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Asymptotic Behaviour in the Time Synchronization Model

Vadim Malyshev and Anatoli Manita

Abstract. There are two types $i = 1, 2$ of particles on the line $\mathbf{R}$, with N_i particles of type i. Each particle of type i moves with constant velocity v_i. Moreover, any particle of type $i = 1, 2$ jumps to any particle of type $j = 1, 2$ with rates $N_j^{-1}\alpha_{ij}$. We find phase transitions in the clusterization (synchronization) behaviour of this system of particles on different time scales $t = t(N)$ relative to $N = N_1 + N_2$.

1. The model

It is a pleasure to contribute to this volume devoted to the half-life birthday of Anatoly Vershik. He contributed a lot to probability theory and asymptotic analysis and these are exactly the topics of this paper.

The simplest formulation of the model which we consider here is in terms of the particle system. On the real line there are N_1 particles of type 1 and N_2 particles of type 2, $N = N_1 + N_2$. Each particle of type $i = 1, 2$ performs two independent movements. First of all, it moves with constant speed v_i in the positive direction. We assume further that v_i are constant and different, thus we can assume without loss of generality that $0 \leq v_1 < v_2$. The degenerate case $v_1 = v_2$ is different and will be considered separately.

Secondly, at any time interval $[t, t + dt]$ each particle of type i independently of the others with probability $\alpha_{ij}dt$ decides to make a jump to some particle of type j and chooses the coordinate of the j-type particle, where to jump, among the particles of type j, with probability $\frac{1}{N_j}$. Here α_{ij} are given nonnegative parameters for $i, j = 1, 2$. Further on, unless otherwise stated, we assume that $\alpha_{11} = \alpha_{22} = 0, \alpha_{12}, \alpha_{21} > 0$.

After such an instantaneous jump, the particle of type i continues the movement with the same velocity v_i. This defines the continuous time Markov chain $\left\{x_k^{(i)}(t)\right\}$, $i = 1, 2$ and $k = 1, \ldots, N_i$, where $x_k^{(i)}(t)$ is the coordinate of k-th particle of type i at time t. We assume that the initial coordinates $x_k^{(i)}(0)$ of the particles at time

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0 are given. We are interested in the long time evolution of this system on various scales with $N \to \infty$, $t = t(N) \to \infty$.

In different terms, this can be interpreted as the time synchronization problem. In general, the time synchronization problem can be presented as follows. There are N systems (processors, units, persons, etc.). There is an absolute (physical) time t, but each processor j fulfills a homogeneous job in its own proper time $t_j = v_j t, v_j > 0$. Proper time is measured by the amount v_j of the job, accomplished by the processor for the unit of the physical time, if it is disjoint from other processors. However, there is a communication between each pair of processors, which should lead to a drastic change of their proper times. In our case the coordinates $x_k^{(i)}(t)$ can be interpreted as the modified proper times of the particles-processors, the nonmodified proper time being $x_k^{(i)}(0) + v_i t$.

There can be many variants of the exact formulation of such a problem; see [**GMP, MSh1, MiMi, BT, MSi**]. We will call the model considered here the basic model, because there are no restrictions on the jump process. Many other problems include such restrictions; for example, only jumps to the left are allowed. Due to the absence of restrictions, this problem, as we will see below, is a "linear problem" in the sense that after scalings it leads to linear equations. Despite this, it has nontrivial behaviour; one sees a different picture at different time scales.

There are, however, other interesting interpretations of this model, related to psychology, biology and physics. For example, in social psychology, perception of time and life tempo strongly depends on the social contacts and intercourse. We will not give the details here.

2. Main results

We show that the process consists of three consecutive stages: initial desynchronization up to the critical scale, critical slow down of desynchronization and final stabilization.

Final stabilization. The first theorem shows that for N_i fixed and $t \to \infty$ there is a synchronization: all particles asymptotically, as $t \to \infty$, move with the same constant velocity v, that is, like vt. However it does not say how fluctuations depend on N_i.

Put

$$m(t) = \min_{i,k} x_k^{(i)}(t) .$$

THEOREM 2.1. *For any fixed N_1, N_2 there exists $v = v(N_1, N_2) > 0$ such that for any $i = 1,2$ and any $k = 1, \ldots, N_i$ a.s.*

$$\lim_{t\to\infty} \frac{x_k^{(i)}(t)}{t} = v .$$

Moreover, the distribution of the vector $\left\{x_k^{(i)}(t) - m(t),\ i = 1,2 \text{ and } k = 1, \ldots, N_i\right\}$ tends to a stationary distribution.

The velocity v will be written down explicitly in terms of this distribution; it depends of course on α_{ij} and v_i. Note that both the velocity and the distribution do not depend on the initial coordinates.

Initial desynchronization. Now we consider the case when $N \to \infty$ but t is fixed. More exactly, we consider a sequence of pairs (N_1, N_2) such that $N_1, N_2 \to \infty$ so that $\frac{N_i}{N} \to c_i$, where $c_1 + c_2 = 1, c_i > 0$. It is convenient here to consider positive measures or generalized functions

$$m^{(N_i)}(t,x) = \frac{1}{N_i} \sum_k \delta(x - x_k^{(i)}(t)), \quad x \in \mathbf{R}_+,$$

defined by the coordinates of N_i particles of type i at time t. We assume that at time $t = 0$ for any bounded C^1-functions $\phi_i(x)$ on $\mathbf{R}$ the sequence $\langle m_i^{(N_i)}(0,.), \phi_i \rangle$ converges to some number.

THEOREM 2.2. *Then for any t there are weak deterministic limits*

$$\lim_{N\to\infty} \frac{1}{N} m_i^{(N_i)}(t,x) = m_i(t,x)$$

where $m_i(t,x)$ satisfy the equations

$$\frac{\partial m_1}{\partial t} + v_1 \frac{\partial m_1}{\partial x} = \alpha_{12}(m_2 - m_1), \tag{2.1}$$

$$\frac{\partial m_2}{\partial t} + v_2 \frac{\partial m_2}{\partial x} = \alpha_{21}(m_1 - m_2). \tag{2.2}$$

Now we want to study the asymptotic behaviour of $m_i(x,t)$ for $t \to \infty$. Denote

$$a_i(t) = \int x m_i(x,t) dx, \qquad d_i(t) = \int (x - a_i(t))^2 m_i(t,x)\, dx.$$

THEOREM 2.3. *There exist constants $v, d > 0$ such that as $t \to \infty$*

$$a_i(t) = vt + a_{i0} + o(1),$$

$$d_i(t) = dt + d_{i0} + o(1)$$

for some constants a_{i0}, d_{i0}. Moreover,

$$\Delta_i(x,t) = \frac{m_i(x,t) - a_i(t)}{\sqrt{d_i(t)}}$$

tends to $\frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})$ pointwise as $t \to \infty$.

Critical point and uniform estimates. Here we assume that $N_1 = [c_1 N]$, $N_2 = [c_2 N]$ for some $c_i > 0, c_1 + c_2 = 1$. Introduce the empirical means (mass centres) for types 1 and 2

$$\overline{x^{(i)}}(t) = \frac{1}{N_i} \sum_{k=1}^{N_i} x_k^{(i)}(t),$$

the empirical variances

$$S_i^2(t) = \frac{1}{N_i} \sum_{k=1}^{N_i} \left(x_k^{(i)}(t) - \overline{x^{(i)}}(t) \right)^2$$

and their means

$$\boldsymbol{\mu}_i(t) = \mathsf{E}\overline{x^{(i)}}(t), \quad \boldsymbol{l}_{12}(t) = \boldsymbol{\mu}_1(t) - \boldsymbol{\mu}_2(t), \quad R_i(t) = \mathsf{E}S_i^2(t).$$

The following asymptotic results hold for any sequence of pairs (N,t) with $N \to \infty$ and $t = t(N) \to \infty$.

THEOREM 2.4. *We have the following asymptotical results as $t \to \infty$:*

$$l_{12}(t) \to \frac{v_1 - v_2}{\alpha_{12} + \alpha_{21}}, \qquad \frac{\mu_i(t)}{t} \to \frac{\alpha_{12} v_2 + \alpha_{21} v_1}{\alpha_{12} + \alpha_{21}}.$$

Assume now that $N_i = c_i N$, where $c_i > 0, c_1 + c_2 = 1$.

THEOREM 2.5. *There are the following three regions of asymptotic behaviour, uniform in $t(N)$ for sufficiently large N:*

- *if* $\frac{t(N)}{N} \to 0$, *then* $R_i(t(N)) \sim h\varkappa_2 t(N)$,
- *if* $t = t(N) = sN$ *for some* $s > 0$, *then* $R_i(t(N)) \sim h\,(1 - e^{-\varkappa_2 s})N$,
- *if* $\frac{t(N)}{N} \to \infty$, *then* $R_i(t(N)) \sim hN$,

where the constant $\varkappa_2 > 0$ *can be explicitly calculated and*

$$h = \frac{2\alpha_{12}\alpha_{21}\,(v_1 - v_2)^2}{\varkappa_2(\alpha_{12} + \alpha_{21})^3}.$$

The proofs for uniform estimate results will be published elsewhere; see also [**MaMa1, MaMa2**].

3. Limit $t \to \infty$

In this section we will prove Theorem 2.1.

Two particles. It is useful to consider first the case when $N_1 = N_2 = 1$. Thus consider the process $(x^{(1)}(t), x^{(2)}(t))$. We will prove that there exist deterministic limits

$$\lim_{t\to\infty} \frac{x^{(i)}(t)}{t} = v$$

for $i = 1, 2$ and some $v > 0$; moreover the distribution of the random variable $\rho(t) = x^{(2)}(t) - x^{(1)}(t)$ tends to some distribution on $\mathbf{R}_+$.

We can assume that $v_1 = 0, v_2 > 0$. The Markov chain $\rho(t) = x^{(2)}(t) - x^{(1)}(t)$ on $\mathbf{R}_+$ satisfies the Doeblin condition; that is, from any $x \in \mathbf{R}_+$ there is a jump rate to 0, bounded away from zero. Here it equals $\alpha_{12} + \alpha_{21}$. It follows that $\rho(t)$ is ergodic. Then as $t \to \infty$ there exists the limiting (invariant) distribution $F(x)$ for $\rho(t)$. Let

$$t_1 < t_2 < \cdots$$

be the time moments when $x^{(1)}(t) = x^{(2)}(t)$. It is clear that $t_k - t_{k-1}$ are independent random variables, exponentially distributed with parameter $\alpha_{12} + \alpha_{21}$. It follows that $F(x)$ is exponential with the density

$$p(x) = \lambda \exp(-\lambda x), \quad \lambda = \frac{\alpha_{12} + \alpha_{21}}{v_2 - v_1}.$$

Thus, if the limits $\lim_{t\to\infty} \frac{x_i(t)}{t}$ exist, then they are equal. Let us prove that they exist and

$$v = v_1 + \alpha_{12} \int x p(x) dx. \tag{3.1}$$

In fact, the particle 1 moves with constant speed v_1 and performs on the time interval $[0, T]$ independent exponentially distributed jumps in the positive direction. As $T \to \infty$, the number of these jumps asymptotically equals $\alpha_{12} T$, and the mean jump asymptotically is $\int x p(x) dx$.

Similarly one can get

$$v = v_2 - \alpha_{21} \int xp(x)dx\,. \tag{3.2}$$

From this and (3.1) we have

$$v = \frac{\alpha_{21}v_1 + \alpha_{12}v_2}{\alpha_{21} + \alpha_{12}}\,.$$

General case. Let us prove first the second statement of the theorem. We can put $v_1 = 0$ and change the coordinate system putting $m(t) = 0$. Consider a configuration of particles at time t. Denote the particle, which has coordinate $m(t) = 0$ at time t, as particle 0. Let $p(t+2)$ be the probability that at time $t+2$ each particle will be inside the interval $[0, 2v_2]$. This probability can be (very roughly) estimated from below as

$$p(t+2) \geq \min(p_{01}p_2p_1, p_{02}p_3p_4)\,.$$

To prove this, consider first the case when the particle 0 has type 1. Under this condition $p(t+2)$ can be estimated from below as $p_{01}p_2p_1$, where p_{01} is the probability that particle 0 does not do any jumps in the time interval $(t, t+2)$, p_2 is the probability that each particle of type 2 jumps at least once to the particle 0 in the time interval $(t, t+1)$ and does not do any more jumps in the time interval $(t, t+2)$, p_1 is the probability that each particle of type 1 jumps to some particle of type 2 in the time interval $(t+1, t+2)$. Similarly, under the condition that the particle 0 has type 2, $p(t+2)$ can be estimated from below as $p_{02}p_3p_4$, where p_{02} is the probability that the particle 0 does not do any jumps in the time interval $(t, t+2)$, p_3 is the probability that each particle of type 1 jumps at least once to the particle 0 in the time interval $(t, t+1)$ and does not do any more jumps in the time interval $(t, t+2)$, p_4 is the probability that each particle of type 2 jumps to some particle of type 1 in the time interval $(t+1, t+2)$.

This means that the Markov chain $\mathcal{L} = \left\{x_k^{(i)}(t) - m(t), i = 1, 2 \text{ and } k = 1, ..., N_i\right\}$ satisfies the Doeblin condition. Then it is ergodic and has some stationary distribution. We will now write the formula for v, assuming however that $\alpha_{ii} = 0$. For this we need some marginals of this stationary distribution.

Let $A_i(t)$ be the event that at time t at the point $m(t)$ there is a particle of type i, and let $q_i = \lim_{t\to\infty} P(A_i(t))$ be the stationary (limiting) probability of A_i. Let $p_i(y)$ be the stationary conditional (under the condition A_i) probability density of the distance from m to the nearest particle. In the time interval $[T, T+dt]$ the particle in $m(t)$ moves with the speed v_i and moreover can make one jump. This gives, for example under the condition A_1, constant movement $v_1 dt$ of m and the jump of m to the nearest point with rate $\alpha_{12}dt$. Thus as $T \to \infty$ we have

$$\begin{aligned} \mathsf{E}(m(T+dt)\,|\,m(T)) - m(T) &= q_1\left(v_1 + \alpha_{12}\int yp_1(y)dy\right)dt \\ &+ q_2\left(v_2 + \alpha_{21}\int yp_2(y)dy\right)dt + o(1) \end{aligned}$$

and then

$$v = q_1\left(v_1 + \alpha_{12}\int yp_1(y)dy\right) + q_2\left(v_2 + \alpha_{21}\int yp_2(y)dy\right).$$

About Doeblin chains. In the standard theory of Doeblin chains, see [**D**], it is assumed that transition probabilities are absolutely continuous with respect to some positive measure μ on the state space.

If at time 0 all $x_k^{(i)}$ are different, then for any t it is true that all $x_k^{(i)}$ are different a.s. Thus transition probabilities (for example, for the embedded chain at times $0, 1, 2, \ldots$) are absolutely continuous with respect to Lebesgue measure on $(\mathbf{R}_+^{N_1-1} \times \mathbf{R}_+^{N_2}) \cup (\mathbf{R}_+^{N_1} \times \mathbf{R}_+^{N_2-1})$. If at time 0 some coordinates coincide, then a.s. in finite time τ they become all different.

4. Limit $N \to \infty$

It is very intuitive to introduce the following continuous model. Let $m_i(0,x)$, $x \in \mathbf{R}$, $i = 1, 2$, be positive smooth functions, $M_i = \int m_i(0,x)dx = 1$. We call them continuous mass distributions of type i at time $t = 0$. The dynamics of the masses is deterministic — during time dt from each element dm_1 of the mass the part $\alpha_{12}\, dt\, dm_1$ goes out and distributes correspondingly to the mass $m_2(x)$, namely it becomes the mass distribution with density $m_2(x)\, \alpha_{12}\, dt\, dm_1$, and vice-versa, interchanging 1 and 2. Moreover each mass element moves with velocities v_1 and v_2, respectively. From this we easily get linear equations (2.1)–(2.2) for mass distribution $m_i(t,x)$ at time t with the initial conditions

$$m_i(0,x) = f_i(x).$$

Now we will prove the convergence of the N particle model to the continuous model.

4.1. Convergence: the martingale problem. Here we prove Theorem 2.2. We consider the continuous time Markov process

$$\xi_{N_1,N_2}(t) = \left(x_1^{(1)}(t), \ldots, x_{N_1}^{(1)}(t); x_1^{(2)}(t), \ldots, x_{N_2}^{(2)}(t)\right) \tag{4.1}$$

with the state space $\mathbf{R}^{N_1+N_2}$. Its generator

$$\begin{aligned}
\left(L_{N_1,N_2} f\right)\left(x^{(1)}; x^{(2)}\right) &= \left[v_1 \sum_{i=1}^{N_1} \frac{\partial}{\partial x_i^{(1)}} + v_2 \sum_{j=1}^{N_2} \frac{\partial}{\partial x_j^{(2)}}\right] f\left(x^{(1)}; x^{(2)}\right) \\
&\quad + \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \frac{\alpha_{12}}{N_2} \left[f\left(\left(x^{(1)}; x^{(2)}\right)_{i \to j}\right) - f\left(x^{(1)}; x^{(2)}\right)\right] \\
&\quad + \sum_{j=1}^{N_2} \sum_{i=1}^{N_1} \frac{\alpha_{21}}{N_1} \left[f\left(\left(x^{(1)}; x^{(2)}\right)_{i \leftarrow j}\right) - f\left(x^{(1)}; x^{(2)}\right)\right],
\end{aligned}$$

where the notation

$$\begin{aligned}
\left(x^{(1)}; x^{(2)}\right) &= \left(x_1^{(1)}, \ldots, x_{N_1}^{(1)}; x_1^{(2)}, \ldots, x_{N_2}^{(2)}\right), \\
\left(x^{(1)}; x^{(2)}\right)_{i \to j} &= \left(x_1^{(1)}, \ldots, x_{i-1}^{(1)}, x_j^{(2)}, x_{i+1}^{(1)}, \ldots, x_{N_1}^{(1)}; x_1^{(2)}, \ldots, x_{N_2}^{(2)}\right), \\
\left(x^{(1)}; x^{(2)}\right)_{i \leftarrow j} &= \left(x_1^{(1)}, \ldots, x_{N_1}^{(1)}; x_1^{(2)}, \ldots, x_{j-1}^{(2)}, x_i^{(1)}, x_{j+1}^{(2)}, \ldots, x_{N_2}^{(2)}\right)
\end{aligned}$$

used is defined on bounded C^1-functions.

We will consider the limiting behaviour of this process when $t = const$ and $N_1, N_2 \to \infty$. It is not convenient to deal with the sequence $\xi_{N_1,N_2}(t)$ of processes because the dimension of the state space changes with N_1, N_2.

Denote

$$M_{N_1,N_2}(t) = \left(\frac{1}{N_1} \sum_{i=1}^{N_1} \delta(\cdot - x_i^{(1)}(t)), \frac{1}{N_2} \sum_{j=1}^{N_2} \delta(\cdot - x_j^{(2)}(t)) \right),$$

where $\delta(x)$, $x \in \mathbf{R}$, is the δ-function. One can see that the generalized functions

$$\frac{1}{N_1} \sum_{i=1}^{N_1} \delta(\cdot - x_i^{(1)}(t)), \qquad \frac{1}{N_2} \sum_{j=1}^{N_2} \delta(\cdot - x_j^{(2)}(t))$$

represent empirical "densities" or masses of (type 1 and 2, respectively) particles at time t. Thus, if $\phi(x) = (\phi_1(x), \phi_2(x))$, where $\phi_i \in S(\mathbf{R})$, then for fixed particle positions $x_1^{(1)}(t), \dots, x_{N_1}^{(1)}(t)$ and $x_1^{(2)}(t), \dots, x_{N_2}^{(2)}(t)$ the vector function $M_{N_1,N_2}(t)$ is a linear functional on the vector test functions ϕ; that is,

$$\langle M_{N_1,N_2}(t), \phi \rangle = \frac{1}{N_1} \sum_{i=1}^{N_1} \phi_1(x_i^{(1)}(t)) + \frac{1}{N_2} \sum_{j=1}^{N_2} \phi_2(x_j^{(2)}(t)).$$

Fix some $T > 0$. Then $(M_{N_1,N_2}(t),\ 0 \le t \le T)$ can be considered as a Markov process taking its values in the space of tempered distributions $S'(\mathbf{R}) \times S'(\mathbf{R})$. In the sequel we consider $S'(\mathbf{R}) \times S'(\mathbf{R})$ as a topological space equipped with the strong topology (see subsection 5.2). Without loss of generality one can assume that the trajectories of the process $M_{N_1,N_2}(t)$ are right continuous functions with left limits. So it is natural to consider the Skorohod space $\Pi^T = D([0,T], S'(\mathbf{R}) \times S'(\mathbf{R}))$ of functions on $[0,T]$ with values in $S'(\mathbf{R}) \times S'(\mathbf{R})$ as a coordinate space of the process $M_{N_1,N_2}(t)$. Subsection 5.2 explains how to introduce topology on this space. Let $\mathcal{B}(\Pi^T)$ be the corresponding Borel σ-algebra. Denote by $P^T_{N_1,N_2}$ the probability measure on $(\Pi^T, \mathcal{B}(\Pi^T))$, induced by the process $(M_{N_1,N_2}(t),\ 0 \le t \le T)$.

Our assumption for the theorem is that for any test function $\phi(x)$ the sequence $\langle M_{N_1,N_2}(0), \phi \rangle$ weakly converges as $N_1, N_2 \to \infty$.

We want to prove that as $N_1, N_2 \to \infty$ the sequence of probability distributions $P^T_{N_1,N_2}$ has a weak limit, and this limit is a one-point measure, that is the only trajectory $(m_1(t), m_2(t))$, $0 \le t \le T$, which is the classical solution of the system (2.1)–(2.2). We split the proof of this result into the next two propositions.

PROPOSITION 4.1. *The family of probability distributions* $\{P^T_{N_1,N_2}\}_{N_1,N_2}$ *on* $(\Pi^T, \mathcal{B}(\Pi^T))$ *is tight.*

PROPOSITION 4.2. *Limit points of the family of distributions* $P^T_{N_1,N_2}$ *are concentrated on the weak solutions of the system* (2.1)–(2.2).

4.1.1. *Tightness.* Before proving Proposition 4.1, we start with some preliminary lemmas. We want to prove that the family of distributions $P^T_{N_1,N_2}$ of the random process $(M_{N_1,N_2}(t),\ 0 \le t \le T)$, with values in the space of generalized functions, is tight. By Theorem 4.1 of **[M]** (see also subsection 5.2), it is sufficient to prove that for any test function $\psi = (\psi_1(x), \psi_2(x))$ the family of random processes $(\langle M_{N_1,N_2}(t), \psi \rangle,\ 0 \le t \le T)$, with values in $\mathbf{R}^1$, is tight. This will be done in Proposition 4.5 below.

Fix some test function $\psi = (\psi_1(x), \psi_2(x))$ and consider the random process

$$\begin{aligned} F_{\psi,N_1,N_2}\left(x^{(1)}(t);x^{(2)}(t)\right) &= \langle M_{N_1,N_2}(t),\psi\rangle \\ &= \frac{1}{N_1}\sum_{i=1}^{N_1}\psi_1(x_i^{(1)}(t)) + \frac{1}{N_2}\sum_{j=1}^{N_2}\psi_2(x_j^{(2)}(t))\,. \end{aligned}$$

This is a function of the Markov process $\xi_{N_1,N_2}(t)$; thus (see [**KL**, Lemma 5.1, p. 330], for example) the following two processes are martingales:

$$\begin{aligned} W_{\psi,N_1,N_2}(t) &= F_{\psi,N_1,N_2}\left(x^{(1)}(t);x^{(2)}(t)\right) - F_{\psi,N_1,N_2}\left(x^{(1)}(0);x^{(2)}(0)\right) \\ &\quad - \int_0^t L_{N_1,N_2}F_{\psi,N_1,N_2}\left(x^{(1)}(s);x^{(2)}(s)\right)ds\,, \end{aligned} \tag{4.2}$$

$$\begin{aligned} V_{\psi,N_1,N_2}(t) &= \left(W_{\psi,N_1,N_2}(t)\right)^2 - \int_0^t L_{N_1,N_2}F^2_{\psi,N_1,N_2}\left(x^{(1)}(s);x^{(2)}(s)\right)ds \\ &\quad + 2\int_0^t F_{\psi,N_1,N_2}\left(x^{(1)}(s);x^{(2)}(s)\right)L_{N_1,N_2}F_{\psi,N_1,N_2}\left(x^{(1)}(s);x^{(2)}(s)\right)ds. \end{aligned}$$

For shortness we will write $F(x^{(1)};x^{(2)})$ instead of $F_{\psi,N_1,N_2}(x^{(1)};x^{(2)})$.

LEMMA 4.3. *The following estimates hold:*

i) $\left|L_{N_1,N_2}F\left(x^{(1)};x^{(2)}\right)\right| \le C_1(\psi,v_1,v_2,\alpha_{12},\alpha_{21})$ *uniformly in* N_1,N_2 *and* $\left(x^{(1)};x^{(2)}\right)$;

ii) *uniformly in* $x^{(1)}$, $x^{(2)}$

$$\left|L_{N_1,N_2}F^2(x^{(1)};x^{(2)}) - F(x^{(1)};x^{(2)})L_{N_1,N_2}F(x^{(1)};x^{(2)})\right| \le \frac{C_{12}(\alpha_{12},\psi_1)}{N_1} + \frac{C_{21}(\alpha_{21},\psi_2)}{N_2}. \tag{4.3}$$

PROOF. Note that

$$\begin{aligned} F\left(\left(x^{(1)};x^{(2)}\right)_{i\to j}\right) - F\left(x^{(1)};x^{(2)}\right) &= \frac{1}{N_1}\left(\psi_1\left(x_j^{(2)}\right) - \psi_1\left(x_i^{(1)}\right)\right), \\ F\left(\left(x^{(1)};x^{(2)}\right)_{i\leftarrow j}\right) - F\left(x^{(1)};x^{(2)}\right) &= \frac{1}{N_2}\left(\psi_2\left(x_i^{(1)}\right) - \psi_2\left(x_j^{(2)}\right)\right). \end{aligned}$$

Thus

$$\begin{aligned} L_{N_1,N_2}F\left(x^{(1)};x^{(2)}\right) &= \frac{v_1}{N_1}\sum_{i=1}^{N_1}\psi_1'\left(x_i^{(1)}\right) + \frac{v_2}{N_2}\sum_{j=1}^{N_2}\psi_2'\left(x_j^{(2)}\right) \\ &\quad + \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\frac{\alpha_{12}}{N_2}\cdot\frac{1}{N_1}\left(\psi_1\left(x_j^{(2)}\right) - \psi_1\left(x_i^{(1)}\right)\right) \\ &\quad + \sum_{j=1}^{N_2}\sum_{i=1}^{N_1}\frac{\alpha_{21}}{N_1}\cdot\frac{1}{N_2}\left(\psi_2\left(x_i^{(1)}\right) - \psi_2\left(x_j^{(2)}\right)\right). \end{aligned} \tag{4.4}$$

Then

$$\begin{aligned} \left|L_{N_1,N_2}F\left(x^{(1)};x^{(2)}\right)\right| &\le |v_1|\,\|\psi_1'\|_C + |v_2|\,\|\psi_2'\|_C \\ &\quad + 2\alpha_{12}\,\|\psi_1\|_C + 2\alpha_{21}\,\|\psi_2\|_C\,, \end{aligned}$$

and the assertion **i)** of the lemma is proved. To prove assertion **ii)**, it is convenient to represent $L_{N_1,N_2} = L^0_{N_1,N_2} + L^1_{N_1,N_2}$ as the sum of the "differential" $L^0_{N_1,N_2}$ and "jump" $L^1_{N_1,N_2}$ parts.

It is easy to see that

$$L^0_{N_1,N_2} F^2(x^{(1)};x^{(2)}) - 2F(x^{(1)};x^{(2)}) L^0_{N_1,N_2} F(x^{(1)};x^{(2)}) = 0.$$

Let us prove that uniformly in $x^{(1)}$, $x^{(2)}$

$$\left| L^1_{N_1,N_2} F^2(x^{(1)};x^{(2)}) - F(x^{(1)};x^{(2)}) L^1_{N_1,N_2} F(x^{(1)};x^{(2)}) \right| \le \frac{4\alpha_{12}\,\|\psi_1\|^2_C}{N_1} + \frac{4\alpha_{21}\,\|\psi_2\|^2_C}{N_2}. \tag{4.5}$$

In fact,

$$\begin{aligned}
F^2&\left(\left(x^{(1)};x^{(2)}\right)_{i\to j}\right) - F^2\left(x^{(1)};x^{(2)}\right) \\
&= \left(F\left(\left(x^{(1)};x^{(2)}\right)_{i\to j}\right) - F\left(x^{(1)};x^{(2)}\right)\right) \\
&\quad\times \left(2F\left(x^{(1)};x^{(2)}\right) + \frac{1}{N_1}\left(\psi_1\left(x_j^{(2)}\right) - \psi_1\left(x_i^{(1)}\right)\right)\right) \\
&= 2F\left(x^{(1)};x^{(2)}\right)\left[F\left(\left(x^{(1)};x^{(2)}\right)_{i\to j}\right) - F\left(x^{(1)};x^{(2)}\right)\right] \\
&\quad+ \left[\frac{1}{N_1}\left(\psi_1\left(x_j^{(2)}\right) - \psi_1\left(x_i^{(1)}\right)\right)\right]^2
\end{aligned}$$

and similarly for expressions with $\left(x^{(1)};x^{(2)}\right)_{i\leftarrow j}$. Thus,

$$\begin{aligned}
L^1_{N_1,N_2}&F^2(x^{(1)};x^{(2)}) \\
&= 2F\left(x^{(1)};x^{(2)}\right)\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\frac{\alpha_{12}}{N_2}\cdot\left[F\left(\left(x^{(1)};x^{(2)}\right)_{i\to j}\right) - F\left(x^{(1)};x^{(2)}\right)\right] \\
&\quad+ \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\frac{\alpha_{12}}{N_2}\cdot\left[\frac{1}{N_1}\left(\psi_1\left(x_j^{(2)}\right) - \psi_1\left(x_i^{(1)}\right)\right)\right]^2 \\
&\quad+ 2F\left(x^{(1)};x^{(2)}\right)\sum_{j=1}^{N_2}\sum_{i=1}^{N_1}\frac{\alpha_{21}}{N_1}\cdot\left[F\left(\left(x^{(1)};x^{(2)}\right)_{i\leftarrow j}\right) - F\left(x^{(1)};x^{(2)}\right)\right] \\
&\quad+ \sum_{j=1}^{N_2}\sum_{i=1}^{N_1}\frac{\alpha_{21}}{N_1}\cdot\left[\frac{1}{N_2}\left(\psi_2\left(x_i^{(1)}\right) - \psi_2\left(x_j^{(2)}\right)\right)\right]^2 \\
&= 2FL^1_{N_1,N_2}F + \frac{\alpha_{12}}{N_1}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\frac{1}{N_2N_1}\left(\psi_1\left(x_j^{(2)}\right) - \psi_1\left(x_i^{(1)}\right)\right)^2 \\
&\quad+ \frac{\alpha_{21}}{N_2}\sum_{j=1}^{N_2}\sum_{i=1}^{N_1}\frac{1}{N_1N_2}\left(\psi_2\left(x_i^{(1)}\right) - \psi_2\left(x_j^{(2)}\right)\right)^2,
\end{aligned}$$

and the estimate (4.5) follows from this. The lemma is proved. □

COROLLARY 4.4.

$$\sup_{t\le T} \mathsf{E}\left(W_{\psi,N_1,N_2}(t)\right)^2 \to 0, \qquad N_1, N_2 \to \infty\,.$$

PROOF. As V_{ψ,N_1,N_2} is a martingale with mean zero, it is sufficient to prove that the expectation of

$$\int_0^t \Big[L_{N_1,N_2}F^2\left(x^{(1)}(s);x^{(2)}(s)\right) \\ -2F\left(x^{(1)}(s);x^{(2)}(s)\right) L_{N_1,N_2}F\left(x^{(1)}(s);x^{(2)}(s)\right)\Big]\,ds$$

tends to zero. This follows from estimate (4.3) of the lemma. □

PROPOSITION 4.5. *The sequence of distributions of the following real-valued random processes*

$$F_{\psi,N_1,N_2}\left(x^{(1)}(t);x^{(2)}(t)\right),\ t\in[0,T],$$

is tight.

PROOF OF PROPOSITION 4.5. Recall that the following representation holds:

$$F_{\psi,N_1,N_2}\left(x^{(1)}(t);x^{(2)}(t)\right) = F_{\psi,N_1,N_2}\left(x^{(1)}(0);x^{(2)}(0)\right) + W_{\psi,N_1,N_2}(t) \\ + \int_0^t L_{N_1,N_2}F_{\psi,N_1,N_2}\left(x^{(1)}(s);x^{(2)}(s)\right)ds\,.$$

Note that our initial assumption is that the sequence $F_{\psi,N_1,N_2}\left(x^{(1)}(0);x^{(2)}(0)\right)$ weakly converges as $N_1, N_2 \to \infty$.

We shall now prove that the sequence

$$\left\{\eta^{N_1,N_2}(t) = \int_0^t L_{N_1,N_2}F\left(x^{(1)}(s);x^{(2)}(s)\right)ds,\ t\in[0,T]\right\}_{N_1,N_2}$$

is tight. We use subsection 5.1 of the Appendix. By assertion **i)** of the lemma

$$\left|\int_0^t L_{N_1,N_2}F\left(x^{(1)}(s);x^{(2)}(s)\right)ds\right| \le C_1(\psi,v_1,v_2,\alpha_{12},\alpha_{21})\cdot T\,;$$

thus, condition 1) of Theorem 5.1 of the Appendix holds. Condition 2) also holds, as one can prove that

$$w'(\eta^{N_1,N_2},\gamma) \le 2\gamma\cdot C_1(\psi,v_1,v_2,\alpha_{12},\alpha_{21})\,.$$

We shall prove that the sequence $\{W_{\psi,N_1,N_2}(t),\ t\in[0,T]\}_{N_1,N_2}$ is tight. Using Kolmogorov's inequality for submartingales with right continuous trajectories (see [**D**]), we have the following estimate, uniform in N_1, N_2:

$$P\left(\sup_{t\le T}|W_{\psi,N_1,N_2}(t)| > C\right) \le \frac{\sup_{t\le T}\mathsf{E}\left(W_{\psi,N_1,N_2}(t)\right)^2}{C^2}\,.$$

Then from Corollary 4.4 condition 1) of the Appendix holds. Thus,

$$\begin{aligned} P\left(|W_{\psi,N_1,N_2}(\tau+\theta) - W_{\psi,N_1,N_2}(\tau)| > \varepsilon\right) &\le \frac{\mathsf{E}\left(W_{\psi,N_1,N_2}(\tau+\theta) - W_{\psi,N_1,N_2}(\tau)\right)^2}{\varepsilon^2} \\ &= \frac{\mathsf{E}\int_\tau^{\tau+\theta} V_{\psi,N_1,N_2}(s)\,ds}{\varepsilon^2} \\ &\le \frac{\theta\cdot\left(C_{12}(\alpha_{12},\psi_1)/N_1 + C_{21}(\alpha_{21},\psi_2)/N_2\right)}{\varepsilon^2}. \end{aligned}$$

Using this estimate, one can check the sufficient condition of Aldous. Then Proposition 4.5 is proved. □

This also concludes the proof of Proposition 4.1.

4.1.2. *Weak solutions.*

Definition 4.6. We say that the pair of functions $M(t) = (m_1(t,x), m_2(t,x))$ is a weak solution of the system (2.1)–(2.2), if for any pair $\phi_1(x), \phi_2(x) \in S(\mathbf{R})$ the identities

$$\begin{aligned} \langle M(t),\phi\rangle &= \langle M(0),\phi\rangle \\ &\quad + \int_0^t \langle M(s), (v_1\phi_1' - \alpha_{12}\phi_1 + \alpha_{21}\phi_2, v_2\phi_2' - \alpha_{12}\phi_1 + \alpha_{21}\phi_2)\rangle\,ds \end{aligned}$$

hold, where $\phi(x) = (\phi_1(x), \phi_2(x))$, and the action of $G(x) = (g_1(x), g_2(x))$ on the test function $\phi(x)$ can be written as

$$\langle G,\phi\rangle = \int g_1(x)\phi_1(x)\,dx + \int g_2(x)\phi_2(x)\,dx\,.$$

Note that from the representation (4.2) and the identity (4.4) it follows that

$$\begin{aligned} \langle M_{N_1,N_2}(t),\phi\rangle &= W_{\phi,N_1,N_2}(t) + \langle M_{N_1,N_2}(0),\phi\rangle \\ &\quad + \int_0^t \langle M_{N_1,N_2}(s), (v_1\phi_1' - \alpha_{12}\phi_1 + \alpha_{21}\phi_2, v_2\phi_2' - \alpha_{12}\phi_1 + \alpha_{21}\phi_2)\rangle\,ds. \end{aligned}$$

Let $h = h(t) \in \Pi^T = D([0,T], S'(\mathbf{R}) \times S'(\mathbf{R}))$. For fixed ϕ define the functional

$$\begin{aligned} J_{\phi,T}(h) = \sup_{t\le T}\Big| &\langle h(t),\phi\rangle - \langle h(0),\phi\rangle \\ &- \int_0^t \langle h(s), (v_1\phi_1' - \alpha_{12}\phi_1 + \alpha_{21}\phi_2, v_2\phi_2' - \alpha_{12}\phi_1 + \alpha_{21}\phi_2)\rangle\,ds\Big|. \end{aligned}$$

In particular,

$$\sup_{t\le T}|W_{\phi,N_1,N_2}(t)| = J_{\phi,T}(M_{N_1,N_2}).$$

The rest of the proof is standard (see [**KL**]) and consists of three steps.

Step 1. From the definition of the topology on Π^T it follows that $J_{\phi,T}(\cdot) : \Pi^T \to \mathbf{R}_+$ is a continuous functional.

Step 2. Note that

$$\forall\varepsilon > 0\quad P\{J_{\phi,T}(M_{N_1,N_2}) > \varepsilon\} \equiv P^T_{N_1,N_2}\{h : J_{\phi,T}(h) > \varepsilon\} \to 0 \qquad (N_1, N_2 \to \infty)$$

by Kolmogorov's inequality and Corollary 4.4.

Step 3. As $J_{\phi,T}(\cdot)$ is continuous, the set $\{h : J_{\phi,T}(h) > 0\}$ is open in Π^T. It follows now that for any limit point P^T_∞ of the family $\{P^T_{N_1,N_2}\}_{N_1,N_2}$ we have

$$P^T_\infty \{h : J_{\phi,T}(h) > \varepsilon\} \le \limsup_{N_1,N_2} P^T_{N_1,N_2} \{h : J_{\phi,T}(h) > \varepsilon\}.$$

That is, for any $\varepsilon > 0$ we have $P^T_\infty \{h : J_{\phi,T}(h) > \varepsilon\} = 0$. In other words, all limit points P^T_∞ of the family $\{P^T_{N_1,N_2}\}_{N_1,N_2}$ have support on the set

$$\{h : J_{\phi,T}(h) = 0\},$$

which consists of weak solutions of (2.1)–(2.2).

This completes the proof of Proposition 4.2.

The problem of uniqueness of the weak solution of (2.1)–(2.2) is quite simple because the system (2.1)–(2.2) is *linear.* In subsection 4.2 we shall see that this system of first-order differential equations has a unique classical solution which can be obtained in an explicit way.

4.2. Time asymptotics for the continuous model. We prove here Theorem 2.3.

Define the mean (mass centrum) $a_i(t) = \int x m_i(t,x)\,dx$ and the variance (momentum of inertia) $d_i(t) = \int (x - a_i(t))^2 m_i(t,x)\,dx$.

From (2.1)–(2.2) we get the following equations for the means:

$$\begin{aligned} \dot{a_1} &= v_1 + \alpha_{12}(a_2 - a_1)\ , \\ \dot{a_2} &= v_2 + \alpha_{21}(a_1 - a_2)\ . \end{aligned}$$

It follows that the equation for $a_2(t) - a_1(t)$ is closed and has the following solution:

$$a_2(t) - a_1(t) = \frac{v_2 - v_1}{\alpha_{12} + \alpha_{21}}\left(1 - e^{-(\alpha_{12}+\alpha_{21})t}\right) + (a_2(0) - a_1(0))\, e^{-(\alpha_{12}+\alpha_{21})t}.$$

Thus

$$a_2(t) - a_1(t) \to \frac{v_2 - v_1}{\alpha_{12} + \alpha_{21}} \qquad (t \to +\infty)\ ,$$

and similarly

$$\frac{d}{dt} a_i(t) \to \frac{\alpha_{21} v_1 + \alpha_{12} v_2}{\alpha_{12} + \alpha_{21}} \qquad (t \to +\infty)\ .$$

The equations for variances are

$$\begin{aligned} \dot{d_1} &= \alpha_{12}(d_2 - d_1) + \alpha_{12}(a_2(t) - a_1(t))^2\ , \\ \dot{d_2} &= \alpha_{21}(d_1 - d_2) + \alpha_{21}(a_1(t) - a_2(t))^2\ . \end{aligned}$$

Or, equivalently,

$$\begin{aligned} \frac{d}{dt}(\alpha_{21} d_1 + \alpha_{12} d_2) &= 2\alpha_{12}\alpha_{21}(a_2(t) - a_1(t))^2, \\ \frac{d}{dt}(d_2 - d_1) &= -(\alpha_{12} + \alpha_{21})(d_2 - d_1) + (\alpha_{21} - \alpha_{12})(a_2(t) - a_1(t))^2\ . \end{aligned}$$

From this we get

$$d_2(t) - d_1(t) \to const = \left(\frac{v_2 - v_1}{\alpha_{12} + \alpha_{21}}\right)^2 \cdot \frac{\alpha_{21} - \alpha_{12}}{\alpha_{12} + \alpha_{21}}$$

and

$$\frac{d}{dt}(\alpha_{21}d_1 + \alpha_{12}d_2) \to 2\alpha_{12}\alpha_{21}\left(\frac{v_2 - v_1}{\alpha_{12} + \alpha_{21}}\right)^2 .$$

Thus the growth of variances is asymptotically linear. Moreover, both are asymptotically equal.

Now we come to the solution of the equations. Define the Fourier transforms

$$m_i(x,t) = \int \exp(ixp) g_i(p,t)dp .$$

We get

$$\frac{\partial g_1}{\partial t} + v_1 ipg_1 = \alpha_{12}(g_2 - g_1) ,$$
$$\frac{\partial g_2}{\partial t} + v_2 ipg_2 = \alpha_{21}(g_1 - g_2)$$

with the initial conditions $m_i(0,x) = m_i(x)$, $i = 1, 2$. We write this system in the vector form

$$\frac{dg}{dt} = Ag ,$$

where

$$A = \begin{pmatrix} -iv_1p - \alpha_{12} & \alpha_{12} \\ \alpha_{21} & -iv_2p - \alpha_{21} \end{pmatrix} .$$

For the eigenvalues we have

$$\lambda_\pm = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b} ,$$

where

$$a = i(v_1 + v_2)p + \alpha_{12} + \alpha_{21}, \qquad b = -v_1v_2p^2 + ip(v_1\alpha_{21} + v_2\alpha_{12}) .$$

One can write the solution as

$$g = C_+\phi_+ \exp(t\lambda_+) + C_-\phi_- \exp(t\lambda_-) ,$$

where $\phi_\pm$ are eigenfunctions. Note that for small p there are two roots. One has $\operatorname{Re}\lambda_- < 0$, thus a strongly decreasing term. The other is

$$\lambda_+ = c_1p + c_2p^2 + O(p^3), \qquad c_2 \neq 0, \tag{4.6}$$

for small p.

Let ξ_t be a random variable with density $m(x,t)$, and let $g(k)$ be its characteristic function. We are interested in $\frac{1}{\sqrt{t}}(\xi_t - a), a = \mathsf{E}\xi_t$, its characteristic function being

$$\exp\left(-ia\frac{k}{\sqrt{t}}\right) g\left(\frac{k}{\sqrt{t}}\right) .$$

Using (4.6), we get the result.

Remark 4.7. One can see that there is no solution of the type

$$m_i(t,x) = f_i(x - vt)$$

as then f_i would be exponents.

Remark 4.8. For the singular initial conditions, that is, when $x_k^{(i)}(0) = 0$ for $k = 1, \dots, N_i$ and $i = 1, 2$, one can get the same asymptotic results.

5. Appendix

5.1. Probability measures on the Skorohod space: tightness. Let us consider a sequence $\{(\xi_t^n, t \in [0,T])\}_{n\in\mathbf{N}}$ of real random processes whose trajectories are right-continuous and admit left-hand limits for every $0 < t \le T$. We will consider ξ^n as random elements with values in the Skorohod space $D_T(\mathbf{R}) := D\left([0,T], \mathbf{R}^1\right)$ with the standard topology. Denote by P_T^n the distribution of ξ^n, defined on the measurable space $(D_T(\mathbf{R}), \mathcal{B}(D_T(\mathbf{R})))$. The following result can be found in [**B**].

THEOREM 5.1. *The sequence of probability measures $\{P_T^n\}_{n\in\mathbf{N}}$ is tight iff the following two conditions hold:*

1) for any $\varepsilon > 0$ there is $C(\varepsilon) > 0$ such that

$$\sup_n P_T^n \left(\sup_{0\le t\le T} |\xi_t^n| > C(\varepsilon) \right) \le \varepsilon\,;$$

2) for any $\varepsilon > 0$

$$\lim_{\gamma\to 0} \limsup_n P_T^n \left(\xi.\, :\ w'(\xi;\gamma) > \varepsilon\right) = 0\,,$$

where for any function $f : [0,T] \to \mathbf{R}$ and any $\gamma > 0$ we define

$$w'(f;\gamma) = \inf_{\{t_i\}_{i=1}^r} \max_{i<r} \sup_{t_i\le s<t<t_{i+1}} |f(t) - f(s)|\,;$$

moreover the inf *is taken over all partitions of the interval $[0,T]$ such that*

$$0 = t_0 < t_1 < \cdots < t_r = T, \qquad t_i - t_{i-1} > \gamma, \quad i = 1,\ldots,r.$$

The following theorem is known as the sufficient condition of Aldous [**KL**].

THEOREM 5.2. *Condition 2) of the previous theorem follows from the condition*

$$\forall \varepsilon > 0 \qquad \lim_{\gamma\to 0} \limsup_n \sup_{\tau\in\mathcal{R}_T,\, \theta\le\gamma} P_T^n \left(|\xi_{\tau+\theta} - \xi_\tau| > \varepsilon\right) = 0\,,$$

where $\mathcal{R}_T$ is the set of Markov moments (stopping times) not exceeding T .

5.2. Strong topology on the Skorohod space. Mitoma theorem. Recall that the Schwartz space $S(\mathbf{R})$ is a Frechet space (complete locally convex space, the topology of which is generated by a countable family of seminorms, which implies metrizability; see [**RS**]). In the dual space $S'(\mathbf{R})$ of tempered distributions there are at least two ways to define topology (both not metrizable):

1) *weak topology* on $S'(\mathbf{R})$, where all functionals

$$\langle\,\cdot\,,\phi\rangle\,, \quad \phi \in S(\mathbf{R}),$$

are continuous;

2) *strong topology* on $S'(\mathbf{R})$, which is generated by the set of seminorms

$$\left\{ \rho_A(M) = \sup_{\phi\in A} |\langle M,\phi\rangle| \ :\ A \subset S(\mathbf{R}) \text{ is bounded} \right\}.$$

We shall consider $S'(\mathbf{R})$ as equipped with the strong topology. Details can be found in [**RS**].

The problem of introducing the Skorohod topology on the space $D_T(S') := D([0,T], S'(\mathbf{R}))$ was studied in [**M**] and [**J**]. The topology on this space is defined as follows. Let $\{\rho_A\}$ be a family of seminorms, which generates the strong topology in $S'(\mathbf{R})$. For each seminorm ρ_A define a pseudometric

$$d_A(y,z) = \inf_{\lambda\in\Lambda}\left\{\sup_t \left|y_t - z_{\lambda(t)}\right| + \sup_{t\neq s}\left|\log\frac{\lambda(t)-\lambda(s)}{t-s}\right|\right\}, \quad y,z\in D_T(S'),$$

where the inf is taken over the set $\Lambda = \{\lambda = \lambda(t),\ t\in[0,T]\}$ of all strictly increasing maps of the interval $[0,T]$ into itself. Equipped with the topology of the projective limit for the family $\{d_A\}$, the set $D_T(S')$ becomes a completely regular topological space.

Let $\mathcal{B}(D_T(S'))$ be the corresponding Borel σ-algebra. Let $\{P_n\}$ be a sequence of probability measures on $(D_T(S'),\mathcal{B}(D_T(S')))$. For each $\phi\in S(\mathbf{R})$ consider a map $\mathcal{I}_\phi : y\in D_T(S')\to y.(\phi)\in D_T(\mathbf{R})$. The following result belongs to I. Mitoma [**M**].

THEOREM 5.3. *Suppose that for any $\phi\in S(\mathbf{R})$ the sequence $\left\{P_n\mathcal{I}_\phi^{-1}\right\}$ is tight in $D_T(\mathbf{R})$. Then the sequence $\{P_n\}$ itself is tight in $D_T(S')$.*

References

[B] P. Billingsley, *Convergence of probability measures,* John Wiley & Sons, 1968.

[BT] D. P. Bertsekas, J. N. Tsitsiklis, *Parallel and Distributed Computation: Numerical Methods,* Athena Scientific, Belmont, MA, 1997.

[D] J. L. Doob, *Stochastic Processes,* John Wiley & Sons, New York.

[GMP] A. Greenberg, V. Malyshev, S. Popov, *Stochastic Model of Massively Parallel Simulation,* Markov Proc. Rel. Fields **1** (1995), 473–490.

[J] A. Jakubowski, *On the Skorohod topology,* Ann. Inst. H. Poincaré: Probabilités et Statistiques **22** (1986), 263–285.

[K] T. Kato, *Perturbation Theory for Linear Operators,* Springer, 1976.

[KL] C. Kipnis, C. Landim, *Scaling Limits of Interacting Particle Systems,* Springer, 1998.

[M] I. Mitoma, *Tightness of probabilities on $C([0,1];S')$ and $D([0,1];S')$,* Ann. Probab. **11** (1983), 989–999.

[MaMa1] V. Malyshev, A. Manita, *Time synchronization model,* Preprint INRIA no. RR-5204, 2004.

[MaMa2] V. A. Malyshev, A. D. Manita, *Phase transitions in the time synchronization model,* Probability Theory and Applications **50** (2005), 150–158.

[MiMi] D. Mitra, I. Mitrani, *Analysis and optimum performance of two message-passing parallel processors synchronized by rollback,* Performance Evaluation **7** (1987), 111–124.

[MSh1] A. Manita, V. Shcherbakov, *Asymptotic analysis of a particle system with mean-field interaction,* Markov Processes Relat. Fields **11** (2005), 489–518.

[MSh2] A. Manita, V. Shcherbakov, *Stochastic particle system with non-local mean-field interaction,* Proceedings of Int. Conf. "Kolmogorov and Contemporary Mathematics", Moscow, June 16–21, 2003, pp. 549–550.

[MSi] A. Manita, F. Simonot, *Clustering in Stochastic Asynchronous Algorithms for Distributed Simulations,* Lecture Notes in Computer Science **3777** (2005), 26–37.

[RS] M. Reed, B. Simon, *Methods of Modern Mathematical Physics. Vol. 1: Functional Analysis,* Academic Press, 1972.

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