



# Harmonic Chain with Weak Dissipation

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Received April 2, 2012

**Abstract.** We consider finite harmonic chain (consisting of  $N$  classical particles) plus dissipative force acting on one particle (called dissipating particle) only. We want to prove that “in the generic case” the energy (per particle) for the whole system tends to zero in the large time limit  $t \rightarrow \infty$  and then in the large  $N$  limit. “In the generic case” means: for almost all initial conditions and for almost any choice of the dissipating particle, in the thermodynamic limit  $N \rightarrow \infty$ .

## 1. Introduction

The energy  $H(t)$  in Hamiltonian systems is conserved. After adding dissipation terms,  $H(t)$  is not conserved anymore but becomes a non-increasing function. We consider finite harmonic chain (consisting of  $N < \infty$  classical particles) plus dissipative force acting on one particle (having number  $n$  and called dissipating particle) only. It appears that  $H(N, n, t)$  is very irregular with respect to  $N$  and  $n$ . However, we want to prove that in the general case the energy per particle  $N^{-1}H(N, n, t)$  tends to zero in the limit where first  $t \rightarrow \infty$  and then  $N \rightarrow \infty$ . “In the general case” means: for almost all initial conditions and for almost any choice of the dissipating particle, see exact definitions below.

Finite linear hamiltonian systems (including also additional linear terms) were studied by many authors, see for example [2–5], but with the goals different from ours. The papers [6, 7] are closer to ours, in [7] the authors study the cases when the dimension of  $L_0$  (see next section) is zero.

Our main formulas concern linear algebra and our main conventions about notation are as follows: all vectors are row vectors, there will be introduced two scalar products  $(\cdot, \cdot)_k, k = 1, 2$ , and we denote the linear span of the set  $R$  of vectors by  $\langle R \rangle$ .

## 2. Main results

We consider the system of  $N$  particles with the phase space

$$L = L_{2N} = \mathbb{R}^{2N} = \{\psi = (q, p) : q = (q_1, \dots, q_N), p = (p_1, \dots, p_N) \in \mathbb{R}^N\},$$

with the scalar product

$$(\psi, \psi')_1 = \sum_{i=1}^N (q_i q'_i + p_i p'_i)$$

and with quadratic Hamiltonian (energy)

$$H(\psi) = T + U, \quad T = \frac{1}{2} \sum_{i=1}^N p_i^2, \quad U = \frac{1}{2} \sum_{i,j} V(i, j) q_i q_j,$$

where  $V = (V(i, j))$  is a (symmetric) positively definite  $(N \times N)$ -matrix. We assume unit masses of the particles. The phase space is the direct sum of the orthogonal coordinate and momentum subspaces,  $l_N^{(q)}$  and  $l_N^{(p)}$  correspondingly, with the induced scalar products  $(q, q')_1$  and  $(p, p')_1$ .

The dynamics of this system is defined by the following system of equations,  $i = 1, \dots, N$ :

$$\begin{aligned} \frac{dq_i}{dt} &= p_i, \\ \frac{dp_i}{dt} &= -\frac{\partial U}{\partial q_i} - \alpha p_i \delta_{in} = -\sum_j V(i, j) q_j - \alpha p_i \delta_{in}, \end{aligned}$$

where we appended a dissipation force for one particle ONLY, that is, we fix particle with number  $n$  and assume that it is subjected to dissipation with fixed  $\alpha > 0$ . More generally, such system can be written in the matrix form

$$\dot{\psi} = A\psi, \quad \psi \in L,$$

with  $(2N \times 2N)$ -matrix

$$A = \begin{pmatrix} 0 & E \\ -V & -D \end{pmatrix}$$

with the matrices  $V, D$  and the unit matrix  $E$  acting in  $\mathbb{R}^N$ . The dissipation matrix  $D \geq 0$  is symmetric and non-negative, in our case it is diagonal with the only non-zero element. The dynamics can also be presented as the equivalent linear second order ODE system in  $\mathbb{R}^N$ :

$$\ddot{q} + D\dot{q} + Vq = 0, \quad q = (q_1, \dots, q_N) \in \mathbb{R}^N.$$

For any initial  $\psi(0) \in L$  the solution is  $\psi(t) = e^{tA}\psi(0)$ . It is well-known (and easy to check) that the energy is non-increasing and

$$\frac{d}{dt}H(\psi(t)) = -(Dp, p)_1. \tag{2.1}$$

**Definition 2.1.**  $L_0 \subset L$  is the subset of elements  $\psi$  of the phase space  $L$  for which the energy is conserved,

$$L_0 = \left\{ \psi \in L : \frac{d}{dt}H(e^{tA}\psi) = 0, \forall t > 0 \right\}.$$

$L_- \subset L$  is the subset of elements  $\psi$  of the phase space  $L$  for which the energy tends to zero as  $t \rightarrow \infty$ ,

$$L_- = \{ \psi \in L : H(e^{tA}\psi) \rightarrow 0, t \rightarrow \infty \}.$$

**Theorem 2.1.** *Subsets  $L_0$  and  $L_-$  are orthogonal linear subspaces of  $L$ , invariant with respect to the dynamics, and  $L$  is their direct sum,*

$$L = L_0 \oplus L_-.$$

*If  $\psi \in L_-$ , then  $H(e^{tA}\psi) \rightarrow 0$  exponentially fast. More exactly, for some constants  $c_1, c_2 > 0$  ( $c_1 = c_1(\psi)$  depends on  $\psi$ ),*

$$\|H(e^{tA}\psi)\|_1 \leq c_1 \exp(-c_2t).$$

Our main result concerns the chain of harmonic oscillators with the Hamiltonian

$$H = \frac{1}{2} \sum_{k=1}^N p_k^2 + \frac{\omega_0}{2} \sum_{k=1}^N (x_k - ka)^2 + \frac{\omega_1}{2} \sum_{k=2}^N (x_k - x_{k-1} - a)^2,$$

where  $\omega_0, \omega_1, a > 0$ . In terms of the deviations

$$q_k = x_k - ka$$

the Hamiltonian becomes

$$H = \frac{1}{2} \sum_{k=1}^N p_k^2 + \frac{\omega_0}{2} \sum_{k=1}^N q_k^2 + \frac{\omega_1}{2} \sum_{k=2}^N (q_k - q_{k-1})^2.$$

We shall denote by  $\text{gcd}(a_1, a_2)$  the greatest common divisor of the natural numbers  $a_1, a_2$ .

**Theorem 2.2.** *The dimension  $D_n(N) = \dim L_{0,n}$  enjoys the properties (2.2)–(2.4).*

$$D_n(N) = \gcd(N, 2n - 1) - 1. \quad (2.2)$$

For any  $\varepsilon > 0$  there exists some  $c(\varepsilon)$  such that

$$S(N) = \frac{1}{N} \sum_{n=1}^N D_n(N) \leq c(\varepsilon)N^\varepsilon. \quad (2.3)$$

as  $N \rightarrow \infty$ .

For some absolute constant  $c > 0$  and any  $N_0 > 0$ ,

$$\frac{1}{N_0} \sum_{N \leq N_0} \frac{S(N)}{N} < c \ln N_0. \quad (2.4)$$

The following examples show that the function  $D_n(N)$  is very irregular in  $n$  and  $N$ :

- If  $N = 2^k$  for some  $k > 0$ , then  $D_n(N) = 0$  for all  $n$ .
- If  $N = 2m - 1$  and  $n = m$ , then  $D_n(N) = 2m - 2 = N - 1$ .

Note that  $S(N)$  is the mean dimension if the particle  $n$  is chosen randomly, and  $N^{-1}S(N)$  is the mean energy per particle. Theorem 2.2 shows that if the initial energies of the particles in the system are uniformly bounded as  $N \rightarrow \infty$ , and thus the initial energy of the whole system is not more than of order  $N$  (that is, proportional to the number of particles), then the energy remaining in the system after a long time is very small, namely of the order  $N^\varepsilon$ , and the energy per particle is of order  $N^{-1+\varepsilon}$ . It can be said, roughly speaking, that in the thermodynamic limit  $N \rightarrow \infty$  the remaining mean energy per particle is zero (even if the kinetic energy is pumping out only of a single particle).

### 3. Proofs

#### 3.1. Proof of Theorem 2.1

Define the second scalar product in  $L$  by

$$(\psi, \psi')_2 = (Vq, q')_1 + (p, p')_1,$$

and denote by  $e_n = (0, \dots, 0, 1, 0, \dots, 0) \in l_N^{(p)}$  the coordinate vector with 1 at the  $n$ th place. Define also the vector  $g_n = (0, e_n) \in L$ .

We shall prove that

$$L_0 = \{\psi \in L : (e^{tA}\psi, g_n)_1 = 0, \forall t > 0\} = \{\psi \in L : (e^{tA}\psi, g_n)_2 = 0, \forall t > 0\}. \quad (3.1)$$

Indeed, we have

$$\frac{d}{dt}H(e^{tA}\psi) = -\alpha p_n^2(t) = -\alpha(e^{tA}\psi, g_n)_1^2.$$

This is just rewriting of the right-hand side of (2.1),

$$(Dp, p)_1 = \alpha p_n^2(t), \quad p_n(t) = (e^{tA}\psi, g_n)_1.$$

Linearity of  $L_0$  follows from (3.1), the invariance of  $L_0$  follows by differentiation of the equality  $(e^{tA}\psi, g_n)_1 = 0$ .

Define a subspace  $l_V \subset \mathbb{R}^N$  as the linear span of the vectors  $V^j e_n$ ,  $j = 0, 1, \dots$ , and introduce the subspaces  $L^{(k)}$  as the orthogonal complements to  $L_0$  with respect to the scalar products  $(\cdot, \cdot)_k$ ,  $k = 1, 2$  correspondingly.

**Lemma 3.1.**

1.  $L^{(1)} = L^{(2)}$ .
2.  $L^{(2)} = \{(q, p) \in L : q, p \in l_V\}$ .
3.  $L^{(2)}$  is invariant with respect to  $A$ .

*Proof.* We will use the following identity

$$A = IQ - \alpha\Gamma,$$

where  $I, Q, \Gamma$  are matrices in  $R^{2N}$  defined by

$$I((q, p)) = (p, -q), \quad Q((q, p)) = (Vq, p), \quad \Gamma\psi = (\psi, g_n)_1 g_n.$$

Note that

$$(\psi, \psi')_2 = (Q\psi, \psi')_1.$$

Let  $A_k^*$  be the adjoint operator to  $A$  with respect to the corresponding scalar products  $(\cdot, \cdot)_k$ ,  $k = 1, 2$ . It is easy to see that

$$A_1^* = -QI - \alpha\Gamma, \\ A_2^* = Q^{-1}A_1^*Q = -IQ - \alpha\Gamma.$$

For  $k = 1, 2$  put

$$M_k = \langle \{(A_k^*)^j g_n\}_{j=0,1,\dots} \rangle.$$

We will prove that the orthogonal complements (with respect to the corresponding scalar products)  $M_k^\perp = L_0$ ,  $k = 1, 2$ . Let  $\psi \in M_k^\perp$ . Since

$$(e^{tA}\psi, g_n)_k = (\psi, e^{tA_k^*} g_n)_k = \sum_{j=0}^{+\infty} \frac{t^j}{j!} (\psi, (A_k^*)^j g_n)_k = 0 \tag{3.2}$$

for any  $t > 0$ , we have  $\psi \in L_0$ . Vice-versa, let  $\psi \in L_0$ . Then for any  $j = 0, 1, \dots$  we have a chain of equalities

$$0 = \frac{d^j}{dt^j} (e^{tA}\psi, g_n)_k |_{t=0} = (A^j \psi, g_n)_k = (\psi, (A_k^*)^j g_n)_k.$$

It follows that  $\psi \in M_k^\perp$ , and thus  $M_k^\perp = L_0$ , which is equivalent to

$$M_1 = L^{(1)}, \quad M_2 = L^{(2)}.$$

Let us use induction in  $m$  to prove that

$$\langle \{(A_2^*)^j g_n\}_{j=0,1,\dots,m} \rangle = \langle \{(IQ)^j g_n\}_{j=0,1,\dots,m} \rangle.$$

For  $m = 0$  the statement is evident. The inductive hypothesis is: for some  $c_j$ ,

$$(A_2^*)^m g_n = \sum_{j=0}^m c_j (IQ)^j g_n.$$

Applying to both sides of this equality the operator  $A_2^* = -IQ - \alpha\Gamma$ , we get

$$(A_2^*)^{m+1} g_n = - \sum_{j=0}^m c_j (IQ)^{j+1} g_n - \alpha((A_2^*)^m g_n, g_n)_2 g_n \in \langle \{(IQ)^j g_n\}_{j=0,1,\dots,m+1} \rangle.$$

Similarly, one can prove the inverse inclusion for the corresponding linear spans. One can prove similarly that

$$L^{(1)} = \langle \{(QI)^j g_n\}_{j=0,1,\dots} \rangle.$$

It is easy to check that the following identities hold for  $j = 0, 1, \dots$

$$\begin{aligned} (IQ)^{2j} g_n &= (-1)^j (0, V^j e_n), \\ (IQ)^{2j+1} g_n &= (-1)^j (V^j e_n, 0). \end{aligned}$$

Thus,  $L^{(2)} = l_V \oplus l_V$ . Similarly for  $j = 0, 1, \dots$  one can check the formulas

$$\begin{aligned} (QI)^{2j} g_n &= (-1)^j (0, V^j e_n), \\ (QI)^{2j+1} g_n &= (-1)^j (V^{j+1} e_n, 0). \end{aligned}$$

It follows that  $L^{(1)} = V(l_V) \oplus l_V$ , where  $V(l_V)$  is the image of  $l_V$  under the mapping  $V$ . Since  $l_V$  is invariant with respect to  $V$  and  $V$  is invertible,  $V(l_V) = l_V$ . Thus,  $L^{(1)} = l_V \oplus l_V = L^{(2)}$ .

Invariance with respect to the operator  $A$  follows from relations

$$\begin{aligned} A(V^j e_n, 0)^T &= (0, -V^{j+1} e_n) \in L^{(2)}, \\ A(0, V^j e_n)^T &= (V^j e_n, 0) - \alpha(V^j e_n, e_n)_1 g_n \in L^{(2)}. \end{aligned}$$

The lemma is proved.  $\square$

Now we will use the classical result about ordinary differential equations. Namely, consider the following system

$$\dot{y} = f(y), \quad y \in \mathbb{R}^n. \tag{3.3}$$

If a function  $v(y) \in C^1$  is given, we will denote its derivatives along the trajectories as

$$\frac{dv}{dt} = \sum_{k=1}^n \frac{\partial v}{\partial y_k}(y) f_k(y).$$

**Theorem 3.1 (Barbashin–Krasovskij’s theorem [1], p. 19, Thm. 3.2).** *Let  $f$  be continuous,  $f(0) = 0$  and for  $|y| \leq \rho$  ( $\rho > 0$ ) there exists a (Lyapounov) function  $v(y) \in C^1$  such that  $v(0) = 0$ ,  $v(y) > 0$  for  $y \neq 0$ ,  $dv/dt \leq 0$ . Assume moreover that the set of all  $y$  for which  $dv/dt = 0$  does not contain any whole trajectory, except  $y = 0$ . Then the zero solution of (3.3) is asymptotically stable.*

**Lemma 3.2.** *The following equality holds*

$$L_- = L^{(2)}.$$

Let us prove that  $L^{(2)} \subset L_-$ . We use Lemma 3.1 and choose  $v(\psi) = H(\psi)$ . Then

$$\frac{dv}{dt}(\psi) = -(Dp, p)_1 = -\alpha(\psi, g_n)_2^2.$$

The trajectories which belong to the set of zeros of  $dv/dt$ , by definition belong to  $L_0$ . As  $L_0 \cap L^{(2)} = \{0\}$ , the zero trajectory is asymptotically stable on  $L^{(2)}$ . This means that for any  $\psi \in L^{(2)}$  we have  $H(e^{tA}\psi) \rightarrow 0$  as  $t \rightarrow +\infty$ . Let us prove the inverse inclusion. Note that for any  $\psi \in L$  we have  $H(\psi) = (\psi, \psi)_2$ . Let  $\psi \in L_-$ . We have

$$\psi = \psi^{(0)} + \psi^{(2)}, \quad \psi^{(0)} \in L_0, \quad \psi^{(2)} \in L^{(2)}.$$

Then  $H(e^{tA}\psi) = H(\psi^{(0)}) + H(\psi^{(2)}) \rightarrow H(\psi^{(0)}) = 0$  by definition of  $L_-$ . It follows that  $\psi^{(0)} = 0$ . Lemma 3.2 and thus Theorem 2.1 are proved.

**3.2. Proof of Theorem 2.2**

First, we need the following general assertion.

**Lemma 3.3.** *Let the spectrum of  $V$  be simple and let  $\{v_1, \dots, v_N\}$  be the eigenvectors, which form the basis of the space  $\mathbb{R}^N$ . Then*

$$\dim L_0 = 2\#\{k \in \{1, \dots, N\} : v_k \perp e_n\},$$

where  $\perp$  is in the sense of  $(\cdot, \cdot)_1$ .

*Proof.* As the subspace  $l_V$  is invariant with respect to  $V$ , one can enumerate the vectors  $\{v_1, \dots, v_N\}$  so that for some  $m$

$$l_V = \langle v_1, \dots, v_m \rangle.$$

Thus,  $e_n \in \langle v_1, \dots, v_m \rangle = \langle v_{m+1}, \dots, v_N \rangle^\perp$ ,  $\perp$  in the sense of  $(,)_1$ .  $\square$

We will calculate the spectrum of the matrix  $V$ , which acts on the vectors  $q$  as follows:

$$(Vq)_i = \begin{cases} \omega_0 q_i - \omega_1 (q_{i-1} + q_{i+1} - 2q_i), & 1 < i < N - 1, \\ \omega_0 q_1 - \omega_1 (q_2 - q_1), & i = 1, \\ \omega_0 q_N - \omega_1 (q_{N-1} - q_N), & i = N. \end{cases}$$

**Lemma 3.4.** *The matrix  $V$  has  $N$  different eigenvalues  $\lambda_0, \dots, \lambda_{N-1}$ , given by*

$$\lambda_k = \omega_0 + 2\omega_1 \left(1 - \cos \frac{\pi k}{N}\right), \quad k = 0, \dots, N - 1.$$

*The corresponding eigenvectors  $y^{(0)}, \dots, y^{(N-1)}$  have the following coordinates:*

$$y_j^{(k)} = \cos \pi \frac{(j - 1/2)k}{N}, \quad j = 1, \dots, N, \quad k = 0, \dots, N - 1.$$

Lemma 3.4 can be proved by direct substitution.

Using Lemmas 3.3 and 3.4 we get

$$\begin{aligned} \dim L_0 &= 2\#\{k \in \{0, \dots, N - 1\} : (y^{(k)}, e_n) = 0\} \\ &= 2\#\left\{k \in \{0, \dots, N - 1\} : \cos\left(\frac{\pi k(n - 1/2)}{N}\right) = 0\right\} = \\ &= 2\#\{k \in \{0, \dots, N - 1\} : a_n(k) \in \mathbb{Z}\}, \end{aligned}$$

where

$$a_n(k) = \frac{1}{2} \left( \frac{2n - 1}{N} k - 1 \right).$$

Denote  $d = \gcd(N, 2n - 1)$ . For some integers  $m_1, m_2$  such that  $\gcd(m_1, m_2) = 1$ , we have

$$2n - 1 = dm_1, \quad N = dm_2, \quad a_n(k) = \frac{1}{2} \left( \frac{m_1}{m_2} k - 1 \right).$$

As the numbers  $d$  and  $m_1$  are odd, the number  $a_n(k)$  is an integer if and only if  $k = (2l - 1)m_2$ ,  $l = 1, 2, \dots$ . From the condition  $k \leq N - 1$  it follows that

$$l \leq \frac{1}{2} \left( \frac{N - 1}{m_2} + 1 \right).$$

Thus

$$\dim L_0 = 2 \left[ \frac{1}{2} \left( \frac{N - 1}{N} d + 1 \right) \right] = 2 \left[ \frac{d}{2} - \frac{d}{2N} \right].$$



Since  $(d/2N) \leq (1/2)$  and  $d$  is odd, we have

$$\left[ \frac{d}{2} - \frac{d}{2N} \right] = \frac{d-1}{2}.$$

This gives the first statement of Theorem 2.2. Statements 2 and 3 of the theorem follow from the results of [8], see theorems 2.1, 3.2, 4.2 therein.

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