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EQUIVALENCES OF C^*-DYNAMICAL SYSTEMS

V. A. MALYSHEV

Faculty of Mechanics and Mathematics, Moscow State University
Moscow, U.S.S.R.

This is an introduction and review of the first results concerning algebraic and unitary equivalence in the noncommutative ergodic theory of infinite particle quantum systems.

1. Introduction

DEFINITION. C^*-dynamical system is a pair $(\mathfrak{A}, \alpha_t)$ where $\mathfrak{A}$ is a C^*-algebra and α_t is a strongly continuous one–parameter group of $*$ – automorphisms of $\mathfrak{A}$. Two such systems $(\mathfrak{A}, \alpha_t)$ and $(\mathfrak{A}', \alpha_t')$ are called (algebraically) equivalent iff there is $*$ – isomorphism γ: $\mathfrak{A} \to \mathfrak{A}'$ such that $\gamma\alpha_t = \alpha_t' \gamma$.

One can easily obtain a large number of examples of equivalence between different C^*-dynamical systems. The simplest examples may be described as follows: let e^{iHt} and $e^{iH't}$ be unitary groups in Hilbert spaces $\mathscr{H}$ and $\mathscr{H}'$. If H and H' are unitarily equivalent then there exists a unitary W: $\mathscr{H} \to \mathscr{H}'$ such that $\gamma(B) = WBW^*$ establishes the algebraic equivalence between $(\mathscr{B}(\mathscr{H}), \alpha_t)$ and $(\mathscr{B}(\mathscr{H}'), \alpha_t')$ where $\mathscr{B}(\mathscr{H})$ is the C^*-algebra of all bounded linear operators in $\mathscr{H}$ and $\alpha_t(B) = e^{iHt}Be^{-iHt}$. We shall call this type of algebraic equivalence the *unitary equivalence*.

Surely enough such examples will have no concern with interacting infinite particle systems. The situation here is quite the same as in classical ergodic theory: many deep theories (which are in a sense finite-dimensional) and no approaches to the systems of statistical mechanics (except of course the free ones). So the famous "ergodic problem" in statistical mechanics from the mathematical point of view has no example to support the so called "ergodic hypothesis" in spite of some recent speculations.

There are two different classes of translation invariant equilibrium states in quantum statistical mechanics: ground states and KMS (temperature) states. One can consider the problem of unitary equivalence for the GNS

representations w.r.t. these states. For ground states, in many cases one can obtain spectral information about GNS Hamiltonian H_{GNS}, which heavily supports the hypothesis of asymptotic completeness: H_{GNS} is unitarily equivalent to the direct sum of free systems. It seems interesting to elaborate the classical ground state dynamics: this can be defined e.g. for oscillators in the sites of a crystall lattice if in the initial moment we excite a finite number of them.

Contrary to the case of quantum and classical translation invariant KMS states there no results in this direction. We note also that the related problem of convergence to the equilibrium can be treated by the same technical means (e.g. perturbation series) as the problem of deriving Boltzmann or Landau equation. For the latter more simple problem there is no mathematical proof even on the formal level, i.e. term-by-term in the perturbation theory series, see [10].

Here we consider only the non-translation invariant case. Heuristically this means that particles interact only if they are in some fixed domain Ω and move freely outside Ω. We note that on the formal level (term-by-term in perturbation theory) some of our results are contained (and in fact are among the most important results) in the classical monographs by Friedrichs [1], Ch. 3 § 9–15, and Hepp [2], Th. 2.8. Actually we have succeeded in the proof of convergence of these series, [3].

2. The main constructions

CAR-algebra. Let $\mathfrak{A} = \mathfrak{A}(l^2(Z^\nu))$ be the CAR-algebra over $l^2(Z^\nu)$, i.e. CAR-algebra generated by $a(f)$, $f \in l^2(Z^\nu)$, where the anticommutation relations are fullfilled

$$(1)\qquad \begin{aligned} a(f)\,a(g)+a(g)\,a(f) &= a^*(f)\,a^*(g)+a^*(g)\,a^*(f) = 0,\\ a^*(f)\,a(g)+a(g)\,a^*(f) &= (f, g)\,1 \end{aligned}$$

and we assume that (f, g) is antilinear in g.

Free dynamics. Let h be a selfadjoint operator in $l^2(Z^\nu)$. Then we define

$$(2)\qquad \alpha_t\big(a(f)\big) = a(e^{ith} f).$$

This uniquely defines a C^*-dynamical system $(\mathfrak{A}, \alpha_t)$. An equivalent definitions is

$$(3)\qquad \alpha_t\big(a(f)\big) = e^{itd\Gamma(h)}\, a(f)\, e^{itd\Gamma(h)}$$

where in the Fock representation

$$(4)\qquad d\Gamma(h) = H_0 = \sum c_{xy}\, a_x^*\, a_y, \qquad c_{xy} = (h\delta_x, \delta_y).$$

Further on we assume that $h = -\Delta+\mu 1$, Δ is the lattice laplacian.

Perturbed dynamics. Let $V = V^*$ be a finite sum of terms

$$a^*(f_1)\ldots a^*(f_m)\,a(g_1)\ldots a(g_n)$$

where f_i, g_j have finite support. Then α_t^V can be formally defined by

$$\alpha_t^V(A) = e^{itH} A e^{-itH}, \qquad H = H_0 + \varepsilon V$$

and non-formally by the series

$$(5) \quad \alpha_t^V(A) = \alpha_t(A) + \sum_{n=1}^{\infty} (i\varepsilon)^n \int_{0<t_1<\ldots<t_n<t} \!\!\ldots\int \big[\alpha_{t_1}(V), \big[\alpha_{t_2}(V), \ldots [\alpha_{t_n}(V), \alpha_t(A)]\ldots\big]\big]\, dt_1 \ldots dt_n$$

which can easiely be proved to converge for any *finite* t.

Möller morphisms. γ (direct) and $\hat{\gamma}$ (inverse) if they exist are defined by

$$\gamma(A) = s-\lim_{t\to\infty} \alpha_{-t}^V(\alpha_t(A)),$$

$$\hat{\gamma}(A) = s-\lim_{t\to\infty} \alpha_{-t}(\alpha_t^V(A)).$$

From the equation

$$\frac{d}{dt}(\hat{\gamma}_t(A)) = i\,[\alpha_{-t}(V), \hat{\gamma}_t(A)]$$

where $\hat{\gamma}_t(A) = \alpha_{-t}(\alpha_t^V(A))$ we get the series

$$(6) \quad \hat{\gamma}_t(A) = A + \sum_{n=1}^{\infty} (i\varepsilon)^n \int_{0<t_1<\ldots<t_n<t} \!\!\ldots\int [\alpha_{-t_n}(V), \ldots [\alpha_{-t_1}(V), A]\ldots]\, dt_1 \ldots dt_n.$$

The main convergence result

THEOREM. *If* $\nu \geqslant 3$, $m+n$ *is even,* ε *is sufficiently small then for any* A *of the form*

$$A = a^*(h_1)\ldots a^*(h_p)\,a(h_{p+1})\ldots a(h_{p+q})$$

where h_i *have finite support the series* (6) *for* $t = \infty$ *and the similar series for* $\gamma(A) = \gamma_\infty(A)$ *are absolutely convergent. Otherwise speaking* $\gamma(A)$ *and* $\hat{\gamma}(A)$ *are analytic in* ε *in a small neibourghood of* 0. *This statement is uniform in* h_i, *i.e. holds for* $|\varepsilon| < \varepsilon_0$ *where* ε_0 *does not depend on* h_i.

For a proof see [3].

3. Algebraic equivalence

From the main convergence result easily follows the existence of direct and inverse Möller morphysms. It is evident that

$$\gamma\hat{\gamma} = \hat{\gamma}\gamma = 1$$

and

$$\alpha_t^V \gamma = \gamma\alpha_t.$$

This gives the desired equivalence between $(\mathfrak{A}, \alpha_t)$ and $(\mathfrak{A}, \alpha_t^V)$.

For any C^*-algebra $\mathfrak{A}$ let us denote by $\mathfrak{A}^*$ the set of states on $\mathfrak{A}$ and for any morphism α: $\mathfrak{A}_1 \to \mathfrak{A}_2$ of C^*-algebras let us denote by α^*: $\mathfrak{A}_2^* \to \mathfrak{A}_1^*$ the dual morphism.

Ergodic and mixing properties

We can reduce the problem of convergence to equilibrium or different mixing properties for α_t^V to the similar problem for the free dynamics α_t. E.g. we have the following

COROLLARY. *If ω and ω' are states on $\mathfrak{A}$ then*

$$(\alpha_t^V)^* \omega \underset{t\to\infty}{\to} \omega'$$

iff

$$(\alpha_t)^* (\gamma^* \omega) \underset{t\to\infty}{\to} \gamma^* \omega'.$$

4. Unitary equivalence

Let ω_0 be the unique β-KMS state w.r.t. α_t and ω_V – the unique β-KMS state w.r.t. α_t^V (see [4]). Then by Theorem 2 of [5] $\omega_V(\gamma(A)) = \omega_0(A)$.

Let us denote by $(\mathscr{H}_{\omega_V}, \pi_{\omega_V}, \Omega_{\omega_V})$ the cyclic GNS representation w.r.t. ω_V and let $\exp(itH_{\omega_V})$ be the unitary group implemented by α_t^V in this representation. Similarly let $\mathscr{H}_{\omega_0}$, π_{ω_0}, Ω_{ω_0}, H_{ω_0} be the same objects for ω_0, α_t.

COROLLARY 1. *H_{ω_0} and H_{ω_V} are unitarily equivalent.*

For a proof see [3]. It is quite easy due to above remark. For the ground states the situation is more complicated.

DEFINITION. We say that V does *not polarize* the vacuum if for all monomials in V we have $m > 0$ (therefore $n > 0$).

The following Corollaries 2 and 3 were proved by Botvich and Aizenstadt in their Candidate Theses.

Let $H_0 = d\Gamma(h)$ be the free Hamiltonian in the Fock representation $(\mathscr{H}, \pi, \Omega)$ of $\mathfrak{A}$.

COROLLARY 2. *If V does not polarize the vacuum Ω then H_0 and $H_0 + V$ are unitarily equivalent in $\mathscr{H}$.*

Proof. We shall prove the existence of u and $\hat{u}$, the direct and inverse Möller morphisms in $\mathscr{H}$. E.g. if $A^* = a^*(f_1)\dots a^*(f_s)$ then (we write $H_0 + V$ instead of $H_0 + \varepsilon V$)

$$\begin{aligned} UA^*\Omega &= \lim_{t\to\infty} e^{-it(H_0+V)} e^{itH_0} A^*\Omega \\ &= \lim(\gamma_t A^*) e^{-itH(H_0+V)} e^{itH_0}\Omega \\ &= \lim(\gamma_t A^*)\Omega = (\gamma A^*)\Omega. \end{aligned}$$

COROLLARY 3. *If $\mu > 0$ (i.e. there is a mass gap) and V does polarize the vacuum then $H_0 + V$ is unitarily equivalent to $H_0 - \varepsilon_V 1$ for some ε_V.*

Proof. By standard perturbation theory of isolated eigenvalues ther is an eigenvector Ω_V of $H_0 + V$ that

$$(H_0 + V)\Omega_V = \varepsilon_V \Omega_V.$$

Any vector of the Fock space is cyclic w.r.t. $\mathfrak{A}$, so the Fock space is the closure of the linear span of the vectors

$$\hat{a}^*(f_1)\dots\hat{a}^*(f_n)\hat{a}(f_{n+1})\dots\hat{a}(f_{n+m})\Omega_V$$

where

$$\hat{a}^\#(f) = \gamma a^\#(f).$$

Let us prove that

$$\hat{a}(f)\Omega_V = 0.$$

In fact,

$$\alpha^V_{-t}\alpha_t a(f)\Omega_V \to \hat{a}(f)\Omega_V$$

but also

$$\|(\alpha^V_{-t}\alpha_t a(f))\Omega_V\| = \|e^{-it(H_0+V)} a(e^{ith}f)\Omega_V\| = \|a(e^{ith}f)\Omega_V\| \to 0$$

So the Fock space is spanned by $\hat{a}^*(f_1)\dots\hat{a}^*(f_n)\Omega_V$.

We define a linear operator U by

$$Ua^*(f_1)\dots a^*(f_n)\Omega = \hat{a}^*(f_1)\dots\hat{a}^*(f_n)\Omega_V.$$

U is unitary since $\hat{a}^{\#}(f)$ satisfy the same CAR relations as $a^{\#}(f)$. Then

$$\begin{aligned} & e^{it(H_0+V)} U a^*(f_1)\dots a^*(f_n)\Omega \\ &= e^{it(H_0+V)} \hat{a}^*(f_1)\dots \hat{a}^*(f_n)\Omega_V \\ &= e^{it(H_0+V)} \gamma(a^*(f_1)\dots a^*(f_n)) e^{-it(H_0+V)} e^{-it\varepsilon_V}\Omega_V \\ &= e^{-it\varepsilon_V} \gamma(e^{itH_0}(a^*(f_1)\dots a^*(f_n)) e^{-itH_0})\Omega_V \\ &= e^{-it\varepsilon_V} \hat{a}^*(e^{ith} f_1)\dots \hat{a}^*(e^{ith} f_n)\Omega_V \\ &= U e^{it(H_0-\varepsilon_V)} a^*(f_1)\dots a^*(f_n)\Omega \end{aligned}$$

or

$$e^{it(H_0+V)} U = U e^{it(H_0-\varepsilon_V)}.$$

One can give another characterization of U. Let us denote P_0 and P_V the orthogonal projections onto Ω and Ω_V respectively. Then by Theorem 2.5 of [2]

$$P_V = \lim_{t\to\infty} e^{-it(H_0+V)} e^{itH_0} P_0 e^{-itH_0} e^{it(H_0+V)} = \gamma P_0$$

Moreover, P_V is analytic in ε. In other words γ can be lifted up to be an $*$-automorphism of the weak closure $\mathfrak{A}'' = \mathscr{B}(\mathscr{H})$ of $\mathfrak{A}$. So γ is unitarily implementable. Moreover γA is analytic in ε for all A from a dense (in the weak topology) subset of $\mathscr{B}(\mathscr{H})$ or UF is analytic in ε for all F from a dense subset of $\mathscr{H}$.

The same question in the situation when there is no mass gap is open.

Remark. These results can be easily translated to the case when we take $L_2(R^\nu)$ instead of $l^2(Z^\nu)$ and V has sufficiently smooth kernel with bounded support (i.e. the case dealt with Hepp's book [2]).

5. Particle in a gas

Now, there are a lot of equivalence results for the case where instead of CAR algebra $\mathfrak{A}$ we consider the tensor product $\mathfrak{A}\otimes\mathfrak{M}_2$ where $\mathfrak{M}_2$ is the C^*-algebra of 2×2 matrices and $\otimes$ can be understood in the usual sense or in the sense of superalgebras. This may be called the spin interacting with the fermi-gas. The results will be published in [6] and elsewhere.

Now we describe the result by Domnenkov about the interaction in a bounded region of a Schrödinger particle on Z^ν with the free fermi-gas. We consider two C^*-dynamical systems $(\mathfrak{A}\otimes\mathrm{Com}, \alpha_t)$ and $(\mathfrak{A}\otimes\mathrm{Com}, \alpha_t^V)$ where Com is the C^*-algebra of compact operators in $l^2(Z^\nu)$. The free dynamics α_t is implemented by the Hamiltonian

$$H_0\otimes 1 + 1\otimes h$$

where H_0 acts in the Fock space $\mathscr{F}$ as $(l^2(Z^\nu))$ and $h = -\Delta$ acts in $l^2(Z^\nu)$. The dynamics α_t^V is implemented by

$$H_0 \otimes 1 + 1 \otimes h + V \otimes V'$$

where $V \in \mathfrak{A}$ is of the type considered earlier and $V' \in \mathrm{Com}$ is of the type $\sum_{i=1}^{m} (f_i, \cdot) f_i$ where $f_i \in l^2(Z^\nu)$ have finite support. In [7] the equivalence of these two C^*-dynamical systems is proved. It is rather interesting that, due to the presence of the compact tensor component, the proof of the corresponding convergence result is essentially simpler.

The above results can be applied to problems concerning finite open systems, see [11].

The proof of equivalence of some translation invariant systems (specifically a particle translation invariantly interacting with the ideal gas) is of ultimate importance. Together with infinite vacuum renormalization we have been mass renormalization. Up to now only analogues of direct Möller morphisms have been constructed (i.e. Haag–Ruelle scattering theory), see [8], [9]. Of interest is also the lattice fermi–gas with translation invariant interaction without vacuum polarization and mass renormalization, i.e. when $n, m \geqslant 2$.

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