Renormalization group convergence for small perturbations of Gaussian random fields with slowly decaying correlations

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(Received 30 May 1979; accepted for publication 6 March 1980)

We find that certain random fields obtained by perturbing Gaussian fields with a self-interaction potential have the same limit properties as do the random fields of statistical mechanics.

PACS numbers: 05.50. + q

INTRODUCTION

The limit theorems of the theory of probability have been applied many times to Gibbs random fields of various kinds.

The integral central limit theorem has been proved by many authors[1,2] and also the local one.[3–5] These theorems are very important because they describe the properties of Gibbs random fields outside the critical region. The critical behavior can be described by other kinds of limit theorems, that is by the existence of fixed points of the renormalization group.[6–10] In this paper we find both types of limit theorems. The first one (the integral theorem) is obtained in the case of a Gaussian random field with slowly decreasing correlations but integrable, perturbed with a small self-interaction potential; the second one is obtained in the case of a Gaussian random field with nonintegrable correlation perturbed by a potential of the same kind as before. In this case we obtain a Gaussian isotropical automodel random field with a correlation asymptotically equal to the correlation of the unperturbed Gaussian random field. As we know this is the first example of renormgroup convergence for non-Gaussian Gibbs translation invariant field to a nontrivial Gaussian field. We proceed now to exact statements.

1. FORMULATION OF RESULTS

Let σ_t be a translation invariant Gaussian random field on a lattice $\mathbb{Z}^\nu$, with

$$\langle\sigma_t\rangle_0 = 0\,,\quad \langle\sigma_t\,\sigma_{t'}\rangle_0 = \varphi(t-t')\,.$$

We denote μ_0 the corresponding probability measure on the space $\Omega = \mathbb{R}^{\mathbb{Z}^\nu}$ of configurations with Borel σ-albegra Σ; $\langle\cdot\rangle_0$ is the expectation w.r.t. μ_0. Let $u(x)$ be a real function on $\mathbb{R}$ bounded from below and such that

$$\langle u^2(\sigma_t)\rangle_0 < \infty\,.$$

Let us consider a new measure μ_Λ on Ω with density

$$d\mu_\Lambda/d\mu_0 = Z_\Lambda^{-1}\exp\Big(-\sum_{t\in\Lambda}\epsilon u(\sigma_t)\Big)$$

w.r.t. μ_0; Λ is a finite subset of $\mathbb{Z}^\nu$, $Z_\Lambda = \Big\langle e^{-\sum_{t\in\Lambda}\epsilon u(\sigma_t)}\Big\rangle_0$, $\langle\cdot\rangle_\Lambda$ is the expectation w.r.t. μ_Λ. We denote Σ_Λ the minimal σ-subalgebra of Σ such that any σ_t, $t\in\Lambda$, is measurable w.r.t. Σ_Λ. We shall write $F = F_\Lambda$ if F is Σ_Λ-measurable and $|\Lambda| < \infty$. We shall use the following:

Theorem 1.1: If

$$d = \sum_{t:0\neq t\in\mathbb{Z}^\nu} |\varphi(t)| < \varphi(0) < \infty\,, \tag{1.1}$$

then there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$ and any bounded F_Λ the following limits

$$\langle F_\Lambda\rangle \equiv \lim_{\Lambda\to\mathbb{Z}^\nu}\langle F_\Lambda\rangle_\Lambda$$

exist and have convergent cluster expansion defined in Sec. 2. The moments $\langle F_\Lambda\rangle$ uniquely define a translation invariant measure μ on Ω.

This theorem was proved in Ref. 11. We give independent proof for the sake of completeness. We use the methods of this proof also in the proof of the following theorems.

Theorem 1.2: Under the conditions of theorem 1.1 the central limit theorem holds for the random field σ_t with probability measure μ. We shall prove this theorem in Sec. 4. Let us put $t = (t^1,\ldots,t^\nu)\in\mathbb{Z}^\nu$ and

$$S_t^{(k)} = \sum_{t^i k < \tilde t^i < (t^i+1)k,\, i=1,\ldots,\nu} \sigma_{\tilde t}\,.$$

$\sigma_t^{(k)} = S_t^{(k)}/\sqrt{DS_t^{(k)}}$. (w.r.t. measure μ); $DS_t^{(k)} = \langle S_t^{(k)2}\rangle$
This transformation is the renormalization group transformation.[6–9] The easy generalization of Theorem 1.2 is the following.

Theorem 1.3: The random field $\sigma_t^{(k)}$ converges weakly (i.e., finite dimensional distributions converge) to the trivial Gaussian field ξ_t with $\langle\xi_t\rangle_0 = 0$, $\langle\xi_t\xi_{t'}\rangle_0 = \delta_{tt'}$. We want now to find conditions when the convergence is to a nontrivial Gaussian field.

Let us consider some Gaussian field with measure μ_0 such that

$$\langle\sigma_t\rangle_0 = 0,\quad \langle\sigma_t\sigma_{t'}\rangle_0 = \psi(t-t')\,.$$

We shall not suppose (1.1), but assume that

$$|t|^{\nu-2}\psi(t)\to C,\quad |t|\to\infty\,. \tag{1.2}$$

This field exists when $\nu\geqslant 3$. Let us define new random variables

$$\hat\sigma_t = \hat\sigma_t^{\,i,j,k} = [(\sigma_t - \sigma_{t+e_i}) - (\sigma_{t+e_j} - \sigma_{t+e_i+e_j})] - [(\sigma_{t+e_k} - \sigma_{t+e_k+e_i}) - (\sigma_{t+e_k+e_j} - \sigma_{t+e_i+e_j+e_k})]\,,$$

where $e_1,\ldots,e_\nu$ are the unit coordinate vectors and e_i, e_j, e_k are three different elements of this basis. Let us consider the new measure

$$\frac{d\mu_A}{d\mu_0} = \hat{Z}_A^{-1}\exp\Big(-\sum_{t\in A}\epsilon u(\hat{\sigma}_t)\Big)$$

$$\hat{Z}_A = \Big\langle \exp\Big(-\sum_{t\in A}\epsilon u(\hat{\sigma}_t)\Big)\Big\rangle_0 .$$

Assumption: We assume that μ_0 is such that $\varphi(t) = \langle \hat{\sigma}_0,\hat{\sigma}_t\rangle$ satisfies (1.1).

It is an easy matter to construct such examples. We can consider arbitrary Gaussian field σ_t satisfying (1.2) and make the transformation

$$\sigma_t \rightarrow \sigma'_{t'} = \sigma_{mt} . \tag{1.3}$$

We note that for all m

$$\sum_t |\langle \hat{\sigma}'_0,\hat{\sigma}'_t\rangle| < \infty , \tag{1.4}$$

and for sufficiently large m (1.1) is valid with

$$\varphi(t) = \langle \hat{\sigma}'_0,\hat{\sigma}'_t\rangle .$$

Theorem 1.4: For $\epsilon > 0$ sufficiently small μ_A tends weakly to some non-Gaussian measure μ. There is convergent cluster expansion in this case. The proof is quite similar to the proof of Theorem 1.1. Let σ_t be the non-Gaussian random field with measure μ constructed in Theorem 1.4. Now let $\nu = 3$.

Theorem 1.5: The sequence of random fields

$$\sigma_t^{(k)} = S_t^{(k)}/(DS_t^{(k)})^{1/2}$$

(w.r.t. measure μ) weakly converges when $k\rightarrow\infty$ to a Gaussian isotropic automodel vector field with correlation function $K(t,s)$ given by

$$K(t,s) = \text{const}\int_{A_t}\int_{A_s} du\, dv \frac{1}{|u-v|^{\nu-2}}, \tag{1.5}$$

where

$$A_t = \{u\in\mathbb{R}^\nu : u = t + u^{(0)}, u^{(0)} = (u_1^{(0)},\dots,u_\nu^{(0)}), 0\leqslant u_k^{(0)}\leqslant 1\} . \tag{1.6}$$

2. VACUUM CLUSTER EXPANSION

Let us denote

$$f(x) = \exp[(-\epsilon u(x))] , \quad f_T = \prod_{t\in T} f(\sigma_t) .$$

Then

$$Z_A = \Big\langle \prod_{t\in A} f(\sigma_t)\Big\rangle_0 = \langle f_A\rangle_0 .$$

and

$$Z_A\langle F_A\rangle_A = \langle F_A f_A\rangle_0 . \tag{2.1}$$

The right-hand side of (2.1) is the moment of $N+1$ random variables

$$F_A, f(\sigma_{t_1}),\dots,f(\sigma_{t_N}), \text{ where } A = \{t_1,\dots,t_N\} .$$

So one can expand this moment in semi-invariants and after resummation we get

$$\langle F_A f_A\rangle = \sum_{T_1,T_2,\dots,T_P} \langle F_A f_{T_1}\rangle_0^C \langle f_{T_2}\rangle_0^C \cdots \langle f_{T_P}\rangle_0^C = \sum_{T_1\subset A} \langle F_A f_{T_1}\rangle_0^C Z_{A-T_1} , \tag{2.2}$$

where T_1 can be empty and where, e.g., $\langle F_A f_{T_1}\rangle_0^C$ is the semi-invariant of T_1+1 random variables $F_A, f_t, t\in T_1$, w.r.t. measure μ_0. This gives the desirable expansion in finite volume A:

$$\langle F_A\rangle_A = \sum_{T\subset A} \langle F_A f_T\rangle_0^C g_T^{(A)} , \qquad g_T^{(A)} = Z_{A-T}/Z_A . \tag{2.3}$$

In order to prove convergence we use the following systems of equations for g_T. For any finite nonempty $A\subset \mathbf{Z}^\nu$ we fix some point $t_A\in A$ and expand $Z_{A-(A-t_A)}$ in semi-invariants

$$Z_{A-(A-t_A)} = \sum_{T_1,\dots,T_k} \langle f_{T_1}\rangle_0^C \cdots \langle f_{T_k}\rangle_0^C ,$$

where the summation is through all partitions $T_1\cup\cdots\cup T_k$ of $A-(A-t_A)$. We can assume that $t_A\in T_1$. After resummation over $T_2,\dots,T_k$ we get

$$Z_{A-(A-t_A)} = \sum_T \langle f_T\rangle_0^C Z_{A-(A\cup T)} , \tag{2.4}$$

where the summation is over all T such that $t_A\in T\subset A-(A-t_A)$.

Let us denote

$$K_T = \langle f_T\rangle_0^C .$$

Then $K_{\{t_A\}} \equiv K_0$ does not depend on t_A and from (2.4) we get

$$g_{A-\{t_A\}}^{(A)} = K_0 g_A^{(A)} + \sum_T{}' K_T g_{A\cup T}^{(A)} ,$$

where in Σ' the summation is over all T such that $|T|>1$, $t_A\in T\subset A-(-A-t_A)$. Alternatively we have a system of equations

$$g_A^{(A)} - K_0^{-1} g_{A-t_A}^{(A)} + K_0^{-1}\sum_T{}' K_T g_{A\cup T}^{(A)} = 0, \quad |A|>1,$$
$$g_A^{(A)} + K_0^{-1}\sum_T{}' K_T g_{A\cup T}^{(A)} = K_0^{-1}, \quad |A|=1 . \tag{2.5}$$

We shall consider also the limiting system of equations

$$g_A - K_0^{-1} g_{A-t_A} + K_0^{-1}\sum_T{}' K_T g_{A\cup T} = 0, \quad |A|>1,$$
$$g_A + K_0^{-1}\sum_T{}' K_T g_{A\cup T} = K_0^{-1}, \quad |A|=1 , \tag{2.6}$$

where in Σ' the summation is over all T such that $|T|>1$ and $t_A\in T\subset \mathbf{Z}^\nu - (A-\{t_A\})$.

We shall prove that (2.5) and (2.6) have unique solutions in the Banach space $\mathscr{B}_A$ (respectively $\mathscr{B}$) of functions $\psi = (\psi_A)$ on the set of all nonempty finite subsets $A\subset A$ (respectively $A\subset \mathbf{Z}^\nu$) with the norm

$$\|\psi\| = \sup_A [(K_0/2)^{|A|}|\psi_A|].$$

Let us consider the linear operator $L = E - R + K$ in, e.g., $\mathscr{B}$, where E is the identity, $R\psi = \psi'$, where $\psi' = (\psi'_A)$, and

$$\psi'_A = \begin{cases} K_0^{-1}\psi_{A-t_A}, & |A|\geqslant 2 \\ 0 & |A| = 1 . \end{cases}$$

It is easy to see that $\|R\| \leqslant 1/2$. The operator K transforms the vector (ψ_A) into (ψ'_A), where

$$\psi'_A = K_0^{-1} \sum_{T:t_A \in T \subset \mathbf{Z}^\nu - (A - t_A),\ |T| > 1} K_T \psi_{A \cup T}$$

Let us note that

$$0 < C_1 \leqslant K_0 = \langle e^{-\epsilon u(\sigma_t)} \rangle_0 \leqslant C_2 < \infty, \tag{2.7}$$

uniformly on $0 \leqslant \epsilon \leqslant \epsilon_0$. It follows that

$$\|K\| \leqslant K_0^{-1} \sum_{T:0 \in T \subset \mathbf{Z}^\nu |T| > 1} |K_T| |2/K_0|^{|T|-1}. \tag{2.8}$$

Lemma 2.1: Under the conditions of Theorem 1.1 for any $\delta > 0$ there exists $\epsilon_0 > 0$ such that for $0 < \epsilon \leqslant \epsilon_0$ one has $\|K\| < \delta$. This lemma will be proved below.

It follows that $L = E - R + K$ is invertible in $\mathscr{B}$ (and in $\mathscr{B}_A$). Thus there exists a unique solution of (2.6) [and (2.5)]. The solution $g_A^{(A)}$ of (2.5) tends to the solution g_A of (2.6) when $A \to \mathbf{Z}^\nu$. We shall not prove this fact here (see Ref. 11), as the proof is standard (see Ref. 12).

If $\|K\| \leqslant 1/4$ then $\|(E - R + K)^{-1}\| \leqslant 4$ and so

$$|g_A| \leqslant 2(K_0/2)^{-|A|}. \tag{2.9}$$

We shall prove below also

Lemma 2.2:

$$\sum_{T:|T| = n,\ T \subset \mathbf{Z}^\nu} |\langle F_A f_T \rangle_0^C| \leqslant C(F_A)(C\epsilon)^n, \tag{2.10}$$

where C does not depend on ϵ and $F_A, C(F_A)$ depends only on F_A.

It follows that the cluster expansion

$$\langle F_A \rangle = \sum_{T \subset A} \langle F_A f_T \rangle_0^C g_T \tag{2.11}$$

is absolutely convergent. Thus Theorem 1.1 is proved.

3. BOUNDS ON SEMI-INVARIANTS OF FUNCTIONS OF A GAUSSIAN RANDOM FIELD

Let us put

$$\hat{f}(x) = f(x) - 1 = \int_0^\epsilon [-u(x)] e^{-\epsilon' u(x)} d\epsilon' \tag{3.1}$$

and note that

$$\langle f(\sigma_{t_1}),\ldots,f(\sigma_{t_m}) \rangle_0^C \equiv \langle \hat{f}(\sigma_{t_1}),\ldots,\hat{f}(\sigma_{t_n}) \rangle_0^C. \tag{3.2}$$

We shall estimate the semi-invariants (3.2) under the conditions of Theorem 1.1. Let us put $t_1 = 0$ and we shall always assume that $t_1, t_2, \ldots, t_n$ are pairwise different. Let $\Sigma^{(n)}$ be the sum over all lexicografically ordered sequences $(t_2,\ldots,t_n)$, the components of which are pairwise different and $0 \neq t_i \in \mathbf{Z}^\nu$, i.e., over all subsets $T = \{t_2,\ldots,t_n\} \subset \mathbf{Z}^\nu - \{0\}$. Lemma 2.1 evidently follows from Lemma 3.1.

Lemma 3.1:

$$\Sigma^{(n)} |\langle f(\sigma_0), f(\sigma_{t_2}),\ldots,f(\sigma_{t_n}) \rangle_0^C| \leqslant (C\epsilon)^n. \tag{3.3}$$

We shall prove this lemma now. Let $h_n(\sigma_t)$ be the normed Hermite polynomials w.r.t. μ_0, i.e., $\langle h_n^2(\sigma_t) \rangle_0 = 1$; we can assume that $\varphi(0) = \langle \sigma_t^2 \rangle_0 = 1$ without loss of generality. Then $d < 1$.

Let us denote

$$:\sigma_t^{(n)}: = \sqrt{n!}\, h_n(\sigma_t)$$

the corresponding Wick polynomials.

We expand $\hat{f}(\sigma_t)$ in Hermite polynomials in $L_2(d\mu_0)$.

$$\hat{f}(\sigma_t) = \sum_{m=0}^\infty a_m h_m(\sigma_t) = \sum_{m=0}^\infty \frac{a_m}{\sqrt{m!}} :\sigma_t{}^m: .$$

Then

$$\sum_{m=0}^\infty (a_m)^2 = \int |\hat{f}|^2 d\mu_0 \leqslant (C\epsilon)^2,$$

and so

$$|a_m| \leqslant C\epsilon. \tag{3.4}$$

Then

$$\langle \hat{f}(\sigma_0), \hat{f}(\sigma_{t_2}),\ldots,\hat{f}(\sigma_{t_n}) \rangle_0^C = \sum_{m_1=1}^\infty \cdots \sum_{m_n=1}^\infty \frac{a_{m_1} \cdots a_{m_n}}{\sqrt{m_1! \cdots m_n!}} \langle :\sigma_0^{m_1}:,\ldots,:\sigma_{t_n}^{m_n}: \rangle_0^C, \tag{3.5}$$

if the series in the right-hand side of (3.5) is absolutely convergent. Let us fix the ordered sequence $(m_1, m_2, \ldots, m_n)$.

Lemma 3.2:

$$\sum_{(t_2,\ldots,t_n)} \left| \langle :\sigma_0{}^{m_1}:, :\sigma_{t_2}{}^{m_2}:, \ldots, :\sigma_{t_n}{}^{m_n}: \rangle_0^C \right| \leqslant (n-1)! d^{N/2} \prod_{i=1}^n m_i(m_i - 2)(m_i - 4)\cdots, \tag{3.6}$$

where $N = m_1 + \cdots + m_n$ and the sum $\Sigma_{(t_2,\ldots,t_n)}$ is over all the *ordered* sequences $(t_2,\ldots,t_n)$ of points $t_i \in \mathbf{Z}^\nu$ such that $t_i \neq 0, t_i \neq t_j$, if $i \neq j$. We shall prove this lemma below and now using (3.6) we shall prove Lemma 3.1.

Proof of Lemma 3.1: We have

$$\begin{aligned} &\Sigma^{(n)} |\langle f(\sigma_0), f(\sigma_{t_2}),\ldots,f(\sigma_{t_n}) \rangle_0^C| \\ &\leqslant \sum_{m_1=1}^\infty \cdots \sum_{m_n=1}^\infty \frac{|a_{m_1}| \cdots |a_{m_n}|}{\sqrt{m_1! \cdots m_n!}\,(n-1)!} (n-1)! d^{N/2} \times \prod_{i=1}^n m_i(m_i-2)(m_i-4) \\ &\leqslant \sum_{m_1=1}^\infty \cdots \sum_{m_n=1}^\infty |a_{m_1}| \cdots |a_{m_n}| 2m_1 \cdots 2m_n d^{(m_1 + \cdots + m_n)/2} \\ &\leqslant (C_1 \epsilon)^n, \end{aligned} \tag{3.7}$$

where C_1 depends on $(1-d)^{-1}$. Thus Lemma 3.1 is proved.

Proof of Lemma 3.2:

We shall use the diagrammatic representation

$$\sum_{(t_2,\ldots,t_n)} \langle :\sigma_0^{m_1}:, :\sigma_{t_2}^{m_2}:, \ldots, :\sigma_{t_n}^{m_n}: \rangle_0^C = \sum I_G, \tag{3.8}$$

which we shall now define exactly.

A diagram G consists of the following objects: (1) An ordered set of vertices $\{1,\ldots,n\}$ and of one-to-one mapping τ of $\{1,\ldots,n\}$ in $\mathbf{Z}^\nu$ such that $\tau(1) = 0, \tau(i) = t_i$. (2) Each vertex has an ordered set of m_i legs $(i,1),(i,2),\ldots,(i,m_i)$. (3) There is a partition of the set of all $N = m_1 + \cdots + m_n$ legs onto pairs such that each pair (line of the diagram G) has legs from different vertices and the resulting graph is connected. (4). The Gaussian random variable $\sigma_{ij} = \sigma_{t_i}$ corresponds to the leg (i, j). The contribution of the diagram G is then

$$I_G = \prod \langle \sigma_{ij}, \sigma_{kl} \rangle_0 ,$$

where the product is taken over all pairs (ij, kl) of contracted legs (i.e., over lines of G). Formula (3.8) is then Wick's theorem for semi-invariants.

Let $\mathfrak{A}_N$ be the set of all sequences $\alpha = (x_1, \ldots, x_{N/2})$ with $0 \neq x_i \in \mathbb{Z}^\nu, N = m_1 + \cdots + m_n$. Let us fix a permutation

$$\Pi = \begin{pmatrix} 2 & \cdots & n \\ \pi(2) & \cdots & \pi(n) \end{pmatrix}$$

and $\hat{m}_1 = m_1, \hat{m}_i = m_{\Pi(i)}, i \geqslant 2$. For each $\alpha \in \mathfrak{A}_N$ and each Π we shall construct a set $\mathcal{G}_\pi(\alpha)$ of connected diagrams with fixed $(m_1, \ldots, m_n)$. A set $\mathcal{G}_\pi(\alpha)$ can be empty or otherwise it contains no more than $\Pi_{i=1}^n m_i(m_i - 2)(m_i - 4)\cdots$ diagrams and moreover α, π uniquely define vertices $t_2, \ldots, t_n$ which are the same for all $G \in \mathcal{G}_\pi(\alpha)$. Moreover the contribution of each $G \in \mathcal{G}_\pi(\alpha)$ *is equal to* $\Pi_{i=1}^{N/2} \varphi(x_i)$. We describe now the algorithm of construction of $\mathcal{G}_\pi(\alpha)$. This algorithm constructs vertices and contracts legs step by step.

First step: We begin with the vertex $t_1 = 0$ and contract leg (1,1) with arbitrary of $\hat{m}_2$ legs of vertex $\hat{t}_2 = x_1$. Thus the first step is finished and we proceed by induction.

Let $T = (t_1, \hat{t}_2, \ldots, \hat{t}_l), l < k$, be the vertices with $\hat{m}_1, \ldots, \hat{m}_l$ legs which have been constructed after k steps. Suppose that $r_1, \ldots, r_l$ legs of the correspondingly $\hat{m}_1, \ldots, \hat{m}_l$ are yet uncontracted. So on each step one line and 0 or 1 new vertex is constructed.

$(k+1)$st *step*: Let us consider the vertex $\hat{t}_i, < i < l$, such that $\hat{t}_i = x_1 + x_2 + \cdots x_k$. It exists by inductive construction. There can be two cases: $r_i > 0$ or $r_i = 0$.

(1) If $r_i > 0$ then consider the point $\hat{t}_i + x_{k+1}$ is not equal to any $\hat{t}_1, \ldots, \hat{t}_l$ then we construct new a vertex $\hat{t}_{l+1} = \hat{t}_i + x_{k+1}$ and we contract the *first* of r_i uncontracted legs of $\hat{t}_i$ with *arbitrary* of $\hat{m}_{l+1}$ legs of $\hat{t}_{l+1}$. If $\hat{t}_i + x_{k+1}$ is equal to some $\hat{t}_j, j < l$, then we contract the *first* of r_i uncontracted legs of $\hat{t}_i$ with *any* of the r_j uncontracted legs of $\hat{t}_j$. If $r_j = 0$ then we define $\mathcal{G}_\pi(\alpha)$ to be empty. Otherwise we proceed to the next step.

(2) If $r_i = 0$ then we take the *first* $\hat{t}_j, 1 < j < l$, such that $r_j \neq 0$ and we take the *first* of its uncontracted legs and we contract it with *any* of the uncontracted legs of the vertex $\hat{t}_p = \hat{t}_j + x_{k+1}$ (as in case 1). If all $r_j, 1 \leqslant j < l$, are equal to 0, then we define $\mathcal{G}_\pi(\alpha)$ to be empty.

We get a combinatorial factor

$$\hat{m}_i(\hat{m}_i - 2)\cdots$$

for each $\hat{t}_i$ since after each *arbitrary* choice of a leg we immediately take the *first* leg of $\hat{t}_i$. Evidently each diagram G with given $(m_1, \ldots, m_n)$ is in at least one $\mathcal{G}_\pi(\alpha)$. Lemma 3.2 is proved.

Proof of Lemma 2.2: It is sufficient to consider the case when

$$F_A = \prod_{i=1}^k f_i(x_i) ,$$

where $|f_i| < 1$ and $A = \{x_1, \ldots, x_k\}$. We use the following formula

$$\langle f_1(\sigma_{x_1}) f_k(\sigma_{x_k}), f(\sigma_{t_1}), \ldots, f(\sigma_{t_n}) \rangle_0^C = \sum \prod_{j=1}^p \langle f_i(\sigma_{x_i}), x_i \in T_j, f(\sigma_t), t \in T_j \rangle_0^C , \tag{3.9}$$

where the sum is over all the partitions $(T_1, \ldots, T_p)$ of the set $\{x_1, \ldots, x_k, t_1, \ldots, t_n\}$ such that each T_j has non empty intersection with $\{x_1, \ldots, x_k\}$. If all the t_i are different from all x_j then as in the proof of Lemma 3.1 we expand f_i and f in Hermite polynomials and we use diagrams. The diagram G will give contribution to the rhs of (3.9) iff each of its vertices t_i is connected by some path with some of the vertices $x_1, \ldots, x_k$. The proof then repeats the proof of Lemma 3.1.

If in

$$\langle f_i(x_i), x_i \in T_j, f(\sigma_t), t \in T_j \rangle_0^C , \tag{3.10}$$

some $x_i \in T_j$ is equal to some $t \in T_j$ then we use the formula

$$\langle \psi(y_1), \ldots, \psi(y_q) \rangle_0^C = \langle \psi(y_1)\psi(y_2), \psi(y_3), \ldots, \psi(y_q) \rangle_0^C - \sum_{T' \subset \{y_3, \ldots, y_q\}} \langle \psi(y_1), \psi(y_j), j \in T' \rangle_0^C \times \langle \psi(y_2), \psi(y_j), y_j \in \{3, \ldots, q\} - T' \rangle_0^C$$

to exclude these cases.

We use (3.11) subsequently for each pair $x_i = t_i$ and for all semi-invariants (3.10).

After this procedure we shall also have the sum of diagrams such that each of its points is connected with some of the points $x_1, \ldots, x_k$. We get also factors depending on k. Our construction is again applicable to this sum of "connected" diagrams.

4. THE CENTRAL LIMIT THEOREM

We shall prove Theorem 2 here. Let $\langle S_A, \ldots, S_A \rangle^C$ be the nth semi-invariant of the random variable $S_A = \Sigma_{t \in A} \sigma_t$. We want to show that

$$\lim_{|A| \to \infty} \frac{1}{(DS_A)^{n/2}} \langle S_A, \ldots, S_A \rangle^C = 0, \quad n \geqslant 3 . \tag{4.1}$$

We have

$$\langle S_A, \ldots, S_A \rangle^C = \sum_{t_1, \ldots, t_n \in A} \langle \sigma_{t_1}, \ldots, \sigma_{t_n} \rangle^C .$$

We shall define new random variables

$$\tilde{\sigma}_t = \sum_{\alpha=1}^n \sigma_t^{(\alpha)} \omega^\alpha ,$$

where $\omega = e^{2\pi i/n}$ and $\sigma_t^{(\alpha)} = 1, \ldots n$, are new random fields which are independent copies of the random field σ_t with measure μ. Then we have (see Ref. 13)

$$\langle \tilde{\sigma}_{t_1} \cdots \tilde{\sigma}_{t_k} \rangle = 0, \quad k < n ,$$

and

$$\langle \sigma_{t_1}, \ldots, \sigma_{t_n} \rangle^C = (1/n) \langle \tilde{\sigma}_{t_1}, \ldots, \tilde{\sigma}_{t_n} \rangle^C = (1/n) \langle \tilde{\sigma}_{t_1} \cdots \tilde{\sigma}_{t_n} \rangle . \tag{4.2}$$

Using the cluster expansion for our n independent copies of μ [it is similar to (2.11)]

$$\langle \tilde\sigma_{t_1} \cdots \tilde\sigma_{t_n} \rangle = \sum_{\{x_1,\ldots,x_k\}\subset \mathbb{Z}^\nu} \langle \tilde\sigma_{t_1} \cdots \tilde\sigma_{t_n}, F(x_1),\ldots,F(x_k)\rangle_0^C, \times (g_{\{x_1,\ldots,x_k\}})^n, \tag{4.3}$$

where

$$F(x_i) = \prod_{\alpha=1}^{n} f(\sigma_{x_i}^{(\alpha)}),$$

we have

$$\langle \tilde\sigma_{t_1} \cdots \tilde\sigma_{t_n}, F(x_1),\ldots,F(x_k)\rangle_0^C, = \langle \tilde\sigma_{t_1},\ldots,\tilde\sigma_{t_n}, F(x_1),\ldots,F(x_k)\rangle_0^C, \tag{4.4}$$

and

$$\langle \tilde\sigma_{t_1} \cdots \tilde\sigma_{t_p}, F(x_1),\ldots,F(x_k)\rangle_0^C = 0, \quad p<n.$$

The proof of (4.4) is similar to that of (4.2) due to the symmetry of $F(x)$ w.r.t. permutations of $\{1,\ldots,n\}$. Then

$$|\langle \tilde\sigma_{t_1},\ldots,\tilde\sigma_{t_n}, F(x_1),\ldots,F(x_k)\rangle_0^C| < n^n \sup_{\alpha'_1,\ldots,\alpha'_n} |\langle \sigma_{t_1}^{(\alpha'_1)},\ldots,\sigma_{t_n}^{(\alpha'_n)}, F(x_1),\ldots,F(x_k)\rangle_0^C|. \tag{4.5}$$

We expand each $f(\sigma_t^{(\alpha)})$ in the series of Wick polynomials and get as in (3.5)

$$\langle \sigma_{t_1}^{(\alpha'_1)},\ldots,\sigma_{t_n}^{(\alpha'_n)}, F(x_1),\ldots,F(x_k)\rangle_0^C = \sum_{m_1^1,\ldots,m_n^1>0} \frac{a_{m_1^1}\cdots a_{m_n^1}}{\sqrt{m_1^1!\cdots m_n^1!}} \cdots \sum_{m_1^k,\ldots,m_n^k>0} \frac{a_{m_1^k}\cdots a_{m_n^k}}{\sqrt{m_1^k!\cdots m_n^k!}} \times \langle \sigma_{t_1}^{(\alpha'_1)},\ldots,\sigma_{t_n}^{(\alpha'_n)}, :(\sigma_{x_1}^{(1)})^{m_1^1}:\cdots :(\sigma_{x_1}^{(n)})^{m_n^1}:,\ldots,:(\sigma_{x_k}^{(1)})^{m_1^k}:\cdots:(\sigma_{x_k}^{(n)})^{m_n^k}:\rangle_0^C. \tag{4.6}$$

From (4.2), (4.3), (2.9) we get

$$|\langle \sigma_{t_1},\ldots,\sigma_{t_n}\rangle^C| \leqslant \sum_{\{x_1,\ldots,x_k\}\subset \mathbb{Z}^\nu} 2^n B^{nk} |\langle \tilde\sigma_{t_1},\ldots,\tilde\sigma_{t_n}, F(x_1),\ldots,F(x_k)\rangle_0^C|, \tag{4.7}$$

$B = 2/K_0$.

We want to represent the semi-invariants in the rhs of (4.6) as the sum of connected diagrams G with $n+k$ vertices $t_1,\ldots,t_n,\ldots,x_k$. We fix $t_1 = 0$ and we want to prove that

$$\sum_{t_2,\ldots,t_n \in \mathbb{Z}^\nu} |\langle \sigma_0,\ldots,\sigma_{t_n}\rangle^C| \leqslant \text{const}. \tag{4.8}$$

(4.8) follows from:

Lemma 4.1:

$$\sum_{t_1,\ldots,t_n\in\Lambda} |\langle \sigma_{t_1},\ldots,\sigma_{t_n}\rangle^C| \leqslant \text{const}|\Lambda|. \tag{4.9}$$

Proof of (4.8): Let us denote by χ the semi-invariants in the rhs of (4.6). It depends on $m_i^j, \alpha'_i, x_i, t_j$. Then using (4.6), (4.7) we can bound (4.8) by

$$\sum_{t_2,\ldots,t_n} \sum_{\{x_1,\ldots,x_k\}\subset \mathbb{Z}^\nu} 2^n B^{nk} \sum_{m_i^j>0} \prod_{i,j} \frac{|a_{m_i^j}|}{\sqrt{m_i^j!}} |\chi| \leqslant \sum_{t_2,\ldots,t_n} 2^n \sum_{\{x_1,\ldots,x_k\}} (BC\epsilon)^{nk} \sum_{m_i^j>0} \prod_{i,j} \cdot \frac{1}{\sqrt{m_i^j!}} |\chi| < \frac{2^n}{k!} \sum_{\substack{t_2,\ldots,t_n,x_1,\ldots,x_k \\ x_i\neq x_j}} (BC\epsilon)^{nk} \sum_{m_i^j>0} \prod_{i,j} \frac{1}{\sqrt{m_i^j!}} |\chi|.$$

We note that for all i

$$\sum_j m_j^i \geqslant 1. \tag{4.10}$$

We fix, n,k,m_i^j, and shall prove that

$$\sum_{\substack{t_2,\ldots,t_n,x_1,\ldots,x_k \\ x_i\neq x_j}} |\chi| < k!k^n(1+d)^n d^{\Sigma m_i^j - n/2} \times \left[\prod_{i,j} m_i^j(m_i^j-2)(m_i^j-4)\cdots\right] \prod_{i,j} m_i^j. \tag{4.11}$$

From (4.10) and (4.11) we get (4.8), (4.9).

In order to complete the proof we must prove (4.11). We show that this proof is quite similar to the proof of Lemma 3.2. In fact there are no lines between legs of the same vertex x_i since $\sigma_{x_i}^{(\alpha)}, \sigma_{x_i}^{(\alpha')}$ are independent for $\alpha\neq\alpha'$. To construct the diagrams using our method we define $\mathfrak{A}_{N,n}$ to be the set of sequences

$$(y_1,\ldots,y_n,z_1,\ldots,z_{N-n/2}),$$

$$N = \sum_{i,j} m_i^j, \; y_i, z_j \in \mathbb{Z}^\nu, \quad z_j \neq 0.$$

Let us note that if we delete vertices $0, t_2,\ldots,t_n$ and delete lines connecting them to the other vertices then the remaining part of the diagram remains connected. Thus we begin with the vertex 0 and construct the second vertex. $x_{\pi(1)} = y_1$ and the line between 0 and $x_{\pi(1)}$. This gives a factor $(1+d)$.

As earlier, we fix a permutation π of $\{1,\ldots,k\}$. This gives a factor $k!$. Then for each $i\in\{2,\ldots,n\}$ we choose one of the k vertices $x_1,\ldots,x_k$ and choose one leg from the chosen vertex to be contracted with the vertex t_i. This gives a factor $k^{n-1}\Pi_{i,j} m_i^j$. Each such a construction generates a factor $(1+d)$. The remaining connected part of the diagram is constructed in the closest similarity to the construction in the proof of Lemma 3.2. The proof of (4.8) and of Lemma 4.1 is thus completed.

Lemma 4.2:

$$\langle S_\Lambda^2\rangle \geqslant Q|\Lambda|, \quad Q>0.$$

The proof is quite similar to the preceding one. We have

$$\langle S_\Lambda^2\rangle = \sum_{t,t'\in\Lambda} \langle \sigma_t, \sigma_{t'}\rangle^C.$$

In order to bound $\langle \sigma_t, \sigma_{t'}\rangle_0^C$ we use cluster expansion. It contains the "main term" $\langle \sigma_t, \sigma_{t'}\rangle_0^C$. The sum of the remaining terms being of order $O(\epsilon)$.

We have for the "main terms"

$$\frac{1}{|\Lambda|} \sum_{t,t'\in\Lambda} \langle \sigma_t, \sigma_{t'}\rangle_0^C = \frac{1}{|\Lambda|} \langle S_\Lambda^2\rangle_0 \underset{|\Lambda|\to\infty}{\to} Q_1.$$

Then Lemma 4.2 follows with $Q \geqslant Q_1 - \epsilon$. It is not difficult to prove that

$$\frac{1}{|\Lambda|}\langle S_\Lambda^2\rangle \underset{|\Lambda|\to\infty}{\to} Q.$$

5. RENORMGROUP CONVERGENCE TO NONTRIVIAL GAUSSIAN AUTOMODEL FIELD

We have to show that $\langle \sigma_{t_1}^{(k)},...,\sigma_{t_n}^{(k)}\rangle^C \underset{k\to\infty}{\longrightarrow} 0$, (5.1)

and

$$\langle \sigma_t^{(k)},\sigma_{t'}^{(k)}\rangle^C \to K(t,t'). \tag{5.2}$$

The equivalence of these statements and of the statements of the theorem is due to the following general facts. Statement (5.1) is equivalent to the statement that the weak limit of $\sigma_t^{(k)}$ is Gaussian if it exists. Moreover if the limit exists then it is automodel and there exists only one isotropic automodel random field with asymptotics (1.5) (see Ref. 14).

We use the following asymptotic properties

$$|\langle \sigma_t,\hat{\sigma}_{t'}\rangle_0^C| < \text{const}/|t-t'|^{\nu+1},$$
$$|\langle \hat{\sigma}_t,\hat{\sigma}_{t'}\rangle_0^C| < \text{const}/|t-t'|^{\nu+1}, \tag{5.3}$$

which are quite evident.

We can write cluster expansion for

$$\langle \sigma_t,\sigma_{t'}\rangle = \langle \sigma_t,\sigma_{t'}\rangle_0 + \sum_{k\geqslant 1,\{x_1,...,x_k\}\subset Z^\nu} \langle \sigma_t\sigma_{t'},F(x_1),...,F(x_k)\rangle_0^C \times g_{\{x_1,...,x_k\}}, \tag{5.4}$$

similar to the earlier cluster expansion. Convergence of this expansion follows from (5.3). From (5.3) it follows that

$$\frac{\langle \sigma_t,\sigma_{t'}\rangle^C}{\langle \sigma_t,\sigma_{t'}\rangle_0^C} \to 1 \text{ if } |t-t'|\to\infty, \tag{5.5}$$

because in the diagrammatic expansion of the second term in the rhs of (5.4) there is no line between vertices t and t'. Statement (5.1) can be proved quite similarly to the proof of the same statement in the preceding section. Theorem (5.1) is proved.

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