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Random Infinite Spin Graph Evolution

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ABSTRACT. We define and study a general class of Markov processes that includes most concrete processes used in applications. This class arose both from computer science applications (grammars and graph grammars) and from physics (quantum gravity). They are quite similar to processes with local interaction, but the underlying space changes together with the spin values. In this paper we prove for all finite times the existence of an infinite cluster local dynamics starting with a countable graph. We establish the clustering properties of this dynamics for small times. The new phenomenon of space disappearance for large times is discussed as well.

1. Introduction

From the probability theory point of view, here we consider a kind of homogeneous stochastic processes with local interaction but without the underlying space. From the computer science point of view, we consider large graph grammars and their nondeterministic trajectories. From the topology point of view, we consider infinite complexes (in this paper they are one-dimensional).

Here we do not speak about physics at all, but it is present behind the scenes in the methods and the mere ideology of the paper. A spin graph corresponds to a field on a lattice in physics, its stochastic dynamics is called the stochastic quantization method. But here the lattice itself changes dynamically in time together with the fields. Already the two-dimensional case is very popular in physics, and we shall study it in detail in [**9**].

The outline of this paper is the following. Section 2 mainly contains definitions and some motivation. In Section 3 we formulate and prove the main result: the existence of the dynamics itself. For $t \to \infty$, however, new unexpected problems arise and we start discussing them there.

2. Random graph grammars

2.1. Spin graphs. Consider a (finite or countable) graph G with the set of vertices $V = V(G)$ and the set of links $L = L(G)$. A *subgraph* of G is a subset $V_1 \subset V$ of vertices together with some links connecting pairs of vertices from V_1 and inherited from L. A *regular subgraph* $G(V_1)$ of G is a subset $V_1 \subset V$ of vertices together with ALL links connecting pairs of vertices from V_1 and inherited from L.

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Links are not directed. There can be any finite number of links between two vertices, loops are allowed as well. A link l connecting the vertices u and v is called *incident* to both of them and vice versa. A link is incident with the subset V' if it is incident with some vertex $v \in V'$. We assume everywhere that each vertex has at most a finite number incident links.

The *distance* $\rho(v, v_1)$ between two vertices v, v_1 of the graph is the minimal length of a path connecting these two vertices, i.e., the minimal number of links in such a path. Vertices connected by a link are called *neighbors*. The *neighborhood* $O_d(v)$, $O_1(v) = O(v)$, of the vertex v in G is the regular subgraph with the set of vertices consisting of v itself and of all vertices at the distance not greater than d from v.

A *spin graph* (also colored graph, marked graph, spin system, etc.) is a pair $\alpha = (G, s)$, where $s = s(\,\cdot\,)$ is a function $s\colon V \to S$ with S the set of "spin values" or alphabet. An *isomorphism* of spin graphs is an isomorphism of graphs respecting the spins. The *empty spin graph* $\varnothing$ is the empty graph with no spin.

If we say that the set V of vertices is given as a set, this means that the graph is labelled, for example, we can enumerate V by positive integers. It will be useful to extend the definition of labelling. Assume that we have some abstract set Ψ, which will be called the *space*. Then a *labelling* is an arbitrary embedding of V into Ψ. We shall often consider also equivalence classes of spin graphs (they are sometimes called nonlabelled graphs). An isomorphism of spin graphs is given by a one-to-one mapping between their sets of vertices respecting spins and links. It will always be clear from the context whether we consider labelled or nonlabelled graphs. One of the difficulties that we shall encounter later will be the absence of natural labellings (or the absence of a natural space) for the set of vertices of graphs evolving in time. One can even say that the set of vertices loses its exact meaning.

2.2. Operations, substitutions, transformations. A spin graph dynamics is a random sequence (where the moments $\ldots < t_k < t_{k+1} < \ldots$, are also random) of spin graphs

$$\alpha_0 = (G_0, s_0), \quad \alpha_{t_1} = (G_{t_1}, s_{t_1}), \quad \ldots, \quad \alpha_{t_k} = (G_{t_k}, s_{t_k}), \quad \ldots$$

to be defined below. The graph α_{t_k} is obtained from $\alpha_{t_{k-1}}$ by a simple transformation from some fixed class of transformations. We start by defining substitutions, which are the generalization of the substitutions in grammars; see [**8**]. Note that grammars correspond to graph grammars when the graphs are linear graphs (chains).

DEFINITION 1. The *substitution* (*production*) $\mathrm{Sub} = (\Gamma; \varphi\colon V_0 \to V(\Gamma'))$ is defined by two "small" spin graphs Γ and Γ', the subset $V_0 \subset V = V(\Gamma)$ and the mapping $\varphi\colon V_0 \to V' = V(\Gamma')$; either of Γ and Γ' can be empty.

DEFINITION 2. The *transformation* $T = T(\mathrm{Sub})$ (which eventually will also be called a substitution) of a spin graph G corresponding to a given substitution Sub is defined in the following way. Fix an isomorphism $\psi\colon \Gamma \to \Gamma_1$ onto a spin subgraph Γ_1 of G. Consider the disjoint union of G and Γ', delete all links belonging to Γ_1, delete all vertices in $\psi(V) \setminus \psi(V_0)$ together with all links incident to them, identify each $\psi(v) \in \psi(V_0)$ with $v' = \varphi(v) \in \Gamma'$. The function s on $V(G) \setminus V(\Gamma_1)$ is inherited from G, and on $V(\Gamma')$ from Γ'. We denote the resulting graph by

$G(\mathrm{Sub}, \psi)$. *Regular transformations* are defined similarly, with the condition that only isomorphisms ψ to regular subgraphs Γ_1 of G are allowed.

This can be formulated differently: one must first find a subgraph of G isomorphic to Γ, delete the subset $V(\Gamma) \setminus V_0$ of vertices together will all incident links, identify some vertices in V_0 and add some links between vertices of V_0.

Note that these definitions are in the same spirit as attaching one topological space T_1 to another T_2 by the continuous mapping $\varphi\colon T_1' \to T_2$, where T_1' is a closed subspace of T_1.

It is easy to see that some of the simplest operations, such as deleting and appending a link, identifying two vertices, changing s in one vertex, etc., are transformations according to the above definition. For example, to append a vertex, we take Γ' consisting of one vertex and Γ empty (we assume that there exists only one empty subgraph for each graph).

2.3. Graph grammars.

DEFINITION 3. A *graph grammar* is defined by a finite set of substitutions Sub_i, $i = 1, \ldots, r$. A graph grammar is said to be *local* if the Γ's corresponding to all Sub_i are connected. The *language* $L(\alpha_0, \{\mathrm{Sub}_i\})$ is defined also by (an initial) spin graph $\alpha_0 = (G_0, s_0)$ (finite or infinite) and is the set of all spin graphs that can be obtained from α_0 by applying transformations an arbitrary number of times in arbitrary order. More exactly, $\alpha_0 \in L(\alpha_0, \{\mathrm{Sub}_i\})$ and if $\alpha \in L(\alpha_0, \{\mathrm{Sub}_i\})$, then $T\alpha \in L(\alpha_0, \{\mathrm{Sub}_i\})$ for arbitrary $T = T(\mathrm{Sub}_i)$.

Further on we consider only local dynamics. There has been recent activity in graph grammars in the computer science framework; see [**2**] and the proceedings of four conferences in graph grammars, the first one of which is [**3**]. Note, however, that graph grammars (more exactly, operations on spin graphs) appeared under another name earlier (see [**11**]). One should realize the existence (and this is quite inevitable) of many different variations on the theme of graph grammars. All of them use different generalizations of spin graphs (see for example, [**2, 3**]). Examples are bipartite graphs (used in Petri nets), hypergraphs with hyperedges instead of edges. The choice of a particular definition depends on local historical tradition, personal taste, and favorite applications. The most general framework for all these notions seems to be category theory.

We want to emphasize that many statements about such generalized graph grammars can be reformulated using only simple spin graphs by enlarging the set of spins, redefining a graph with new vertices and new edges, etc. In the general theory it is convenient to choose the simplest formal definition without introducing an infinite number of evident generalizations.

2.4. Markov process. We introduce a general class of Markov processes that we call *random graph grammars*. The *states* are spin graphs, that is, pairs (G, s), where G is a graph and $s = s(v)$ is a function on the vertices of G taking values in some "spin space" S. Contrary to processes with local interactions, where the graph (normally a lattice) is fixed, in our definition the graph changes (locally) together with the spin values. The process is specified completely by the initial state and by the rates $c(\alpha, \beta)$ of transformations from the state $\alpha = (G, s)$ to the state $\beta = (G_1, s_1)$.

Let us fix some graph grammar. The only positive rates are those that correspond to the transformations defined by substitutions from the fixed graph grammar. We assume the analog of translation invariance, which has a more general form here: the rates $c(\alpha, \beta)$ depend only on the substitution done but not on the place (that is not on the choice of ψ) where it is done. Thus $c(\alpha, \beta)$ take only a finite number of different values $c_i = c(\mathrm{Sub}_i)$.

Thus each allowed morphism ψ waits a corresponding exponential time (independently of the others). When the first of these succeeds, it performs a transition (transformation), and the procedure starts again, and so on. To give conditions when this process can be defined globally (for the entire time interval $[0, \infty]$) we discuss two cases separately. When the initial graph is finite, we get a countable Markov chain and we must only prove that it is nonexploding. When the initial graph is countable, the Markov chain has a continuum number of states and the construction of the dynamics is a more complicated problem with new unexpected thermodynamic limit aspects.

We refer the reader to [**5, 6**] for conditions under which, for finite initial graphs, the minimal nonexploding process exists. Our further assumptions, however, include these conditions.

2.5. Examples. It is worth noticing that the introduced class of processes covers many famous processes.

• Branching processes. For the Galton–Watson process, the graphs consist of isolated vertices, no links. Vertices represent particles, and spins represent particle types.

• Processes with local interaction on a lattice. It is a local grammar and the graph is not changed at all. Only spins undergo local transitions.

• A queue is a random string of symbols, where the symbols are the customer types. For example in LIFO and FIFO protocols, the changes may be only at one or at both ends of the string. At the same time, this is a particular case of a random grammar, because one can put the special "nonterminal symbols" L (at the end of the queue) and F (at the beginning of the queue). The substitutions have the form $L \to xL$ (customer x arrived) and $Fy \to F$ (customer y is served). In this case the grammar is context free: customers arrive and are served independently of the context. Generalization to queueing networks is discussed in [**7**].

• Random walks. For example, states of a random walk on $\mathbb{Z}_+$ are strings with one symbol (the length of the string is the position of the random walker). In other words, a random walk is a one-sided linear grammar with one terminal symbol and one nonterminal symbol (at the end of the word). Random walks on a free noncommutative group are strings with particular substitutions at one end.

• An embedded graph grammar is a graph grammar with additional structure: for a given embedding of the graph G into some topological space X (normally $\mathbb{R}^n$), the embeddings of all its transforms $G(\mathrm{Sub}, \psi)$ are given as well.

• Random fractals: embedded graph grammars with or even without spins; see [**8**].

• Stochastic geometry. In stochastic geometry only embedded cell complexes are considered; see [**12**].

• Random graph theory. This beautiful field with many deep results was always very popular in combinatorics. The evolution of random graphs was studied already by Erdös and Rényi in 1960, and continued later by many authors; see [**14, 4, 13**].

Unfortunately, in fact, the only model considered was the following: the number of vertices held fixed, on each step we append a new link, choosing it with equal probability among all absent links. In [**5, 6**] we gave many examples of more general evolution.

• Quantum gravity models in physics, which now are the most popular fundamental models of nature. We shall consider such models in detail in [**9**].

3. Dynamics of infinite spin graphs

3.1. Labelled infinite graphs. We shall consider probability distributions on the set of all countable spin graphs (with fixed spin space S). However, we immediately encounter difficulties here. The probability space Ω and the σ-algebra are quite standard, as long labelled graphs are considered and, moreover, the space Ψ of labels is fixed (it is not assumed equipped with a metric or a topology). Then formally one can define the probability space Ω to be the set of all triples (Φ, s, L) (labelled spin graphs), where $\Phi \subset \Psi$ is the set of labels corresponding to the vertices of the graph (all vertices have different labels), s is a function on Φ with values in S, and L is a symmetric function on $\Phi \times \Phi$ with values in $\mathbb{Z}_+$, the number of links connecting a given pair of vertices.

Then the set of all spin graphs is a subset of the space $(S \cup \{\varnothing\})^\Psi \times \mathbb{Z}_+^{\Psi \times \Psi}$ with the product topology. Let Σ be the Borel σ-algebra in this space. Probability measures on this σ-algebra are defined via the Kolmogorov theorem and are generated by finite-dimensional distributions $P(O_d(v) = \alpha)$ for different d and v. Below we make some assumptions on the graph grammar. It will follow that $O_d(v)$ are all finite a.s.

This approach will be sufficient for us in the next section, where the finite times dynamics is considered. When time tends to infinity, the labels are lost due to the possible disappearance and appearance of vertices. Below other approaches will be discussed.

3.2. Existence. Here we construct a Markov dynamics for arbitrary initial countable graphs and prove that it is the thermodynamic limit of finite dynamics. Below we give exact definitions. Moreover we shall show that, for small times, this dynamics has a remarkable property, called the cluster property in [**8**]. Some ideas of this sections are similar to [**8**] and we sometimes refer to this paper.

Fix some graph grammar $\mathcal{G}$ and fix some initial spin graph $\alpha(0) = (G = G(0), s(0))$. Choose a vertex in $G(0)$ and denote it 0. Let $O_N = O_N(0) = G^N(0)$ be the N-neighborhood of 0 in G. We make the following assumptions. Let $A(D)$ be the set of all graphs with the property that each vertex has at most D incident links.

ASSUMPTION. We assume that the initial graph belongs to $A(D)$ and that $TA(D) \subset A(D)$ for all transformations T of our graph grammar.

We assume that the radii of all Γ entering the definition of substitutions of our graph grammar do not exceed 1. The latter assumption is needed only to make the presentation shorter.

Denote by $\xi_\beta(t)$ the Markov process starting at a finite spin graph β. Consider an increasing family $G^N(0) = O_N$ of finite spin graphs $O_1 \subset \cdots \subset O_N \subset \cdots$. We want to study the $N \to \infty$ limit for the family $\xi^N(t) = \xi_{O_N}(t)$ of Markov processes on the probability spaces Ω_N.

We shall need some auxiliary notation. Consider a graph G and a subset $V' \subset V = V(G)$. Introduce the *boundary of width d*

$$\partial_d V' = \{v \in V \setminus V' : 0 < \rho(v, V') \leqslant d\}, \qquad \partial V' = \partial_1 V'.$$

A subset V_1 of vertices is *connected* if the regular subgraph with this set of vertices is connected.

Now let us fix N and t. For the process $\xi^N(t)$ and a trajectory ω, we say that the vertex $v \in O_N$ *was touched* during the time interval $[0, t]$ if it belongs to one of the graphs $\psi(\Gamma)$ in the random sequence of transformations. Denote by $Q(N, t; \omega)$ the set of all vertices in $G^N(0) = O_N$ that were touched by transformations of the process ξ^N during this time interval. Denote by $\eta^N(v)$ the following random field on $V(G(0))$: $\eta^N(v) = 1$ if $v \in \overline{Q}(N, t; \omega) = Q(N, t; \omega) \cup \partial_1 Q(N, t; \omega)$ and $\eta^N(v) = 0$ otherwise. For a finite $T \subset V$, define the correlation functions

$$\langle \eta_T^N \rangle = \left\langle \prod_{v \in T} \eta^N(v) \right\rangle.$$

Limiting correlation functions (we shall prove that they exist) define, by the Kolmogorov theorem, a probability measure on $\{0, 1\}^{V(G(0))}$ (i.e., the limiting random field $\eta(v)$) or on the set of all subsets of $V(G(0))$. For any initial graph $\alpha = \alpha(0)$ let us define the random point set $E(\omega, t, \alpha) = \{v : \eta(v) = 1\}$.

THEOREM 1. *There exists a $t_0 > 0$ such that for any initial spin graph α and any fixed $0 \leqslant t < t_0$ the sequence of random fields $\eta^N(v)$ tends weakly to a random field $\eta(v) = \eta(v; \alpha)$ on $V(\alpha)$ as $N \to \infty$, i.e., $\langle \eta_T^N \rangle \to \langle \eta_T \rangle = \langle \prod_{v \in T} \eta(v) \rangle$. Moreover, the random set $E(\omega, t, \alpha)$ consists of a countable number of finite connected components a.s.*

PROOF. Let $Q_v(N, t; \omega)$ be the connected component of $Q(N, t; \omega)$ containing the point v. It is empty if v itself was not touched. Let $B_v(N, t; \omega)$ be the connected component of $\overline{Q}(N, t; \omega)$ containing the point v. For some v these components can coincide. Note that each $B_v(N, t; \omega)$ is the union of some $Q_w(N, t; \omega) \cup \partial_d Q_w(N, t; \omega)$. The union of the corresponding $Q_w(N, t; \omega)$ is called the *core* of $B_v(N, t; \omega)$. □

LEMMA 2. *Let $B \in G(0)$ be a finite set of vertices such that $0 \in B$. As $N \to \infty$, the probabilities $P(Q_0(N, t; \omega) \cup \partial Q_0(N, t; \omega) = B)$ tend to a limit, which we denote $P(B)$. Moreover $\sum_B P(B) = 1$.*

To prove this, put $P^N(A) = P(\overline{Q}(N, t; \omega) = A)$. First, we obtain a special representation for $P^N(A)/P^N(\varnothing)$. Note first that $P^N(\varnothing)$ is easy to calculate,

$$P^N(\varnothing) = \prod_{\Gamma, \psi} P(\Gamma, \psi),$$

where the product is taken over all Γ and all morphisms ψ of Γ into G^N, $P(\Gamma, \psi) = \exp(-\lambda(\Gamma)t)$ is the probability that the subgraph $\psi(\Gamma)$ does not undergo transition. Note that all $P(\Gamma, \psi)$ are close to 1 if t is small enough. However, formulas for $P^N(A)$ are not so explicit. Let $p(B; N)$ be the probability that in the finite chain starting at B all points in the core of B are touched and all vertices of the boundary are not touched.

LEMMA 3. *For A the following formula holds:*

$$P^N(A) = \prod_B p(B,N) \prod P(\Gamma,\psi),$$

where the product $\prod_B$ is over all connected components of A, the last product is over all (Γ,ψ) such that $\psi(\Gamma)$ intersects $\Lambda \setminus A$. Note that then it cannot intersect the core of A.

The proof of this lemma is similar to the proof of the corresponding lemma in [**8**] (namely Lemma 2 on page 11).

From this lemma we have the following cluster representation:

$$\frac{P^N(A)}{P^N(\varnothing)} = \prod_B k_B^N, \qquad k_B^N = \frac{p(B;N)}{\prod^B P(\Gamma,\psi)},$$

where the product $\prod_B$ is over all connected components of A, the product $\prod^B$ is over all pairs (Γ,ψ) which touch the core of B. Note that if $\operatorname{dist}(B, G \setminus G^N)$ is larger than $d = 1$, then $p(B;N)$ does not depend on N. In this case we denote it by $p(B)$.

We need also a cluster estimate.

LEMMA 4. *We have $k_B^N < a^{|B|}$ uniformly in N for some $a = a(t) \to 0$ as $t \to 0$.*

Let $P_0^N(B)$ be the probability that B is the connected component of A containing 0. We can write the following cluster representation for it:

$$P_0^N(B) = \frac{\sum_A^B P^N(A)}{1 = \sum_A P^N(A)} = \frac{P^N(\varnothing)\sum_A^B P^N(A)}{P^N(\varnothing)\sum_A P^N(A)} = \frac{k_B \sum_{B_1 \dots B_n}^B k_{B_1} \dots k_{B_n}}{\sum_{B_1 \cdots B_n} k_{B_1} \cdots k_{B_n}},$$

where $\sum_A^B$ is over all A such that B is one of its connected components, $\sum_{B_1 \dots B_n}$ is over all admissible (for example, corresponding to some A) systems of connected components, $\sum_{B_1 \dots B_n}^B$ is over all systems of connected components such that $(B, B_1, \dots, B_n)$ is an admissible system. The system is said to be *admissible* if the distance between each two components B_i, B_j, $i \neq j$, is larger than 1.

This expansion is written in a form that fits exactly into the general scheme of cluster expansions as presented in [**10**, Chapter 3.1]. It is shown in [**10**] how to get a convergent series for the probability $P_0^N(B)$ using this scheme. Note that all results from [**10**] hold for general graphs instead of the lattice $\mathbb{Z}^d$ provided the main assumption holds for some a D. Note also in our case that the parameter $k_{\{v\}}$ in [**10**, Chapter 3.1], is identically 1.

The terms of the convergent series as defined in [**10**] do not depend on N up to the terms of order c^N for some $c = c(D) < 1$. The exponential convergence (as $N \to \infty$) of $P^N(B)$ to some $P(B)$ follows.

Similarly one can obtain the cluster expansion for all correlation functions $\langle \eta_E^N \rangle$, $E \subset V(G^N)$ and for all limiting correlation functions $\langle \eta_E \rangle = \lim_{N\to\infty} \langle \eta_E^N \rangle$, where $E \subset V(G(0))$.

We shall use the sets $E(\omega, t, \alpha)$ for constructing the infinite dynamics. For infinite dynamics, the vertices of the core of E are interpreted as the vertices that were touched by substitutions on the time interval $[0, t]$.

Let us consider the probability space $\Omega_1 = \{0,1\}^{G(0)}$ on which the limiting random field $\eta(v)$ is defined. Denote by μ_1 the corresponding probability measure

on Ω_1. For $\omega_1 \in \Omega_1$ let $B_k(\omega_1)$, $k = 1, 2, \dots$, be all the connected components of $E(\omega_1, t, \alpha)$.

For a given B consider the process that starts at a subgraph B of $G(0)$, and coincides with conditional finite dynamics with the same graph grammar under the condition that all vertices in the core of B are touched but the vertices on the boundary of the core of B are not touched. Denote by $\zeta(t, B)$ the conditional process and by μ_B the corresponding conditional measure. The trajectories of this process on the time interval $[0, t]$ are denoted by $\omega(B)$. Denote the probability space for this process by $(\Omega_B, \Sigma_B, \mu_B)$.

The limiting probability space for the time interval $[0, t]$ is the set Ω of all arrays

$$\omega = (\omega_1, \omega(B_k(\omega_1)),\ k = 1, 2, \dots).$$

Take some subsets $D_1, \dots, D_n$ of $G(0)$ and some measurable subsets $E_i \subset \Omega_{D_i}$. Let $C(E_1, D_1; \dots; E_n, D_n)$ be the set of all ω such that each D_i is equal to one of the $B_k(\omega_1)$ and moreover $\omega(B_k(\omega_1)) \in E_k$. Consider the minimal σ-algebra Σ generated by all finite connected subsets $C(E_1, D_1; \dots; E_n, D_n)$ of Ω.

The probability distribution μ is uniquely defined by the following conditions: the projection of μ on $\{0, 1\}^V$ is μ_1, and

$$\mu(C_{D_1,\dots,D_n}) = \mu_1(A(D_1, \dots, D_n)) \prod_{i=1}^{n} \mu_{D_i}(E_i),$$

where $A(D_1, \dots, D_n)$ is the set of all ω_1 such that D_i are its connected components. Note that for a given N the conditional distributions of trajectories on B are independent for different connected components.

Points of the probability space (Ω, Σ, μ) are, by definition, the trajectories of the infinite dynamics. The previous formula constitutes the cluster property. Similar representations can be obtained for each finite N. In this sense the infinite dynamics is the limit of the finite dynamics as $N \to \infty$. Now we can formulate the final result.

THEOREM 5. *For each t there exists a dynamics on the set of spin graphs which is the thermodynamic limit of a sequence of countable chains. Moreover, it has the cluster property.*

We have proved everything for small t. However, contrary to the one-dimensional case, we can construct cluster dynamics only for small enough t. It is easy to construct examples where after a finite time the set of touched vertices has an infinite component. Note that for all $t_1 < t_2$ a.s.

$$E(\omega, t_2, \alpha) \subset E(\omega, t_1, \alpha).$$

But the dynamics itself can be constructed by using the semigroup property just by gluing together the dynamics for consecutive time intervals $[nt_0, (n+1)t_0]$ of length t_0. But this dynamics becomes more and more complicated as $n \to \infty$. Some properties of the dynamics that hold uniformly under all initial conditions for $t < t_0$ can be obtained for each finite t as well. For larger times, control over labels seems to be finally lost.

REMARK 1. In many examples (see [**8, 5, 9**]), the main assumption does not hold. It seems, nevertheless, that for all those examples similar results are valid.

3.3. Space reconstruction for large times. This part contains generalizations of some problems that were earlier discussed in more detail and for concrete examples in the case of grammars; see [8].

Normally time evolution is considered in a fixed space. Here the space itself (that is the graph) evolves in a quite unpredictable way: during the time evolution vertices (that is, parts of the space) can appear and disappear. While the cluster property of dynamics holds, the space still keeps traces from time 0: there is a set of vertex labels that do not change, thus providing a coordinate system that allows us to establish some order related to evolving finite connected components. If a vertex was not touched, then the label it had at time zero does not change. If it has been touched, then for small t this label can be assigned as an appropriate function of the countable Markov chain corresponding to the finite cluster.

Unlabelled spin graphs. Now we shall discuss how to define probability distributions on unlabelled countable graphs. For this, one could choose our probability space Ω as the set of all equivalence classes of countable spin graphs, which differ only by labelling. What is then very unsatisfactory is that the notion of finite-dimensional distribution (or correlation function) is lost. To have such a notion, one should be able to specify the vertices at which this correlation function is taken. This would be easy if there existed a function $f(A)$ on the set Δ of all countable labelled spin graphs A (the set of labels is not fixed), the value of $f(A)$ being some vertex in A. This specified vertex could have been a reference point (the origin) in our space. For example, if $\alpha = (G, s)$ with G imbedded into some fixed space, it is easy to construct $f(A)$: just choose $v_0 = f(A)$ closest to some fixed space point. It is very plausible that in the general case we have NO natural embedding; any attempt to fix a reference vertex will change the stochastic space structure in the vicinity of the reference point. This resembles the effects produced by measurements in quantum mechanics.

The only general way to find such a function is to use the Zermelo axiom of choice, but this is highly nonconstructive.

We could also restrict the number of events. Then the σ-algebra could be defined as the minimal σ-algebra containing all sets $R(\Gamma)$, where $R(\Gamma)$ is the set of all equivalence classes containing (somewhere) the fixed finite spin graph Γ as a subgraph. Alternatively, one could define a topology on the set of equivalence classes and take the Borel σ-algebra. For example, for grammars (linear graphs) Ω becomes the quotient space of $S^{\mathbb{Z}}$, i.e., Ω consists of all orbits of the translations in $S^{\mathbb{Z}}$. But then in the homogeneous case $P(R(\Gamma))$ are either 0 or 1. Thus such direct approaches fail.

There are several ways out of this desperate situation.

Local observer. If we want to define a random spin graph as the limit of some time evolution, there is another way to choose a reference point. If we start with a fixed or random initial spin graph (finite or infinite), we can introduce a local observer, i.e., a specified vertex $v_0(t)$. The evolution of the local observer is defined by a function f on the pairs (v, Sub) taking values in $V(\Gamma') \cup O_1(\psi(\Gamma))$. Its interpretation is the following: if the vertex v disappears (if it belongs to $\psi(\Gamma)$), then after the substitution Sub the local observer jumps from v to the vertex $f(v, \mathrm{Sub}) \in V(\Gamma')$ if $V(\Gamma') \neq \varnothing$ and to $f(v, \mathrm{Sub}) \in O_1(\psi(\Gamma))$ otherwise.

Stochastic local homogeneity for labelled spin graphs. Another approach is to define an analog of the space homogeneous situation in which one need not specify the place where the correlation function is considered. If the vertices are labelled

by a fixed set Ψ, then there are no problems. The classical Kolmogorov approach, where the distribution is defined by the finite-dimensional distribution, holds. We discussed this in the introduction to this section. However, when there is no space, the notion of finite-dimensional distribution itself disappears.

DEFINITION 4. We say that a random spin graph is *locally stochastically homogeneous* (LSH) if for each d the distribution on the graphs $O_d(v)$ does not depend on v, i.e., for each d and each spin graph α, the probability $p(d,\alpha) = P(O_d(v) = \alpha)$ does not depend on v.

LEMMA 6. *If the spin graph is LSH at time* 0, *then it is LSH for at each finite time.*

Here the distance is not related to any metric on the set of labels, but is an intrinsic characteristic of the graph. If there is a group U acting transitively on the set of labels (it can be the group of shifts as in grammars, see [8], or the permutation group on the set of labels), then one can define homogeneity with respect to the embedding space. For grammars, for example, this coincides with the above definition.

Another inconvenience is that the space of spin graphs with labelled vertices is too huge: labels do not carry useful information, also there is an amount of arbitrariness in the choice of labels for appearing vertices. Also we shall see below that labels do not survive during dynamics. That is why one would like to consider unlabelled graphs.

Unlabelled distributions. For the unlabelled space homogeneous case, we shall define a set of numbers that could be an analog of correlation functions. Let us consider the partially ordered system T of all finite unlabelled spin graphs with respect to inclusion $\Gamma \subset \Gamma'$.

For finite unlabelled spin graphs Γ, let numbers $p(\Gamma)$ be given satisfying the following (compatibility) conditions:

$$0 \leqslant p(\Gamma) \leqslant 1,\ \Gamma \subset \Gamma' \implies p(\Gamma) \leqslant p(\Gamma'),\ \sum_{\Gamma'} p(\Gamma') = p(\Gamma)$$

for all Γ, where the sum is over all Γ', $\Gamma \subset \Gamma'$, $|V(\Gamma') \setminus V(\Gamma)| = 1$ such that any two pairs (Γ, Γ') are not isomorphic. Assume also that $p(\varnothing) = 1$. Such a system is called a *space homogeneous distribution* on countable unlabelled spin graphs, or briefly, an *unlabelled distribution.*

We now give examples of how such systems can be obtained.

First example: take a statistically homogeneous labelled spin graph; for some fixed vertex v and unlabelled connected spin graph Γ take the probability $p(\Gamma)$ that there exists a connected spin subgraph isomorphic to Γ and containing the vertex v.

Second example: for any unlabelled graph Γ, let $n(\Gamma, N)$ be the number of equivalence classes of nonisomorphic pairs $(\gamma, O_v(N))$ such that γ is a subgraph of $O_v(N)$ isomorphic to Γ. Then define $p(\Gamma) = n(\Gamma, N)/|O_v(N)|$. In particular, let a local observer $v_0(t)$ be given. Consider the probabilities $p(t,\gamma) = P(O_d(v_0(t)) = \gamma)$ for finite spin graphs γ. Normally they are different for different local observers (as we saw in [8]), but the distributions far away from the local observer, as for example,

$$p(\beta) = \lim_{D\to\infty} \frac{\natural\{\gamma \subset \alpha(t) : \gamma \sim \beta,\ \gamma \subset O_D(v_0)\}}{|O_D(v_0)|}$$

(for unlabelled β) can be the same for different local observers.

These limits define an unlabelled distribution. Since everything depends on t, this also defines the dynamics of unlabelled distributions.

Time as a reference frame. Time also can serve as a reference frame if we start with a finite spin graph. Assume first that the vertices cannot disappear. Then let v_i be the ith appearing vertex (in time order). Consider the probability $P_t(O_d(v_i) = \alpha)$ that at time t its d-neighborhood is isomorphic to a fixed finite spin graph α. Then we can consider different functions $i = i(t)$ and the limits

$$\lim_{t\to\infty} P_t(O_d(v_{i(t)})) = \alpha;$$

it would be interesting to compare them.

If vertices can disappear, then a more general approach is possible. Let $C_t(\omega)$ be the spin graph at time t for the trajectory ω. For a fixed random trajectory ω one can consider the disjoint union $C(\omega) = \bigcup_t C_t(\omega)$ of all graphs appearing during the time evolution. The vertices of $C(\omega)$ are the pairs (v, t), $v \in C_t(\omega)$.

Let us construct a reasonable connected graph, identifying some vertices in $C(\omega)$. Each appearing vertex can be characterized by its life time interval $[t_1, t_2]$, where t_1 is the time when it first appeared and t_2, when it disappeared. Then we identify all points (v, t) in $C(\omega)$ for fixed v and all $t \in [t_1, t_2]$.

Denote the resulting spin graph (space-time) $C_{\text{con}}(\omega)$. Note that the time (of appearance of the fixed vertex v) could be deduced from $C_{\text{con}}(\omega)$ as the scaled distance (of v) from the initial vertex.

In [**8, 5, 6**] we considered nontrivial examples of graph evolution, where, moreover, the unlabelled correlation functions were calculated.

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