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ERGODICITY OF INFINITE SYSTEMS OF STOCHASTIC EQUATIONS

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We consider the system of random variables ξ_z^t, indexed by the lattice points $z \in \mathbf{Z}^\nu$, $t \in \mathbf{R}_+$, satisfying the system of Ito stochastic equations:

$$d\xi_z^t = [b_0(\xi_z^t) + \varepsilon b_1(\xi_{z+D}^t)]\, dt + dW_z^t, \tag{1}$$

where $D = \{l_1 = 0, l_2, \ldots, l_N\} \subset \mathbf{Z}^\nu$, $\xi_{z+D}^t = (\xi_{z+l_1}^t, \ldots, \xi_{z+l_N}^t)$, while the functions b_0: $\mathbf{R} \to \mathbf{R}$ and b_1: $\mathbf{R}^N \to \mathbf{R}$ satisfy the conditions:

a) there exist $\beta_0 > 0$, $Q > 0$ such that

$$\begin{aligned} b_0(u) &> \beta_0, \quad \text{if} \quad u < -Q, \\ b_0(u) &< -\beta_0, \text{if} \quad u > Q; \end{aligned} \tag{2}$$

b) the function b_0 has continuous derivatives up to and including the third order and for some $B_0 > 0$ one has

$$|b_0(u)|, \quad |b_0^{(n)}(u)| < B_0 \ (n = 1, 2, 3), \ u \in \mathbf{R}; \tag{3}$$

c) the function b_1 has continuous partial derivatives up to and inlcuding the third order and for some $B_1 > 0$ one has

$$|b_1(\bar{u})|, \left|\frac{\partial}{\partial u_i} b_1(\bar{u})\right|, \left|\frac{\partial^2}{\partial u_i \partial u_j} b_1(\bar{u})\right|, \left|\frac{\partial^3 b_1(\bar{u})}{\partial u_i \partial u_j \partial u_k}\right| < B_1$$
$$(1 \leqslant i, j, k \leqslant N). \tag{4}$$

Assume, in addition, that

$$\xi_z^0 = \eta_z, \quad z \in \mathbf{Z}^\nu, \tag{1'}$$

where η_z are random variables such that for some $a > 0$, $C > 0$ we have

$$\mathsf{M} \exp\{a(|\eta_{z_1}| + \ldots + |\eta_{z_k}|)\} \leqslant C^k$$

for all $k \in \mathbf{N}$; $z_1, \ldots, z_k \in \mathbf{Z}^\nu$; $z_i \neq z_j$, $i \neq j$.

The fundamental feature of this system is its invariance with respect to the lattice shifts. Existence and uniqueness theorems for such systems have been considered in [1, 2]. However, ergodicity has been investigated only for linear systems [3], for systems with monotonicity properties [4], or for similar systems with discrete time [5, 6].

Here we consider the case of a small perturbation of a system without interaction, i.e. a system of processes, indexed by the points $z \in \mathbf{Z}^\nu$ and independent for various t for any fixed realization $\bar{\eta} = \{\eta_z\}$. This system (1) admits complete control: an explicit series for finite-dimensional limit distributions, exponential convergence, etc.

The fundamental result of this paper consists in the following theorem.

THEOREM 1. Let ξ_z^t, $z \in \mathbf{Z}^\nu$, $t > 0$, be the process defined by the system (1), (1') and assume that the conditions (2)-(4) are satisfied. Then there exist $\varepsilon_0 > 0$, $\vartheta > 0$, $t_0 > 0$, such that for any finite subset X of $\mathbf{Z}^\nu$, any ε, $|\varepsilon| < \varepsilon_0$, any $t \geqslant t_0$, $t' > 0$, $v_X \in \mathbf{R}^{|X|}$ we have

$$|p^X(t + t', v_X) - p^X(t, v_X)| < \Theta(|X|)\, e^{-\vartheta t},$$

where $\Theta(|X|) > 0$, $p^X(t, v_X)$ is the probability density of the process $(\xi_z^t, z \in X)$.

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We note that for an infinite system the transition probabilities $P(\vec{\xi}, d\vec{\xi}', t)$ are singular measures. Therefore, the standard methods for proving ergodicity cannot be applied here. Our methods combine the methods of Lyapunov functions for chains with transition densities, coupling methods, the parametric method, and the method of cluster expansions.

1. The Exponential Convergence of the Density of an "Unperturbed" Process. THEOREM 2. Let $\xi(t)$, $t \in [0, \infty)$ be a one-dimensional homogeneous Markov process, for which there exist densities p_{uv}^t of the probabilities of the transition in time t from the point u to the point v, and assume that there exist $0 < t_1 < t_2$, $Q > 0$ such that for some nonnegative function f, monotonically increasing on $[0, \infty)$, the following conditions hold:

A) there exist B, $\beta > 0$ such that

$$p_{uv}^\tau < Be^{\beta(f(|u|)-f(|v|))}, \quad p_{uv}^\tau < B, \; u, \; v \in \mathbf{R}, \; t_1 \leqslant \tau \leqslant t_2; \tag{5}$$

B) there exist $E > 0$, $\varepsilon > 0$ such that

$$M\{f(|\xi(t+\tau)|) - f(|\xi(t)|) \mid f(|\xi(t)|) = u\} < -\varepsilon, \tag{6}$$

$$M\{[f(|\xi(t+\tau)|) - f(|\xi(t)|)]^2 \mid f(|\xi(t)|) = u\} < E \tag{7}$$

for any $u > Q$, $t_1 \leqslant \tau \leqslant t_2$.

Then there exist $\Delta > 0$, $\delta > 0$, $0 < h < \beta$, $t_0 > 0$ such that

$$|p_{uw}^{t+t'} - p_{vw}^t| < \Delta e^{-\delta t} e^{hf(\max(|u|, |v|))}, \tag{8}$$

$$\int_{|x|>|w|} |p_{ux}^{t+t'} - p_{vx}^t| \, dx < \Delta e^{-\delta t} e^{h[f(\max(|u|, |v|))-f(|w|)]} \tag{9}$$

for all $u, v, w \in \mathbf{R}$, $t' \geqslant 0$, $t \geqslant t_0$.

The proof of Theorem 2 is based on the following result which is, basically, an analogue of Lemma 1.1 of [7]; we give it without proof.

LEMMA 1. Let $\eta(t)$, $t \in [0, \infty)$ be a homogeneous Markov process on a line, for which there exist transition probability densities π_{uv}^t and such that for some $0 < t_1 < t_2$ one has:

A) there exist $B > 0$, $\beta > 0$ such that

$$\pi_{uv}^\tau < Be^{\beta(u-v)}, \; u, \; v \in \mathbf{R}, \quad t_1 \leqslant \tau \leqslant t_2;$$

B) there exist $E > 0$, $\varepsilon > 0$ such that

$$M\{\eta(t+\tau) - \eta(t) \mid \eta(t) = u\} < -\varepsilon,$$

$$M\{[\eta(t+\tau) - \eta(t)]^2 \mid \eta(t) = u\} < E$$

for any $u \in \mathbf{R}$, $t_1 \leqslant \tau \leqslant t_2$.

Then there exist $A > 0$, $\alpha > 0$, $0 < h < \beta$ such that

$$P\{\eta(t) > v \mid \eta(0) = u\} < Ae^{-\alpha t} e^{h(u-v)}, \quad u, \; v \in \mathbf{R}, \; t > 0.$$

For the proof of Theorem 2 we consider the process $(\xi_1(t), \xi_2(t))$ on $\mathbf{R}^2$, $t = 0, t_1, 2t_1, \ldots$, defined by the densities of the probabilities of transition in time t_1 from the point (u, v) into the point (u', v'):

$$p(t_1; u, v; u', v') = \begin{cases} p_{uu'}^{t_1} \delta_{u'v'}, & \text{if } u = v, \\ p_{uu'}^{t_1} \cdot p_{vv'}^{t_1}, & \text{if } u \neq v, \; (u, v) \notin [-Q, Q]^2, \\ \chi(u, v; u', v') + \frac{1}{1 - X(u, v)}(p_{uu'}^{t_1} - \chi(u, v; u', u')) \cdot \\ \cdot (p_{vv'}^{t_1} - \chi(u, v; v', v')), & \text{if } u \neq v, \\ & (u, v) \in [-Q, Q]^2, \end{cases}$$

where

$$\chi(u, v; u', v') = \min\{p_{uu'}^{t_1}, p_{vv'}^{t_1}\},$$

$$X(u, v) = \int_{\mathbf{R}} \chi(u, v; u', u') \, du',$$

δ_{uv} is the δ-function on $\mathbf{R}$ if one of the variables (u, v) is fixed ($\delta_{uv} = 0$ if $u \neq v$).

We mention some properties of the process $(\xi_1(t), \xi_2(t))$:

1) $\int_{\mathbf{R}} p(t_1; u, v; u', v')\,dv' = p_{uu'}^{t_1}, \quad \int_{\mathbf{R}} p(t_1; u, v; u', v')\,du' = p_{vv'}^{t_1};$ (10)

2) if $(\xi_1(t), \xi_2(t)) = (u, v) \in [-Q, Q]^2$, then

$$\mathrm{P}\{\xi_1(t+t_1) = \xi_2(t+t_1) \mid (\xi_1(t), \xi_2(t)) = (u, v)\} = \int_{\mathbf{R}} p(t_1; u, v; u', u')\,du' = \mathrm{X}(u, v) > 0; \tag{11}$$

3) if $\xi_1(t) = \xi_2(t)$ then $\xi_1(t+t_1) = \xi_2(t+t_1)$.

LEMMA 2. Let λ be the moment when the process $(\xi_1(t), \xi_2(t))$ hits for the first time the set $q = \{(u, v) \in \mathbf{R}^2\colon u = v\}$. Then there exist Γ, γ, $h > 0$ such that

$$\mathrm{P}\{\lambda > t, |\xi_1(t)| > U, |\xi_2(t)| > V \mid (\xi_1(0), \xi_2(0)) = (u, v)\} \leqslant \Gamma e^{-\gamma t} e^{h[f(\max\{|u|, |v|\}) - f(\max\{U, V\})]}, \quad \forall U, V \geqslant 0, \; u, v \in \mathbf{R}. \tag{12}$$

Proof. Let $t = nt_1$. We set

$$\bar{p}(t; u_0, v_0; u_n, v_n; T_1, \ldots, T_k) = \begin{cases} \int_{G_1} \cdots \int_{G_{n-1}} \left[\prod_{i=1}^{n} p(t_1; u_{i-1}, v_{i-1}; u_i, v_i)\right]\left[\prod_{j=1}^{n-1} du_j\, dv_j\right], & u_n \neq v_n; \\ 0, & u_n = v_n; \end{cases}$$

where

$$G_i = (\mathbf{R}^2 \setminus [-Q, Q]^2) \setminus q, \quad \text{if} \quad it_1 \notin \{T_1, \ldots, T_k\},$$
$$G_i = [-Q, Q]^2 \setminus q, \quad \text{if} \quad it_1 \in \{T_1, \ldots, T_k\};$$
$$\bar{p}(t; u_0, v_0; u_n, v_n) = \sum_{k=1}^{n} \sum_{0<T_1<\ldots<T_k\leqslant t} \bar{p}(t; u_0, v_0, u_n, v_n; T_1, \ldots, T_k);$$
$$\bar{p}(t; u_0, v_0; G; T_1, \ldots, T_k) = \int_{G\setminus q} \bar{p}(t; u_0, v_0; u, v; T_1, \ldots, T_k)\,du\,dv; \tag{13}$$
$$\bar{p}(t; u_0, v_0; G) = \sum_{k=1}^{n} \sum_{0<T_1<\ldots<T_k\leqslant t} \bar{p}(t, u_0, v_0; G; T_1, \ldots, T_k).$$

We show that for

$$(u_0, v_0) \in [-Q, Q]^2, \; \tilde{Q} = [-Q, Q]^2, \; \bar{p}(t; u_0, v_0; \tilde{Q}) \leqslant \Gamma_1 e^{-\gamma_1 t}, \quad \Gamma_1, \gamma_1 > 0. \tag{14}$$

Applying Lemma 1, it is easy to show that

$$\mathrm{P}\{(\xi_1(\tau), \xi_2(\tau)) \notin \tilde{Q}, \tau = t_1, 2t_1, \ldots, t, \; |\xi_1(t)| > U, |\xi_2(t)| > V \mid (\xi_1(0), \xi_2(0)) = (u, v)\} \leqslant A e^{-\alpha t} e^{h[f(\max\{|u|, |v|\}) - f(\max\{U, V\})]}, \quad \forall u, v \in \mathbf{R}; \; U, V > 0. \tag{15}$$

From here, in particular, we obtain: for some $\tilde{A}$, $\tilde{\alpha} > 0$ we have

$$\bar{p}(t; u_0, v_0; \tilde{Q}, t) \leqslant \tilde{A} e^{-\tilde{\alpha} t}, \quad (u_0, v_0) \in \tilde{Q}. \tag{16}$$

From the positive recurrence of the process $(\xi_1(t), \xi_2(t))$ and (11) there follows

$$\sum_{n=1}^{\infty} \bar{p}(nt_1; u_0, v_0; \tilde{Q}; nt_1) \leqslant 1 - \varepsilon_1, \text{where } \varepsilon_1 > 0. \tag{17}$$

From (16) and (17) there follows the existence $z_0 > 1$, $\varepsilon_2 > 0$ such that

$$\sum_{n=1}^{\infty} z_0^n \bar{p}(nt_1; u_0, v_0; \tilde{Q}; nt_1) < 1 - \varepsilon_2. \tag{18}$$

Taking into account (13) and making use of (18) one can show that

$$\sum_{n=1}^{\infty} z_0^n \bar{p}(nt_1; u_0, v_0; \tilde{Q}) \leqslant \sum_{n=1}^{\infty} (1 - \varepsilon_2)^k < \infty,$$

which proves (14).

Now, from (14), (15), and (16) it is easy to derive

$$\iint_{|x|>U,\ |y|>V} \bar{p}(t; u_0, v_0; x, y)\,dx\,dy \leqslant \Gamma_2 e^{-\gamma_2 t} e^{-hf(\max\{U, V\})} \tag{19}$$

for some $\Gamma_2, \gamma_2 > 0$, $(u_0, v_0) \in \tilde{Q}$. From here and from (15) we obtain the estimate (12).

We proceed to the <u>proof of Theorem 2</u>. From the properties of the process $(\xi_1(t), \xi_2(t))$ we have

$$\int_{|x|>|w|} |p_{ux}^t - p_{vx}^t|\,dx \leqslant 2\iint_{|x|>0,\ |y|>|w|} \bar{p}(t: u, v; x, y)\,dx\,dy \leqslant 2\Gamma e^{-\gamma t} e^{h[f(\max\{|u|, |v|\}) - f(|w|)]}.$$

Then, for $t_1 \leqslant \tau \leqslant t_2$ we have

$$\int_{|x|>|w|} |p_{ux}^{t+\tau} - p_{vx}^t|\,dx = \int_{|x|>|w|} \left| \int_{\mathbf{R}} p_{uy}^{\tau} p_{yx}^t\,dy - \int p_{uy}^{\tau} p_{vx}^t\,dy \right| dx$$
$$\leqslant \int_{\mathbf{R}} p_{uy}^{\tau} \cdot 2\Gamma e^{-\gamma t} \exp\{h[f(\max\{|y|, |v|\}) - f(|w|)]\}\,dy. \tag{20}$$

Making use of condition (5), it is easy to derive

$$\int_{|x|>|w|} |p_{ux}^{t+\tau} - p_{vx}^t|\,dx \leqslant \tilde{\Gamma} e^{-\gamma t} \exp\{h[f(\max\{|u|, |v|\}) - f(|w|)]\} \tag{21}$$

for $t_1 \leqslant \tau \leqslant t_2$ and for some $\tilde{\Gamma} > 0$.

If $t' > t_0$, where t_0 satisfies : $t_1 \leqslant t_0 \left(\left[\frac{t_0}{t_2}\right] + 1\right)^{-1} < t_2$, $T = \left[\frac{t'}{t_2}\right] + 1$, then

$$\int_{|x|>|w|} |p_{ux}^{t+t'} - p_{vx}^t|\,dx \leqslant \int_{|x|>|w|} \sum_{k=1}^{T} |p_{ux}^{t+kt'/T} - p_{vx}^{t+(k-1)t'/T}|\,dx$$
$$\leqslant \Delta_1 e^{-\gamma t} \exp\{h[f(\max\{|u|, |v|\}) - f(|w|)]\}, \text{where } \Delta_1 > 0.$$

In a similar manner, for t', $t'' > t_0$ we obtain

$$\int_{|x|>|w|} |p_{ux}^{t+t''} - p_{vx}^{t+t'}|\,dx \leqslant \Delta_2 e^{-\gamma t} \exp\{h[f(\max\{|u|, |v|\}) - f(|w|)]\},$$

from where there follows the validity of (9).

Statement (8) follows from the boundedness of p_{uv}^{τ} for $t_1 \leqslant \tau \leqslant t_2$ and from the inequality

$$|p_{uw}^{t+t'} - p_{vw}^t| \leqslant \int_{\mathbf{R}} |p_{ux}^{t+t'-\tau} - p_{vx}^{t-\tau}|\, p_{xw}^{\tau}\,dx.$$

The theorem is proved.

<u>2. Properties of the Density of the Transition Probability of the "Unperturbed" Process.</u> We consider the equation

$$d\xi_t = b_0(\xi_t)\,dt + dW_t, \tag{22}$$

where the function $b_0(u)$, $u \in \mathbf{R}$ satisfies the conditions (2), (3). It is known (see, for example, [8]) that under the indicated restrictions on b_0 the process ξ_t has transition density $p_0(t, u, v)$, $t > 0$, $u, v \in \mathbf{R}$ such that there exist $(\partial/\partial t)p_0(t, u, v)$, $(\partial/\partial u)p_0(t, u, v)$, $(\partial^2/\partial u^2)p_0(t, u, v)$, $t > 0$, $u, v \in \mathbf{R}$, and $p_0(t, u, v)$ is bounded on any segment $[t_1, t_2]$, $0 < t_1 < t_2 < \infty$.

<u>LEMMA 3</u>. There exist $\sigma_1, \sigma_2 > 0$ such that

$$p_0(t, u, v) < e^{\sigma_1 t} \frac{1}{\sqrt{t}} e^{-(u-v)^2/2t\cdot\sigma_2}, \tag{23}$$

$$\left| \frac{\partial}{\partial u} p_0(t, u, v) \right| < e^{\sigma_1 t} \frac{1}{t} e^{-(u-v)^2/2t\cdot\sigma_2} \tag{24}$$

for all $t \geqslant 0$, $u, v \in \mathbf{R}$.

<u>LEMMA 4</u>. There exist $h > 0$, δ, $\sigma_3 > 0$, $T > 0$ such that

$$p_0(t, u, v) < \sigma_3,$$
$$\left|\frac{\partial}{\partial u} p_0(t, u, v)\right| < \sigma_3 e^{-\delta t} e^{h|u|}$$

for $t > T$, $u, v \in \mathbf{R}$.

LEMMA 5. Let T be the quantity defined by Lemma 4. Then there exist α, h, A > 0 such that

$$\int_{\mathbf{R}} p_0(t, u, v) e^{h|v|} dv < Ae^{h|u|}, \quad t > 0; \tag{25}$$

$$\int_{\mathbf{R}} \left|\frac{\partial}{\partial u} p_0(t, u, v)\right| e^{h|v|} dv < A \frac{1}{\sqrt{t}} e^{h|u|}, \quad 0 < t \leqslant T; \tag{26}$$

$$\int_{\mathbf{R}} \left|\frac{\partial}{\partial u} p_0(t, u, v)\right| e^{h|v|} dv < Ae^{-\alpha t} e^{h|u|}, \quad t > T. \tag{27}$$

The proofs of these lemmas carry a purely technical character and we omit them.

3. Existence of the Densities of the Transition Probabilities for ξ_z^t, $z \in \mathbf{Z}^\nu$. Let Λ be a bounded subset of $\mathbf{Z}^\nu$, and let $D \subset \Lambda$. We consider the system

$$\begin{cases} \xi_z^t = 0, & z \notin \Lambda; \\ d\xi_z^t = [b_0(\xi_z^t) + b_1(\xi_{z+D}^t)]\, dt + dW_z^t, & z \in \Lambda; \\ \xi_z^0 = u_z; \end{cases} \tag{28}$$

where $|u_z| < C$, $z \in \Lambda$. This system defines a process on $\mathbf{R}^{|\Lambda|}$, whose transition function has density $p^\Lambda(t, u, v)$, $u = (u_z, z \in \Lambda)$, $v = (v_z, z \in \Lambda)$, satisfying the inverse Kolmogorov equation (the proof of this fact for $|\Lambda| < \infty$ is similar to the proof in the one-dimensional case (see, for example, [8]). Thus,

$$\frac{\partial}{\partial t} p^\Lambda(t, u, v) = (H_0 + \varepsilon H_1) p^\Lambda(t, u, v), \tag{29}$$

where $H_0 = \sum_{z \in \Lambda} b_0(u_z) \frac{\partial}{\partial u_z} + \frac{1}{2} \sum_{z \in \Lambda} \frac{\partial^2}{\partial u_z^2}$, $H_1 = \sum_{z \in \Lambda} b_1(u_{z+D}) \frac{\partial}{\partial u_z}$. The density of the "unperturbed" process, satisfying the equation ($\varepsilon = 0$)

$$\frac{\partial}{\partial t} p^\Lambda(t, u, v) = H_0 p^\Lambda(t, u, v),$$

will be denoted by p_0^Λ. Since each solution of the equation

$$p^\Lambda = p_0^\Lambda + \varepsilon \Phi p^\Lambda, \tag{30}$$

where $(\Phi p^\Lambda)(t, u, v) = \int_0^t \int_{\mathbf{R}^{|\Lambda|}} p_0^\Lambda(t-s, u, u^1) H_1 p^\Lambda(s, u^1, v)\, du^1 ds$, is a solution of equation (29), it follows that the solution of (30) is unique (see, for example, [9]). It is clear that in the case of the convergence of the series

$$\sum_{k=0}^\infty \varepsilon^k \Phi^k p_0^\Lambda, \tag{31}$$

where $\Phi^0 p_0^\Lambda = p_0^\Lambda$, $\Phi^k p_0^\Lambda = \Phi\Phi^{k-1} p_0^\Lambda$ $(k = 1, 2, \ldots)$, this solution has the form

$$p^\Lambda = \sum_{k=0}^\infty \varepsilon^k \Phi^k p_0^\Lambda.$$

LEMMA 6. The series (31) converges absolutely for each ε, $|\Lambda| < \infty$ and uniformly with respect to $u, v \in \mathbf{R}^{|\Lambda|}$ for all $t_0 > 0$, $t \geqslant t_0$.

Proof. By definition,

$$\Phi^k p_0^\Lambda(t, u, v) = \sum_{\bar{z}=(z_1, \ldots, z_k) \in \Lambda^k} \int_0^{s_0} \cdots \int_0^{s_{k-1}} p_0^\Lambda(s_0 - s_1; u^0, u^1) \cdot \\ \cdot \prod_{i=1}^k \left[b_1(u_{z_i+D}^i) \frac{\partial}{\partial u_{z_i}^i} p_0^\Lambda(s_i - s_{i+1}, u^i, u^{i+1}) \right] du^1 \ldots du^k\, d\bar{s}, \tag{32}$$

where $s_0 \equiv t$, $s_{k+1} \equiv 0$, $u^0 \equiv u$, $u^{k+1} \equiv v$, $d\bar{s} \equiv ds_k \ldots ds_1$.

From (32) it is easy to obtain

$$|\Phi^k p_0^\Lambda(t, u, v)| \leqslant \sum_{\bar{z}\in\Lambda^k} \int_0^{s_0} \cdots \int_0^{s_{k-1}} \Big(\prod_{y\in\{z_1, \ldots, z_k\}} \int_{\mathbf{R}^k} p_0(t - s_1, u_y, u_y^1)\cdot$$
$$\cdot \Big\{\prod_{i:\, z_i = y} B_1 \Big|\frac{\partial}{\partial u_y^i} p_0(s_i - s_{i+1}, u_y^i, u_y^{i+1})\Big|\Big\}\Big\{\prod_{i:\, z_i\neq y} p_0(s_i - s_{i+1}, u_y^i, u_y^{i+1})\Big\}\cdot du_y^1 \ldots du_y^k\Big) d\bar{s}.$$

From here and from (25)-(27) we find

$$|\Phi^k p_0^\Lambda(t, u, v)| \leqslant \sum_{\bar{z}\in\Lambda^k} \int_0^t \cdots \int_0^{s_{k-1}} A_1 \sigma_3^{|\Lambda| - |\{z_1, \ldots, z_k\}|}\cdot$$
$$\cdot \Big\{\prod_{y\in\{z_1, \ldots, z_k\}} A e^{h|u_y|} \Big(\prod_{i:\, z_i = y} B_1 A F(s_i - s_{i+1}) A\Big) \frac{e^{-\frac{(u_y - v_y)^2}{2t}\sigma_1}}{\sqrt{t}}\Big\} d\bar{s}, \tag{33}$$

where $A_1 > 0$,

$$F(\tau) = \begin{cases} 1/\sqrt{\tau}, & \tau < T; \\ e^{-\alpha\tau}, & \tau \geqslant T; \end{cases}$$

therefore,

$$|\Phi^k p_0^\Lambda(t, u, v)| \leqslant |\Lambda|^k e^{khC} S^k \sigma_3^{|\Lambda|} \frac{t^{k/2}}{\left[\frac{k}{2}\right]!} \tag{34}$$

for some $S = S(t_0) > 0$, from where we obtain the assertion of the lemma. The lemma is proved.

Assume further that $X = \{x_1, \ldots, x_m\}$. We consider

$$p^{\Lambda, X}(t, u_\Lambda, v_X) = \int_{\mathbf{R}^{|\Lambda| - m}} p^\Lambda(t, u, v) \prod_{z\in\Lambda\setminus X} dv_z,$$
$$v_X \in \mathbf{R}^{|X|}, \quad u_\Lambda \in \mathbf{R}^{|\Lambda|}.$$

<u>LEMMA 7</u>. There exists

$$\lim_{\Lambda\uparrow \mathbf{Z}^\nu} p^{\Lambda, X}(t, u_\Lambda, v_X) \equiv p^X(t, u, v_X),$$
$$u = (u_z, z\in \mathbf{Z}^\nu), \quad |u_z| < C, \ z\in \mathbf{Z}^\nu. \tag{35}$$

<u>Proof</u>. By virtue of Lemma 6 we have

$$\int_{\mathbf{R}^{|\Lambda| - m}} p^\Lambda(t, u, v) \prod_{z\in\Lambda\setminus X} dv_z = \sum_{k=0}^\infty \varepsilon^k \int_{\mathbf{R}^{|\Lambda| - m}} \Phi^k p_0^\Lambda(t, u, v)\cdot$$
$$\cdot \prod_{z\in\Lambda\setminus X} dv_z = \sum_{k=0}^\infty \varepsilon^k \sum_{\bar{z}\in\Lambda^k} \int_0^t \cdots \int_0^{s_{k-1}} \varphi(\bar{s}, \bar{z}, k, u, v) \cdot \Big[\prod_{z\in\Lambda\setminus X} dv_z\Big] d\bar{s}, \tag{36}$$

where

$$\varphi(\bar{s}, \bar{z}, k, u, v) = \int_{\mathbf{R}^{|\Lambda|\cdot k}} p_0^\Lambda(t - s_1, u, u^1) \Big(\prod_{i=1}^k \Big[b_1(u^i_{z_i + D}) \frac{\partial}{\partial u^i_{z_i}} \cdot p_0^\Lambda(s_i - s_{i+1}, u^i, u^{i+1}\Big]\Big) du^1 \ldots du^k.$$

<u>Definition</u>. By a cluster of power k with vertex X we shall mean any vector $\bar{z} = (z_1, \ldots, z_k)$ such that $\bar{z} \in \mathbf{Z}^{\nu\cdot k}$ and

1) $z_k \in X$,

2) $z_i \in \{\bigcup_{j=i+1}^k \{z_j + D\}\} \cup X \quad (i = 1, \ldots, k-1)$.

Clearly, the summation in (36) is carried out in fact only over those $\bar{z}$ which are clusters, the number of which does not exceed M^k, M being a constant that depends on m, ν, N (see [10]).

Thus, taking into account (34), we obtain

$$\left|\int_{\mathbf{R}^{|\Lambda|-m}} \Phi^k p_0^{\Lambda}(t, u, v) \prod_{z\in\Lambda\setminus X} dv_z\right| \leqslant M^k S_1^k \sigma_3^m e^{hCk} \frac{t^{k-1/2}}{\left[\frac{k}{2}\right]!} \tag{37}$$

for some $S_1 > 0$, i.e., the series (36) converges uniformly with respect to $|\Lambda| > 0$. From here there follows the possibility of taking the limit $\lim_{|\Lambda|\to\infty}$ under the summation sign of the series (36); further, from the fact that

$$\int_{\mathbf{R}^{|\Lambda|-m}} \Phi^k p_0^{\Lambda}(t, u, v) \prod_{z\in\Lambda\setminus X} dv_z = \int_{\mathbf{R}^{|\tilde\Lambda|-m}} \Phi^k p_0^{\tilde\Lambda}(t, u, v) \prod_{z\in\tilde\Lambda\setminus X} dv_z, \tag{38}$$

for all sufficiently large $|\Lambda|$, $|\tilde\Lambda|$, we obtain the assertion of the lemma.

4. The Exponential Convergence of $p^X(t, u, v_X)$ for $t \to \infty$.

THEOREM 3. There exist $\varepsilon_0 > 0$, $\vartheta > 0$, such that for any finite subset X of Z^ν and any ε, $|\varepsilon| < \varepsilon_0$, there exist $t_0 > 0$, $\Theta = \Theta(|X|) > 0$, such that

$$|p^X(t + t', u, v_X) - p^X(t, u, v_X)| \leqslant \Theta e^{-\vartheta t},$$

where $p^X(t, u, v_X)$ is the density of the finite-dimensional distribution of the process ξ_z^t, $z \in X$, satisfying the system of equations (1) with initial conditions $\xi_z^0 = u_z$, $|u_z| < C$, $z \in \mathbf{Z}^\nu$.

Proof. By virtue of the absolute convergence of the series (36) and the equality (38), we have

$$|p^X(t + t', u, v_X) - p^X(t, u, v_X)| \leqslant \left|\prod_{z\in X} p_0(t + t', u_z, v_z) - \prod_{z\in X} p_0(t, u_z, v_z)\right|$$
$$+ \sum_{k=1}^{\infty} \varepsilon^k \left|\int_{\mathbf{R}^{|\Lambda_k|-m}} (\Phi^k p_0^{\Lambda_k}(t + t', u, v) - \Phi^k p_0^{\Lambda_k}(t, u, v)) \prod_{z\in\Lambda_k\setminus X} dv_z\right|,$$

where $\Lambda_k = \bigcup_{\bar z} (\bigcup_{i=1}^k \{z_i + D\})$, while $\bar z$ is a cluster of power k with vertex X. Now we estimate the difference

$$\left|\int_{\mathbf{R}^{|\Lambda_k|-m}} (\Phi^k p_0^{\Lambda_k}(t + t', u, v) - \Phi^k p_0^{\Lambda_k}(t, u, v)) \prod_{z\in\Lambda_k\setminus X} dv_z\right| \leqslant \sum_{\bar z\text{-cluster}} (2\sigma_3)^{k-1} \int_{\mathbf{R}} \cdots \int_{\mathbf{R}} \left[\int_0^t \cdots \int_0^{s_{k-1}} \Big(\prod_{y\in\{z_1,\ldots,z_k\}\cup X}\right.$$
$$\cdot \Big(\int_{\mathbf{R}^k} |p_0(t + t' - s_1, u_y, u_y^1) - p_0(t - s_1, u_y, u_y^1)| \cdot \Big\{\prod_{i:\, z_i = y} B_1 \left|\frac{\partial}{\partial u_y^i} p_0(s_i - s_{i+1}, u_y^i, u_y^{i+1})\right|\Big\} \cdot$$
$$\cdot \Big\{\prod_{i:\, z_i \neq y} p_0(s_i - s_{i+1}, u_y^i, u_y^{i+1})\Big\} du_y^1 \ldots du_y^k\Big) d\bar s + \int_t^{t+t'} \cdots \int_0^{s_{k-1}} \Big[\prod_{y\in\Lambda_k} \int_{\mathbf{R}^{|\Lambda_k|-m}} p_0(t + t' - s_1, u_y, u_y^1)\cdot$$
$$\cdot \Big[\prod_{i:\, z_i = y} B_1 \left|\frac{\partial}{\partial u_y^i} p_0(s_i - s_{i+1}, u_y^i, u_y^{i+1})\right|\Big]$$
$$\left.\cdot \Big\{\prod_{i:\, z_i \neq y} p_0(s_i - s_{i+1}, u_y^i, u_y^{i+1})\Big\} du_y^k \ldots du_y^1\Big] d\bar s\right] \prod_{y\in\Lambda_k\setminus X} dv_y.$$

With the use of the previous estimates we obtain

$$\left|\int_{\mathbf{R}^{|\Lambda_k|-m}} (\Phi^k p_0^{\Lambda_k}(t + t', u, v) - \Phi^k p_0^{\Lambda_k}(t, u, v)) \prod_{z\in\Lambda_k\setminus X} dv_z\right|$$
$$\leqslant M^k e^{hCk} \sigma_3^m S_2^k \left\{\int_0^t \cdots \int_0^{s_{k-1}} \Big(\prod_{i=0}^k F(s_i - s_{i+1})\Big) d\bar s + \int_t^{t+t'} \cdots \int_0^{s_{k-1}} \Big(\prod_{i=1}^k F(s_i - s_{i+1})\Big) d\bar s\right\}, \tag{39}$$

where $S_2 > 0$ is some constant, which, in turn, does not exceed

$$M^k (\exp\{hCk\}) \sigma_3^m S_3^k \left(e^{-\vartheta_1 t} + \frac{(k+1)^{3/2}}{\sqrt{t}} T^{k/2}\right) \text{ for } k \geqslant \frac{t}{T}\quad S_3 > 0,\quad \vartheta_1 > 0,$$

and

$$M^k e^{hCk} \sigma_3^m S_3^k e^{-\vartheta_1 t} \text{ for } k < \frac{t}{T},$$

where T is defined in Lemma 4.

From here

$$|p^{\lambda}(t+t', u, v_X) - p^X(t, u, v_X)| \leqslant \Delta m\sigma_3^{m-1} e^{-\delta t} e^{hC} + \sigma_3^m A' e^{-\vartheta' t},$$
$$\Delta, A', \vartheta' > 0,$$

which concludes the proof of Theorem 3.

Proof of Theorem 1. We note that

$$p^X(t, v_X) = \mathsf{M}_{\bar{\eta}} p^{\mathbf{X}}(t, \xi^0, v_X).$$

Further, from the assertion of Lemma 1 we find that the constant h can be selected so that $0 < h < a$. Now it is easy to obtain the assertion of the theorem: for this it is sufficient to consider the system (1) with an arbitrary fixed initial condition $\bar{\eta}'$ and then to repeat the proof of Theorem 3, making use, instead of the estimates

$$\prod_{z=z_1, \ldots, z_k} e^{h|u_z|} \leqslant e^{hCk}$$

in the formulas (33), (34), and (37), the estimates

$$\mathsf{M}\left\{\prod_{z=z_1, \ldots, z_k} e^{h|u_z|}\right\} \leqslant \mathsf{M}\left\{\prod_{z=z_1, \ldots, z_k} e^{a|u_z|}\right\} \leqslant C^k.$$

The theorem is proved.

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