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Random walk in dynamic environment with mutual influence

C. Boldrighini*
Dipartimento di Matematica e Fisica, Università di Camerino, Camerino, Italy

I.A. Ignatyuk
Moscow State University, Moscow, Russia

V.A. Malyshev*†
Dipartimento di Matematica e Fisica, Università di Camerino, Camerino, Italy

A. Pellegrinotti*
Dipartimento di Matematica, Università 'La Sapienza', Rome, Italy

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We study a discrete time random walk in $\mathbb{Z}^\nu$ in a dynamic random environment, when the evolution of the environment depends on the random walk (mutual influence). We assume that the unperturbed environment evolves independently at each site, as an ergodic Markov chain, and that the interaction is strictly local. We prove that the central limit theorem for the position X_t of the random walk (particle) holds, whenever one of the following conditions is met: (i) the particle cancels the memory of the environment and the influence of the environment on the random walk is small; (ii) the exponential relaxation rate of the environment is large; (iii) the mutual interaction of the environment and the random walk is small. We also prove convergence of the distribution of the 'environment as seen from the particle'. Proofs are obtained by cluster expansion techniques.

random walk in random environment * mutual influence * central limit theorem

Introduction

The expression 'random walk in random environment' has more than one meaning. Let $X_t \in \mathbb{Z}^\nu$, $t = 0, 1, \ldots$, be the path of a particle performing a discrete-time random walk on the lattice $\mathbb{Z}^\nu$, and $\xi_t(x)$ the random variable describing the value of the environment at the space point x and at time t. (We assume for simplicity that

Correspondence to: Dr. C. Boldrighini, Dipartimento di Matematica e Fisica, Università di Camerino, Via Madonna delle Carceri, 62032 Camerino, Italy.
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† Permanent address: Moscow State University, 119808 Moscow, Russia.

$\xi_t(x) \in S$ where S is a finite set.) Different situations arise depending on the behavior of the environment and on the interaction between the particle and the environment.

The situation which has been mostly investigated is that of a static environment: $\xi_t(x) = \xi(x)$, i.e. the variables $\xi(x)$ are chosen at random, according to some distribution, and kept fixed. There are fairly general mathematical results only in the one-dimensional case: it was proved that under some natural assumptions the random walk can behave in a quite nonuniform way, and the central limit theorem for X_t can fail [10]. For $\nu \geq 2$ the central limit theorem can be proved when the influence of the environment on the particle is weak and some symmetry condition is imposed [1, 6]. There are by now several results for various types of symmetry conditions, but little is known for the general nonsymmetric case. The reader is referred to the comprehensive paper [5] and to the references quoted there.

The analysis of the random walk in dynamic (i.e. nonstatic) environments could begin from the case in which the environment is described by some stationary process $\xi_t(x)$ with good mixing properties in space and time. If the motion of the particle has no influence on the environment it should be possible to get a central limit theorem by proving some sort of mixing conditions for the increments $\eta_t \equiv X_t - X_{t-1}$, or some bounds for the corresponding semiinvariants [8], or by some other probabilistic technique. Up to now there are, as far as we know, only results for special situations [7]. (A proof for the special case when the processes $\xi_t(x)$ are identical and independent, for fast exponential relaxation of the environment, is a consequence of our results: see Remark 2.3.)

The most general situation is that of mutual influence, when the transition probabilities of the random walk depend on the environment, and the particle that performs the random walk induces in its turn changes in the transition probabilities of the environment. In general one can expect to get the central limit theorem for X_t (or the Wiener behavior) only if the interaction between particle and environment is in some sense small.

In the present paper we consider random walks in dynamic environments with mutual influence in a simple situation: we assume that the environment in absence of the particle relaxes to a stationary distribution exponentially fast, uniformly in space and time. Such a uniform exponential relaxation cannot hold, for instance, in the more physical situation of a particle in a lattice gas.

Our results are based on cluster expansion techniques, which give a good control over the situation by disentangling the recollision mechanism. The main aim of the present paper is to work out cluster expansion techniques for the investigation of random walks in dynamical random media, that might be generalized in various directions.

Different techniques for special cases of random walks in dynamic environments are considered in the papers [2, 4].

Our main result is expressed for brevity in the form of a central limit theorem for X_t. Other important properties, such as the existence of a stationary process for the environment 'as seen from the particle's point of view', can be derived by

applying the cluster expansion techniques developed in the proofs. This is discussed in the concluding remarks.

1. Definitions and formulation of the results

We denote the random walk by $X_t \in \mathbb{Z}^\nu$, $t \in \mathbb{Z}_+$. The 'free' transition probability P for the reference random walk (independent of the environment) is subject to the natural conditions

(pi) $P(x, x') = P(x' - x)$;

(pii) $P(y) = 0$ if $|y| > d$, for some $d > 0$;

(piii) P is nondegenerate, i.e. $P(y) < 1$ for all $y \in \mathbb{Z}^\nu$.

The environment at the space site $x \in \mathbb{Z}^\nu$ and at time $t \in \mathbb{Z}_+$ is described by a variable $\xi_t(x) \in S$, where S is a finite set. The whole environment at time t is denoted by $\xi_t \equiv \{\xi_t(x), x \in \mathbb{Z}^\nu\}$. The free (in absence of the particle) evolution of the environment is described by a process $\xi_\cdot \equiv \{\xi_\cdot(x), x \in \mathbb{Z}^\nu\}$ with the following properties:

(ξi) $\xi_\cdot(x)$ is a stationary ergodic Markov chain with transition probabilities $p(s, s')$, $s, s' \in S$ independent of $x \in \mathbb{Z}^\nu$;

(ξii) the Markov chains $\xi_\cdot(x)$, $x \in \mathbb{Z}^\nu$ are independent.

We denote by $\pi(s)$, $s \in S$ the (unique) stationary distribution of the Markov chains $\xi_\cdot(x)$. By assumption (ξi) there are constants c, $\gamma > 0$ such that

$$r(t) \equiv \max_{s,s'} |\pi(s) - p^t(s', s)| \leq c\, \mathrm{e}^{-\gamma t}, \quad t \in \mathbb{Z}_+, \tag{1.1}$$

where p^t denotes the tth iteration of the matrix p.

The random walk in random environment with mutual influence that we want to study is the Markov process $\zeta_t \equiv (X_t, \xi_t)$ with space state $\mathbb{Z}^\nu \times S^{\mathbb{Z}^\nu}$, defined by the following conditions:

(i) (*Conditional independence.*) For fixed ζ_t the random variables X_{t+1}, $\xi_{t+1}(x)$, $x \in \mathbb{Z}^\nu$, are mutually independent;

(ii) $\mathbb{P}(X_{t+1} = x' | X_t = x, \xi_t) = P(x' - x) + \delta c(x' - x, \xi_t(x))$, for some $\delta > 0$;

(iii) $\mathbb{P}(\xi_{t+1}(x) = s' | X_t = x', \xi_t) = (1 - \delta_{xx'})p(s, s') + \delta_{xx'}\tilde{p}(s, s')$, where $\delta_{x,x'}$ is the Kronecker δ-function, $s = \xi_t(x)$, and $\tilde{p}$ is any stochastic matrix;

(iv) (*Initial condition.*) $X_0 = 0$, and at time $t = 0$ the environment is distributed according to some initial distribution p_0 on $S^{\mathbb{Z}^\nu}$.

The function c is subject to the following conditions:

(ci) $P(\cdot) + \delta c(\cdot, s)$ is a probability distribution for any $s \in S$, which implies that for any $s \in S$, $y \in \mathbb{Z}^\nu$, $P(y) + \delta c(y, s) \in [0, 1]$ and $\sum_y c(y, s) = 0$;

(cii) $c(y, s) = 0$ if $|y| > d$, $s \in S$;

(ciii) $\sum_y \max_s |c(y, s)| = 1$;

(civ) $\sum_s \pi(s) c(y, s) = 0$, $y \in \mathbb{Z}^\nu$.

Condition (ciii) is an obvious normalization condition. As for condition (civ) it can always be enforced by redefining the free random walk transition probabilities

as $P(y) + \delta \sum_s \pi(s)c(y, s)$. This would just change the drift of the reference random walk. It can be regarded as a sort of normalization condition: if the distribution of the environment is the stationary one and the particle does not act on the environment ($\tilde{p} = p$) then the (average) transition probabilities of the random walk are equal to the free ones.

We impose an additional condition to simplify the proofs:

(a) $P(0) = c(0, s) = 0$, $s \in S$.

If we do not impose condition (a) then the 'effective rate' of relaxation of the environment is not controlled by the parameter γ alone, one should take into account the probability of the random walk to stay at the same place as an additional parameter. This would actually cause no real difficulty, only more complicated estimates.

It is convenient to single out a special case:

Case A. $\tilde{p}(s, s') = \tilde{p}(s')$, where $\tilde{p}$ is any probability distribution on S.

In Case A the particle performing the random walk 'cancels the memory' of the environment.

In the general case we set

$$e(s, s') \equiv \tilde{p}(s, s') - p(s, s'), \qquad \varepsilon \equiv \max_s \sum_{s'} |e(s, s')|. \tag{1.2}$$

$\varepsilon \in [0, 2]$ is the maximum of the variation distances between $\tilde{p}(s, \cdot)$ and $p(s, \cdot)$, and is a measure of the influence of the particle on the environment.

It is to be expected that in general the central limit theorem for X_t will hold only if the influence of the environment on the random walk and the inverse influence of the random walk on the environment are weak (small δ and small ε), or the speed of relaxation of the environment is large (large γ). There are actually two additional relevant parameters. One of them, the probability of the random walk to stay at its place, which changes the effective relaxation rate, was removed by condition (a). The other one is the constant c appearing in (1.1), which will be assumed to be fixed: the role that it plays is clear from the estimates.

We shall use the following notation. The jump of the random walk at time $t \in \mathbb{Z}_+$ is denoted by $\eta_t = X_{t+1} - X_t$, and its expected value and dispersion are denoted by $m_t = \mathbb{E}\eta_t$ and $\sigma_t^2 = \mathbb{E}(\eta_t - m_t)^2$, respectively. $\mathbb{P}$ will always denote the probability distribution induced by the process $\zeta_\cdot$.

Our main results can be formulated as follows. If the process $\zeta_\cdot$ satisfies the conditions listed above, then

(I) in Case A for any fixed $\gamma > 0$ and δ small enough, or for any fixed $\delta \geq 0$ compatible with condition (ci) and γ large enough,

(II) in the general case for any fixed γ and δ, and ε small enough, or for any fixed values of $\delta \geq 0$ compatible with condition (ci) and of $\varepsilon \in [0, 2]$, and γ large enough,

the following statements hold:

Theorem 1.1. *There are constants* c_1, $c_2 > 0$ *and* $\beta \in (0, 1)$ *such that for all* $t \in \mathbb{Z}_+$,

$$|m_{t+1} - m_t| \leqslant c_1 \beta^t, \tag{1.3a}$$

$$|\sigma_{t+1} - \sigma_t| \leqslant c_2 \beta^t. \tag{1.3b}$$

Theorem 1.2. *There are absolute constants* $c > 0$, *and* $\beta \in (0, 1)$, *such that for any choice of* T, H, $n \in \mathbb{Z}_+$, *of the finite sequences of times* $t_1, t_2, \ldots, t_n \in \mathbb{Z}_+$ *such that* $t_n > t_{n-1} > \cdots > t_1 > H$, *and sites* $y_1, \ldots, y_n \in \mathbb{Z}^\nu$, *and of the condition* $\zeta_T = \hat{\zeta} = (\hat{x}, \hat{\xi})$, *the following inequality holds*:

$$\begin{aligned} &|\mathbb{P}(\{\eta_{t_1+T} = y_1, \eta_{t_2+T} = y_2, \ldots, \eta_{t_n+T} = y_n\} \,|\, \zeta_T = \hat{\zeta}) \\ &\quad - \mathbb{P}(\{\eta_{t_1+T} = y_1, \eta_{t_2+T} = y_2, \ldots, \eta_{t_n+T} = y_n\})| \leqslant c\beta^H. \end{aligned} \tag{1.4}$$

Theorem 1.2 states that for the process $\{\eta_t\}_{t=1}^{\infty}$ the strong mixing (or ψ-mixing) property holds, with exponential decay of the coefficient.

Theorem 1.3. *The central limit theorem holds for the process* X_t.

The proof of these results in the general case for any γ and ε fixed and δ small enough cannot be obtained by a straightforward application of the methods of the present paper. (See however Remark 2.4 below.)

2. Proofs

The first part of the present paragraph contains some preliminary notions and results. To make the exposition more transparent it is divided into 5 points. Point 2 contains an easy proof of Theorem 1.1 for large γ, which can be regarded as a guideline to the proof in the general case.

In what follows, if no confusion can arise, we write for brevity ξ in the arguments of functions depending on the environment.

2.1. Graphs and clusters

The main technical tool in proving the cluster expansion estimates that we need is the representation of probabilities as sums of contributions associated to graphs in $\mathbb{Z}^\nu \times \mathbb{Z}_+$. A trajectory of the random walk up to time T,

$$\Gamma = \{(0, 0), (1, x_1), \ldots, (T, x_T)\}$$

is represented by a graph with vertices at $z_i = (i, x_i)$, $i = 0, \ldots, T$, and bonds $b_i = (z_i, z_{i+1})$. The path Γ can be understood as a sequence of bonds $\{b_i\}_{i=0}^{T-1}$, and its time length is denoted by $|\Gamma|$: in this case $|\Gamma| = T$. To take account of the evolution

of the environment we associate to each $b_i \in \Gamma$, $i > 1$, a bond parallel to the time axis (as the environment evolves independently at each site), having as upper (in time) vertex z_i, and as lower vertex the one corresponding to the last visit at the same space site x_i, before time i, along the path Γ of the random walk. If x_i was never visited before time i the lower vertex is $(0, x_i)$. The bonds $b \in \Gamma$ (random walk transitions) will be called 'particle bonds' (or 'p-bonds'), the other ones 'environment bonds' (or 'e-bonds'). In what follows vertices will be always denoted by the letter z with different affixes and suffixes.

All bonds can be understood as vectors on $\mathbb{Z}^\nu \times \mathbb{Z}_+$, pointing 'upwards', i.e. with positive time component, and having as 'application point' the 'lower' (in time) vertex. We denote the lower vertex of a bond b by $z(b) = (t(b), x(b))$, and the 'upper' vertex by $\hat{z}(b) = (\hat{t}(b), \hat{x}(b)) \equiv z(b) + b$. $|b|$ will denote its (time) 'length', i.e. the absolute value of the time difference of its vertices (for e-bonds it is equal to the total length). The shorthand notation $\xi_b \equiv \xi_{t(b)}(x(b))$, $\hat{\xi}_b \equiv \xi_{\hat{t}(b)}(\hat{x}(b))$, for the values of the environment at both ends of a bond b, will also be used. For p-bonds $\eta(b) \in \mathbb{Z}^\nu$ will denote the jump, i.e. the difference of the space components of the upper and lower vertices of b.

The collection of p-bonds and e-bonds makes up a graph G_Γ (Figure 1). The probability of the path Γ for a fixed initial configuration of the environment ξ_0, $p(\Gamma | \xi_0)$, is associated to G_Γ in the following way. To each p-bond $b \in G_\Gamma$ we associate a random walk transition probability $P(\eta(b)) + \delta c(\eta(b), \xi_b)$, and to each e-bond $a \in G_\Gamma$ a transition probability of the environment $q(a; \xi) \equiv q(a; \xi_a, \hat{\xi}_a)$,

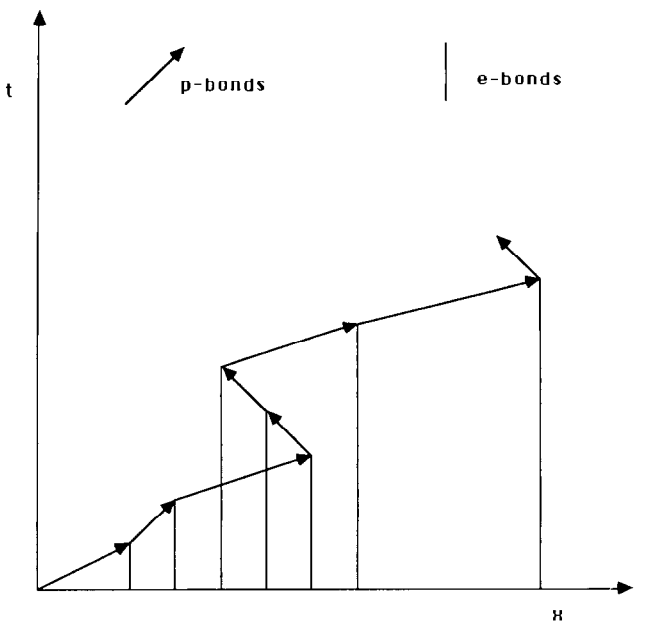

Fig. 1.

given by

$$q(a; s, s') = \begin{cases} p^{|a|}(s, s') & \text{if } t(a) = 0,\ x(a) \neq 0, \\ \sum_{s_1} \tilde{p}(s, s_1) p^{|a|-1}(s_1, s') & \text{otherwise.} \end{cases} \tag{2.1}$$

There is a one-to-one correspondence between e-bonds a and p-bonds b, since the lower vertex of a p-bond is the upper vertex of an e-bond (except for b_0, which has no e-bond: we can agree that it has a degenerate e-bond, corresponding to the factor 1). Denoting by $a(b)$ the e-bond associated to the p-bond $b \in \Gamma$, the conditional probability $p(\Gamma \mid \xi_0)$ of the path Γ for a fixed choice of the initial environment ξ_0 can be written as

$$p(\Gamma \mid \xi_0) = \sum_{\xi}^{*} \prod_{b \in \Gamma} (P(\eta(b)) + \delta c(\eta(b), \xi_b)) q(a(b); \xi), \tag{2.2}$$

where $\sum_{\xi}^{*}$ means, here and in the following, that summation goes over all values of the environment $\xi_t(x)$ on which the expression that is summed depends, keeping $\xi_0 = \{\xi_0(x), x \in \mathbb{Z}^\nu\}$ fixed. The contribution of an e-bond is decomposed as

$$q(a; s, s') = \alpha(a; s, s') + \pi(s'), \tag{2.3a}$$

$$\alpha(a; s, s') = \begin{cases} p^{|a|}(s, s') - \pi(s') & \text{if } t(a) = 0,\ x(a) \neq 0, \\ \sum_{s_1} \tilde{p}(s, s_1)(p^{|a|-1}(s_1, s') - \pi(s')) & \text{otherwise.} \end{cases} \tag{2.3b}$$

(Here the notation $\alpha(a; \xi)$ is used in analogy with $q(a; \xi)$ above.)

By 'opening the brackets' of the factors $q = \alpha + \pi$ corresponding to e-bonds (except for the degenerate e-bond associated to b_0), we get a sum over the subsets $B \subset \Gamma \setminus b_0$ of the bonds $b \in \Gamma$ for which the factor $\alpha(a(b); \xi)$ appears:

$$p(\Gamma \mid \xi_0) = \sum_{B \subseteq \Gamma \setminus b_0} \hat{p}(\Gamma, B \mid \xi_0), \tag{2.4a}$$

$$\begin{aligned} \hat{p}(\Gamma, B \mid \xi_0) = {} & (P(\eta(b_0)) + \delta c(\eta(b_0), \xi_0(0))) \\ & \times \sum_{\xi}^{*} \prod_{b \in B} (P(\eta(b)) + \delta c(\eta(b), \xi_b)) \alpha(a(b); \xi) \\ & \times \prod_{b \in \Gamma \setminus B} (P(\eta(b)) + \delta c(\eta(b), \xi_b)) \pi(\xi_b). \end{aligned} \tag{2.4b}$$

For fixed Γ and B the e-bonds for which the factor α appears will be called α-bonds.

2.2. *Proof of Theorem 1 for large γ*

Let T, Γ, $|\Gamma| = T$ and B be fixed. We say that the time t, $0 < t < T$, is separating for (Γ, B) if in expression (2.4b) there is no α-bond a such that $t(a) < t \leq \hat{t}(a)$.

Consider the quantities

$$Q_0(t, x \mid \xi_0) = \sum_{(t,x)}^{0} \hat{p}(\Gamma', B' \mid \xi_0)(x - X_{t-1}),$$

$$Q(t, x \mid \xi_0) = \sum_{(t,x)} \hat{p}(\Gamma', B' \mid \xi_0)(x - X_{t-1}).$$

Here $\sum^0_{(t,x)}$ denotes summation over all Γ', B' such that $|\Gamma'| = t$, $X_t = x$ (X_{t-1} is of course a function of Γ'), and (Γ', B') has no separating times, and $\sum_{(t,x)}$ denotes the same sum with the additional condition on (Γ', B') that there are no α-bonds a with lower vertex $(t(a), x(a))$ such that $t(a) = 0$, $x(a) \neq 0$. Clearly the quantity

$$Q_0(T|\xi_0) = \sum_{x\in\mathbb{Z}^\nu} Q(t, x|\xi_0)$$

is the contribution to $\mathbb{E}(\eta_{T-1}|\xi_0)$ of all pairs (Γ', B'), $|\Gamma'| = T$, with no separating times. Moreover, setting

$$Q(t, x|\pi) = \int d\Pi(\xi_0) Q(t, x|\xi_0), \qquad Q(t|\pi) = \sum_{x\in\mathbb{Z}^\nu} Q(t, x|\pi),$$

where Π denotes the environment equilibrium distribution (which can be formally written as $\Pi(\xi_0) = \prod_x \pi(\xi_0(x))$), it is not hard to see, by translation invariance of the transition probabilities, that the contribution to $\mathbb{E}(\eta_{T-1}|\xi_0)$ of all pairs (Γ', B'), $|\Gamma'| = T$, which have τ as maximal separating time is given by

$$\sum_{x,y\in\mathbb{Z}^\nu} \mathbb{P}(X_\tau = x|\xi_0) Q(T-\tau, y-x|\pi)$$
$$= \sum_{x\in\mathbb{Z}^\nu} \mathbb{P}(X_\tau = x|\xi_0) Q(T-\tau|\pi) = Q(T-\tau|\pi).$$

To see this note that for all α-bonds a with $t(a) > 0$ the lower vertex $(t(a), x(a))$ lies on the trajectory. Hence if τ is a maximal separating time and $X_\tau = x$ there is at most one α-bond a with $t(a) = \tau$, and for it $x(a) = x$. The process after the separating time τ depends only on the position $X_\tau = x$, and is independent of the distribution of the environment before time τ.

We find

$$\mathbb{E}(\eta_{T-1}|\xi_0) = Q(T|\xi_0) + \sum_{\tau=2}^{T-1} Q(T-\tau|\pi), \tag{2.5a}$$

$$\mathbb{E}(\eta_T|\xi_0) - \mathbb{E}(\eta_{T-1}|\xi_0) = Q(T+1|\xi_0) - Q(T|\xi_0) + Q(T|\pi). \tag{2.5b}$$

Now the number of different pairs (Γ, B) with $|\Gamma| = T$ is bounded by c^T, where c is a constant depending on ν and d. Moreover, by assumption (a) of Section 1, $P(0) = c(0, s) = 0$, hence all α-bonds have length not smaller than 2, except for the α-bond associated to b_1, which gives a contribution to $Q_0(t, x|\xi_0)$. Hence, by position (2.3b) and condition (1.1) for each pair (Γ, B) with $|\Gamma| = T$ and no separating times we have $\hat{p}(\Gamma, B|\xi_0) < (c' e^{-\gamma})^{T/2}$, where c' is some constant independent of γ. Hence for γ large enough the first assertion of the theorem is proved. The second assertion follows easily by changing $x - X_{T-1}$ for $(x - X_{T-1})^2$.

2.3. Clusters

By 'opening the brackets' of the factors $(P(\eta(b)) + \delta c(\eta(b), \xi_b))$ in equations (2.4a), (2.4b), $p(\Gamma|\xi_0)$ can be written as a double sum over the subset B and over the

subset A of the bonds $b \in \Gamma$ for which the factor $\delta c(\eta(b), \xi_b)$ appears:

$$p(\Gamma | \xi_0) = \sum_{A \subseteq \Gamma} \sum_{B \subseteq \Gamma \setminus b_0} p(\Gamma, A, B | \xi_0). \tag{2.6a}$$

$p(\Gamma, A, B | \xi_0)$ is given by

$$\begin{aligned} p(\Gamma, A, B | \xi_0) = \sum\nolimits_{\xi}^{*} & \prod_{b \in B^c \setminus b_0} \pi(\xi_b) \\ & \times \prod_{b \in A \cap B} \delta c(\eta(b), \xi_b) \alpha(a(b); \xi) \prod_{b \in A \cap B^c} \delta c(\eta(b), \xi_b) \\ & \times \prod_{b \in A^c \cap B} P(\eta(b)) \alpha(a(b); \xi) \prod_{b \in A^c \cap B^c} P(\eta(b)), \end{aligned} \tag{2.6b}$$

where A^c, B^c denote the complements of the sets A, B with respect to Γ.

We now fix Γ, A, B and consider the right-hand side of the expression (2.6b). At a given site $x \in \mathbb{Z}^\nu$ one can have more than one α-bond. They can be grouped into a certain number of maximal connected sequences. (A sequence of α-bonds $a_1, \ldots, a_k$ at the site x, labeled in order of increasing time, is connected if $t(a_{i+1}) = \hat{t}(a_i)$, $i = 1, \ldots, k-1$.) If $a_1, \ldots, a_k$ is a maximal connected sequence $a_k(a_1)$ is the 'final bond' ('initial bond') and $\hat{z}(a_k)(z(a_1))$ is the 'final vertex' ('initial vertex') of the sequence.

When summing over the environment ξ we note the following.

Remark 2.1. (i) If $b \in A \cap B$, then $z(b)$ can be either a final or an intermediate vertex of a maximal connected sequence of α-bonds.

(ii) If $b \in A \cap B^c$ we can assume that $z(b)$ is an initial vertex of a maximal connected sequence of α-bonds. In fact, if $z(b)$ is not the lower vertex of an α-bond, the only factor depending on ξ_b is $\pi(\xi_b) c(\eta(b), \xi_b)$, which gives 0 when summed over s', by condition (civ).

(iii) If $b \in A^c \cap B$ we can assume that $z(b)$ is an intermediate vertex of a maximal connected sequence of α-bonds. Otherwise the only factor depending on ξ_b would be $\alpha(a(b); \xi_{a(b)}, \xi_b)$, which gives 0 when summed over ξ_b, since $\alpha(a; s, \cdot)$ is the difference of two probability distributions.

(iv) If $b \in A^c \cap B^c$ then $z(b)$ can be the initial vertex of a maximal connected sequence of α-bonds. If it is not, then the only factor that is left is $P(\eta(b))$. This is obvious for $b = b_0$. If $b \neq b_0$, i.e. $t(b) > 0$, summing over ξ_b we get $P(\eta(b)) \sum \pi(\xi_b) = P(\eta(b))$.

If $z = z(a) = (t, x)$ is the initial vertex of a maximal connected sequence of α-bonds, then if $t = 0$, $x \neq 0$ the e-bond a gives the relaxation of the environment at x from the initial fixed value $\xi_0(x)$, whereas if $t > 0$, or $t = 0$, $x = 0$, a describes the relaxation after the perturbation induced by the particle at x, in which case z lies on the path Γ. We denote by $\underline{Z}$ the set of the initial vertices z of maximal connected sequences of α-bonds that belong to the path Γ. Note that the α-bonds are uniquely determined by $\underline{Z}$ and $Z_B \equiv \{z(b), b \in B\}$.

For a fixed choice of $0 < t_1 < \cdots < t_n$, $y_j \in \mathbb{Z}^\nu$, $j = 1, \ldots, n$ we have

$$\mathbb{P}(\{\eta_{t_1} = y_1, \ldots, \eta_{t_n} = y_n\} \mid \xi_0) = \sum_{\Gamma} p(\Gamma \mid \xi_0) \prod_{j=1}^{n} \chi(\eta_{t_j} = y_j), \tag{2.7}$$

and the sum can be limited to paths with time length $|\Gamma| = T \equiv t_n + 1$. We insert expressions (2.6a,b) into (2.7) and sum over all paths Γ, $|\Gamma| = T$ which are compatible with a given choice of A, Z_B, $\underline{Z}$, and of the jumps $\eta_{t_j} = y_j$. The sum is over all compatible values of $P(\eta(b))$, $b \in A^c$, excpet for b such that $t(b) = t_j$, $j = 1, \ldots, n$ (for which $\eta(b)$ is fixed). As a result we get factors corresponding to transition probabilities for the free random walk. They are the probabilities of paths that connect the bonds of A, go through the vertices of Z_B and $\underline{Z}$, avoid intersection of the α-bonds at other points, and have the prescribed jumps y_j at the times t_j, $j = 1, \ldots, n$. (We keep the notation Z_B even though we are summing over some $\eta(b)$, $b \in B$: Z_B denotes the set of the lower vertices of the bonds of B in all the terms $p(\Gamma, A, B \mid \xi_0)$ which give contribution to the sum.) In the graphical representation these paths are represented by bonds which connect either two subsequent (in time) vertices of $Z \equiv Z_B \cup \underline{Z}$ or a vertex $\hat{z}(b)$, $b \in A$ with the subsequent (in time) vertex of Z. We call these bonds 'p-bonds of type 0', since they refer to free transition probabilities, and denote the collection of such bonds by L. The bonds of A will be called 'p-bonds of type I'. Denoting by E the set of the α-bonds (which determines both $\underline{Z}$ and Z_B), we can write

$$\mathbb{P}(\{\eta_{t_1} = y_1, \ldots, \eta_{t_n} = y_n\} \mid \xi_0) = \sum_{A,E} W(A, E; y_1, \ldots, y_n), \tag{2.7a}$$

$$W(A, E; y_1, \ldots, y_n) = \sum_{\xi}^{*} \prod_{(t,x) \in Z \setminus (0,0)} \pi(\xi_t(x)) \times \prod_{c \in L} P(c) \prod_{a \in E} \alpha(a; \xi) \prod_{b \in A} \delta c(\eta(b), \xi_b), \tag{2.7b}$$

where $P(c)$ denotes the probability of the path represented by c.

$W(A, E; y_1, \ldots, y_n)$ is then described by a graph $G_{A,E;y_1,\ldots,y_n}$ with three types of bonds: (i) the p-bonds of type 0, $c \in L$, associated to the free transition probabilities $P(c)$; (ii) the p-bonds of type I, $b \in A$, corresponding to factors δc; (iii) the e-bonds of E corresponding to factors α (α-bonds). In addition there are factors π associated to the vertices of $\underline{Z} \setminus \{(0, 0)\}$.

In what follows we shall omit as a rule to write $y_1, \ldots, y_n$ in the notation.

For the topological classification of the graph $G_{A,E}$ we identify all vertices $(0, x)$ with $(0, 0)$. A 'subgraph' of $G_{A,E}$ is any collection of bonds of $G_{A,E}$ containing at least one p-bond.

In order to avoid confusion with the definition of clusters some preliminary remarks are in order. Note that we can still identify a trajectory in the graph $G_{A,E}$: the curve obtained by joining the p-bonds (of type 0 or I), which is connected. Hence it is clear what we mean by subsequent (in time) p-bonds. By 'disconnecting' two subsequent (in time) p-bonds we mean that the connection between the two

bonds at their common vertex z is removed. The graph which arises from the operation is ambiguous if z is also a vertex of α-bonds: we remove the ambiguity by prescribing that the connection at z between the p-bond which is higher in time (i.e. the one which has z as lower vertex) and the α-bonds is not removed. We define:

Definition 1. $G_{A,E}$ is said to be *simple* if it cannot be decomposed into two connected subgraphs just by disconnecting two subsequent p-bonds. (So, e.g., the graph in Figure 2 is not simple.)

Definition 2. A *cluster graph* of $G_{A,E}$ is any maximal simple subgraph of $G_{A,E}$.

In what follows there will be some abuse of language. We shall apply the term 'cluster' sometimes to cluster graphs, sometimes to equivalence classes of cluster

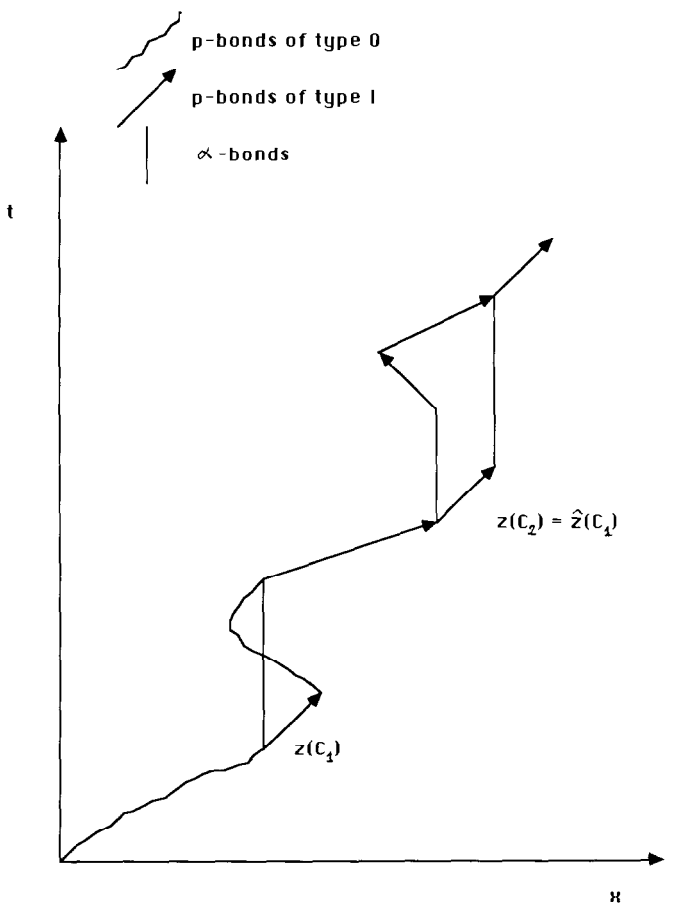

Fig. 2. A graph with two clusters.

graphs with respect to translations in $\mathbb{Z}^\nu \times \mathbb{Z}_+$. Moreover, although by our definition a cluster graph might consist of a single p-bond of type 0, in what follows we always mean by that cluster graphs which contain at least one p-bond of type I.

$G_{A,E}$ is decomposed into cluster subgraphs, and a cluster subgraph C is identified by subsets $A' \subset A$, $E' \subset E$ (or equivalently by $A' \subset A$, $Z_{B'} \subset Z_B$ and $\underline{Z}' \subset \underline{Z}$): $C = G_{A',E'}$. Figure 2 shows a graph with two cluster graphs.

The 'initial' and 'final' vertices of C (i.e. the ones with lowest and highest time coordinate), are denoted by $z(C) \equiv (\tau(C), x(C))$ and $\hat{z}(C) \equiv (\hat{\tau}(C), \hat{x}(C))$ respectively, and their coordinates are the 'initial' and 'final' times and positions. The 'height' of the cluster C is $h(C) \equiv \hat{\tau}(C) - \tau(C)$. The 'initial' bond b_C ('final' bond $\hat{b}_C$) of C is the p-bond with lower vertex $z(C)$ (with upper vertex $\hat{z}(C)$).

We label the cluster subgraphs of $G_{A,E}$ in order of increasing time: $\{C_i\}_{i=1}^N$. We denote by l_i, $i > 1$, the p-bonds of type 0 connecting the $(i-1)$th and the ith cluster: l_i is the path connecting $\hat{z}(C_{i-1})$ and $z(C_i)$ (if they do not coincide), subject only to the condition of having the prescribed jumps η_{t_j}, for $i > 1$. We denote by $P(l_i)$ the free random walk probability of l_i. If $\hat{z}(C_{i-1}) = z(C_i) l_i$ is by convention a 'degenerate' bond with $P(l_i) = 1$. l_1 is the degenerate bond at $(0, 0)$, $P(l_1) = 1$.

$W(A, E)$ can then be written as

$$W(A, E) = \prod_{i=1}^{N} P(l_i) W_{C_i}. \tag{2.8}$$

The contribution of each cluster $W_{C_i} = W(A_i, E_i)$ is computed using equation (2.7b), except of course for obvious changes: in particular the product $\prod_{c \in L} P(c)$ goes into $\prod_{c \in L_i} P(c)$, the product of the probabilities of the p-bonds of type 0 of C_i (which are prescribed to have the jumps η_k for $t_k \in [\tau(C_i), \hat{\tau}(C_i))$). An important remark is that if $\tau(C_i) > 0$, then by translation invariance W_{C_i} does not depend on the initial vertex $z(C_i) \in \underline{Z}$.

The important point with clusters is factorization: if we sum up expression (2.8) over all possible initial positions $x(C_i)$, which corresponds to summing over all admissible final positions of the bonds l_i, we get for the contribution to $\mathbb{P}(\{\eta_{t_1} = y_1, \ldots, \eta_{t_n} = y_n\} | \xi_0)$ of all graphs with clusters $C_1, \ldots, C_n$,

$$W(C_1, \ldots, C_n) \equiv \prod_{i=1}^{n} W_{C_i} \prod_{t_j \in \Delta} P(y_j) \tag{2.9}$$

where

$$\Delta = \bigcup_{i=1}^{n-1} [\hat{\tau}(C_{i-1}), \tau(C_i)) \cup [\hat{\tau}(C_n), T).$$

2.4. *Estimate of cluster contributions: Case A*

We estimate the contribution of all clusters graphs C with a given initial vertex $z(C) = z$ and height $h(C) = H + 1$, $H > 0$. Suppose first that $\tau(C) > 0$, and let A, Z_B, $\underline{Z}$ be the sets of bonds and vertices defining C. Going back to Remark 2.1(iii),

we see that in case A the set $A^c \cap B$ is empty. In fact if $b \in A^c \cap B$, $z(b) = (t, x)$ is the upper vertex of the α-bond $a(b)$ and, since the factor $\tilde{p}$ of the 'next' α-bond does not depend on $s' \equiv \xi_b$, the only term depending on s' is $\alpha(a(b); s, s')$ which gives 0 when summed over s'. This corresponds to the fact that when the particle goes through some site it cancels the memory of the previous history. Hence each α-bond can be associated to a bond of $B_1 \equiv B \cap A \subset A$. According to equation (2.7b) we can write

$$|W_C| \leq \delta^{|A|} \prod_{b \in A} F(b) \prod_{c \in L} P(c), \tag{2.10a}$$

where, setting $\hat{c}(y) = \max_s |c(y, s)|$, we have

$$F(b) = \begin{cases} \sum_s \pi(s)|c(\eta(b), s)| \leq \hat{c}(\eta(b)), & b \in A \setminus B_1, \\ \max_x \sum_{s_1} |\alpha(a(b); s, s_1)||c(\eta(b), s_1)| \leq 2r(|a(b)| - 1)\hat{c}(\eta(b)), & b \in B_1. \end{cases} \tag{2.10b}$$

(Here and in the following we use for any finite set A the notation $|A| = \text{card } A$.)

The estimate of the total contribution of all clusters C such that $|A| = n$, $|B_1| = m$, and the times of the bonds of A and B_1, $\mathcal{T}_A \equiv \{t(b), b \in A\}$, $\mathcal{T}_{B_1} \equiv \{t(b), b \in B_1\} \subset \mathcal{T}_A$, are specified, is obtained by summing expression (2.10a) over all transition probabilities $P(c)$ and $F(b)$ over all jumps $\eta(b)$. If $b \in A \setminus B_1$, then $z(b)$ is the initial vertex of a maximal connected sequence of α-bonds (Remark 2.1(ii)). Hence there are at least $n - m$ such sequences. If $n - m > 1$ any one of them must have a superposition in time with some other sequence, hence the total length of the α-bonds is not less than $H + (n - m - 1)$. Since moreover all α-bonds have length not smaller than 2, inequality (2.10b) shows that the total contribution for $\mathcal{T}_A$, $\mathcal{T}_{B_1}$ fixed is bounded by

$$\delta^n (2c)^m \, e^{-(\gamma/2)(n-m-1)} \, e^{-(\gamma/2)H}. \tag{2.11}$$

It is not hard to see that the estimate for the corresponding sum of clusters contributions for $\tau(C) = 0$, is the same, with the only change of the factor $e^{-(\gamma/2)(n-m-1)}$ which becomes $e^{-(\gamma/2)(n-m-2)}$, due to the fact that there may be one α-bond with length 1. The final estimate is now obtained by summing up the contributions for all possible choices of $\mathcal{T}_{B_1}$ for fixed $\mathcal{T}_A$ and for all possible choices of $\mathcal{T}_A$:

$$e^{-(\gamma/2)H} \sum_{n=0}^{H} \binom{H}{n} \sum_{m=0}^{n} \binom{n}{m} \delta^n (2c)^m \, e^{-(\gamma/2)(n-m-2)}.$$

The result is stated as a lemma:

Lemma 2.1. *In Case A for the total contribution of cluster graphs with a given initial vertex z and height $h(C) = H + 1$ the following estimate holds:*

$$\sum_{C: z(C) = z, h(C) = H+1} |W_C| \leq \bar{c} \beta^H, \qquad \beta = e^{-\gamma/2}(1 + \delta(2c + e^{-\gamma/2})), \tag{2.12}$$

where $\bar{c}$ is a constant independent of z and H. □

2.5. Estimate of cluster contributions: General case

By (1.2) we decompose $\tilde{p}$ as $\tilde{p} = p + e$, and substituting into formula (2.3b) the product over the α-bonds in equation (2.7b) can be written as

$$\prod_{a \in E} \alpha(a; \xi) = \sum_{E_1 \subseteq E} \prod_{a \in E_1} \alpha_\varepsilon(a; \xi) \prod_{a \in E \setminus E_1} \alpha'(a; \xi), \tag{2.13a}$$

$$\alpha_\varepsilon(a; s, s') = \sum_{s_1} e(s, s_1)(p^{|a|-1}(s_1, s') - \pi(s')),$$
$$\alpha'(a; s, s') = p^{|a|}(s, s') - \pi(s'). \tag{2.13b}$$

Clearly

$$\max_s \sum_{s'} |\alpha_\varepsilon(a; s, s')| \leqslant \varepsilon r(|a|-1), \qquad \max_s \sum_{s'} |\alpha'(a; s, s')| = r(|a|). \tag{2.13c}$$

Two types of α-bonds arise, the 'α_ε-bonds' of E_1, and the 'α'-bonds', of $E \setminus E_1$, and the contribution $W(A, E)$ of $G_{A,E}$ decomposes into a sum over the subsets E_1,

$$W(A, E) = \sum_{E_1 \subseteq E} W(A, E, E_1).$$

Let $Z_A \equiv \{z = z(b): b \in A\}$ be the set of the lower vertices of the bonds of A. When summing over the environment the following should be noted.

Remark 2.2. (i) If $z = (t, x) \in \mathbb{Z} \setminus Z_A$, $t > 0$ is the lower vertex of an α'-bond, i.e. $z = z(a)$, $a \in E \setminus E_1$, then the sum over $s = \xi_a$ gives a factor $\sum_s \pi(s)\alpha'(a; s, s')$, which is 0, since π is the stationary measure for the transition matrix p. Hence we can assume that the vertices of $\mathbb{Z} \setminus Z_A$ are lower vertices of α_ε-bonds.

(ii) If $z \in Z_B \setminus Z_A$, then by Remark 2.1(iii), z is an intermediate vertex of a maximal connected sequence of α-bonds: $z = z(a) = \hat{z}(a')$, $a, a' \in E$. If $a \in E \setminus E_1$ then the sum over $s = \xi_a$ gives by the composition rule

$$\sum_s \bar{\alpha}(a'; s', s)\alpha'(a; s, s'') = \bar{\alpha}(a''; s', s''),$$

where $\bar{\alpha}$ can be either α_ε ($a' \in E_1$) or α' ($a' \in E \setminus E_1$), and a'' is the e-bond obtained by 'joining' a' and a.

For a given choice of A, E, $E_1 \subseteq E$ we introduce a new set of α-bonds $\bar{E}$, obtained by 'glueing together' the α-bonds of E which have a common vertex $z \in \tilde{Z} \equiv \{z \in Z_B \setminus Z_A: z = z(a), a \in E \setminus E_1\}$, as described in Remark 2.2(ii). $\bar{E}_1 \subseteq \bar{E}$ will denote the set of the new α_ε-bonds. We sum up all contributions $W(A, E, E_1)$ for A, E, E_1 which differ only by the choice of $\tilde{Z}$ (and hence correspond to the same sets of new α-bonds $\bar{E}$, $\bar{E}_1 \subseteq \bar{E}$). Summation over the intermediate vertices of $\tilde{Z}$ gives rise to new p-bonds of type 0 (which have, as before, the prescribed jumps y_j at the times t_j). Note that by Remark 2.2, z can be a lower vertex of an α'-bond only if $z \in Z_A$.

The total contribution for a given choice of A, $\bar{E}$, $\bar{E}_1 \subseteq \bar{E}$, $\tilde{W}(A, \bar{E}, \bar{E}_1)$, is represented by a new graph, $G_{A,\bar{E},\bar{E}_1}$, with p-bonds of type 0; p-bonds of type I, and two types of α-bonds, the α_ε-bonds of $\bar{E}_1$, and the α'-bonds of $\bar{E}\setminus\bar{E}_1$. We extend the notion of simple and cluster graph to the new graphs by stating that $G_{A,\bar{E},\bar{E}_1}$ is simple or cluster whenever $G_{A,\bar{E}}$ (in the sense of the previous definition) is simple or cluster. A cluster subgraph is now identified by subsets $A' \subseteq A$, $\bar{E}' \subseteq \bar{E}$, $\bar{E}'_1 \subseteq \bar{E}_1$. In analogy with (2.8) we have $\tilde{W}(A, \bar{E}, \bar{E}_1) = \prod P(l_i)\tilde{W}_{C_i}$, where C_i are the cluster subgraphs, and the paths l_i are as before.

We now come to the estimate of the contribution of all cluster graphs with fixed initial vertex $z(C) = (t, x)$, $t > 0$, and height $h(C) = H+1$, $H > 0$. It is not hard to see (by (2.10b), (2.13b)) that the contribution of such a cluster graph G_{A,E,E_1} is bounded by

$$\delta^{|A|} \prod_{b\in A} \hat{c}(\eta(b)) \prod_{g\in L} P(g) c^{|E|}\varepsilon^{|E_1|} \prod_{a\in E_1} \mathrm{e}^{-\gamma(|a|-1)} \prod_{a\in E\setminus E_1} \mathrm{e}^{-\gamma|a|}. \tag{2.14a}$$

As before we have, by Remark 2.1, that the final bonds of the maximal connected sequences of α-bonds are in A. We denote them by $\hat{A}$, and by $Z_{\hat{A}} = \{z = z(a): a \in \hat{A}\}$ their lower vertices. We introduce furthermore the set of the lower vertices of the α_ε-bonds $Z^\varepsilon = \{z = z(a), a \in E_1\}$, and of the α'-bonds $Z' = \{z = z(a), a \in E\setminus E_1\}$. By Remark 2.2, $Z' \subseteq Z_A \setminus Z_{\hat{A}}$. $Z \equiv Z_{\hat{A}} \cup Z^\varepsilon \cup Z'$ is the set of all lower vertices of the bonds of the graph. We set $|A| = n$, $|\hat{A}| = m \leq n$, $|Z'| = |E\setminus E_1| = p \leq n-m$, $|Z^\varepsilon| = |E_1| = n-m-p+k$, where k is the number of the vertices of $Z\setminus Z_A$ (which by Remark 2.2 are all in Z^ε). Summing up, as before, all contributions (2.14a) of the graphs for which the times of all vertices of the various kinds are fixed (i.e. for fixed $\mathcal{T}_A$, $\mathcal{T}_{\hat{A}} = \{t = t(b): b \in \hat{A}\}$, $\mathcal{T}' = \{t = t(a): a \in E\setminus E_1\}$, $\mathcal{T}^\varepsilon = \{t = t(a): a \in E_1\}$), and taking into account that there are at least m maximal connected sequences of α-bonds (which must overlap in time if $m > 1$), we find

$$\delta^n c^{n-m+k} \varepsilon^{n-m-p+k} \mathrm{e}^{-\gamma p} \mathrm{e}^{-(\gamma/2)(m-1)} \mathrm{e}^{-(\gamma/2)H}. \tag{2.14b}$$

Multiplying by the number $\binom{n-m}{p}$ of the possible choices of $\mathcal{T}' \subseteq \mathcal{T}_A \setminus \mathcal{T}_{\hat{A}}$, and by the number $\binom{n}{m}$ of the possible choices of $\mathcal{T}_{\hat{A}} \subseteq \mathcal{T}_A$, and summing over p and m we find

$$(\delta^*)^n (c\varepsilon)^k \mathrm{e}^{-(\gamma/2)(H-1)}, \qquad \delta^* = \delta(c(\mathrm{e}^{-\gamma} + \varepsilon) + \mathrm{e}^{-(\gamma/2)}). \tag{2.14c}$$

This is again multiplied by the number $\binom{H-n}{k}$ of all possible choices of the times of the vertices of $Z\setminus Z_A$ and by the number $\binom{H}{n}$ of all possible choices of $\mathcal{T}_A$, and summed over k and n. For clusters with $\tau(C) = 0$ one has to apply the same modification as in the previous case.

Note that the sum of all contributions $W(A, E)$ for A, E corresponding to cluster graphs with a given initial vertex z and height H is the same as the sum of all $\tilde{W}(A, \bar{E}, \bar{E}_1)$ corresponding to cluster (in the new sense, with two kinds of α-bonds) with initial vertex z and height H. In fact the new graphs are obtained by joining some α-bonds of the previous graphs (as explained in Remark 2.2(ii)), without canceling any bond.

Hence we can state the result:

Lemma 2.2. *In the general case for the total contribution of cluster graphs with a given initial vertex z and height $h(C) = H + 1$ the following estimate holds:*

$$\sum_{C: z(C)=z, h(C)=H+1} |W_C| \leq \bar{c}\beta^H, \quad \beta = e^{-(\gamma/2)}[1 + c\varepsilon + \delta^*], \tag{2.15}$$

where $\bar{c}$ is a constant independent of z and H, and δ^* is given in formula (2.14c). □

Remark 2.3. The estimate for $\varepsilon = 0$ (no influence of the particle on the environment) is similar to the one for Case *A*. This is due to the fact that there is only one kind of α-bonds (there are no α_ε-bonds).

Remark 2.4. The result for the case when γ and ε are fixed and δ is small follows from Lemma 2.2 if $(1 + c\varepsilon)\, e^{-\gamma/2} < 1$. The restriction can probably be removed by a more detailed analysis of the graph contributions.

Proof of Theorem 1.1. We have

$$\mathbb{E}(\eta_t \mid \xi_0) = \sum_y y \mathbb{P}(\{\eta_t = y\} \mid \xi_0).$$

In the cluster expansion of $\mathbb{P}(\{\eta_t = y\} \mid \xi_0)$ only clusters with height $h(C) \leq t+1$ appear. Consider first the contribution of all *N*-cluster graphs $W(A, E)$ for which the last bond b_t is above the final cluster C_N, i.e. such that $\hat{\tau}(C_N) \leq t$. By (2.9),

$$W(C_1, \ldots, C_N) = P(y) \prod_{i=1}^{N} W_{C_i}.$$

Let $b = \hat{b}_{C_N}$ be the final bond of the cluster C_N. By Remark 2.1, $b \in A$, and hence it is associated to a factor $\delta c(\eta(b), \xi)$. $\eta(b)$ is 'free', i.e., as the upper vertex of *b*, $\hat{z}(b)$, cannot be the lower vertex of an α-bond, there is no term in W_{C_N} depending on $\eta(b)$, except $c(\eta(b), \xi)$. Summing over $\eta(b)$ (which corresponds to summing over a subclass of graphs C_N) we get 0, as $\sum_y c(y, s) = 0$. Hence only the term with $N = 0$ gives a contribution, and by (2.8) we find

$$\sum_{A,E: \hat{\tau}(C_n) \leq t} W(A, E) = P(y).$$

On the other hand if we sum up all contributions for which the clusters $C_1, \ldots, C_{N-2}$ and the last cluster C_N are fixed, and $\hat{\tau}(C_N) = t+1$, we get a sum over all possible compatible choices of the cluster C_{N-1}, which, by (2.9), gives

$$W_{C_N} \prod_{i=1}^{N-2} W_{C_i} \sum_{C_{N-1}} W_{C_{N-1}}.$$

(If $\tau(C_{N-1}) > 0$ the sum is meant as a sum over cluster graphs with a fixed initial vertex, i.e. on equivalence classes with respect to shifts of the initial vertex.) We

see again that this is 0 by summing over the jump of the last bond of C_{N-1}, $\eta(\hat{b}_{C_{N-1}})$. Hence the sum of the contributions $W(A, E)$ for which the last cluster $\bar{C}$ is fixed and $\hat{\tau}(\bar{C}) = t+1$ is simply $W_{\bar{C}}$. Therefore, with obvious notation

$$\mathbb{P}(\{\eta_t = y\}|\xi_0) = P(y) + \sum_{C: b_t \in C} W_C. \tag{2.16}$$

Since for $\tau(C) > 0$, W_C does not depend on the initial vertex, (2.16) implies that the contribution to the difference $\mathbb{P}(\{\eta_t = y\}|\xi_0) - \mathbb{P}(\{\eta_{t+1} = y\}|\xi_0)$ is bounded by the total contribution of the graphs with height $t+2$ and $t+1$. Hence by Lemma 2.1 and Lemma 2.2 we have

$$|\mathbb{P}(\{\eta_t = y\}|\xi_0) - \mathbb{P}\{\eta_{t+1} = y\}|\xi_0)|$$
$$\leqslant \sum_{C: h(C) = t+2} |W_C| + 2 \sum_{C: h(C) = t+1} |W_C| \leqslant \bar{c}(\beta^{t+1} + 2\beta^t).$$

For $\beta < 1$, since the sum over y is finite, this implies

$$|\mathbb{E}(\eta_{t+1}|\xi_0) - \mathbb{E}(\eta_t|\xi_0)| \leqslant c_1\beta^t, \quad |\mathbb{E}(\eta_{t+1}^2|\xi_0) - \mathbb{E}(\eta_t^2|\xi_0)| \leqslant c_2\beta^t, \tag{2.17}$$

for some constants c_1, $c_2 > 0$. The theorem is proved. □

Remark 2.5. As it follows from (2.16), by taking out the contribution of the graphs with at least one p-bond of type I (i.e. with $A \neq \emptyset$), we can write

$$m_t = \sum_y yP(y) + \delta m_t^{(1)}, \qquad \sigma_t^2 = \sum_y y^2 P(y) + \delta\sigma_t^{(1)},$$

where $m_t^{(1)}$, $\sigma_t^{(1)}$ are bounded quantities. Hence $m_t = \mathcal{O}(\delta)$ if $\sum_y yP(y) = 0$.

Proof of Theorem 1.2. Since the transition probabilities defining the Markov process ζ_t do not depend on time and space, we have

$$\mathbb{P}(\{\eta_{t_1+T} = y_1, \ldots, \eta_{t_n+T} = y_n\}|\zeta_T = \hat{\zeta})$$
$$= \mathbb{P}(\{\eta_{t_1} = y_1, \ldots, \eta_{t_n} = y_n\}|\xi_0), \tag{2.18}$$

where $\xi_0 = S_{\hat{x}}\hat{\xi}$, and $S_\cdot$ denotes the space translation: $S_{\hat{x}}(x) = \xi(x - \hat{x})$. By the cluster decomposition, $\mathbb{P}(\{\eta_{t_1} = y_1, \ldots, \eta_{t_n} = y_n\}|\xi_0)$ is written as a sum of N-cluster contribution of the form (2.9). Applying the same argument as above one sees that the contributions $W(A, E)$ for which the initial cluster C_1 is such that $\hat{\tau}(C_1) \leqslant H+1$ are absent. Similarly the cluster decomposition applied to $\mathbb{P}(\{\eta_{t_1+T} = y_1, \ldots, \eta_{t_n+T} = y_n\}|\xi_0^*)$, for any choice of the initial environment ξ_0^*, is given by the sum of the N-cluster contributions of the type (2.9) with the first cluster such that $\hat{\tau}(C_1) > T+H+1$. It is readily seen that for any N-cluster contribution $W(C_1, \ldots, C_N)$ to $\mathbb{P}(\{\eta_{t_1} = y_1, \ldots, \eta_{t_n} = y_n\}|\xi_0)$ with $\tau(C_1) > 0$, $\hat{\tau}(C_1) > H+1$ there is a corresponding contribution $W(C_1', \ldots, C_N')$ with $\tau(C_1') > T$, $\hat{\tau}(C_1') > T+H+1$ to $\mathbb{P}(\{\eta_{t_1+T} = y_1, \ldots, \eta_{t_n+T} = y_n\}|\xi_0^*)$ such that the C_i''s differ from the C_i's by a translation of the initial vertex. It is actually a one-to-one correspondence and the contributions

are independent of ξ_0^*, ξ_0 and equal. So the only contribution to the difference is given by the N-cluster contributions for which $h(C_1) \geq H+1$. If C_1 is such a cluster then it is easy to see that the contribution to $\mathbb{P}(\{\eta_{t_1+T} = y_1, \ldots, \eta_{t_n+T} = y_n\} | \xi_0^*)$ of all N-cluster graphs with initial cluster equal to C_1 is equal to W_{C_1} times the probability that some fixed jumps occur in the time interval $[\hat{\tau}(C_1), t_n+1)$. Hence, by Lemma 2.1 and Lemma 2.2, we have, for $\beta < 1$,

$$|\mathbb{P}(\{\eta_{t_1+T} = y_1, \ldots, \eta_{t_n+T} = y_n\} | \zeta_T = \hat{\zeta}) - \mathbb{P}(\{\eta_{t_1+T} = y_1, \ldots, \eta_{t_n+T} = y_n\})| \leq \bar{c}\beta^H.$$

The theorem is proved. □

The following Corollary is a simple consequence of the proof above.

Corollary 2.1. *The shifted process* $\eta_\cdot^{(T)}$: $\eta_k^{(T)} = \eta_{T+k}$, $k = 1, 2, \ldots$, *tends, as* $T \to \infty$, *to a limiting process* $\eta_\cdot^*$, *which is again a strong mixing process with exponentially decaying coefficient.*

Proof. In proving Theorem 1.2 we found that for any choice of the initial environment ξ_0 and of the sequences $k_1, \ldots, k_n \in \mathbb{Z}_+$, $y_1, \ldots, y_n \in \mathbb{Z}^\nu$, the probability

$$\mathbb{P}(\{\eta_{T+k_1} = y_1, \ldots, \eta_{T+k_n} = y_n\} | \xi_0)$$

is equal, except for a term of order $0(\beta^T)$, to a sum of terms r_k, corresponding to N-cluster contributions with $\tau(C_1) = T-k$, $\hat{\tau}(C_1) \geq T$ $(k < T)$, which are independent of T and ξ_0. Since $|r_k| <$ constant β^k the series $\sum_{k=1}^{\infty} r_k$ defines the limiting probability $\mathbb{P}(\{\eta_{t_1}^* = y_1, \ldots, \eta_{t_n}^* = y_n\})$, and it is easy to see that for some constant $\tilde{c}$

$$|\mathbb{P}(\{\eta_{T+k_1} = y_1, \ldots, \eta_{T+k_n} = y_n\} | \xi_0) - \mathbb{P}(\{\eta_{t_1}^* = y_1, \ldots, \eta_{t_n}^* = y_n\})| \leq \tilde{c}\beta^T. \tag{2.19}$$

The strong mixing property for the process $\eta_\cdot^*$ now follows from the strong mixing property of the process $\eta_\cdot$. □

Proof of Theorem 1.3. The proof could be deduced by Theorems 1.1 and 1.2, using, for instance, the semiinvariant method [8], or some other probabilistic technique. An immediate proof is provided by Corollary 2.1 above. In fact the limiting distributions, as $T \to \infty$, of the random variables

$$w_T = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \hat{\eta}_t, \quad w_T^* = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \hat{\eta}_t^*, \quad \hat{\eta}_t = \eta_t - \mathbb{E}\eta_t, \quad \hat{\eta}_t^* = \eta_t^* - \mathbb{E}\eta_t^*,$$

coincide. To see this one can, for instance, set

$$w_T = \frac{1}{\sqrt{T}} \sum_{t=1}^{M} \hat{\eta}_t + \frac{1}{\sqrt{T}} \sum_{t=M+1}^{T} \hat{\eta}_t \equiv w_T^0 + \tilde{w}_T,$$

where $M=[\sqrt{T}]$ ($[\cdot]$ denotes the integer part). w_T^0 tends to 0 in distribution, as $T\to\infty$. (This follows for instance, from Theorem 1.1.) If we denote by $\mathcal{M}$ the σ-algebra of the process $\eta=\{\eta_t\}_{t=1}^{\infty}$, clearly $\tilde{w}_T$ is measurable with respect to the sub-σ-algebra $\mathcal{M}_M\subset\mathcal{M}$ generated by the random variables $\{\eta_{M+k}\}_{k=1}^{\infty}$. For $A\in\mathcal{M}$ we denote by S_MA its translate: $S_MA=\{\eta: S_{-M}\eta\in A\}$ (where S_{-t} is the shift, $(S_{-t}\eta)_k=\eta_{t+k}$, $k=1,2,\ldots$). It follows from (2.19) that for any choice of the initial environment ξ_0,

$$|\mathbb{P}(S_MA\,|\,\xi_0)-P^*(A)|\leqslant\tilde{c}\beta^M,$$

where P^* denotes the probability associated to the process η^*. Hence the probability of the event $\{\tilde{w}_T<x\}$ is reduced, as $T\to\infty$ to the probability of the event $\{\tilde{w}_T^*<x\}$, where

$$\tilde{w}_T^*=\sum_{t=M+1}^{T}\eta_t^*,$$

which is asymptotically equal to the probability of the event $\{w_T^*<x\}$.

The result now follows by applying the classical theorems for stationary processes with the strong mixing property (see, for instance [3]). The proof of the invariance principle is also straightforward. □

3. Concluding remarks

We have seen (Corollary 2.1) that the process $\eta_k^{(T)}=\eta_{T+k}$, $k=1,\ldots$, tends, as $T\to\infty$, to a limiting process η^*. One could also consider the limiting behavior of the process $\zeta_\cdot$ 'as seen from the particle position', i.e. in a moving neighborhood around the particle position X_T for a large time T. This amounts to studying the limit as $T\to\infty$ of the finite dimensional distributions

$$\mathbb{P}(\{\eta_T=y_0,\ldots,\eta_{T+m}=y_m;\ \xi_T(X_T+x_1^{(0)})=s_1^{(0)},\ldots,\xi_T(X_T+x_{n_0}^{(0)})=s_{n_0}^{(0)};\ldots;$$
$$\xi_{T+m}(x_{T+m}+x_1^{(m)})=s_1^{(m)},\ldots,\xi_{T+m}(X_{T+m}+x_{n_m}^{(m)})=s_{n_m}^{(m)}\}). \tag{3.1}$$

The limiting behavior of the probabilities (3.1) can be studied by using a simple modification of the cluster technique introduced above: one should add new vertices at the points $(T+k,\ X_{T+k}+x_j^{(k)})$, and the corresponding e-bonds. Carrying out the estimates, which are essentially the same as before, except for more clumsy formulas, one can see that in the range of the parameters for which $\beta<1$, as $T\to\infty$ the probabilities (3.1) tend exponentially fast to limiting probabilities

$$Q(x_1^{(0)},\ldots,x_{n_0}^{(0)};\ldots;x_1^{(m)},\ldots,x_{n_m}^{(m)};s_1^{(0)},\ldots,s_{n_0}^{(0)};\ldots;s_1^{(m)},\ldots,s_{n_m}^{(m)}).$$

That is, there is a stationary distribution for the environment as seen from the particle.

Deeper results can be obtained by using a refined estimate of the semiinvariants. The semiinvariants [9] corresponding to the random variables η_{t_j}, $j=1,\ldots,n$,

$t_1 < t_2 < \cdots < t_n$, and to the vectors $k^{(j)} \in \mathbb{Z}^\nu$, $j = 1, \ldots, n$, with nonnegative components $k_i^{(j)} \in \mathbb{Z}_+$, are linear combinations of the expected values of all possible products of the factors $(\eta_{t_j})_i^{k_i^{(j)}}$, $i = 1, \ldots, \nu$, and are denoted by

$$\langle \eta_{t_1}^{k^{(1)}} \cdots \eta_{t_n}^{k^{(n)}} \rangle.$$

We claim that under the same conditions for which we proved Theorems 1, 2 above the following estimate holds for some constants $C > 0$ and $\beta \in (0, 1)$,

$$|\langle \eta_{t_1}^{k^{(1)}} \cdots \eta_{t_n}^{k^{(n)}} \rangle| \leq C^{|k^{(1)}| + \cdots + |k^{(n)}|} |k^{(1)}|! \cdots |k^{(n)}| ! \beta^{t_n - t_1}. \tag{3.2}$$

(Here $|k| = \sum_{i=1}^{\nu} k_i$.) For large γ (both in Case A and in the general case) the estimate (3.2) is straightforward since this case admits a regular cluster expansion [9].

The estimates (3.2) and similar ones for the environment are the main ingredients in investigating the spectrum of the system, in analogy to what is done for the transfer matrix of the Gibbs fields. In particular it is important to find the renormalized one-particle space and the spectrum in this space.

In conclusion we may say that the present paper is intended to be a first contribution to the study of random walks in dynamic environments with mutual influence. As a next step one could study the situation in which the environment has nonzero space correlations, which decay exponentially, as, for instance, in the case of Glauber dynamics in the high temperature region. This case can probably be treated in analogy to what is done in the present paper. A further more difficult step is the extension to the situation when the decay of the space correlations of the environment is slow (nonsummable).

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