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Translation invariant quantum master equation

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Abstract. We give the first rigorous derivation of a kinetic equation in a nontrivial translation invariant situation. We consider the heavy particle (mass scaled as $m\varepsilon^{-\delta}$, $\delta \geqslant 0$) in the ideal fermi gas with translation invariant interaction $\varepsilon\mathbb{V}$ between them. In the Heisenberg representation we prove convergence to the strongly continuous completely-positive semigroup T_s for any value of scaled time s, $t = s/\varepsilon^2$ and for $\delta > 2$. When $\delta = 2$ T_s is not a semigroup. For $0 \leqslant \delta < 2$ we announce some more weak results.

0. Introduction

Rigorous derivation of the quantum kinetic equations from the first principles is one of the central problems of nonequilibrium statistical mechanics. The first results (initiated by Van Hove [29]) were obtained for the so called weak coupling limit between system and reservoir, where the system was described by finite-dimensional Hilbert space $\mathscr{H}_p$ and the reservoir was the ideal quantum gas [10, 34]. Further progress was connected mainly with such finite-dimensional $\mathscr{H}_p$ for the following limits:

1. weak coupling limit [8–10, 12–16, 18];
2. singular coupling limit [28, 24, 26–27, 33];
3. low density limit [21, 22].

See also review of Spohn [37].

Recent progress for the case when the system is a Schroedinger particle, and $\mathscr{H}_p = L_2(\mathbb{R}^\nu, dx)$, see in [19, 1–4].

The main limitation of all these results is that the interaction is not translation invariant. Otherwise speaking system interacts only with a bounded region of the ideal gas.

In this paper we consider the weak translation invariant interaction between Schroedinger particle and the ideal quantum gas. There are also some other novelties in this paper;

(a) A quantum kinetic equation for an observable A_p can be conventionally written as

$$\frac{\partial A_p(s)}{\partial s} = i\varepsilon^{\delta-2}[h_p, A_p(s)] + L_s(A_p)$$

where $\delta > 2$,

$$L_s(A_p) \equiv e^{i(s\varepsilon^{\delta}-2)h_p} L(A_p(s)) e^{-i(s\varepsilon^{\delta}-2)h_p}$$

L is the generator of a quantum semigroup, h_p is the free hamiltonian (e.g. $h_p = -\Delta$). We discuss the arising mass renormalisation problem (see remark 1).

(b) In all earlier papers the degree of interaction (see below) did not exceed 2. This was connected with number of diagrams estimation problems. Here we consider arbitrary degree of interaction in the fermi case.

(c) The generator of our quantum semigroup is unbounded and so it does not fit into the known classification results [16, 17, 23, 28, 35].

(d) We consider also the case when the kinetic equation has non Markovian character.

Our scaling of particle mass for $\delta > 2$ corresponds to the case when the test particle has infinite mass (see Remark 1). In the quantum case such particles can move because of the influence of the heat reservoir. The limiting behavior of this particle does not depend on the parameter $\delta > 2$ (see the Theorem 1 and 2).

We think that for $0 \leqslant \delta < 2$ a picture must be different from one described above. In the weak coupling limit only the influence of the free kinetic motion of the particle remains (see Remark 1′). It means that in the weak coupling limit in the interaction representation the particle does not move. Now we can prove a weak variant of this hypothesis (see the Theorem 3). This triviality is closely connected with the unitary equivalence of the hamiltonians H_0 and H for sufficiently small $|\varepsilon|$, $\varepsilon \in \mathbb{R}$, when $\delta = 0$ (see [6, 7]).

In the case $\delta = 2$ the influence of the kinetic motion of the particle and the heat reservoir remain, but reduced dynamic has a memory.

In this paper we use an original method of diagram resummation. Already similar methods were applied for the proof of the asymptotic completeness (see [5–7, 20, 32]).

Boson case for the degree interaction 1 or 2 can be treated as well by the similar methods.

1. Definitions and notations

Hilbert spaces are: $\mathscr{H}_p^{(1)} = L_2(\mathbb{R}^\nu)$ for the particle, and for reservoir it is the antisymmetric Fock space $\mathscr{H}_r = \mathscr{F}_a(\mathscr{H}_r^{(1)})$, where $\mathscr{H}_r^{(1)} = L_2(\mathbb{R}^\nu)$. Let $\mathfrak{A}_p$ *be* C^*-algebra of compact operators in $\mathscr{H}_p \equiv \mathscr{H}_p^{(1)}$ with 1_p, $\mathfrak{A}_r$ be C^*-algebra of CAR in $\mathscr{F}_a(\mathscr{H}_r^{(1)})$,

$$\{a^*(f), a(g)\} = (g,f)I_r,$$

$$\{a^*(f), a^*(g)\} = \{a(f), a(g)\} = 0,$$

f, $g \in \mathscr{H}_r^{(1)}$, (g,f) is assumed to be antilinear in g. The corresponding operator valued distributions $a^{\#}(x)$ are introduced by

$$a^*(g) = \int g(x)a^*(x)\,dx, \qquad a(g) = \int \bar{g}(x)a(x)\,dx.$$

We define the free hamiltonian in $\mathscr{H}_p \otimes \mathscr{H}_r$

$$H_o = \varepsilon^\delta h_p \otimes 1_r + 1_p \otimes H_r,$$

$$h_p = -\Delta, \quad H_r = d\Gamma(h_r), \quad h_r = -\Delta + \mu 1_r, \quad \mu \in \mathbb{R}, \quad \delta \geqslant 0, \quad \varepsilon > 0,$$

and the free dynamics on $\mathfrak{A}_p$ and $\mathfrak{A}_r$

$$\tau_t^p(A_p) = \exp(it\varepsilon^\delta h_p)A_p \exp(-it\varepsilon^\delta h_p), \qquad A_p \in \mathfrak{A}_p,$$

$$\tau_t^r(A_r) = \exp(itH_r)A_r \exp(-itH_r), \qquad A_r \in \mathfrak{A}_r.$$

We use the same symbol for

$$\tau_t^p = \tau_t^p \otimes 1_r, \qquad \tau_t^r = 1_p \otimes \tau_t^r,$$

and define on $\mathfrak{A} = \mathfrak{A}_p \otimes \mathfrak{A}_r$

$$\tau_t = \tau_t^p \otimes \tau_t^r$$

Let ω_β be β-KMS state [8] on $\mathfrak{A}_r$ w.r.t. the free dynamics τ_t^r. For any f, $g \in \mathscr{H}_r^{(1)}$

$$\omega_\beta(a^*(f)a(g)) = (f, B_\beta g),$$

$$\omega_\beta(a^*(f)a^*(g)) = \omega_\beta(a(f)a(g)) = 0,$$

where $B_\beta = \exp(-\beta h_r)(1_r + \exp(-\beta h_r))^{-1}$, $0 \leqslant B_\beta \leqslant 1$.

We define the bounded linear map $\omega\colon \mathfrak{A} \to \mathfrak{A}_p$ by

$$\omega(A_p \otimes A_r) = A_p \omega_\beta(A_r), \qquad A_p \in \mathfrak{A}_p, \qquad A_r \in \mathfrak{A}_r. \tag{1.1}$$

We note that

$$\omega \circ \tau_t = \tau_t^p \circ \omega \tag{1.2}$$

It is convenient to consider $\mathscr{H}_p$ as 1-particle subspace of the Fock space $\mathscr{F}_\pm(\mathscr{H}_p)$ with creation-annihilation operators $b^*(f)$, $b(f)$ which can be either bose or fermi. In $\mathscr{H} \subseteq \mathscr{F} \otimes \mathscr{F}_a$ we consider the unitary group of translations $\Gamma(U_x)$ generated by the corresponding group U_x in $L_2(\mathbb{R}^\nu)$:

$$f_x(y) = (U_x f)(y) = f(y-x), \qquad f \in L_2(\mathbb{R}^\nu).$$

E.g.

$$\Gamma(U_x)(b^\#(g) \otimes a^\#(f))\Gamma(U_{-x}) = b^\#(g_x) \otimes a^\#(f_x)$$

where $\#$ means * or the absence of *.

For any $f \in L_2$ we will denote by $\hat{f}$ the Fourier transform of the function f.

We define the perturbed Hamiltonian formally by

$$H = H_o + \varepsilon \mathbb{V}, \quad \mathbb{V} = \int_{\mathbb{R}^\omega} V_x\, dx, \tag{1.3}$$

$$H_\Lambda = H_o + \varepsilon \mathbb{V}_\Lambda, \qquad \mathbb{V}_\Lambda = \int_\Lambda V_x\, dx, \tag{1.3'}$$

$$V_x = \Gamma(U_x) V \Gamma(U_x)^{-1}$$

for two classes $\mathscr{A}$ and $\mathscr{B}$ of operators V in $\mathscr{F} \otimes \mathscr{F}_a$:

We say that $V \in \mathscr{A}_o$ ($V \in \mathscr{A}_S$) if V has the following form

$$\text{(A)}\quad V = \sum_{m,n} \int \mathscr{V}_{m,n}(y_0, z_0; y_1, \ldots, y_m, z_n, \ldots, z_1) b^*(y_0) b(z_0) \tag{1.4}$$
$$a^*(y_1) \cdots a^*(y_m) a(z_n) \cdots a(z_1)\, dy_0\, dz_0\, dy_1 \cdots dy_m\, dz_n \cdots dz_1$$

where m,n in (1.4) are such that $m + n$ is even and

$$\hat{\mathscr{V}}_{m,n} \in C_0^\infty(\mathbb{R}^{\nu(m+n+2)}) \; (\hat{\mathscr{V}}_{m,n} \in S(\mathbb{R}^{\nu(m+n+2)}));$$

We say that $V \in \mathscr{B}_o$ ($V \in \mathscr{B}_S$) if V has the following form

$$\text{(B)}\quad V = \sum_{m,n} \int \mathscr{V}_{m,n}(y_0; y_1, \ldots, y_m, z_n, \ldots, z_1) b^*(y_0) b(y_0) \tag{1.5}$$
$$a^*(y_1) \cdots a^*(y_m) a(z_n) \cdots a(z_1)\, dy_0\, dy_1 \cdots dy_m\, dz_n \cdots dz_1$$

where m,n in (1.5) are such that $m + n$ is even and

$$\hat{\mathscr{V}}_{m,n} \in C_o^\infty(\mathbb{R}^{\nu(m+n+1)}) \; (\hat{\mathscr{V}}_{m,n} \in S(\mathbb{R}^{\nu(m+n+1)}));$$

$\Sigma_{m,n}$ is a finite sum and it is assumed that the bounded operator V is symmetric on $\mathscr{F} \otimes \mathscr{F}_a$.

Then the (unbounded) operator $\mathbb{V}$ is symmetric on the domain

$$\mathscr{H}^o = \left\{ \int F(x_0, x_1, \ldots, x_n) b^*(x_0) a^*(x_1) \cdots a^*(x_n) \Omega : \; F \in S(\mathbb{R}^{\nu(n+1)}),\, n = 0, 1, 2, \ldots \right\},$$

Ω is the vacuum vector in $\mathscr{F} \otimes \mathscr{F}_a$. Let us denote $\mathfrak{A}_p^o(\mathfrak{A}_p^S)$, $\mathfrak{A}_r^0(\mathfrak{A}_r^S)$—subalgebras of $\mathfrak{A}$ generated by 1, and $b^\#(f)$, $a^\#(f)$ correspondingly for $\hat{f} \in C_o^\infty(\mathbb{R}^\nu)$ ($\hat{f} \in S(\mathbb{R}^\nu)$); $\mathfrak{A}^o = \mathfrak{A}_p^o \otimes \mathfrak{A}_r^o$, $\mathfrak{A}^S = \mathfrak{A}_p^S \otimes \mathfrak{A}_r^S$.

The perturbed dynamics α_t on $\mathfrak{A}$ can be defined e.g. using the suitable generalization of the Robinson's argument [8]

$$\alpha_t(A) = \lim_{\Lambda \to \mathbb{R}^\nu} \alpha_t^\Lambda(A) = \lim_{\Lambda \to \mathbb{R}^\nu} e^{it(H_o + \varepsilon \mathbb{V}_\Lambda)} A e^{-it(H_o + \varepsilon \mathbb{V}_\Lambda)}, \tag{1.6}$$

using Dyson-Schwinger series in ε. We do not stay on this as our expansions below are uniform in Λ.

2. Main results

Let us consider the map (similar to an inverse Möller morphism)

$$\gamma_t^\varepsilon : \mathfrak{A}_p \to \mathfrak{A}_p : \gamma_t^\varepsilon(A_p) = \omega(\tau_{-t}\alpha_t(A_p \otimes 1_r)), \quad A_p \in \mathfrak{A}_p, \quad t \in \mathbb{R}_+. \tag{2.1}$$

Theorem 1. *Let $\nu \geqslant 3$, $\delta \geqslant 2$, $\mu \in \mathbb{R}$ and V belongs to either $\mathscr{A}_S$ or $\mathscr{B}_S$, and $\omega(V) = 0$.*

Then for any $s \in \mathbb{R}_+$ *there exists the limit*

$$\underset{\varepsilon \to 0}{s\text{-lim}}\, \gamma^{\varepsilon}_{s/\varepsilon^2} = T_s, \qquad \| T_s \| \leqslant 1. \tag{2.2}$$

and T_s *is a strongly continuous completely positive map from* $\mathfrak{A}_p$ *into* $\mathfrak{A}_p$.

Theorem 2. *Under the conditions of the Theorem 1 for* $\delta > 2$ T_s *is a semigroup on* $\mathfrak{A}_p$. *The generator* L *of the semigroup* $\{T_s, s \geqslant 0\}$ *which exists by strong continuity of* T_s *(see [8]) is given by* ($A_p \in \mathfrak{A}_p^o$)

$$L(A_p) = -\iint_{\mathbb{R}^{2\nu}} dx_1\, dx_2 \int_0^{\infty} dt\, \omega([\tau_t^r(V_{x_2}), [V_{x_1}, A_p \otimes 1_r]]) \tag{2.3}$$

Corollary. *Under the conditions of the Theorem 1 for* $\delta > 2$ *and when* V *has the form*

$$V = \sum_i V_i^p \otimes V_i^r, \qquad V_i^p \in \mathfrak{A}_p^S, \qquad V_i^r \in \mathfrak{A}_r^S,$$

Then T_s *is a semigroup and its generator* L *is given by* ($A_p \otimes \mathfrak{A}_p^o$)

$$L(A_p) = -\sum_{i,j=1}^{M} \iint_{\mathbb{R}^{2\nu}} dx_1\, dx_2\, \{g_{ij}(x_2 - x_1) V_{i,x_2}^p [V_{j,x_1}^p, A_p]$$
$$- \bar{g}_{ij}(x_2 - x_1)\, [V_{j,x_1}^p, A_s] V_{i,x_2}^p\}$$

where

$$g_{ij}(x) = \int_0^{\infty} dt\, \omega_\beta(V_{i,t,x}^r V_j^r)$$

For $\delta = 2$ T_s is not a semigroup and it has more complicated structure.

For the case $0 \leqslant \delta < 2$ we can prove the following more weak than the Theorems 1 and 2 result. Let us consider the map $\gamma_t^{\varepsilon,\Lambda} : \mathfrak{A}_p \to \mathfrak{A}_p$:

$$\gamma_t^{\varepsilon,\Lambda}(A_p) = \omega(\tau_{-t} \alpha_t^{\Lambda}(A_p \otimes 1_r)), \tag{2.4}$$

$$A_p \in \mathfrak{A}_p, \qquad t \in \mathbb{R}_+.$$

where Λ is a domain in $\mathbb{R}^\nu$.

Theorem 3. *Let* $\nu \geqslant 3$, $0 \leqslant \delta < 2$, $\mu \in \mathbb{R}$, $\Lambda \subset \mathbb{R}^\nu$ *be a finite domain and* V *belongs to either* $\mathscr{A}_S$ *or* $\mathscr{B}_S$, *and*

$$\omega(V) = 0.$$

Then for any $s \in \mathbb{R}_+$ *there exists the limit*

$$\underset{\varepsilon \to 0}{s\text{-lim}}\, \gamma^{\varepsilon,\Lambda}_{s/\varepsilon^2} = T_s,$$

where T_s *is trivial, i.e.* $T_s(A_p) = A_p$ *for any* $s \in \mathbb{R}_+$, $A_p \in \mathfrak{A}_p$.

It means that for $0 \leqslant \delta < 2$ the limit

$$s\text{-}\lim_{\Lambda \to \mathbb{R}^\nu} \lim_{\varepsilon \to 0} \gamma^{\varepsilon,\Lambda}_{s/\varepsilon^2} \tag{2.5}$$

is trivial for any $s\,\mathbb{R}_+$. Our hypothesis is that the same is true in the translation invariant case, i.e. when we change the order of the limits in

$$s\text{-}\lim_{\varepsilon \to 0} \lim_{\Lambda \to \mathbb{R}^\nu} \gamma^{\varepsilon,\Lambda}_{s/\varepsilon^2} \equiv s\text{-}\lim_{\varepsilon \to 0} \gamma^{\varepsilon}_{s/\varepsilon^2}$$

as in the Theorems 1 and 2. Then the limiting semigroup will be trivial. But now we can not prove this hypothesis because of some technical difficulties. The Theorem 3 follows from the results of [38]. Also we note that for $\delta \geqslant 2$ it is possible to change the order of the limits in (2.5) and the limits will be the same. This follows from the proofs of the Theorems 1 and 2.

Remark 1. In other words for $\delta \geqslant 2$, $A_p \in \mathfrak{A}_p$

$$\|\omega(\alpha_{s/\varepsilon^2}(A_p \otimes 1_r)) - \tau^p_{s/\varepsilon^2} T_s(A_p)\| \to 0 \quad \text{if } \varepsilon \to 0. \tag{2.6}$$

Putting

$$A^{(\varepsilon)}(s) = \tau^p_{s/\varepsilon^2} T_s(A_p), \qquad A \in \mathfrak{A}^S_p,$$

heuristically we get the equation

$$\frac{\partial A^{(\varepsilon)}(s)}{\partial s} = i\varepsilon^{\delta-2}[h_p, A^{(\varepsilon)}(s)] + L_s(A^{(\varepsilon)}(s)) \tag{2.7}$$

where

$$L_s \equiv \tau^p_{s/\varepsilon^2}(L) = e^{is(\varepsilon^\delta - 2h_p)} L e^{-is(\varepsilon^\delta - 2h_p)}$$

For $\delta > 2$ $\tau^p_{s/\varepsilon^2}(A_p) \to A_p$ if $\varepsilon \to 0$ for any $s \in \mathbb{R}^\nu$, $A_p \in \mathfrak{A}_p$. It means that we can rewrite (2.6) as

$$\|\omega(\alpha_{s/\varepsilon^2}(A_p \otimes 1_r)) - T_s(A_p)\| \to 0 \quad \text{if } \varepsilon \to 0. \tag{2.8}$$

Remark 1′. In the case when $0 \leqslant \delta < 2$ it follows from the Theorem 3 that for any fixed domain $\Lambda \subset \mathbb{R}^\nu$

$$\|\omega(\alpha^\Lambda_{s/\varepsilon^2}(A_p \otimes 1_r)) - \tau^p_{s/\varepsilon^2}(A_p)\| \to 0 \quad \text{if } \varepsilon \to 0. \tag{2.9}$$

for any $s \in \mathbb{R}^\nu$, $A_p \in \mathfrak{A}_p$.

Now consider the case when $\omega(V) \neq 0$. Putting

$$h'_p = \omega(\mathbb{V}) \tag{2.10}$$

it is easy to see that in the case (A) the operator h'_p is the multiplication operator (in the k-representation) by a function $\hat{h}'_p(k)$ and in the case (B)

$$h'_p = \lambda(V, \beta, \mu) 1_p \tag{2.11}$$

where $\lambda(V, \beta, \mu)$ is a constant.

Let us consider the map (similar to renormalised inverse Möller morphism) $\tilde{\gamma}_t^\varepsilon : \mathfrak{A}_p \to \mathfrak{A}_p$:

$$\tilde{\gamma}_t^\varepsilon(A_p) = \omega(\tau_{-t}^\varepsilon \alpha_t(A_p \otimes 1_r)), \quad A_p \in \mathfrak{A}_p, \qquad t \in \mathbb{R}_+. \tag{2.12}$$

where

$$\tau_t^\varepsilon = \tau_t^{\varepsilon;p} \otimes \tau_t^r$$

and the free dynamics $\tau_t^{\varepsilon;p}$ on $\mathfrak{A}_p$ is defined by

$$\tau_t^{\varepsilon;p}(A_p) = \exp(it(\varepsilon^\delta h_p + \varepsilon h_p'))A_p \exp(-it(\varepsilon^\delta h_p + \varepsilon h_p')), \qquad A_p \in \mathfrak{A}_p.$$

Theorem 1′. *Let $v \geqslant 3$, $\delta > 2$, $\mu \in \mathbb{R}$ and V belongs to $\mathscr{B}_S$. Then for any $s \in \mathbb{R}_+$ there exists the limit*

$$s\text{-}\lim_{\varepsilon \to 0} \tilde{\gamma}_{s/\varepsilon^2}^\varepsilon = \tilde{T}_s, \qquad \|\tilde{T}_s\| \leqslant 1. \tag{2.13}$$

and $\tilde{T}_s$ is a strongly continuous completely positive map from $\mathfrak{A}_p$ into $\mathfrak{A}_p$.

Theorem 2′. *Under the conditions of the Theorem 1′ for $\delta > 2$ and when V belongs to $\mathscr{B}_S$ $\tilde{T}_s$ is a semigroup on $\mathfrak{A}_p$. The generator $\tilde{L}$ of the semigroup $\{T_s, s \geqslant 0\}$ is given by ($A_p \in \mathfrak{A}_p^\circ$)*

$$\tilde{L}(A_p) = -\iint_{\mathbb{R}^{2v}} dx_1\, dx_2 \int_0^\infty dt\, \omega([\tau_t^r(\tilde{V}_{x_2}), [\tilde{V}_{x_1}, A_p \otimes 1_r]]) \tag{2.14}$$

where $\tilde{V} = V - \omega(V)$.

For $\delta = 2$ $\tilde{T}_s$ is not a semigroup and has more complicated structure.

Remark 2. If $\mathbb{V}$ is the two-particle interaction (formally):

$$\mathbb{V} = \int_{\mathbb{R}^{2v}} \mathscr{V}(x - y)\delta_y \otimes a^*(x)a(x)\, dy\, dx$$

where $\delta_y = \delta(x - y)$, there exist the difficulties in the proof of the Theorem 1 due to more singular nature of this interaction.

Remark 3. The proofs of Theorems 1′ and 2′ repeat the proofs of the Theorems 1 and 2.

3. Proof of the Theorem 1

The strategy of the proof of the Theorem 1 is the following. First we prove the existence of the limit

$$\lim_{\varepsilon \to 0} \gamma_{s/\varepsilon^2}^\varepsilon(A_p),$$

more exactly, the convergence of (2.2) on the dense subset of $\mathfrak{A}_p$, where V has a special form (see (4.1), (4.2)). From here we get the convergence of (2.2) for V of the general form. As for any $t \in \mathbb{R}$, $\varepsilon \in \mathbb{R}$

$$\|\gamma_t^\varepsilon\| \leqslant 1$$

the convergence on $\mathfrak{A}_p$ will follow.

The map γ_t^ε is completely positive as the maps τ_t, α_t and ω are completely positive for any $t \in \mathbb{R}$. Then, T_s has this property.

Now we prove the convergence of (2.2) on some dense subset of $\mathfrak{A}_p$.

We shall use the following series for γ_t^ε

$$\gamma_t^\varepsilon(A_p) = A_p + \sum_{n=1}^{\infty} (i\varepsilon)^n \int_{\Delta_n^t} dt_1 \cdots dt_n \int_{(\mathbb{R}^\nu)^n} dx_1 \cdots dx_n$$
$$\omega([V_{t_n,x_n}, [\ldots, [V_{t_1,x_1}, A_p \otimes 1_r] \cdots]]) \tag{3.1}$$

where

$$\Delta_n^t = \{(t_1, \ldots, t_n) : 0 < t_1 < \cdots < t_n < t\} \subset \mathbb{R}^n,$$
$$V_{t,n} \equiv \tau_t(V_x) \equiv V_{t,x}^p \otimes V_{t,x}^r.$$

Also we shall use the following notations

$$f_t^p \equiv \exp(it\varepsilon^\delta h_p) f, \qquad f_{t,x}^p \equiv \exp(it\varepsilon^\delta h_p) f_x;$$
$$f_t^r \equiv \exp(ith_r) f; \qquad f_{t,x}^r \equiv \exp(ith_r) f_x. \tag{3.2}$$

for $f \in \mathscr{H}_p$ or $f \in \mathscr{H}_r^{(1)}$, $t \in \mathbb{R}$, $x \in \mathbb{R}^\nu$.

Unfolding of the brackets of all commutators in the n-th term of the series (3.1) gives

$$\omega([V_{t_n,x_n}, [\ldots, [V_{t_1,x_1}, A_p \otimes 1_r] \cdots]]) = \sum_{\sigma \in P_n} \omega(W_{\bar{t},\bar{x}}^{n;\sigma}(A_p \otimes 1_r))$$

where

$$\bar{t} = (t_1, \ldots, t_n), \qquad \bar{x} = (x_1, \ldots, x_n),$$
$$\omega(W_{\bar{t},\bar{x}}^{n;\sigma}(A_p \otimes 1_r)) = \theta(\sigma) V_{t_{\sigma(n)},x_{\sigma(n)}} \cdots V_{t_{\sigma(j)},x_{\sigma(j)}}(A_p \otimes 1_r)$$
$$V_{t_{\sigma(j-1)},x_{\sigma(j-1)}} \cdots V_{t_{\sigma(1)},x_{\sigma(1)}} \tag{3.3}$$

P_n is the subset of the symmetric group S_n, $|P_n| = 2^n$; $\theta(\sigma) = \pm 1$ and is determined uniquely by σ; $j = j(\sigma)$ in (3.3) satisfy the inequalities

$$\sigma(n) > \sigma(n-1) > \cdots > \sigma(j),$$
$$\sigma(j-1) < \sigma(j-2) < \cdots < \sigma(1).$$

We can write the series (3.1) in the form

$$\gamma_t^\varepsilon(A_p) = A_p + \sum_{n=1}^{\infty} (i\varepsilon)^n \sum_{\sigma \in P_n} \int_{\Delta_n^t} d\bar{t} \int_{(\mathbb{R}^\nu)^n} d\bar{x}\, \omega(W_{\bar{t},\bar{x}}^{n;\sigma}(A_p \otimes 1_r)) \tag{3.4}$$

where

$$d\bar{t} = dt_1 \cdots dt_n, \qquad d\bar{x} = dx_1 \cdots dx_n.$$

The existence of the limit (2.2) in the Theorem 1 will follow from the following estimate for n-th term of the series (3.1).

Lemma 1. *Under the conditions of the Theorem 1 there exists a constant $C = C(V, v) > 0$ such that for any $n \in \mathbb{N}$, $s_o \in \mathbb{R}_+$, $\sigma \in P_n$ the following bound holds if $\varepsilon^2 t \in [0, s_o]$, $A_p \in \mathfrak{A}_p^o$*

$$\left\| \int_{\Delta_n^t} d\bar{t} \int_{(\mathbb{R}^v)^n} d\bar{x}\omega(W_{\bar{t},\bar{x}}^{n;\sigma}(A_p \otimes 1_r)) \right\| \leqslant C^n C(A_p) s_o^{dn} \left(\sum_{k=0}^{[n/2]} \frac{t^k}{k!} \right) \tag{3.5}$$

where $d = 2v$, $C(A_p)$ does not depend on n, s_o, σ.

Indeed, from (3.5) we have

$$\begin{aligned}
\|\gamma_t^\varepsilon(A_p)\| &\leqslant C(A_p) + \sum_{n=1}^{\infty} 2^n |\varepsilon|^n C^n C(A_p) s_o^{nd} \left(\sum_{k=0}^{[n/2]} \frac{t^k}{k!} \right) \\
&= \left(1 + \sum_{n=1}^{\infty} \sum_{k=0}^{[n/2]} 2^n |\varepsilon|^{n-2k} C^n s_o^{nd} \frac{|\varepsilon^2 t|^k}{k!} \right) C(A_p) \\
&\leqslant \left(1 + \sum_{n=1}^{\infty} \sum_{k=0}^{[n/2]} 2^n |\varepsilon s_o^d|^{n-2k} C^n \frac{(s_o^{(2d+1)})^k}{k!} \right) C(A_p)
\end{aligned}$$

If $|\varepsilon s_o^d| < 1$ we have

$$\begin{aligned}
\|\gamma_t^\varepsilon(A_p)\| \leqslant \Bigg(1 + \sum_{n=1}^{\infty} \Bigg(&\sum_{k=0}^{[n/4]} (2C^2 |\varepsilon s_o^d|)^{n/2} \frac{(s_o^{(2d+1)})^k}{k!} \\
&+ \sum_{k=[n/4]+1}^{[n/2]} \frac{(2C)^n}{(n/4)!} \frac{(s_o^{(2d+1)})^k}{(k!)^{1/2}} \Bigg) C(A_p)
\end{aligned} \tag{3.6}$$

It is evident that the series in the right-hand side of (3.6) converges if

$$|\varepsilon| < \min\left(\frac{1}{s_o^d}, \frac{1}{4C^2 s_o^d} \right).$$

Therefore the expansions (3.1) for γ_t^ε is norm convergent. The Theorem 1 is proved.

In the sections 4–6 we shall prove the Lemma 1.

4. Diagrams and graphs

To simplify notations in the proof of the Lemma 1 we shall consider the case when V has the form

$$V = V^p \otimes V^r,$$
$$V^p = b^*(f) b(f)|_{\mathscr{H}_p}, \tag{4.1}$$

$$V^r = a^*(f_1)\cdots a^*(f_N)a(f_N)\cdots a(f_1) - \omega_\beta(a^*(f_1)\cdots a^*(f_N)a(f_N)\cdots a(f_1)) \tag{4.2}$$
$$\omega_\beta(V^r) = 0,$$

where $\hat{f}, \hat{f}_i \in C_o^\infty(\mathbb{R}^\nu)$, $i = 1, \ldots, N$.

For $A = b^*(g_1)b(g_2)|_{\mathscr{H}_p} \in \mathfrak{A}_p^o$, $\hat{g}_1, \hat{g}_2 \in C_o^\infty(\mathbb{R}^\nu)$ we have

$$W_{\bar{t},\bar{x}}^{n;\sigma}(A_p \otimes 1_r) = U_{\bar{t},\bar{x}}^\sigma(A_p)\omega_\beta(V^r_{t_{\sigma(n)},x_{\sigma(n)}}\cdots V^r_{t_{\sigma(1)},x_{\sigma(1)}}) \tag{4.3}$$

where

$$U_{\bar{t},\bar{x}}^\sigma(A_p) = \theta(\sigma)V^p_{t_{\sigma(n)},x_{\sigma(n)}}\cdots V^p_{t_{\sigma(j)},x_{\sigma(j)}}(A_p \otimes 1_r) V^p_{t_{\sigma(j-1)},x_{\sigma(j-1)}}\cdots V^p_{t_{\sigma(1)},x_{\sigma(1)}}, \tag{4.4}$$

and $j = j(\sigma)$.

We put $x_0 \equiv 0$, $t_0 \equiv 0$.

Lemma 2. *Let $d \in \mathbb{N}$, $s_o \in \mathbb{R}_+$. For $\varepsilon^2 t \in [0, s_o]$, $0 < \varepsilon < 1$, $\sigma \in P_n$, $\bar{x} \in (\mathbb{R}^\nu)^n$ uniformly on $\bar{t} \in \Delta_n^t$ the following bound holds:*

$$\left\| U_{\bar{t},\bar{x}}^\sigma(A_p) \right\| \leqslant \frac{s_o^{dn}C^nC(A_p)}{(|x_{\sigma(j)}|+1)^d(|x_{\sigma(j-1)}|+1)^d} \prod_{\substack{i=1\\ i\neq j-1,j}}^{n} \frac{1}{(|x_{\sigma(i)} - x_{\sigma(i-1)}|+1)^d} \tag{4.5}$$

where $C = C(V, \nu, d) > 0$ is a constant depending only on V, ν, d; $C(A_p)$ does not depend on n, $\bar{t}$, $\bar{x}$, σ.

Proof of the Lemma 2. For simplicity we consider the case when $\sigma = \sigma_o \in P_n$, $\sigma_o(i) = i$, $i = 1, \ldots, n$. We have

$$U_{\bar{t},\bar{x}}^\sigma(A_p) = \prod_{i=2}^{n}(f^p_{t_i,x_i}, f^p_{t_{i-1},x_{i-1}})(f^p_{t_1,x_1}, g_2)b^*(f^p_{t_n,x_n})b(g_1)|_{\mathscr{H}_p}$$

The bound (4.5) can be proved by integration by parts d times in each variable $k^{(i)}$, $k = (k^{(1)}, \ldots, k^{(\nu)})$ in

$$(f^p_{t_i,x_i}, f^p_{t_{i-1},x_{i-1}}) = \int_{\mathbb{R}^\nu} dk|\hat{f}(k)|^2 e^{i\varepsilon^\delta(t_i - t_{i-1})h_p(k) + i(x_i - x_{i-1},k)}$$

Then we have

$$|(f^p_{t_i,x_i}, f^p_{t_{i-1},x_{i-1}})| \leqslant \frac{C(f,d)\,(|\varepsilon^\delta(t_i - t_{i-1})|)^d}{(1+\|x_i - x_{i-1}\|)^d} \leqslant \frac{C(f,d)s_o^d}{(1+\|x_i - x_{i-1}\|)^d} \tag{4.6}$$

for $i = 2, \ldots, n$ and

$$|(f^p_{t_1,x_1}, g_1)| \leqslant \frac{C(f,A,d)s_o^d}{(1+\|x_1\|)^d} \tag{4.7}$$

where $C(f,d)$ depends on f, d and $C(f,A,d)$ depends on f, A, d.

From (4.6), (4.7) it follows (4.5).

Remark 4. The bound (4.5) allows us to integrate in (3.5) over the space variables $x_1, \ldots, x_n$.

To integrate in the time variables $t_1, \ldots, t_n$ we estimate the expression

$$\omega_\beta(V^r_{t_{\sigma(n)},x_{\sigma(n)}} \cdots V^r_{t_{\sigma(1)},x_{\sigma(1)}}) \tag{4.8}$$

For this it is convenient to use the diagrams.

Let $\langle \circ \rangle$ be a quasi free state on $\mathfrak{A}_r$. Graphically the monomial

$$V_v = a^\#(f_{v,1}) \cdots a^\#(f_{v,N_v}), \qquad v = 1, \ldots, n$$

will be drawn as the vertex v and N_v free legs $(v, 1), \ldots, (v, N_v)$.

Lemma 3. *Let (v_o, p_o) be any fixed leg of V_{v_o}. For any quase-free state $\langle \circ \rangle$ the following formula takes place:*

$$\langle V_1 \cdots V_n \rangle = \sum_{\substack{(v,p) \\ (v,p) \neq (v_o,p_o)}} g_{(v_o,p_o),(v,p)} \langle V_1 \cdots \check{V}_{v;p} \cdots \check{V}_{v_o,p_o} \cdots V_n \rangle$$

where

$$\check{V}_{v,p} = a^\#(f_{v,1}) \cdots a^\#(f_{v,p-1}) a^\#(f_{v,p+1}) \cdots a^\#(f_{v,N_v}),$$

$$g_{(v_o,p_o),(v,p)} = \begin{cases} (-1)^q \langle a^\#(f_{v_o,p_o}) a^\#(f_{v,p}) \rangle, & \text{if } v > v_o, \\ & \text{or} \quad v = v_o \quad \text{and } p > p_o \\ (-1)^q \langle a^\#(f_{v,p}) a^\#(f_{v_o,p_o}) \rangle, & \text{if } v < v_o, \\ \text{or} \quad v = v_o \quad \text{and } p < p_o & \end{cases}$$

and q is the number of creation-annihilation operators in $V_1 \cdots V_n$ between $a^\#(f_{v_o,p_o})$ and $a^\#(f_{v,p})$.

Proof of the Lemma 3 is similar to the integration-by part formulae in the classical case [31].

We say that in (4.9) the term

$$g_{(v_o,p_o),(v,p)} \langle V_1 \cdots \check{V}_{v;p} \cdots \check{V}_{v_o,p_o} \cdots V_n \rangle$$

in $\langle V_1 \cdots V_n \rangle$ corresponds to the coupling of the leg (v_o, p_o) of the vertex v_o with the leg (v, p) of the vertex v.

With the line $l = ((v_o, p_o), (v, p))$ we associate the number

$$g_{(v_o,p_o),(v,p)}$$

which one calls the contribution of this line.

In other words the formula (4.9) means that a quasi-free state of the product of the monomials may be represented as the sum of the terms in which the fixed leg (v_o, p_o) is connected with some leg (v, p).

We shall use the formula (4.9) many times to represent (4.8) as a sum of admissible diagrams which we define below.

We can assume that the vertices of a diagram are the points $t_1, \ldots, t_n$ on the real line. The vertices are numerated in the order of increasing of the times t_i, $0 < t_1 < \cdots < t_n < t$ and this ordering does not depend on $\sigma \in P_n$. Then the line of a diagram is the pair $(t_v, t_{v'})$. We identify the line $(t_v, t_{v'})$ with the coupling of the leg (v, p) and the leg (v', p'), for any $1 \leqslant p \leqslant N_v$, $1 \leqslant p' \leqslant N_{v'}$, which appears in the right-hand side of (4.9) for (4.8). We say that a contribution of this line $(t_v, t_{v'})$ is equal to

$$g_l(t_v - t_{v'}, x_v - x_{v'})$$

from (4.9), where $l = ((v, p), (v', p'))$. We note that our estimates of $g_l(\circ)$ will be uniform in p, p'.

We define the algorithm which constructs all admissible diagrams and only them. The set of all admissible diagrams thus constructured is denoted by A_n. We shall define also some distinguished subset $D_n \subset A_n$ which we call surviving admissible diagrams. We shall prove that exactly these diagrams survive.

The algorithm consists of not more than $2Nn$ steps. We numerate these steps by

$$(1, 1), \ldots, (1, 2N), \ldots, (n, 1), \ldots, (n, 2N).$$

On the steps $(1, 1)$ we construct a line from the leg $(1, 1)$ to some leg (v', p'). We call one of the special type if $v' = v$ or $v' = v + 1$. Then we proceed by induction. Let the lines $l_1, \ldots, l_k$ be already constructed and we are on the step (v, p). The rules of the algorithm are the following:

1. On each step (v, p) we construct a line from the leg (v, p) if the following conditions 2–3, 5 are satisfied;
2. If on the step (v, p) we did not construct a line then on the steps (v, p'), $p' > p$ we also do not construct lines, i.e. we begin with the new vertex $v + 1$;
3. If on the step (v, p) we construct a line which is not special then on the steps $(v, p'), p' > p$, we do not construct lines;
4. If on the step (v, p) we construct a line from the leg (v, p) to some leg (v', p') then we call v' "used on the step (v, p)";
5. If the vertex v was used on the earlier steps then on the step (v, p), we do not construct a line;
6. The algorithm stops on the step $(n, 2N)$.

The admissible diagram is the graph without free legs and its lines are constructed by this algorithm.

We denote the set of all the admissible diagrams by A_n and the set of all the diagrams only with the lines of the special type by D_n, $D_n \subset A_n$. Let us call the elements of D_n surviving admissible diagrams. From the condition

$$\omega_\beta(V^r) = 0$$

and from the algorithm of the construction of admissible diagrams it follows that for any vertex of a diagram from A_n there exists at least one line from this vertex which is not a loop.

We remark that the number of the connected components of any admissible diagram $G \in A_n$ is bounded by $n/2$ for even n and $(n-1)/2$ for odd n.

We denote by $A_{n,k}$ the subset of A_n of diagrams with k the connected components.

Lemma 4. *$D_n \subset A_{n,n/2}$ if n is even and $D_n = \emptyset$ if n is odd.*

Lemma 5. *For any $\sigma \in P_n$ the following formula takes place*

$$\omega_\beta(V^r_{t_{\sigma(n)},x_{\sigma(n)}} \cdots V^r_{t_{\sigma(1)},x_{\sigma(1)}}) = \sum_{G \in D_n} J^\sigma_G + \sum_{G \in A_n \setminus D_n} J^\sigma_G \omega_\beta(W_G) \tag{4.11}$$

where

$$J^\sigma_G = \prod_{l \in G} g^\sigma_l(t_v - t_{v'}, x_v - x_{v'}) \tag{4.12}$$

$$l = ((v, p), v', p')$$

and W_G is the set of all the free legs (which are not coupled) in $V^r_{t_{\sigma(n)},x_{\sigma(n)}} \cdots V^r_{t_{\sigma(1)},x_{\sigma(1)}}$.

Proofs of the Lemma 4 and the Lemma 5 evidently follow from the description of the algorithm of A_n and from the fomula (4.9).

Lemma 6. *Let $v \geqslant 3$, $\delta \geqslant 2$, $d = 2v$ and $\omega_\beta(V^r) = 0$. Then there exists some constant $C = C(V, v) > 0$ such that for any $n \in \mathbb{N}$, $s_o > 0$, $\sigma \in P_n$, $A_p \in \mathfrak{A}^o_p$ the following estimate holds if $\varepsilon^2 t \in [0, s_o]$*

$$\left\| \int_{(\mathbb{R}^v)^n} d\bar{x}\, \omega(W^{n;\sigma}_{\bar{t},\bar{x}}(A_p \otimes 1_r)) \right\| \leqslant C^n C(A_p) s_o^{dn} \sum_{\substack{r \in G \\ r = (v, v')}} \frac{1}{(1 + |t_v - t_{v'}|)^{v/2}} \tag{4.13}$$

where $C(A_p)$ does not depend on n, s_o, $\bar{t} \in \Delta^t_n$, ε, σ.

Lemma 7. *Let $g \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$, $g \geqslant 0$ symmetric and Riemann integrable. Then there exists $C > 0$ not depending on n, k, t such that*

$$\sum_{G \in A_{n,k}} \int_{\Delta^t_n} dt_1 \cdots dt_n \prod_{\substack{r \in G \\ r = (v, v')}} g(t_v - t_{v'}) \leqslant \frac{C^n t^k}{k!} (\|g\|_1)^{n-k} \tag{4.14}$$

Remark 5. It is evident that for the function

$$g(t) = \frac{1}{(1 + |t|)^{v/2}} \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R}), \qquad g \geqslant 0,$$

The lemma 1 follows from the Lemma 6 and the Lemma 7.

Proof of the Lemma 6. For $x_v, x_{v'} \in \mathbb{R}^\nu$, $t_v, t_{v'} \in [0, t]$ if $v, v' > 0$ we have

$$\begin{aligned}|g_l^\sigma(t_v - t_{v'}, x_v - x_{v'})| &= |\omega_\beta(a^*(f^r_{t_v, x_v})a(f^r_{t_{v'}, x_{v'}}))| \\ &= |(f^r_{t_v, x_v}, \tilde{B}_\beta f^r_{t_{v'}, x_{v'}})| \\ &= C\left|\int \exp\left((i(t_v - t_{v'})h_r(k) + i(x_v - x_{v'}, k)\right)\hat{f}_p(k)\bar{\hat{f}}_{p'}(k)\tilde{b}_\beta(k)\,dk\right| \\ &\leqslant \frac{C(V)}{(1 + |t_v - t_{v'}|)^{\nu/2}}, \end{aligned} \tag{4.15}$$

where the operator $\tilde{B}_\beta$ may be equal to either B_β or $1_r^{(1)} - B_\beta$ and $C(V)$ does not depend on n, x_v, $x_{v'}$, A_p.

If v or v' is equal 0 then we have

$$| g_l^\sigma(t_v - t_{v'}, x_v - x_{v'}) \leqslant \frac{C(V, A_p)}{(1 + |t_v - t_{v'}|)^{\nu/2}} \tag{4.16}$$

where $C(V, A_p)$ does not depend on n, x_v, $x_{v'}$.

The Lemma 6 follows from the bounds (4.15), (4.16) and the bound (4.13) for $\langle \circ \rangle \equiv \omega_\beta$.

5. Proof of the Lemma 7

We write the inequality (4.14) in the form

$$\sum_{G \in A_{n,k}} \int_{\Delta_n^t} dt_1 \cdots dt_n \prod_{\substack{r \in G \\ r = (v, v')}} g(t_v - t_{v'}) \leqslant C^n \int_{\Delta_k^t} ds_1 \cdots ds_k \left(\int_{\mathbb{R}} g(r)\,dr\right)^{n-k} \tag{5.1}$$

where $C > 0$ does not depend on n.

We shall call the lines of G old lines. For any diagram $G \in A_{n,k}$ we do the following procedure. We add the vertex 0 to the diagram G and we add k lines of the form $(v, v+1)$ (which we shall call new) to G so that the diagram G will become connected. It is easy to see that such k new lines exist. The contribution of any new line will be equal to 1, by definition. We denote the resulting diagram by $\tilde{G}$. Let $\tilde{A}_{n,k}$ be the set of all the diagrams $\tilde{G}$, $G \in A_{n,k}$.

It is evident that in the left-hand side of (5.1) we can change $\sum_{G \in A_{n,k}}$ on $\sum_{\tilde{G} \in \tilde{A}_{n,k}}$.

We approximate both sides of (5.1) by Riemannian sums

$$\delta^n \sum_{0 < t_1 < \cdots < t_n < t} \left(\sum_{\tilde{G} \in \tilde{A}_{n,k}} \prod_{\substack{r \in \tilde{G} \\ r = (v, v')}} g(t_v - t_{v'}) \leqslant C^n \delta^n \left(\sum_{\substack{0 < s_1 + \cdots + s_k < t \\ s_i > 0}} \cdots \sum 1\right)\right.$$
$$\left(\sum_{r_1 \neq 0} \cdots \sum_{r_{n-k} \neq 0} \prod_{i=1}^{n-k} g(r_i)\right) \tag{5.2}$$

where $t_i, r_i, s_i \in \mathbb{Z}_\delta$ ($\mathbb{Z}_\delta$ is one-dimensional δ-lattice).

We omit δ^n in (5.2) and we define for fixed $(s_1, \ldots, s_k, r_1, \ldots, r_{n-k})$, $0 < s_1 + \cdots + s_k < t, s_i > 0, i = 1, \ldots, k, 0 < r_i < t, i = 1, \ldots, n-k$, the algorithm by which one can construct not more than C^n admissible diagrams from $\tilde{A}_{n,k}$ for some $0 < t_1 < \cdots < t_n < t$ with the contribution equal to

$$g(r_1) \cdots g(r_{n-k})$$

The algorithm consists of not more than $2N(n+1)$ steps. We numerate these steps by

$$(0, 1), \ldots, (0, 2N), \ldots, (n, 1), \ldots, (n, 2N).$$

On the step (0, 1) we take s_1 and construct a new line from $t_0 = 0$ to s_1. This line connects vertices $t_o = 0$ and t_1 in the left-hand side of (5.2) which has the contribution equal to 1. The following step has the number (1, 1). We take r_1 and construct an old line from s_1 to the vertex $s_1 + r_1$. This line has the contribution $g(r_1) = g((s_1 + r_1) - s_1)$. This old line connects t_1 and some vertex in the left-hand side of (5.3). Then we proceed by induction.

Let the old lines $r_1, \ldots, r_a$ and the new line $s_1, \ldots, s_b$ be already constructed and we are on the step (i, j), then the rules of the algorithm are the following.

1. On each step (i, j) we decide whether to construct 1 old line r_{b+1} or 1 new line s_{a+1} or not to construct lines at all on this step;
2. If on the step (i, j) we decided not to construct a line then on the steps (i, j'), $j' > j$ we also do not construct lines;
3. The new line can be constructed only on the step $(i, 1)$ and then on the steps (i, j'), $j' > 1$, we do not construct lines;
4. On the step $(i, 1)$ we choose one of the constructed vertices v_i and on the steps $(i, 1), \ldots (i, 2N)$ we can draw the lines from v_i. We call v_i "used on the step $(i, 1)$";
5. The choice of v_i is uniquely defined by the rule: v_i is the first already constructed vertex not used in the earlier steps.
6. The algorithm stops either on the step $(n, 2N)$ or when there are used vertices or when $(n-k)$ old lines and k new lines are constructed.

It is evident that each admissible diagram G will be constructed and each array $(s_1, \ldots, s_k, r_1, \ldots, r_{n-k})$ is used not more than C^n times.

The Lemma 7 is proved.

6. The interaction of the general form

We have proved the Theorem 1 for the interaction of the special form (4.1), (4.2). Here we say briefly about a proof of the Theorem 1 for the interaction of the general form.

Lemma 8. *$V \in \mathscr{A}_o$ can be represented as*

$$V = \sum_{m,n} \sum_{L} \sum_{\bar{N}} C_{m,n}(\bar{L}, \bar{N}) b^*(e_{L_1}) b(e_{L_2})|_{\mathscr{H}_p} \otimes a^*(e_{N_1}) \cdots a^*(e_{N_m}) a(e_{N_{m+1}}) \cdots a(e_{N_{m+n}}) \tag{6.1}$$

where $\bar{L} = (L_1, L_2)$, $\bar{N} = (N_1, \ldots, N_{m+n})$, $\|e_K\| = 1$, $e_K \in C_o^\infty(\mathbb{R}^\nu)$, K, L_i, $N_j \in \mathbb{Z}^\nu$, $i = 1, 2$, $j = 1, \ldots, m+n$,

$$\sum_{m,n} \sum_{\bar{L}} \sum_{\bar{N}} |C_{m,n}(\bar{L}, \bar{N})| < \infty \tag{6.2}$$

and if we fixed $\theta > 0$ *then for any* $\hat{f} \in C_o^\infty(\mathbb{R}^\nu)$, N, $N' \in \mathbb{Z}^\nu$, x, $x' \in \mathbb{R}^\nu$, t, $t' \in [0, s_o/(\varepsilon^2)]$ *the following bounds hold*

$$|(e^p_{N;t,x}, e^p_{N';t',x'})| \leqslant \frac{C(d)s_o^d}{(1 + \|(x+N) - (x'+N')\|)^d} \tag{6.3}$$

$$|(e^p_{N;t,x}, f)| \leqslant \frac{C(f, d)s_o^d}{(1 + \|x+N\|)^d} \tag{6.4}$$

$$|(e^r_{N;t,x}, \tilde{B}_\beta e^r_{N';t',x'})| \leqslant \frac{C(\theta)}{(1 + |t - t'|)^{\nu/2 - \theta}} \tag{6.5}$$

where the operator $\tilde{B}_\beta$ *may be equal to either* B_β *or* $1_r^{(1)} - B_\beta$ *or* 0_r *and* $C(\theta)$ *does not depend on* N, N', x, x', t, t'.

Remark 6. It is easy to see that we can use the bounds (6.3), (6.4), (6.5) instead of the bounds (4.6), (4.7), (4.15), correspondingly, and (6.2) and prove the Theorem 1.

Proof of the Lemma 8. Let the Fourier transforms of the kernels $\mathscr{V}_{m,n}$ be

$\hat{\mathscr{V}}_{m,n} \in C_o^\infty(\mathbb{R}^{\nu E})$, $\qquad E = m + n + 2.$

Then there exist $M \in \mathbb{R}_+$ such that

$\text{supp}\, \hat{\mathscr{V}}_{m,n} \subseteq [-M, M]^{\nu E}$

for any m, n.

We put

$$e_n(k) = \frac{\kappa(k) \exp(2\pi i n k/A)}{d_n}, \quad k \in \mathbb{R} \tag{6.6}$$

where the function $\kappa(k)$ is from $C_o^\infty(\mathbb{R})$, $0 \leqslant \kappa(k) \leqslant 1$, $\kappa(k) = 1$ for $|k| \leqslant B$ and $\kappa(k) = 0$ for all $|k| \geqslant B + 1$. The constants d_n are chosen so that $\|e_n\| = 1$.

We choose A and B so that $M < B < 2M < A$. Then we have

$$\hat{\mathscr{V}}_{m,n} = \sum_{\bar{L},\bar{N}} \sum C_{m,n}(\bar{L}, \bar{N}) \prod_{j=1}^{\nu E} e_{n_j}(k_j) \tag{6.7}$$

where $\bar{L} = (L_1, L_2)$, $\bar{N} = (N_1, \ldots, N_{m+n})$.

It is evident that for any $q \in \mathbb{N}$ there exists $C(q)$ such that

$$|C_{m,n}(\bar{L}, \bar{N})| \leqslant \frac{C(q)}{(1 + |\bar{N}| + |\bar{L}|)^q}, \qquad |\bar{L}| = \sum_i |L_i|, \qquad |\bar{N}| = \sum_i |n_i| \tag{6.8}$$

and for $q > \nu E$ the following bound takes place

$$\sum_{\bar{L}} \sum_{\bar{N}} |C_{m,n}(\bar{L}, \bar{N})| < \infty \tag{6.9}$$

Therefore, the functions e_N and $e_{N'}$

$$e_N = \prod_{j=1}^{\nu} e_{n_j}(k_j) \qquad e_{N'} = \prod_{j=1}^{\nu} e_{n'_j}(k_j)$$

satisfy the bounds (6.3), (6.4), (6.5).

If $\hat{\mathscr{V}}_{m,n} \in S(\mathbb{R}^{\nu E})$, we shall use the partition of unity

$$\sum_{\gamma} \alpha_{\gamma}(k) = 1 \tag{6.10}$$

where

$$\operatorname{diam} \operatorname{supp} \alpha_{\gamma} \leqslant \text{const}$$

uniformly in γ.

Then we represent the kernels $\hat{\mathscr{V}}_{m,n}$ as $\Sigma_{\gamma} \hat{\mathscr{V}}_{m,n}\alpha_{\gamma}$ of the kernels from $C_o^{\infty}(\mathbb{R}^{\nu E})$ and by using an expansion similar to (6.6) for $\hat{\mathscr{V}}_{m,n}\alpha_{\gamma}$ (with correspondingly shifted functions e_N^{γ}) we repeat the proof of the Lemma 8.

7. The estimation of the contribution of all nonsurviving admissible diagrams

In this section we prove that in the weak coupling limit all the admissible diagrams from $A_{n,k} \backslash D_n$, $k \leqslant n$, give the zero contribution to T_s.

Definition. We define $L_1^{\kappa}(\mathbb{R})$ for $\kappa > 0$ by:

$$\|g\|^{\kappa} \equiv \int_{\mathbb{R}} (1 + |t|)^{\kappa} |g(t)| \, dt < \infty.$$

Lemma 9. *Let $g \in L_1^{\kappa}(\mathbb{R}) \cap L_{\infty}(\mathbb{R})$, $\kappa > 0$, $g \geqslant 0$ and $n = 2m$. Then there exist $C > 0$, $\theta > 0$ not depending on n, t such that for $t > 1$*

$$\sum_{G \in A_{n,m} \backslash D_n} \int_{\Delta_n^t} dt_1 \cdots dt_n \prod_{\substack{r \in G \\ r = (v,v')}} g(t_v - t_{v'}) \leqslant \frac{C^n t^{n/2 - \theta}}{(n/2)!} \tag{7.1}$$

Proof of the Lemma 9. This lemma follows from Lemma 7. Indeed, for each diagram $G \in A_{n,m} \backslash D_n$, $n = 3m$, there exist some line (v', v'') and $v \in \{1, \ldots, n-1\}$, $v' \leqslant v$, $v'' \leqslant v + 1$, such that the line (v', v'') belongs to the diagram G and if we add the line $(v, v+1)$ to G then G will have $(m-1)$ connected components

Let $0 < \gamma < 1$, $\gamma + \kappa > 1$ then we have

$$g(t_{v'} - t_{v''}) = (1 + |t_{v'} - t_{v''}|)^{\kappa} g(t_{v'} - t_{v''}) \frac{(1 + |t_{v'} - t_{v''}|)^{\gamma}}{(1 + |t_{v'} - t_{v''}|)^{\kappa + \gamma}}$$

$$\leqslant (1 + |t_{v'} - t_{v''}|)^{\kappa} g(t_{v'} - t_{v''})\, t^{\gamma} \frac{1}{(1 + |t_{v'} - t_{v''}|)^{\kappa + \gamma}}$$

We add the line $(v, v + 1)$ to G and define its contribution as

$$\frac{1}{(1 + |t_{v'} - t_{v''}|)^{\kappa + \gamma}}$$

Then, for the function

$$\hat{g}(t) = \max \left\{ (1 + |t|)^{\kappa} g(t), \frac{1}{(1 + |t_{v'} - t_{v''}|)^{\kappa + \gamma}} \right\} \in L_1(\mathbb{R}) \cap L_{\infty}(\mathbb{R}), \qquad \hat{g} \geqslant 0,$$

it follows from the Lemma 7 that the left-hand side of (7.1) is bounded by

$$\frac{C^n t^{m-1} t^{\gamma}}{m!} \tag{7.2}$$

where C, γ do not depend on n, m.

For $\theta = 1 - \gamma$ the bound (7.1) holds. Lemma 9 is proved.

Remark 7. By using the bounds (4.5), (4.14), (7.1) and by repeating the proof of the bound (3.6) we see that the contribution of all nonsurviving admissible diagrams can be bounded by

$$\varepsilon^{\theta/2} C(A_p) \tag{7.3}$$

if $0 < \varepsilon < \min\{1/4C^2 s_o^d, 1/s_o^d\}$. Hence only surviving admissible diagrams may have the nonzero limit if $\varepsilon \to 0$, $\varepsilon^2 t \to s$, $s > 0$.

8. Case $\delta > 2$. The generator of the semigroup T_s

In this section we prove that for $\delta > 2$ T_s is a semigroup. Also we calculate the generator L of T_s.

We denote

$$\omega([V_{t_n, x_n}, [\dots, [V_{t_1, x_1}, A_p \otimes 1_r] \cdots]])_G$$

$$\stackrel{\text{def}}{=} \sum_{\sigma \in P_n} \theta(\sigma) V^p_{t_{\sigma(n)}, x_{\sigma(n)}} \cdots V^p_{t_{\sigma(j)}, x_{\sigma(j)}} (A_p \otimes 1_r) V^p_{t_{\sigma(j-1)}, x_{\sigma(j-1)}}$$

$$\cdots V^p_{t_{\sigma(1)}, x_{\sigma(1)}} \omega_{\beta}(V^r_{t_{\sigma(n)}, x_{\sigma(n)}} \cdots V^r_{t_{\sigma(1)}, x_{\sigma(1)}})_G$$

where $\theta(\sigma)$ and $j = j(\sigma)$ are the same as in (3.3); $\omega_{\beta}(\circ)$ denotes the term $J^{\sigma}_G \omega_{\beta}(W_G)$ in the expression (4.9).

Lemma 10. *For $n = 2m$ the following formula takes place*

$$\sum_{G \in D_{2m}} \omega([V_{t_{2m},x_{2m}}, [\ldots, [V_{t_1,x_1}, A_p \otimes 1_r]\ldots]])_G$$
$$= \omega(V_{t_{2m},x_{2m}}, [V_{t_{2m-1},x_{2m-1}}, \omega([\ldots, \omega([V_{t_2,x_2}, [V_{t_1,x_1} A_p \otimes 1_r])\cdots])]]) \quad (8.1)$$

Proof of the Lemma 10 follows from the definitions of the map ω and of surviving admissible diagrams.

For $n = 2m$ and $A_p \in \mathfrak{A}_p^o$ let us put

$$L_t^{(n)}(A_p) = (i\varepsilon)^n \int_{\Delta^t} d\bar{t} \int_{(\mathbb{R}^\nu)^n} d\bar{x}\, \omega([V_{t_n,x_n}, [\ldots, [V_{t_1,x_1}, A_p \otimes 1_r]\cdots]])$$

Lemma 11. *Under the conditions of the Theorem 2, for $s \in \mathbb{R}_+$, $A_p \in \mathfrak{A}_p^o$ there exist the limits*

$$\lim_{\substack{\varepsilon^2 t = s \\ \varepsilon \to 0}} L_t^{(2)}(A_p) = sL(A_p) \tag{8.2}$$

$$\lim_{\substack{\varepsilon^2 t = s \\ \varepsilon \to 0}} L_t^{(n)}(A_p) = \frac{s^m L^m(A_p)}{m!} \tag{8.3}$$

where

$$L(A_p) = -\iint_{\mathbb{R}^{2\nu}} dx_1\, dx_2 \int_0^\infty dt\, \omega([\tau_t^r(V_{x_2}), [V_{x_1}, A_p \otimes 1_r]]) \tag{8.4}$$

Moreover, the convergence in (8.2), (8.3) is uniform on $[0, s_o]$ for any $s_o \in \mathbb{R}_+$ and T_s is the strongly continuous semigroup and the operator L is its generator, i.e. ($A_p \in \mathfrak{A}_p^o$)

$$T_s(A_p) = \sum_{m=1}^{\infty} \frac{s^m}{m!} L^m(A_p) = \exp(sL)(A_p) \tag{8.5}$$

Proof of the Lemma 11. We begin with (8.2), (8.3). At first we prove the Lemma 11 for the case when $V \in \mathscr{A}^o$.

Let $u_x \in L_1^\kappa(\mathbb{R})$, $\kappa > 0$, $x \in \mathbb{R}$ and for some $C > 0$

$$\|u_x\|^\kappa \leqslant C$$

holds.

Then for $\delta > 2$, $i \in \{1, 2\}$ and for any compact set $\mathscr{K} \subset \mathbb{R}$ we have

$$\lim_{\substack{\varepsilon^2 t = s \\ \varepsilon \to 0}} \varepsilon^2 \int_{\Delta_2^t} u_x(t_2 - t_1) \exp(i\varepsilon^\delta t_i K)\, dt_1\, dt_2$$
$$= \int_0^\infty u_x(t)\, dt \begin{cases} s & \text{if } K = 0 \\ \lim\limits_{\substack{\varepsilon^2 t = s \\ \varepsilon \to 0}} \varepsilon^2 \dfrac{(e^{i\varepsilon^{\delta-2} sK} - 1)}{i\varepsilon^\delta K}, & \text{if } K \neq 0 \end{cases} = s \int_0^\infty u_x(t)\, dt \tag{8.6}$$

uniformly over $K \in \mathscr{K}$.

For $\delta > 2$, $n = 2m$, $m \geqslant 1$ it is easy to show that for any compact set $\mathscr{K} \subset \mathbb{R}^\nu$ the n-dimensional analog of (8.6) takes place

$$\lim_{\substack{\varepsilon^2 t = s \\ \varepsilon \to 0}} \varepsilon^{2m} \int_{\Delta_n^t} \prod_{i=1}^{m} u_{x_i}(t_{2i} - t_{2i-1}) \exp\left(i\varepsilon^\delta \sum_{i=0}^{2m-1} (t_{\sigma(i+1)} - t_{\sigma(i)}) K_{i+1} \right) dt_1 \cdots dt_m$$

$$= \prod_{i=1}^{m} \left(\int_o^\infty u_{x_i}(t)\, dt \right) \frac{s^m}{m!} \tag{8.7}$$

uniformly over $K \in \mathscr{K}$, $K = (K_1, \ldots, K_n)$ (to prove (8.7) one can use the following substitution $s_i = \varepsilon^2 t_i$, $i = 1, \ldots, n$).

By changing the order of integration in (8.3) and by using (8.6), (8.7) for

$$u_x(t) = \omega_\beta(V^r_{t,x} V^r)$$

we get (8.2), (8.3) and (8.4), correspondingly.

The formula (8.5) follows from (8.2), (8.3) and (8.4). The Theorem 2 is proved for $V \in \mathscr{A}^o$.

When $V \in \mathfrak{A}^S$ the proof of this Lemma is similar. In this case we can use the following bound

$$|\exp(i\varepsilon^\delta t h(\bar{k})) - 1| \leqslant C|\varepsilon|^{2-\delta} s_o |\bar{k}|^\theta \tag{8.8}$$

for any $t \in [0, s_o/(\varepsilon^2)]$, $k \in \mathbb{R}^\nu$, $\theta > 0$ is constant not depending on ε, t, s_o, k. For $h = (\bar{k}, \bar{k})$ it is evident that $\theta = 2$.

9. Case $\delta = 2$

In the case $\delta = 2$ the family of the maps T_s is not a semigroup. The following formula takes place

$$\lim_{\substack{\varepsilon^2 t = s \\ \varepsilon \to 0}} \varepsilon^2 \int_0^t e^{i\varepsilon^2 t_1 K}\, dt_1 = \begin{cases} s, & \text{if } K = 0 \\ \dfrac{(e^{isK} - 1)}{iK}, & \text{if } K \neq 0 \end{cases} \stackrel{\text{def}}{=} F_2(K, s) \tag{9.1}$$

where $F_2(K, s)$ is a nonlinear function on s and depends essentially on K. This means that T_s is not a semigroup.

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