



Phase diagram for one-way traffic flow with local control



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HIGHLIGHTS

- Control protocol with two parameters only for deterministic movement of infinite chain of cars with the leader is studied in detail.
- Complete phase diagram with three regions: stability, partial stability, instability is studied in detail.
- In stability region both absence of collisions and uniformly positive density are proven.
- In two regions of stability, we studied fine dependence on the initial conditions and the movement of the leader.
- Mathematically rigorous proofs are presented in all cases.

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ABSTRACT

We consider one-way road deterministic traffic model with N particles. The simplest local control protocol, which reminds physical interaction is considered. We obtain complete phase diagram uniformly in N and study in detail its stable and unstable domains.

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1. Introduction

Theoretical modeling and computer simulation of transportation systems is a very popular field, see very impressive review [1]. There are two main directions in this research – macro and micro models. Macro approach does not distinguish individual transportation units and uses analogy with the fluid flow in hydrodynamics, see [2,3]. Stochastic micro models are most popular and use almost all types of stochastic processes: mean field, queuing type and local interaction models. We consider here completely deterministic transportation flows. Although not as popular as stochastic traffic, there is also a big activity in this field, we give short review (with comparative analysis) in the last section.

We consider the one-way road traffic model organized as follows. At any time $t \geq 0$ there is finite or infinite number of point particles (may be called also cars, units etc.) with coordinates $z_k(t)$ on the real axis, enumerated as follows

$$\dots < z_n(t) < \dots < z_1(t) < z_0(t). \quad (1)$$

We assume that the rightmost car (the leader) moves “as it wants”, that is the trajectory $z_0(t)$ is often assumed to have non-negative velocity.

Our problem is to find the simplest possible local protocol (control algorithm) which would guarantee both safety (no collisions), stable (or even maximal) density of the flow or maximal current. Otherwise speaking, we try to find control mechanism which guarantees that the distance between any pair of neighboring cars is close (on all time interval $(0, \infty)$) to some (given a priori) fixed number, that defines the density of the flow.

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More exactly, denoting $r_k(t) = z_{k-1}(t) - z_k(t)$, and

$$I = \inf_{k \geq 1} \inf_{t \geq 0} r_k(t), \quad S = \sup_{k \geq 1} \sup_{t \geq 0} r_k(t),$$

we try to get the bounds – lower positive bound on I and upper bound on S – as close as possible. Namely, we want to prove the following inequalities

$$0 < I \leq S < \infty. \quad (2)$$

That is by stability we mean that the cars do not become too close to each other, and simultaneously the distances between neighbor cars do not become too large. As for the general mathematical definitions of stability, we use the following: the system is stable if the inequalities (2) hold, otherwise it is unstable. Then one could give more detailed definitions, see Section 5, depending strongly on the class of initial conditions and of allowed motion of the leader.

Locality (of the control) means that the “driver” of the k th car, at any time t , knows only its own velocity $v_k(t)$ and the distance $r_k(t)$ from the previous car. Thus, for any $k \geq 1$ the trajectory $z_k(t)$, being deterministic, is uniquely defined by the trajectory $z_{k-1}(t)$ of the previous particle.

Using physical terminology one could say that if, for example, $r_k(t)$ becomes larger than d , then some virtual force F_k increases acceleration of the particle k , and vice-versa. Thus the control mechanism is of the physical nature, like forces between molecules in crystals but our “forces” are not symmetric. Thus our system is not a Hamiltonian system. Nevertheless, our results resemble the dynamical phase transition in the model of the molecular chain rapture under the action of external force, see [4]. However here we do not need the double scaling limit used in [4].

We will see however that for the stability, besides F_k , also friction force $-\alpha v_k(t)$, restraining the growth of the velocity $v_k(t)$, is necessary, where the constant $\alpha > 0$ should be chosen appropriately. Taking F_k to be simplest possible

$$F_k(t) = \omega^2(z_{k-1}(t) - z_k(t) - d) \quad (3)$$

we get that the trajectories are uniquely defined by the system of equations for $k \geq 1$

$$\frac{d^2 z_k}{dt^2} = F_k(t) - \alpha \frac{dz_k}{dt} = \omega^2(z_{k-1}(t) - z_k(t) - d) - \alpha \frac{dz_k}{dt}. \quad (4)$$

Note that general “Microscopic follow the leader model” can be defined in various ways. See for example equation (5) in [1] and also the models listed in the last section. Our goal was to obtain most detailed results for the simplest model (4) of this kind.

Stability depends not only on the parameters α , ω , d but also on the initial conditions and on the movement of the leader (on its velocity and acceleration). This is easy to understand for the case of $N + 1$ particles. For example, for $N = 1$, where the calculations are completely trivial, assume also the simplest leader movement

$$z_0(t) = vt, \quad t \geq 0. \quad (5)$$

Then, if the initial conditions for the second particle are

$$z_1(0) = -a = -\left(d + \frac{\alpha}{\omega^2}v\right), \quad \dot{z}_1(0) = v,$$

then $z_1(t) = -a + vt$ for any d , α , ω . However, if we change only the initial velocity $\dot{z}_1(0) = w$ to some $w > 0$, then for any α , ω there exists $w_1 = w_1(\alpha, \omega, d)$ such that for any $w \geq w_1$ collision occurs.

For $N = 2, 3, \dots$ the situation becomes more and more complicated, and its study has no much sense. That is why we study, in the space of two parameters α , ω (for fixed d), stability conditions, which are uniform in N and in large class of reasonable initial conditions and reasonable movement of the leader.

Natural (reasonable) initial conditions are as follows: at time 0 it should be

$$0 < \inf_{k \geq 1} r_k(0) \leq \sup_{k \geq 1} r_k(0) < \infty.$$

As for the leader movement, it is sometimes sufficient to assume that the function $z_0(t)$ were continuous, but in other cases it is assumed to be twice differentiable and has the following bounds on the velocity and acceleration of the leader:

$$\sup_{t \geq 0} |\dot{z}_0(t)| = v_{\max}, \quad \sup_{t \geq 0} |\ddot{z}_0(t)| = a_{\max}. \quad (6)$$

It appears that under these conditions there are 3 sectors in the quarter-plane $R_+^2 = \{(\alpha, \omega)\}$: 1) $\alpha > 2\omega$, where we can prove stability, 2) $\alpha < \sqrt{2}\omega$, where we can prove instability, and the sector 3) $\sqrt{2}\omega \leq \alpha \leq 2\omega$, where we can prove stability only for more restricted classes of initial conditions and of the leader motion.

The paper is organized as follows. Section 2 contains definitions and formulation of the main results, Sections 3 and 4 are devoted to the detailed proof of stability and instability conditions correspondingly, Section 5 is a short review of earlier results in comparison with our model.

2. Results

2.1. Stability

Here we consider the region $\alpha > 2\omega$, and this is always assumed in this section.

Movement close to stationary. For any given $d > 0$ there are special initial conditions (which can be called equilibrium configuration) when the force acting on any particle is zero:

$$z_k(0) = -ka, \dot{z}_k(0) = v, k = 0, 1, 2, \dots, \tag{7}$$

with

$$a = a(\alpha, \omega, d, v) = d + \frac{\alpha}{\omega^2}v. \tag{8}$$

If the leader moves as (5), then for any $k > 0$ also

$$z_k(t) = z_k(0) + vt.$$

Such movement we call stationary. Now consider a perturbation of this situation

Theorem 1. For given d and v assume

$$|z_k(0) + ka| \leq \theta a, \quad |\dot{z}_k(0) - v| \leq \beta v, \tag{9}$$

$$\sup_{t \geq 0} |z_0(t) - vt| \leq \delta a \tag{10}$$

for $0 \leq \delta < \frac{1}{2}$ and $\theta, \beta \geq 0$ such that

$$\epsilon = 2 \max \left\{ \delta, \frac{\theta + \frac{2v\beta}{\alpha a}}{\sqrt{1 - \left(\frac{2\omega}{\alpha}\right)^2}} \right\} < 1. \tag{11}$$

Then

$$(1 - \epsilon)a \leq I \leq S \leq (1 + \epsilon)a. \tag{12}$$

Now we want to show that in some cases the sufficient conditions of this theorem are not very far from necessary. For example, from the theorem follows that the condition

$$\theta < \frac{1}{2} \sqrt{1 - \left(\frac{2\omega}{\alpha}\right)^2}, \delta = \beta = 0 \tag{13}$$

is sufficient for $I > 0$. This means that the deviations of the particle from the equilibrium (that is the distances between particles are equal to a) do not exceed $\frac{a}{2}$.

Remark 1. The movement of any first N particles does not depend on other particles. Thus for any N the system of Eqs. (4) for $k = 1, \dots, N$, is finite-dimensional, linear and, for any $\alpha, \omega > 0$, the spectrum of the corresponding linear operator is inside the left half-plane. One could look at it formally, neglecting possible collisions between particles. In this case it is asymptotically stationary, that is, as $t \rightarrow \infty$, its solution converges to stationary for any initial conditions and if $z_0(t) = vt$. However we do not know whether collisions occur before it becomes stationary. In the following theorem we get such conditions.

Because of this one could think that the spectrum does not play big role in the stability problems. This is not quite true if we consider the problems uniform in N , or the corresponding infinite-dimensional operator, see below.

Note also that for the random movement of the leader the same results hold. More exactly, assume that $z_0(t)$ is a stationary process with smooth trajectories, satisfying bounds (6) with probability one. Then the mean velocity of any particle converges to the mean velocity of the leader.

Theorem 2. (1) Assume (7) and (10). Then for all $k = 1, 2, \dots$ we have:

$\sup_{t \geq 0} |\dot{z}_k(t) - v| \leq \sup_{t \geq 0} |\dot{z}_0(t) - v|$. If the right hand side is sufficiently small then $v_k(t) = \dot{z}_k(t) > 0$ for any k, t .

(2) Assume (5), (9) and that for some parameters $\theta, \beta > 0$

$$\zeta = 2 \frac{\theta + \frac{2v\beta}{\alpha a}}{\sqrt{1 - \left(\frac{2\omega}{\alpha}\right)^2}} < 1. \tag{14}$$

Then for all k

$$\lim_{t \rightarrow \infty} (z_k(t) - (vt - ka)) = 0$$

where a is in (8) and

$$(1 - \zeta)a \leq I \leq S \leq (1 + \zeta)a.$$

Non-stationary initial conditions. In Theorems 1 and 2 we considered initial coordinates of the particles close to the fixed lattice points $-ka$. Here we consider more natural initial conditions with restrictions only on the distances between particles. Denote

$$d^* = \frac{a_{\max} + \alpha v_{\max}}{\omega^2}. \tag{15}$$

Theorem 3. Let the initial conditions be

$$(1 - \theta)d \leq r_k(0) \leq (1 + \theta)d, \quad |\dot{z}_{k-1}(0) - \dot{z}_k(0)| \leq \beta, \quad k = 1, 2, \dots \tag{16}$$

for some $\beta \geq 0, 0 \leq \theta < 1$. Assume moreover that

$$\eta = \max \left(\frac{d^*}{d}, \frac{\theta + \frac{2\beta}{\alpha d}}{\sqrt{1 - (\frac{2\omega}{\alpha})^2}} \right) < 1. \tag{17}$$

Then we have the following stability bounds

$$(1 - \eta)d \leq I \leq S \leq (1 + \eta)d. \tag{18}$$

Remark 2. If $\beta = \theta = 0$, one can prove the same result for the case $\alpha = 2\omega$.

Corollary 1. Assume that

$$r_k(0) = z_{k-1}(0) - z_k(0) = d, \quad \dot{z}_k(0) = \dot{z}_0(0), \quad k = 1, 2, \dots \tag{19}$$

If moreover

$$d^* < d$$

then

$$d - d^* \leq I \leq S \leq d + d^*. \tag{20}$$

Remark 3. It is natural to consider the stability problem for more general initial conditions. Assume we know the initial conditions and the parameters v_{\max}, a_{\max} of the leader. We want to know whether the parameters α, ω, d such that $0 < I \leq S < \infty$, exist. Namely, we will prove that for any $z_0(t)$, satisfying (6) and the initial conditions such that:

$$\min_{k \geq 1} (z_{k-1}(0) - z_k(0)) = A > 0, \quad \max_{k \geq 1} (z_{k-1}(0) - z_k(0)) = B < \infty, \quad \max_{k \geq 1} |\dot{z}_{k-1}(0) - \dot{z}_k(0)| = C < \infty,$$

the parameters α, ω, d , such that $0 < I \leq S < \infty$, always exist.

Theorems 1 and 3 and Corollary 1

- (1) guarantee that there are no collisions between particles (by the lower bound),
- (2) provide lower bound for the density of the flow (by the upper bound), that is the mean distance between particles remains bounded.
- (3) do not guarantee that the velocities remain positive. Such conditions were obtained above in [Theorem 2](#).

Flow density. Let $n(t, I)$ be the number of units on the interval $I \subset R$ at time t . Instead of the density ($|I|$ is the length of I)

$$\rho(t) = \liminf_{|I| \rightarrow \infty} \frac{n(t, I)}{|I|}$$

it is more convenient to consider the inverse density, or the mean distance between cars $0, 1, \dots, N$

$$L_N(t) = \frac{1}{N} \sum_{k=1}^N r_k(t) = \frac{z_0(t) - z_N(t)}{N}.$$

Theorem 4. For any $\alpha > 0, \omega > 0$ assume that the initial conditions (16) are such that the following finite limits exist

$$\lim_{N \rightarrow \infty} L_N(0) = L(0), \quad \lim_{N \rightarrow \infty} \dot{L}_N(0) = \dot{L}(0),$$

Then for any t there exists

$$\lim_{N \rightarrow \infty} L_N(t) = L(t),$$

and moreover

$$L(t) = L(0) + \frac{1}{\alpha}(1 - e^{-\alpha t})\dot{L}(0).$$

Note that if moreover $\frac{dz_k(0)}{dt}$ are uniformly bounded then $\frac{dL}{dt}(0) = 0$, that is the mean length does not change with time.

Due to convergence to stationary movement, in the case when $z_0(t) = vt$, the flow current converges, at any point, to the current of the stationary flow

$$va^{-1} = \frac{\omega^2 v}{\omega^2 d + \alpha v}.$$

Restricted stability. Here we consider the region $\sqrt{2}\omega \leq \alpha \leq 2\omega$, where we can prove stability only for asymptotically homogeneous initial conditions.

Theorem 5. Let $\sqrt{2}\omega \leq \alpha \leq 2\omega$ and let $z_0(t)$ be such that

$$\omega \int_0^\infty |z_0(t) - vt| dt = \sigma a < \infty$$

for some $\sigma \geq 0$. Assume also that the initial conditions are “summable”, that is

$$\sum_{k=1}^\infty |z_k(0) + ka| \leq \theta a, \quad \sum_{k=1}^\infty |\dot{z}_k(0) - v| \leq \beta v$$

for some $\theta, \beta \geq 0$. Then

$$I \geq (1 - 2\eta)a, \quad S \leq (1 + 2\eta)a,$$

where

$$\eta = 2 \left(\theta + \frac{\beta v}{a\omega} + \sigma \right).$$

It follows from this theorem that the upper bound $S < \infty$ holds for all parameters, but the lower (safety) bound $I > 0$ holds if

$$\theta + \frac{\beta v}{a\omega} + \sigma < \frac{1}{4}.$$

Theorem 6. Assume again that $\sqrt{2}\omega \leq \alpha \leq 2\omega$, the initial conditions satisfy

$$\sum_{k=1}^\infty |r_k(0) - d| \leq \theta d, \quad \sum_{k=1}^\infty |\dot{z}_{k-1}(0) - \dot{z}_k(0)| \leq \beta, \quad k = 1, 2, \dots$$

for some $\beta \geq 0, 0 \leq \theta \leq 1$, and the leader moves as

$$\frac{1}{\omega} \int_0^{+\infty} |\ddot{z}_0(t) + \alpha \dot{z}_0(t)| dt = \sigma d < \infty$$

for some $\sigma \geq 0$. Then

$$I \geq d(1 - \eta), \quad S \leq d(1 + \eta),$$

where

$$\eta = 2 \left(\theta + \frac{\beta}{\omega d} + \sigma \right).$$

It follows that the safety condition $I > 0$ holds if

$$\theta + \frac{\beta}{\omega d} + \sigma < \frac{1}{2}.$$

2.2. Instability

Here we will prove instability for the region $\alpha < \sqrt{2}\omega$. The first reason for the instability is the absence of dissipation, that is if $\alpha = 0$. The following result shows this even for the most favorable initial conditions.

Theorem 7. Assume that $\alpha = 0$ and

$$z_k(0) = -ka, \quad \frac{dz_k}{dt}(0) = v, \quad k \geq 0 \quad (21)$$

$$z_0(t) = tv + \sin \omega_0 t, \quad v > 0, \quad \omega_0 \neq 0.$$

Then for any $k \geq 2$ we have due to resonance

$$\inf_{t \geq 0} r_k(t) = -\infty.$$

Now we show that, even under the smallest perturbation of the initial conditions (21) and even simpler leader trajectory, we get much more general instability condition.

Theorem 8. Assume $\alpha < 2\omega$, $z_0(t) = vt$ and initial conditions such that:

$$z_k(0) = -ka, \quad \dot{z}_k(0) = \begin{cases} v, & k > 1, \\ v + \epsilon, & k = 1 \end{cases} \quad (22)$$

where ϵ is some real number. Then for any $\mu > \frac{1}{\tau}$, $\tau = \sqrt{\omega^2 - \frac{\alpha^2}{4}}$ the following asymptotic formula takes place:

$$z_{k+1}(t) - (vt - (k+1)a) \sim \frac{c}{\sqrt{k}} e^{kf(\mu)} \sin(\Omega(\mu)k + \phi_0(\mu)), \quad \text{when } t = \mu k, \quad k \rightarrow \infty$$

where

$$f(\mu) = -\frac{\alpha\mu}{2} + 1 - \ln\left(\frac{2\tau}{\mu\omega^2}\right),$$

$$\phi_0(\mu) = \arctan(v), \quad v = \sqrt{\mu^2\tau^2 - 1}, \quad \Omega(\mu) = v - \phi_0(\mu) = v - \arctan(v), \quad c = \epsilon \sqrt{\frac{2\tau}{\pi v \mu}}.$$

The necessary properties of the function $f(\mu)$ will be obtained in Lemma 6.

Corollary 2. Assume $\alpha < \sqrt{2}\omega$ and $z_0(t) = vt$. Assume the initial conditions (22). Then

$$I = -\infty, \quad S = \infty.$$

While proving the theorem we will see that corollary 2 holds even for more general initial conditions:

$$\max_k |z_k(0) + ka| \leq \epsilon_1, \quad \max_k |\dot{z}_k(0) - v| \leq \epsilon_2,$$

with some non negative $\epsilon_1, \epsilon_2 \geq 0$.

3. Stability: proofs

3.1. Theorem 3, Corollary 1, Remark 1

It is very convenient to use new variables

$$x_0(t) = \frac{\ddot{z}_0(t) + \alpha\dot{z}_0(t)}{\omega^2}, \quad x_k(t) = z_{k-1}(t) - z_k(t) - d = r_k(t) - d, \quad k = 1, 2, \dots \quad (23)$$

Using

$$\ddot{z}_k = \omega^2 x_k - \alpha\dot{z}_k = \omega^2 x_k - \alpha(\dot{z}_{k-1} - \dot{x}_k), \quad \ddot{z}_k = \ddot{z}_{k-1} - \ddot{x}_k$$

we get the main equations (4) in terms of these variables:

$$\ddot{x}_k + \alpha\dot{x}_k + \omega^2 x_k = \ddot{z}_{k-1} + \alpha\dot{z}_{k-1} = \omega^2 x_{k-1}, \quad x_0 = \frac{\ddot{z}_0 + \alpha\dot{z}_0}{\omega^2}. \quad (24)$$

Lemma 1. Assume that

$$\sup_{t \geq 0} |x_0(t)| = Q < \infty.$$

Let also the initial conditions be such that

$$\sup_{k \geq 1} |x_k(0)| = A < \infty, \quad \sup_{k \geq 1} |\dot{x}_k(0)| = C < \infty.$$

If $\alpha > 2\omega$ then

$$\sup_{k,t} |x_k(t)| \leq Q' = \max \left\{ Q, \frac{\alpha A + 2C}{\sqrt{\alpha^2 - 4\omega^2}} \right\}.$$

Proof. The solution of the linear equation (24) for any fixed k is of course well-known. Namely, if the roots

$$\lambda_{\pm} = -\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - \omega^2}$$

of the characteristic equation

$$G(\lambda) = \lambda^2 + \alpha\lambda + \omega^2 = 0 \tag{25}$$

are different, then the solution can be written as

$$x_k(t) = x_{k,+}(t) + x_{k,-}(t)$$

where

$$\begin{aligned} x_{k,\pm}(t) &= C_{k,\pm} e^{\lambda_{\pm} t} + \omega^2 \frac{e^{\lambda_{\pm} t}}{G'(\lambda_{\pm})} \int_0^t e^{-\lambda_{\pm} t_1} x_{k-1}(t_1) dt_1 \\ &= C_{k,\pm} e^{\lambda_{\pm} t} + \omega^2 \frac{e^{\lambda_{\pm} t}}{2\lambda_{\pm} + \alpha} \int_0^t e^{-\lambda_{\pm} t_1} x_{k-1}(t_1) dt_1. \end{aligned}$$

Using the initial conditions one finds

$$C_{k,\pm} = \frac{1}{2\gamma} \left(\left(\pm \frac{\alpha}{2} + \gamma \right) a_k \pm b_k \right), \quad \gamma = \sqrt{\frac{\alpha^2}{4} - \omega^2},$$

where $x_k(0) = a_k, k \in \mathbb{N}, \dot{x}_k(0) = b_k, k \in \mathbb{N}$. As $\lambda_+ > \lambda_-$ we get:

$$|x_k(t)| \leq |C_{k,+}| \exp(\lambda_+ t) + |C_{k,-}| \exp(\lambda_- t) + \sup_{s \geq 0} |x_{k-1}(s)| \frac{\omega^2}{2\gamma} \int_0^t |\exp(\lambda_+(t-t_1)) - \exp(\lambda_-(t-t_1))| dt_1.$$

Let us prove that for $y_k = \sup_{s \geq 0} |x_k(s)|$ the following inequalities hold

$$y_k \leq \max \left\{ y_{k-1}, \frac{\alpha A + 2C}{2\gamma} \right\}. \tag{26}$$

Putting $t - t_1 = s$ and using $\omega^2 = \lambda_+ \lambda_-$ and

$$|C_{k,\pm}| \leq \frac{(\frac{\alpha}{2} \pm \gamma)A + C}{2\gamma}$$

we get:

$$\begin{aligned} |x_k(t)| &\leq |C_{k,+}| \exp(\lambda_+ t) + |C_{k,-}| \exp(\lambda_- t) + \sup_{s \geq 0} |x_{k-1}(s)| \frac{\omega^2}{2\gamma} \int_0^t (\exp(\lambda_+ s) - \exp(\lambda_- s)) ds \\ &\leq \frac{1}{2\gamma} \left(\left(\frac{\alpha}{2} + \gamma \right) A + C + y_{k-1} \lambda_- \right) \exp(\lambda_+ t) + \frac{1}{2\gamma} \left(\left(\frac{\alpha}{2} - \gamma \right) A + C - y_{k-1} \lambda_+ \right) \exp(\lambda_- t) + y_{k-1}. \end{aligned} \tag{27}$$

To finish the proof we need the following simple lemma.

Lemma 2. Let us consider the function

$$f(t) = a \exp(\lambda_+ t) + b \exp(\lambda_- t) + c$$

for some constants $b, c > 0, a \in \mathbb{R}, \lambda_- < \lambda_+ < 0$. For every $t \geq 0$ the following statement is true:

$$|f(t)| \leq \max\{c, a + b + c\}.$$

Proof. There are two cases:

1. $a > 0$, then it is obvious that $\sup_{s \geq 0} |f(s)| = f(0) = a + b + c$.
2. $a < 0$, in this case the set of such t , that the derivative of the function is positive, is:

$$t > \frac{1}{\lambda_- - \lambda_+} \ln \left(-\frac{a \lambda_+}{b \lambda_-} \right).$$

The point $t_0 = \frac{1}{\lambda_- - \lambda_+} \ln \left(-\frac{a \lambda_+}{b \lambda_-} \right)$ is the minimum point. If $t_0 < 0$, then $\sup_{s \geq 0} |f(s)|$ tops in the point $+\infty$ and is equal to c . If $t_0 > 0$, then $\sup_{s \geq 0} |f(s)|$ tops in the point 0 or in the point $+\infty$.

Having applied this lemma to (27) we get (26) and

$$|x_k(t)| \leq \max\{Q, B\} = Q'$$

where $B = \frac{\alpha A + 2C}{2\gamma}$. Lemma 1 is thus proved.

To prove theorem 3 it is sufficient to put $A = \theta d$, $C = \beta$, $Q = d^*$. To prove Corollary 1 we can put $\theta = \beta = 0$.

Remark 3. We use here Theorem 3 and the notation therein. From

$$(1 - \theta)d = A, \quad (1 + \theta)d = B, \quad \beta = C, \tag{28}$$

we find first the parameters d, θ ,

$$d = \frac{A + B}{2}, \quad \theta = \frac{B - A}{2d} = \frac{B - A}{A + B}.$$

Then we find α, ω such that $\alpha > 2\omega$ and $\eta < 1$. Namely, we choose α sufficiently large and take any ω so that

$$\frac{1}{\sqrt{d}} \sqrt{a_{\max} + \alpha v_{\max}} < \omega < \frac{\alpha}{2}.$$

Then the inequality (17) evidently holds.

3.2. Theorem 1

Note that the functions $q_k(t), k = 1, 2, \dots$, defined by

$$z_k(t) = vt - ka + q_k(t), \tag{29}$$

satisfy the system of Eqs. (24) with the initial conditions:

$$q_0(t) = z_0(t) - vt, \quad q_k(0) = z_k(0) + ka, \quad \dot{q}_k(0) = \dot{z}_k(0) - v, \quad k = 1, 2, \dots$$

Then

$$\sup_k |q_k(0)| \leq \theta a, \quad \sup_k |\dot{q}_k(0)| \leq \beta v, \quad \sup_t |q_0(t)| = \delta a.$$

Using Lemma 1 one has the lower bound

$$r_k(t) = z_{k-1}(t) - z_k(t) = a + q_{k-1} - q_k(t) \geq a - 2 \sup_{k,t} |q_k(t)| = a(1 - \epsilon).$$

The upper bound for $r_k(t)$ can be obtained similarly.

3.3. Theorem 2

Differentiating the system (24) we get the equations for velocities $v_k(t) = \dot{z}_k(t)$.

$$\ddot{v}_k = \omega^2(v_{k-1} - v_k) - \alpha \dot{v}_k, \tag{30}$$

with initial conditions:

$$v_k(0) = v, \quad \dot{v}_k(0) = 0.$$

So for every $k = 0, 1, 2, \dots$:

$$v_k(t) = v + u_k(t),$$

where u_k satisfy equations (30) with zero initial conditions $u_0(t) = v_0(t) - v, u_k(0) = \dot{u}_k(0) = 0, k \geq 1$. So from Lemma 1 for u_k we get that for any $k = 1, 2, \dots$ and $t \geq 0$

$$|u_k(t)| \leq \sup_{t \geq 0} |u_0(t)| = \sup_{t \geq 0} |\dot{z}_0(t) - v|.$$

This proves assertion (1) of the Theorem.

Let us prove now assertion (2). The functions $q_k(t)$ defined in (29) satisfy the system (24) with initial conditions:

$$q_k(0) = z_k(0) + ka, \quad \dot{q}_k(0) = \dot{z}_k(0) - v.$$

Moreover $q_0(t) = z_0(t) - vt = 0$ for all $t \geq 0$. By induction we get:

$$\left(\frac{d^2}{dt^2} + \alpha \frac{d}{dt} + \omega^2 \right)^k q_k = 0,$$

which gives

$$q_k(t) = e^{\lambda_1 t} P_k(t) + e^{\lambda_2 t} Q_k(t),$$

where P_k, Q_k are the polynomials of degree $k - 1$ and λ_1, λ_2 are the roots of Eq. (25). Note that, for any $\omega, \alpha > 0$, the real parts of λ_1, λ_2 are negative.

The last statement follows if we put $\delta = 0$ in (11), which gives (14).

3.4. Theorems 5 and 6

Before proving them, we need new notation. Introduce the Banach space

$$X = \{ \psi = (q, p), q = \{q_k\}_{k=1,2,\dots}, p = \{p_k\}_{k=1,2,\dots} : \|q\| = \sup_k |q_k| < \infty, \|p\| = \sup_k |p_k| < \infty, q_k, p_k \in \mathbb{R} \} \quad (31)$$

with the norm

$$\|\psi\| = \max\{\|q\|, \|p\|\}.$$

Define also the bounded (non-selfadjoint) linear operator in X

$$A\psi = \psi', \quad \psi = (q, p), \quad \psi' = (q', p'), \quad (32)$$

where

$$q' = p, \quad p'_k = \omega^2(q_{k-1} - q_k) - \alpha p_k, \quad q_0 = 0.$$

Then the dynamics (24), if $q_k = x_k, p_k = \dot{x}_k, k \geq 1$ and $x_0(t) = 0$, can be written as

$$\dot{\psi} = A\psi \implies \psi(t) = e^{At} \psi(0). \quad (33)$$

Proof of Theorems 5 and 6 is similar to the proof of Theorem 1 and 3, but one should use, instead of Lemma 1, the following lemma.

Lemma 3. Consider the system of equations

$$\frac{d^2 q_k}{dt^2} + \alpha \frac{dq_k}{dt} + \omega^2 q_k = \omega^2 q_{k-1}, \quad k = 1, 2, \dots \quad (34)$$

that is the same as (24), where $q_0(t)$ is continuous and absolutely integrable on \mathbb{R}_+ :

$$\int_0^{+\infty} |q_0(t)| dt = Q < \infty.$$

Assume the following initial conditions

$$\sum_{k=1}^{\infty} |q_k(0)| = a < \infty, \quad \sum_{k=1}^{\infty} |\dot{q}_k(0)| = b < \infty.$$

If $\sqrt{2} \leq \alpha \leq 2\omega$ then

$$\sup_{k,t} |q_k(t)| \leq Q' = 2a + \frac{2b}{\omega} + 2\omega Q.$$

Proof. The system (34) can be written in the operator form, as in

$$\dot{\psi} = A\psi + \omega^2 q_0(t) g_1, \quad \psi(0) = \psi, \quad (35)$$

where $g_1 = (0, e_1)^T \in X, e_1$ is the vector $(1, 0, 0, 0 \dots)$. Then the solution of (35) is quite standard, see [5],

$$\psi(t) = e^{tA} \psi(0) + \omega^2 \int_0^t q_0(s) e^{(t-s)A} g_1 ds,$$

and

$$e^{tA} = -\frac{1}{2\pi i} \int_{\Gamma} e^{zt} R(z) dz, \tag{36}$$

where Γ is a closed contour around the spectrum of A , and $R(z)$ is the resolvent of A , see (45). Using Lemma 5 we get

$$q_k(t) = q_k^{(L)}(t) + q_k^{(N)}(t), \tag{37}$$

where

$$q_k^{(L)}(t) = (e^{tA} \psi(0))(q_k) = -\frac{1}{2\pi i} \sum_{j=0}^{k-1} \int_{\Gamma} \phi^j(z) l_{k-j}(z) e^{tz} dz, \tag{38}$$

$$q_k^{(N)}(t) = \frac{\omega^2}{2\pi i} \int_{\Gamma} \frac{\phi^{k-1}(z)}{z^2 + \alpha z + \omega^2} Q(t, z) dz, \quad Q(t, z) = \int_0^t e^{(t-s)z} q_0(s) ds. \tag{39}$$

Then change of variables gives

$$Q(t, z) = \int_0^t e^{sz} q_0(t-s) ds = \sum_{k=0}^{\infty} \frac{z^k}{k!} \int_0^t s^k q_0(t-s) ds.$$

As for any $t \geq 0$

$$\left| \int_0^t s^k q_0(t-s) ds \right| \leq t^k Q,$$

the $Q(t, z)$ is entire for any $t \geq 0$. Note that the integrands in the formulas (38)–(39) are meromorphic functions in the complex plane, having exactly two poles z_1, z_2 , corresponding to the multiplicity of the roots of the polynomial $G(z)$. By Theorem 9, the points z_1, z_2 belong to the spectrum of A , and by the same theorem, we can to choose the contour Γ so that the following 3 conditions hold

1. if $z \in \Gamma$ then $\text{Re}(z) \leq 0$;
2. $\sigma(A) \cap \Gamma = \{0\}$;
3. the points z_1, z_2 lie inside the domain bounded by Γ .

Using formulas (38)–(39) for the chosen Γ , we will get estimates of the functions $q_k^{(L)}, q_k^{(N)}$. It is clear that $|\phi(z)| \leq 1$ for all $z \in \Gamma$. Then

$$|q_k^{(L)}(t)| \leq \frac{1}{2\pi} \sum_{j=0}^{k-1} \int_{\Gamma} |\phi^j(z)| |l_{k-j}(z)| \|e^{tz}\| |dz| \leq c'_q a + c'_p b,$$

where the constants

$$c'_q = \frac{|\Gamma|}{2\pi} \frac{1}{\omega^2} \max_{z \in \Gamma} |\alpha + z|, \quad c'_p = \frac{|\Gamma|}{2\pi} \frac{1}{\omega^2}.$$

Similarly we get estimate for $q_k^{(N)}$:

$$|q_k^{(N)}(t)| \leq Q \frac{|\Gamma|}{2\pi}.$$

We want to estimate the length of the contour Γ , satisfying the conditions 1–3. Assuming that $\alpha < 2\omega$, let the points z_1, z_2 are enumerated so that

$$z_1 = -\frac{\alpha}{2} + ir, \quad z_2 = -\frac{\alpha}{2} - ir, \quad r = \sqrt{\omega^2 - \frac{\alpha^2}{4}}.$$

Then $z^2 + \alpha z + \omega^2 = (z - z_1)(z - z_2)$, and if $z \in \sigma(A)$, then at least one of the inequalities $|z - z_1| \leq \omega$ or $|z - z_2| \leq \omega$ holds, see Theorem 9 below. It follows that $\sigma(A)$ belongs to the union of two circles

$$K_1 = \{z \in \mathbb{C} : |z - z_1| \leq \omega\}, \quad K_2 = \{z \in \mathbb{C} : |z - z_2| \leq \omega\}.$$

Note that both circles intersect the real axis at the points $-\alpha$ and 0 . That is why Γ can be presented as the union of a part of the circle bounding K_1 , segment of the imaginary axis and a part of the circle bounding K_2 . Then

$$|\Gamma| \leq 4\pi\omega, \quad \max_{z \in \Gamma} |z + \alpha| = 2\omega.$$

and finally

$$|q_k^{(L)}(t)| \leq 2a + \frac{2b}{\omega}, \quad |q_k^{(N)}(t)| \leq 2\omega Q.$$

The proof of [Theorem 6](#) is similar to the proof of [Theorem 5](#), only instead of the variables $q_k(t)$, introduced therein, one should use variables $x_k(t)$, introduced in (23).

3.5. Theorem 4

Rewriting L_N in terms of x_k

$$L_N(t) = \frac{z_0(t) - z_N(t)}{N} = \frac{\sum_{k=1}^N x_k(t)}{N} + d, \quad t \geq 0 \tag{40}$$

and using Eqs. (24), we get the linear equation for L_N

$$\ddot{L}_N + \alpha \dot{L}_N + \omega^2 L_N = \frac{\omega^2}{N} \sum_{k=1}^N x_{k-1} + d\omega^2 = \omega^2 \left(\sum_{k=1}^N \frac{x_k}{N} + d + \frac{x_0 - x_N}{N} \right) = \omega^2 L_N + \omega^2 \frac{x_0 - x_N}{N}$$

or

$$\ddot{L}_N + \alpha \dot{L}_N = f_N(t) = \omega^2 \frac{x_0 - x_N}{N}. \tag{41}$$

It is easy to see that the solution of this equation is

$$L_N(t) = L_N(0) + \frac{1}{\alpha}(1 - e^{-\alpha t})\dot{L}_N(0) + g_N(t)$$

where

$$g_N(t) = \int_0^t e^{-\alpha(t-s)} h_N(s) ds, \quad h_N(s) = \int_0^s f_N(s') ds'.$$

Lemma 4. *There exists smooth function $c(t)$ on R_+ such that for any $k \geq 1$ and any $t \geq 0$*

$$|x_k(t)| \leq c(t).$$

Proof. The solution can be written as in (37)

$$x_k(t) = q_k^{(L)}(t) + q_k^{(N)}(t).$$

We can use formula (37), choosing any contour Γ , which does not intersect the spectrum of A . It is clear that for any $z \in \Gamma$ we have $|\phi(z)| < q < 1$ for some q . Then

$$|q_k^{(L)}(t)| \leq \frac{|\Gamma|}{2\pi} \frac{\theta dc + \beta}{1 - q} \frac{1}{\omega^2} u(t), \quad u(t) = \max_{z \in \Gamma} |e^{zt}|, \quad c = \max_{z \in \Gamma} |z + \alpha|.$$

Similarly one can get the bound for $q_k^{(N)}(t)$.

It follows from this lemma that for any $0 < T < \infty$ the sequence $f_N(t) \rightarrow 0$, as $N \rightarrow \infty$, uniformly in $t \in [0, T]$. Thus for any $t > 0$

$$\lim_{N \rightarrow \infty} g_N(t) = 0$$

and the theorem follows.

4. Instability: proofs

4.1. Theorem 7

This is just an exercise to see that the reason for this is the resonance effect: already $x_1(t)$ obtains harmonic components with frequencies ω, ω_0 , that implies resonance for the particle 2 as its proper frequency is ω . Here

$$\frac{d^2 x_1(t)}{dt^2} + \omega^2 x_1(t) = f(t) = -\frac{1}{(\omega_0 \omega)^2} \frac{d^2 z_0(t)}{dt^2} \tag{42}$$

with $x_1(0) = \dot{x}_1(0) = 0$. In case $\omega = \omega_0$ the proper frequency coincides with the frequency of the external force, and already the first particle will have resonance behavior that is $\sup_{t \geq 0} |x_1(t)| = \infty$. Consider now the case when $\omega \neq \omega_0$. The Eq. (42) has the solution

$$x_1(t) = \frac{1}{\omega^2 - \omega_0^2} \left(\sin(\omega_0 t) - \frac{\omega_0}{\omega} \sin(\omega t) \right). \tag{43}$$

That is why already for the particle 2 the resonance occurs. Note however, that the first collision can occur between particles k and $k - 1$ for $k > 2$.

4.2. Spectrum, Theorem 8 and Corollary 2

Remarks concerning spectrum.

Theorem 9. The spectrum $\sigma(A)$ of A is the set

$$\sigma(A) = \{z \in \mathbb{C} : |z^2 + \alpha z + \omega^2| \leq \omega^2\}. \tag{44}$$

As a corollary we have:

- (1) if $\alpha \geq \sqrt{2}\omega$ then for any $z \in \sigma(A)$ we have $\text{Re}(z) \leq 0$ and the equality takes place only if $z = 0$.
- (2) if $\alpha < \sqrt{2}\omega$ then $\sigma(A)$ contains some segment of the imaginary axis together with a neighborhood.

Proof. Let

$$R(z) = (A - zE)^{-1}, \quad z \in \mathbb{C} \tag{45}$$

be the resolvent of A . We use the following Lemma.

Lemma 5. Let $\psi = (q, p) \in X$, $\psi' = (q', p') \in X$ and

$$R(z)\psi = \psi'.$$

Then for any $k = 1, 2, \dots$

$$q'_k = \sum_{j=0}^{k-1} \phi^j(z) l_{k-j}, \quad p'_k = q_k + zq'_k$$

where

$$\phi(z) = \frac{\omega^2}{z^2 + \alpha z + \omega^2}, \quad l_k(z) = -\frac{(\alpha + z)q_k + p_k}{z^2 + \alpha z + \omega^2}. \tag{46}$$

Proof. From the definition

$$\psi = (A - zE)\psi'.$$

Then $q = p' - zq'$ and $p' = q + zq'$. Moreover, we have the system of equations

$$p_k = \omega^2(q'_{k-1} - q'_k) - (\alpha + z)p'_k = \omega^2(q'_{k-1} - q'_k) - (\alpha + z)(q_k + zq'_k)$$

which, using (46), can be rewritten as

$$q'_k = \frac{\omega^2}{z^2 + \alpha z + \omega^2} q'_{k-1} - \frac{(\alpha + z)q_k + p_k}{z^2 + \alpha z + \omega^2} = \phi(z)q'_{k-1} + l_k(z).$$

Finally

$$q'_k = \sum_{j=0}^{k-1} \phi^j(z) l_{k-j}. \tag{47}$$

Proof of Theorem 9. Let us consider two cases. Let $|\phi(z)| = q < 1$. Note that for any k

$$|l_k(z)| \leq \left| \frac{(\alpha + z) + 1}{z^2 + \alpha z + \omega^2} \right| \|\psi\| = c.$$

It follows that for any k

$$|q'_k| \leq c \frac{1 - q^k}{1 - q} \leq \frac{c}{1 - q}.$$

Then the operator $R(z)$ is bounded and $z \notin \sigma(A)$. Another possibility is $|\phi(z)| > 1$. Consider the sequence $\psi_n = (q^n, p^n) \in X$, where

$$q^n = 0$$

$$p^n = \begin{cases} 1, & k < n \\ 0, & k \geq n. \end{cases}$$

Denote $\psi'_n = R(z)\psi_n$, $\psi'_n = (\hat{q}^n, \hat{p}^n)$. From formula (47) we get

$$\hat{q}^n = -\frac{1}{z^2 + \alpha z + \omega^2} \frac{1 - \phi^n(z)}{1 - \phi(z)}.$$

That is why for $|\phi(z)| > 1$

$$\|R(z)\psi_n\| \geq \frac{1}{|z^2 + \alpha z + \omega^2|} \left| \frac{1 - \phi^n(z)}{1 - \phi(z)} \right| \rightarrow \infty, n \rightarrow \infty.$$

However, it is evident that $\|\psi_n\| = 1$ and thus $R(z)$ is not bounded. We conclude then that $z \in \sigma(A)$. As the spectrum is a closed subset, we get the proof of the first part of the theorem. Now let us prove the two other assertions.

For $z = a + ib$, $a, b \in \mathbb{R}$ we have

$$z^2 + \alpha z + \omega^2 = a^2 - b^2 + \alpha a + \omega^2 + i(2ab + \alpha b).$$

Then the inequality $|z^2 + \alpha z + \omega^2| \leq \omega^2$ is equivalent to the inequality

$$(a^2 + \alpha a + \omega^2 - b^2)^2 + b^2(2a + \alpha)^2 \leq \omega^4$$

or

$$b^4 + b^2((2a + \alpha)^2 - 2(a^2 + \alpha a + \omega^2)) + (a^2 + \alpha a + \omega^2)^2 - \omega^4 \leq 0. \tag{48}$$

If we denote

$$A = (2a + \alpha)^2 - 2(a^2 + \alpha a + \omega^2) = 2a^2 + 2\alpha a + \alpha^2 - 2\omega^2$$

$$B = (a^2 + \alpha a + \omega^2)^2 - \omega^4 = a(a + \alpha)(a^2 + \alpha a + 2\omega^2).$$

the inequality (48) can be rewritten as

$$h(a, b) = b^4 + Ab^2 + B \leq 0.$$

Consider the case $\alpha \geq \sqrt{2}\omega$. Then for any $a > 0$ and $b \in \mathbb{R}$ we have evidently $h(a, b) > 0$, that is $z \notin \sigma(A)$. If $a = 0$ then $B = 0$ and $A \geq 0$. It follows that the unique b such that $h(0, b) \leq 0$ is $b = 0$. Thus the first assertion is proved.

Let now $\alpha < \sqrt{2}\omega$. For $a = 0$ we have $A = \alpha^2 - 2\omega^2 < 0$, $B = 0$. It follows that there exists $b \in [-\sqrt{-A}, \sqrt{-A}]$ such that $h(0, b) < 0$. From continuity of h it follows that there is some interval of the imaginary axis around this b , belonging to the spectrum.

Remark 4. Stability problems, considered here, cannot be completely treated by spectral methods. In fact, the stability concern the value of $\sup_{t \geq 0} \|q(t)\|$. This question was studied by many authors, in particular, by Scottsdale and Rein, see [5]. Their results concern the case of ϵ -dichotomous operators, i.e. when the spectrum of the operator A is the union of two subsets σ_+ , σ_- of the complex plane, such that σ_+ belongs to the open right half-plane and σ_- belongs to the open left half-plane. We have shown that for any parameters the spectrum contain the point 0 and thus A is not ϵ -dichotomous operator.

Proof of Theorem 8. Using formula (36), Lemma 5 and the theorem assumptions one gets:

$$q_{k+1}(t) = -\frac{\epsilon}{2\pi i} \int_{\Gamma} e^{zt} \phi^k(z) dz.$$

The integrand has two poles (roots of the characteristic equation). Because of $\alpha < \sqrt{2}\omega$ the roots are:

$$\lambda_1 = -\frac{\alpha}{2} + i\tau, \quad \lambda_2 = -\frac{\alpha}{2} - i\tau, \quad \tau = \sqrt{\omega^2 - \frac{\alpha^2}{4}}.$$

By Jordan lemma, for any $a > -\frac{\alpha}{2}$

$$q_{k+1}(t) = -\frac{\epsilon}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{zt} \phi^k(z) dz.$$

For $t = \mu k$ we can rewrite this as

$$q_{k+1}(\mu k) = F(\mu, k) = -\frac{\epsilon}{2\pi i} \int_{a-i\infty}^{a+i\infty} \exp(k(\mu z - \ln(z^2 + \alpha z + \omega^2) + \ln \omega^2)) dz \tag{49}$$

$$= -\frac{\epsilon}{2\pi i} \int_{a-i\infty}^{a+i\infty} \exp(kS(z)) dz, \quad S(z) = \mu z - \ln(z^2 + \alpha z + \omega^2) + \ln \omega^2. \tag{50}$$

The mapping $z^2 + \alpha z + \omega^2$ transforms the line $a + ib$, $b \in \mathbb{R}$ to parabola. Thus one can select holomorphic branch of the logarithm on any such line with $a > -\frac{\alpha}{2}$

$$\ln(z^2 + \alpha z + \omega^2) = \ln|z^2 + \alpha z + \omega^2| + i \arg(z^2 + \alpha z + \omega^2).$$

Now we use the saddle point method. Saddle points are found out from the equation:

$$S'(z) = \mu - \frac{2z + \alpha}{z^2 + \alpha z + \omega^2} = 0,$$

or from the quadratic equation

$$\mu z^2 + (\alpha\mu - 2)z + \omega^2\mu - \alpha = 0. \tag{51}$$

The discriminant

$$D = (\alpha\mu - 2)^2 - 4\mu(\omega^2\mu - \alpha) = \alpha^2\mu^2 + 4 - 4\mu^2\omega^2 = -4\mu^2\tau^2 + 4.$$

and we assume that

$$\mu > \frac{1}{\tau}.$$

Then $D < 0$ and there are two complex saddle points:

$$z_{\pm} = z_{\pm}(\mu) = \frac{2 - \alpha\mu \pm i\sqrt{-D}}{2\mu} = -\frac{\alpha}{2} + \frac{1}{\mu} \pm \frac{i}{\mu}\sqrt{\mu^2\tau^2 - 1}.$$

To check that $z_{\pm}(\mu)$ are simple saddle points, we should find the second derivative

$$\begin{aligned} S''(z_{\pm}(\mu)) &= -\frac{2}{z^2 + \alpha z + \omega^2} + \frac{(2z + \alpha)^2}{(z^2 + \alpha z + \omega^2)^2} \Big|_{z=z_{\pm}(\mu)} = -\frac{2\mu}{2z_{\pm} + \alpha} + \mu^2 \\ &= -\frac{\mu}{\frac{1}{\mu} \pm \frac{i}{\mu}\sqrt{\mu^2\tau^2 - 1}} + \mu^2 = \mu^2 \left(\frac{\pm i\sqrt{\mu^2\tau^2 - 1}}{1 \pm i\sqrt{\mu^2\tau^2 - 1}} \right). \end{aligned}$$

This shows that $S''(z_{\pm}(\mu)) \neq 0$. We shall put $a = a(\mu) = -\frac{\alpha}{2} + \frac{1}{\mu}$ in the formula (50) for $F(\mu, k)$

$$F(\mu, k) = -\frac{\epsilon}{2\pi i} \int_{a(\mu)-i\infty}^{a(\mu)+i\infty} \exp(kS(z)) dz.$$

To use the line $a(\mu) + ib$, $b \in \mathbb{R}$ as the contour for the saddle point method one should check two following conditions

(1) The function

$$H(y) = \text{Re}(S(a + iy))$$

reaches its maximum only at two points: $\text{Im}(z_{\pm}) = \pm \frac{1}{\mu}\sqrt{\mu^2\tau^2 - 1}$. In fact

$$\begin{aligned} H(y) &= \mu a - \frac{1}{2} \ln((a^2 - y^2 + \alpha a + \omega^2)^2 + ((2a + \alpha)y)^2) + \ln \omega^2 \\ &= \mu a + \ln \omega^2 - \frac{1}{2} \ln h(y^2), \quad h(s) = (s - (a^2 + \alpha a + \omega^2))^2 + (2a + \alpha)^2 s. \end{aligned}$$

And it is sufficient to check that the graph of $h(s)$ is the parabola with the vertex:

$$\begin{aligned} s_0 &= -\frac{(2a + \alpha)^2 - 2(a^2 + \alpha a + \omega^2)}{2} = -\frac{\frac{4}{\mu^2} - 2(\frac{\alpha^2}{4} - \frac{\alpha}{\mu} + \frac{1}{\mu^2} - \frac{\alpha^2}{2} + \frac{\alpha}{\mu} + \omega^2)}{2} \\ &= -\frac{\frac{2}{\mu^2} + \frac{\alpha^2}{2} - 2\omega^2}{2} = \tau^2 - \frac{1}{\mu^2} = \text{Im}(z_{\pm})^2. \end{aligned}$$

(2) In some neighborhoods of the saddle point z_{\pm} the contour goes through two different sectors where $\text{Re}(S(z)) < \text{Re}(S(z_{\pm}))$. To prove this put

$$v = \sqrt{\mu^2\tau^2 - 1}$$

and rewrite the expression for $S''(z_{\pm})$:

$$S''(z_{\pm}) = \mu^2 \frac{\pm iv}{1 \pm iv} = \mu^2 \frac{v^2 \pm iv}{1 + v^2} = \frac{v}{\tau^2} (v \pm i). \tag{52}$$

Further on we agree that the arguments of complex numbers take values in the interval $(-\pi, \pi]$. Using (52) we obtain:

$$\arg(S''(z_+)) \in \left(0, \frac{\pi}{2}\right), \quad \arg(S''(z_-)) \in \left(-\frac{\pi}{2}, 0\right).$$

The argument of the axis of the saddle point (p. 84 [6]) z_{\pm} is

$$\phi_{\pm} = \frac{\pi}{2} - \frac{1}{2} \arg(S''(z_{\pm})).$$

It follows that

$$\phi_{+} \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right), \quad \phi_{-} \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right).$$

In both cases the angle between the integration contour (i.e. the line $a(\mu) + ib$) and the saddle point axis is less than $\frac{\pi}{4}$. That means that the line $a(\mu) + ib$ can be used as the saddle point method contour (see for example, p. 89 of [6]).

So by formula (5.7.2), p. 88 of [6], we have the asymptotic formula

$$F(\mu, k) \sim -\frac{\epsilon}{2\pi i} (a_{+} e^{kS(z_{+})} + a_{-} e^{kS(z_{-})}), \text{ as } k \rightarrow \infty, \tag{53}$$

where the constants

$$a_{\pm} = \sqrt{\frac{2\pi}{k|S''(z_{\pm})|}} s_{\pm}, \quad s_{\pm} = \exp i\phi_{\pm}.$$

We choose the signs of the constants s_{\pm} so that the angle between the saddle point axis (this angle is determined with an accuracy to π) and the contour of integration were acute (see [6]).

One can rewrite the right part of the formula (53)

$$S(z_{\pm}) = \mu z_{\pm} - \ln\left(\frac{2z_{\pm} + \alpha}{\mu}\right) + \ln \omega^2 = \mu z_{\pm} - \ln\left(\frac{2}{\mu^2}(1 \pm i\sqrt{\mu^2\tau^2 - 1})\right) + \ln \omega^2.$$

Obviously,

$$\exp(kS(z_{-})) = \overline{\exp(kS(z_{+}))},$$

and $\arg(S''(z_{-})) = -\arg(S''(z_{+}))$. Then

$$\overline{s_{-}} = \exp(-i\phi_{-}) = \exp\left(-i\left(\frac{\pi}{2} + \arg(S''(z_{+}))\right)\right) = \exp(-i\pi) \exp(i\phi_{+}) = -s_{+}.$$

Here using (52) we get

$$|S''(z_{+})| = |S''(z_{-})| = \frac{\nu}{\tau^2} \sqrt{\nu^2 + 1} = \frac{\nu\mu}{\tau}$$

and

$$F(\mu, k) \sim -\frac{\epsilon}{2\pi i} (a_{+} \exp(kS(z_{+})) - \overline{a_{+} \exp(kS(z_{+}))}) = -\frac{\epsilon}{\pi} \text{Im}(a_{+} \exp(kS(z_{+}))) \tag{54}$$

$$\text{Re}(S(z_{+})) = \mu a(\mu) - \ln\left|\frac{2}{\mu^2}(1 + i\sqrt{\mu^2\tau^2 - 1})\right| + \ln \omega^2 = \mu a(\mu) - \ln\left(\frac{2\tau}{\mu}\right) + \ln \omega^2 = f(\mu)$$

$$\arg\left(1 + i\sqrt{\mu^2\tau^2 - 1}\right) = \arg S''(z_{-}) + \frac{\pi}{2} = \phi_{+}$$

$$\text{Im}(S(z_{+})) = \sqrt{\mu^2\tau^2 - 1} - \phi_{+} = \nu - \phi_{+}.$$

Substituting this to formula (54), we get

$$F(\mu, k) \sim -\epsilon \sqrt{\frac{2\tau}{\pi k\nu\mu}} e^{kf(\mu)} \text{Im}(e^{i(\phi_{+} + k(\nu - \phi_{+}))}) = \epsilon \sqrt{\frac{2\tau}{\pi k\nu\mu}} e^{kf(\mu)} \sin(\Omega(\mu)k + \phi_0(\mu))$$

where

$$\phi_0(\mu) = \phi_{+} = \arctan(\nu), \quad \Omega(\mu) = \nu - \phi_{+} = \nu - \arctan(\nu).$$

Corollary 2. We need the following

Lemma 6.

1. If $\alpha < \sqrt{2}\omega$ then.

$$\sup_{\mu > \frac{1}{\tau}} f(\mu) > 0$$

2. If $\sqrt{2}\omega \leq \alpha < 2\omega$ then

$$\sup_{\mu > \frac{1}{\tau}} f(\mu) \leq 0.$$

Proof. We have:

$$f'(\mu) = -\frac{\alpha}{2} + \frac{1}{\mu}.$$

It follows that on the interval $(0, \frac{1}{\tau})$ the function $f(\mu)$ is increasing, and for $\mu > \frac{1}{\tau}$ it is decreasing. Thus the function f obtains its maximum at the point $\mu^* = \frac{1}{\alpha}$:

$$f(\mu^*) = -\ln\left(\frac{1}{\alpha} \frac{2\tau}{\omega}\right) = -\ln\left(2x\sqrt{1-x^2}\right), \quad x = \frac{\alpha}{2\omega} < 1.$$

As for all $x \in (0, 1)$ we have $x^2(1-x^2) \leq \frac{1}{4}$, with equality only for $x = \frac{1}{\sqrt{2}}$, we get

$$f(\mu^*) \geq 0$$

where the equality is obtained only for $\alpha = \sqrt{2}\omega$. Consider two cases:

1. $\mu^* > \frac{1}{\tau}$. This is equivalent to the inequality $\alpha < \sqrt{2}\omega$, from where the first assertion of the Lemma follows.
2. $\mu^* \leq \frac{1}{\tau}$. This is equivalent to the inequality $\sqrt{2}\omega \leq \alpha$. As f is decreasing for $\mu > \mu^*$, then

$$\sup_{\mu > \frac{1}{\tau}} f(\mu) = f\left(\frac{1}{\tau}\right).$$

We have:

$$f\left(\frac{1}{\tau}\right) = -\frac{\alpha}{2\tau} - \ln\left(\frac{\tau^2}{\omega^2}\right) + 1 - \ln 2 = 1 - \ln 2 - \frac{x}{\sqrt{1-x^2}} - \ln(1-x^2) = h(x), \quad x = \frac{\alpha}{2\omega} \geq \frac{1}{\sqrt{2}}$$

$$\begin{aligned} h'(x) &= -\frac{1}{\sqrt{1-x^2}} - \frac{x^2}{(1-x^2)\sqrt{1-x^2}} + \frac{2x}{1-x^2} = \frac{-1+2x\sqrt{1-x^2}}{(1-x^2)\sqrt{1-x^2}} \\ &= -\frac{(x-\sqrt{1-x^2})^2}{(1-x^2)\sqrt{1-x^2}} < 0, \quad \text{as } x < 1. \end{aligned}$$

It follows that the function h is monotone decreasing on $[0, 1)$. Also, it follows that

$$f\left(\frac{1}{\tau}\right) \leq h\left(\frac{1}{\sqrt{2}}\right) = 0.$$

This gives the second assertion of the Lemma.

For $t = \mu k, k \rightarrow \infty$ we have:

$$\begin{aligned} R_k &= r_{k+1}(\mu k) = z_k(t) - z_{k+1}(t) = a + q_k - q_{k+1} \\ &\sim a + \frac{c}{\sqrt{k-1}} e^{(k-1)f(\mu)} \left(\sin(\Omega(\mu)(k-1) + \phi_0) - \sqrt{\frac{k-1}{k}} e^{f(\mu)} \sin(\Omega(\mu)k + \phi_0) \right). \end{aligned}$$

We chose μ so that $f(\mu) > 0$ and $\Omega(\mu)$ is irrational. There exists a sub-sequence $k_n, n = 1, 2, \dots$, such that $k_n \rightarrow \infty$ as $n \rightarrow \infty$ and

$$\liminf_{n \rightarrow \infty} \sin(\Omega k_n + \phi_0) = -1, \quad \limsup_{n \rightarrow \infty} \sin(\Omega k_n + \phi_0) = 1.$$

Obviously then:

$$\liminf_{n \rightarrow \infty} R_{k_n} = -\infty, \quad \limsup_{n \rightarrow \infty} R_{k_n} = +\infty.$$

Corollary 2 is proved.

5. Related results and possible perspective

Deterministic chains of cars were considered in many papers where interesting wide range results were obtained for sufficiently general control protocols. We give here a short review of some of these results and explain important difference between these results and our estimates. Speaking shortly, we consider the simplest model of the infinite chain of cars, with two control parameters only but with general initial conditions and general motion of the leader. We obtain for this case very detailed results and, in particular, the complete phase diagram. Moreover, with rigorous proofs of necessary and sufficient conditions for two definitions of stability.

In the following review we use mostly the notation of our paper. General overview one can find in the book [7].

Integral functionals. In [8,9] and in references therein, the following model is considered

$$\ddot{z}_n + \alpha \dot{z}_n = u_n, \quad n = 1, \dots, M,$$

where $\alpha > 0$, $z_n = z_n(t)$ is the coordinate of the n th car on the real axis, $u_1(t), u_2(t), \dots, u_M(t)$ – control functions, which should be chosen to minimize the functional

$$J = \frac{1}{2} \int_0^\infty \sum_{n=1}^{M+1} a(z_{n-1}(t) - z_n(t) - d)^2 + b(\dot{z}_n(t) - v)^2 + c(u_n(t) - \alpha v)^2 dt, \quad u_{M+1} = \alpha v$$

where a, b, c are some-positive constants, v is the safe velocity and d is the safe distance between cars. Moreover, it is assumed that

$$z_0(t) = vt, \quad z_{M+1}(t) = vt - (M + 1)d.$$

This functional has an integral character, and does not guaranty that cars will not collide.

On the contrary, in our paper we consider local conditions under which that following inequalities hold

$$I = \inf_{t,n} (z_{n-1}(t) - z_n(t)) > 0, \quad S < \infty \tag{55}$$

which guaranty the absence of collisions and everywhere positive density.

String stability. Another cluster of papers deals with the so called “string stability” [10–27]. Common definition of the string stability one can find, for example in [10,11] and reviews in [12,13]. In some cases the string stability and our conditions (55) seem to be equivalent. Most papers consider zero initial conditions for separation distance errors: as our conditions (19), but also, see for example [14], the analog of our stationary conditions.

Among the papers where non-zero initial conditions are studied, we discuss in more detail the results of the papers [10, 11,15,16]. In [10,11] a general model was considered. The equations for this model are the following (in our notation)

$$M_k \ddot{z}_k = \frac{1}{1+a} (\ddot{z}_{k-1} + a \ddot{z}_0 + (b + \lambda)(\dot{z}_{k-1} - \dot{z}_k) + (c + \lambda a)(\dot{z}_0 - \dot{z}_k) + Z_k(t))$$

where

$$Z_k(t) = \lambda b(z_{k-1} - z_k) + \lambda c(z_0 - z_k + D_k),$$

$a, b, c \geq 0, \lambda, M_k, D_k$ are fixed constants. Under some conditions on the movement $z_0(t)$ of the leader, some bounds from above, uniform in k, t , for $|r_k(t)| = |z_{k-1}(t) - z_k(t)|$ were obtained. From these formulas it is clear that our model is not a particular case of this one.

In [16] the chain, infinite to both sides, is considered. The control for the car k assumes the knowledge of coordinates and velocities of some number of other cars. The author defines “practical string stability” and obtains sufficient condition for string stability and for this kind of stability which is similar to our “restricted stability”.

In [15] one-side infinite chain of cars is considered under the initial conditions where only one distance between cars is deviated from zero condition.

It is important obviously to consider more general control protocols, for arbitrary “information graph”, see [17,18].

Now we want to explain why the zero initial conditions case is technically easier to study (one does not need recurrent procedures). We will use our model as an example. Let $u_k(\lambda)$ denote the Laplace transform of $x_k(t) = z_{k-1}(t) - z_k(t) - d$

$$u_k(\lambda) = \int_0^{+\infty} e^{-\lambda t} x_k(t) dt, \quad \lambda \in \mathbb{C}.$$

Zero initial conditions give the following simple formula

$$u_k(\lambda) = H(\lambda)u_{k-1}(\lambda), \quad H(\lambda) = \frac{\omega^2}{\lambda^2 + \alpha\lambda + \omega^2}.$$

From this it is easy to get the stability conditions as

$$|H(is)| \leq 1, \quad h(t) \geq 0$$

for all $s \in R$, $t \geq 0$, where $h(t)$ is the inverse Laplace transform of H , see [7,19]. This implies our condition $\alpha \geq 2\omega$, that can be seen from Theorem 9. Note that there are similar results for more general protocols in [20–23].

Model similar to ours is [14], however, it is assumed that $z_0(t) = z_N(t)$, that is the system is considered on the circle. In [14] for zero initial conditions the sufficiency of the stability condition $\alpha > 2\omega$ was rigorously proven.

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