2^{(1-u)S} - z_2^{uS}}{2z_2^S} + \beta^{-1} O(S^{-1} \cdot \psi(S)e^{-\psi(S)}) \rightarrow \frac{\rho}{2} \quad \forall u \in (0, 1). \quad (26)$$

and the limiting profile is very far from being laminar. Indeed, it can be checked that the limiting profile here is the same as in the case when there is no boundary conditions (formally, when $\beta = 0$), that is called free boundary conditions. We can conclude that if the random perturbations are strong then the boundaries do not influence the inner layers. This is typical for the turbulent phase.

The case $\psi(S) = \text{const}$ separates the above two phases. For convenience let us put $\psi(S) = K$, $K > 0$. Then

$$\begin{aligned}\mu_{[uS]}^V &\sim \frac{\rho}{2} - \rho \frac{(e^{K(1-u)} - e^{-K(1-u)}) - (e^{Ku} - e^{-Ku})}{2(e^K - e^{-K})} \\ &= \frac{\rho}{2} \left(1 + \frac{\sinh(K(u - 1/2))}{\sinh(K/2)} \right), \quad u \in (0, 1).\end{aligned}$$

Let us remark in conclusion that the assumption $\psi(S)S^{-1} \rightarrow 0$ is not essential for the strong perturbation asymptotics (26). In particular, this results holds in the important case $\varepsilon_S = \varepsilon$. Indeed, in this case z_2 does not depend on S , and, using the explicit formula of Lemma 3, we get for any $u \in (0, 1)$

$$\mu_{[uS]}^V - \frac{\rho}{2} \sim \rho \frac{\text{const}}{b(z_2)z_2^{(1-\min(1-u,u))S}} \rightarrow 0 \quad (S \rightarrow \infty).$$

We stress that all limiting expressions do not depend on the parameter β .

Theorem 3 is proved.

References

- [1] *Strange attractors*. Mir Moscow, 1981. (Mathematics. News of foreign science, V. 22.)
- [2] CHORIN A. J., MARSDEN J. E. *A Mathematical Introduction to Fluid Mechanics*. New York: Springer-Verlag, 1993, 169 p.
- [3] MARCHIORO C., PULVIRENTI M. *Mathematical Theory of Incompressible Non Viscous Fluids*. New York: Springer-Verlag. 1994, 283 p.
- [4] SHIRYAEV A. *On the classical, statistical, and stochastic approaches to the hydrodynamic turbulence*. Research Report N.2. Aarhus: Thiele Centre for Applied Mathematics in Natural Science, 2007.
- [5] KOLMOGOROV A.N. *Mathematical models of turbulent motion of an incompressible viscous fluid*. RUSS MATH SURV, 2004, v. 59, N.1, p. 3-9.
- [6] MESHALKIN L. D., AND SINAI IA. G. *Investigation of the stability of a stationary solution of a system of equations for the plane movement of an incompressible viscous liquid*. Journal of Applied Mathematics and Mechanics, 1961, v. 25, N.6, p. 1700-1705

- [7] LANDAU L. D., AKHIEZER A. I., LIFSHITZ E. M. *General Physics: Mechanics and Molecular Physics*. Pergamon Press, 1970, 372 p.
- [8] LIGGETT T. *Interacting Particle Systems*. Springer, 2005, 496 p.
- [9] KIPNIS C., LANDIM C. *Scaling Limits of Interacting Particle Systems*. Berlin: Springer-Verlag, 1999, 442 p.
- [10] SPOHN H. *Large Scale Dynamics of Interacting Particles*. Berlin: Springer-Verlag, 1991, 342 p.
- [11] ESPOSITO R., MARRA R., YAU H. T. *Navier–Stokes equations for stochastic particle systems on the lattice*. *Comm. Math. Phys.*, 1996, v. 182, N.2, p. 395–456.
- [12] IGNATYUK I. A., MALYSHEV V. A., MOLCHANOV S. A. *Moment-closed processes with local interaction*. *Selecta Math. Soviet.*, 1989, v. 8, N.4, p. 351–384.
- [13] MÉLÉARD S. *Monte-Carlo approximations for 2d Navier–Stokes equations with measure initial data*. *Probab. Theory Related Fields*, 2001, v. 121, N.3, p. 367–388.
- [14] FONTBONA J. *A probabilistic interpretation and stochastic particle approximations of the 3-dimensional Navier–Stokes equations*. *Probab. Theory Related Fields*, 2006, v. 136, N.1, p. 102–156.
- [15] GOLINELLI O., MALLICK K. *Derivation of a matrix product representation for the asymmetric exclusion process from algebraic Bethe ansatz*. *J. Phys. A.*, 2006, v. 39, N.34, p. 10647–10658.
- [16] DERRIDA B., LEBOWITZ J. L., SPEER E. R. *Entropy of open lattice systems*. *J. Statist. Phys.*, 2006, v. 126, N.4/5, p. 1083–1108.