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WHY CURRENT FLOWS: A MULTIPARTICLE ONE-DIMENSIONAL MODEL

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In all known microscopic models of electric current including the basic Drude model, charged particles are accelerated by an external force and some random environment retards them. We introduce a classical multiparticle deterministic one-dimensional model on an interval with nearest-neighbor interaction, explaining how current can flow if the external force acts only on the ends of the passive part (*i.e., outside the generator, battery, etc.*) *of the conductor. We obtain a family of explicit solutions.*

Keywords: electric current, multiparticle system, solid state physics, classical dynamics

1. Introduction

1.1. The Drude Model. In 1900, Drude proposed the first microscopic model of electric current, which can be found in all textbooks on solid state physics (see, e.g., [1]). It is based on the representation of electric current as the motion of charged particles, for example, electrons. We present an exact mathematical formulation. At the time t, there are identical particles at the points $x_i(t)$ on the real line. Their dynamics are based on the following assumptions:

1. The particles do not interact with each other.
2. The dynamics include a random element: at random instants $t_{j,1} < t_{j,2} < \cdots < t_{j,i} < \dots$, the particle j obtains a random velocity $v_{j,i}$ due to interaction with the random environment. All $t_{j,i} - t_{j,i-1}$ are independent and identically distributed, all $v_{j,i}$ are independent and identically distributed and have a zero mean. During the interval between collisions, each jth particle satisfies Newton's equation
$$m\frac{d^2x_j}{dt^2} = F,$$
where the force F is constant.
3. The force F is an external force, i.e., it is independent of the positions $x_i(t)$ of the other particles.

The law of large numbers says that under broad assumptions about the distributions of $t_{j,i} - t_{j,i-1}$ and $v_{j,i}$, there almost certainly exists the limit
$$\lim_{t\to\infty}\frac{x_j(t)}{t} = v = \frac{F\tau}{2m}, \qquad \tau = \langle t_{j,i} - t_{j,i-1}\rangle,$$
for any particle. If the charges e of all particles are the same and the density ρ of particles is constant, then the mean number of particles passing a given point of the line in unit time is $v\rho = F\tau\rho/(2m)$, which gives Ohm's law $U = IR$ for a conductor of unit length with
$$I = \rho v e, \qquad U = \frac{F}{e}, \qquad R^{-1} = \frac{\tau\rho e^2}{2m}.$$

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Newton's equation with friction for any microparticle,

$$m\frac{d^2x_i(t)}{dt^2} = F - \mu\frac{dx_i(t)}{dt},$$

could be used as another (deterministic) alternative for the Drude model.

1.2. Generalizations of the Drude model. The Drude model has been developed in various directions. First, its quantum refinements and generalizations appeared: the Bloch theory, the Sommerfeld theory, the Landau Fermi liquid theory, etc. We note that these theories still exist only at the physical level of rigor although there is a sufficiently developed mathematical theory of the one-particle spectrum in periodic, almost periodic, and random potentials. The point is that the question of introducing a randomness of the Drude type into closed quantum systems does not have a natural, generally accepted answer.

A whole series of mathematical papers on classical generalizations of the Drude model have been published. In these papers, attention has basically been focused on relaxing assumption 2. The external random environment has been introduced in various ways such that $v_{j,i}$ and $t_{j,i}$ interacting with it may even become non-Markovian (see [2] [3] for the latest details on this topic).

In this paper, we consider assumption 3 of the Drude model; it is most essential from our standpoint. The point is that an accelerating *external force is absent* on the entire length of the conductor, except possibly some part, for example, inside a generator. We call this (small) part of the conductor the active part and the remainder the passive part. Even in the physical literature, little attention has been given to the question of why the electrons move in the absence of an external force. This problem is not mentioned at all in many textbooks and monographs and only very briefly in the others. But we must note that the question of where the force F comes from has been actively discussed for many years in the *American Journal of Physics*, basically as a purely methodological question for teaching (see, e.g., [4], [5]). A suggested explanation is that the force moving the electrons is due to a tangential components of charges that somehow appear on the surface of the conductor. In other words, the accelerating force F is some effective force arising from the charges themselves. But if this is so, then we should understand how this occurs. Clearly, the model of independent electrons loses its meaning by definition. Moreover, if there is tangential component of the force, i.e., the charge on one side of an arbitrary cross section is larger than on the other side, then the charge density cannot be constant along the conductor. Furthermore, the charges on the surface cannot stand still and even cannot move with a constant mean velocity. We do not know any mathematical model for this. But a well-defined model is absolutely necessary in order to understand this.

Our goal in this paper is to present a mathematically rigorous model in the simplest one-dimensional case, i.e., for an infinitely thin conductor. Even for this example, all the abovementioned problems receive a satisfactory explanation.

We obtain a solution of the multiparticle problem on an infinite time interval simultaneously satisfying several seemingly contradictory requirements:

a. The particles move to the right because of the gradient of their proper density.

b. The particle velocities are constant on the entire conductor, i.e., Ohm's law holds.

c. The density of particles is constant everywhere on the conductor.

In the exact sense, such a solution cannot exist of course, but it turns out to hold asymptotically for large densities.

Intuitively, the idea is natural and the only possible one. We consider the particles situated at an approximately equal and very small distance r (of the order of the inverse density) from each other. But each distance between two consecutive particles is a little bit larger by some $\epsilon \ll r$ than the preceding distance. Because of the repulsive force, each particle is subjected to a force to the right. The quantity ϵ is adjusted such that this force is of the order $O(1)$. The particle velocities are therefore also of the order $O(1)$. The difficulty of this construction is that if the distance between particles increases, then the particles accelerate. Therefore, the solution is nonstationary but with a very slowly changing velocity. We say that such a model is quasistationary.

It is interesting that such a fundamental physical phenomenon as electric current seems rather delicate.

2. The model and the result

We represent the closed conductor as the interval $[0, L+M]$ with identified endpoints 0 and $L+M$. The two intervals $[0, L)$ and $[L, L+M)$ are respectively called the passive and active parts. The active part corresponds to a generator, battery, etc. We consider only the passive part, and the influence of the active part is taken into account in the boundary conditions on the passive interval.

On the interval $[0, L)$, there are initially $N(L)$ particles at the points

$$0 = x_0(0) < x_{-1}(0) < \cdots < x_{-N(L)+1}(0) < L.$$

We define the energy of interaction (of nearest neighbors) for the particle system as

$$U(\{x_i\}) = \sum_{\langle i,i-1\rangle} V(x_i - x_{i-1}),$$

where the summation ranges all pairs of nearest neighbors inside the interval $[0, L]$. We assume that $V(x) = V(-x) > 0$, $f(r) = -dV(r)/dr \sim c_1 r^{-a}$, $a > 2$, as $r = |x| \to 0$, and $V(x) \to 0$ as $x \to \infty$.

Remark 1. For Coulomb's law in a three-dimensional but very thin conductor, it should be $a = 2$; for a Lennard-Jones potential, $a = 6$; and it is therefore natural to consider the more general case of arbitrary a.

The dynamics of this particle system are described by Newton's equations with friction

$$\frac{d^2 x_i}{dt^2} = -\frac{\partial U}{\partial x_i} - A\frac{dx_i}{dt} + F(x_i, t), \tag{1}$$

where we assume that the particles have unit mass, $m = 1$. It hence follows that the particles cannot change their order on the interval and can be enumerated from right to left $\cdots < x_i(t) < x_{i-1}(t) < \cdots$ at any fixed instant.

Remark 2. If F is constant on the entire interval (the Drude case), then system (1) has a trivial solution on the circle of length L, where the particles are at the equal distance $L/N(L)$ from their neighbors and have the same constant velocity $v = FA^{-1}$. We must construct a more complicated solution that is close to the equilibrium motion for large particle densities.

The boundary conditions are given by some conditions on the force F and by the conditions on the input and output of particles at the endpoints of the passive part of the conductor:

1. If a particle with a positive velocity reaches the point L, then it disappears (enters the active part) at that instant.

2. The external force $F(x,t)$ is bounded and is nonzero only in some small neighborhoods of the endpoints and moreover acts only on the leftmost and rightmost particles $i_-(t)$ and $i_+(t)$. Moreover, we assume that

$$F(x_{i_-(t)}(t)) > \left|f\big(x_{i_-(t)-1}(t) - x_{i_-(t)}(t)\big)\right|, \qquad F(x_{i_+(t)}(t)) < \left|f\big(x_{i_+(t)}(t) - x_{i_+(t)+1}(t)\big)\right| \tag{2}$$

for all t. We thus replace the Drude hypothesis of the external force on the entire conductor with the hypothesis of the external force on only the endpoints.

3. At any instant $t_k = ks$, $k = 0, 1, 2, \ldots$, a particle (from the active part) enters the interval $[0, L]$ at the point 0 with some positive velocity $v_0 > 0$. This is not contradictory because the leftmost particle moves always to the right by assumption 2.

Lemma. *Conditions* 1–3 *uniquely define the dynamics of the system on each time interval* $[ks, (k+1)s]$ *and therefore on the entire interval* $[0,\infty)$*. Moreover, the number of particles on the interval* $[0, L]$ *is finite at each instant.*

It is intuitively obvious that the number of particles on the interval is even uniformly bounded because they are pushed out through the right end.

We first give a rougher formulation of the result in the form of an existence theorem where the conditions a, b, and c in Sec. 1.2 are formulated exactly. We construct the solution explicitly below.

We rewrite Eqs. (1) as

$$\frac{d^2x_i}{dt^2} = f(x_i - x_{i+1}) - f(x_{i-1} - x_i) - A\frac{dx_i}{dt} + F(x_i, t). \tag{3}$$

We let

$$G_i(t) = f\big(x_i(t) - x_{i+1}(t)\big) - f\big(x_{i-1}(t) - x_i(t)\big) + F(x_i, t)$$

denote the force acting on the particle i at the instant $t \in [0, L]$. We recall that F_i is nonzero only for the rightmost and leftmost particles.

Theorem 1. *There exists a sufficiently wide class of potentials V satisfying the above conditions and such that the following holds for any potential in this class. There exists $L_0 > 0$ such that for any $L < L_0$, any sufficiently large $N = [N(L)/L]$, $s = 1/N$, and arbitrary $v_0 > 0$, there exists an $F^{(N)}$ satisfying condition 2 such that the corresponding solution $x_i^{(N)}(t)$ of system (1) satisfies the following conditions for all $0 \le t < \infty$ as $N \to \infty$:*

1. *The forces $G_i^{(N)}(t) \to \mathrm{const} > 0$ for any t and any particle i on the interval at the instant t.*

2. *For any particle i on the interval $[0, L]$ at the instant t, the velocity $v_i^{(N)}(t) \to v_0$.*

3. *For all t and any interval of length l belonging to $[0, L]$, $N(l,t)/N \to l$, where $N(l,t)$ is the number of particles on that interval at the instant t.*

This theorem follows from the more exact theorem below.

3. Explicit solution

3.1. Quasistationary solutions. We say that a solution of system (1) is quasistationary if there exists an increasing smooth function $x(t)$ on the interval $[0, T_L]$ such that $x(0) = 0$, $x(T_L) = L$, and

$$x_k(t) = x(t - ks) \tag{4}$$

for all integers $k > -N(L)$. Thus, $x_k(t)$ is defined for $t \in [ks, ks + T_L]$. The function $x(t)$ is called the *generating function*. Below, we in fact fix the density N of particles, take $s = N^{-1}$ and $T_L = N(L) = NT_L$ for some $T_L = O(1)$, and choose $L = x(T_L)$. It is then easy to see that one particle enters and one particle leaves at the instants ks. It follows that the number $N(L)$ of particles on the interval $[0, L)$ is conserved, and the vector $\bigl(\dots, x_i(t), \dots\bigr)$ is periodic on the interval $[0, L]$ with period s if we take only the configuration and not the enumeration of particles into account.

We note that Eq. (3) can be rewritten for each particle as

$$\frac{d^2 x_i}{dt^2} = G_i(t) - A\frac{dx_i}{dt}.$$

In quasistationary situations, the forces $G_i(t) = G(t)$ are independent of i. Moreover, the force $G(t)$ acting on the particle at the point $x(t) \in [0, L]$ at the instant t under the condition that it is not an extreme particle is determined by the following functional equation (obtained by substituting the quasistationary solution in (1))

$$f\bigl(x(t) - x(t - s)\bigr) - f\bigl(x(t + s) - x(t)\bigr) = G(t), \quad t \in [0, T_L]. \tag{5}$$

The generating function $x(t)$ should therefore satisfy two equations simultaneously, Eq. (5) and the equation

$$\frac{d^2 x}{dt^2} = G(t) - A\frac{dx}{dt}. \tag{6}$$

The question of the boundary conditions remains. We note that Eq. (6) together with the conditions

$$x(0) = 0, \qquad x'(0) = v_0$$

uniquely determines the function $x(t)$ on the entire real axis. This allows fixing the force F. But we solve Eq. (5) on the interval $[0, T_L]$.

It is convenient to introduce the proper time $\tau_k = t - ks$, $0 \le \tau \le T_L$, for each particle k starting from the instant it enters the interval. For the extreme particles i_- and i_+, we then have

$$F(x_{i_-}(\tau_{i_-})) = f\bigl(x(\tau_{i_-} - s) - x_{i_-}(\tau_{i_-})\bigr), \qquad F(x(\tau_{i_+})) = f\bigl(x(\tau_{i_+} + s) - x_{i_+}(\tau_{i_+})\bigr).$$

It follows from this definition that Eq. (5) also holds for both extreme particles. We note that such a specific choice of the boundary conditions is dictated only by the specific explicit solution. Apparently, for more general boundary conditions, there exist solutions that are already not quasistationary.

Our plan for solving functional equation (5) is as follows. We find $x(t)$ from Eq. (6), where we take the function in the form

$$G(t) = w + g_0(t) + g(t),$$

where we choose the seed constant w and function $g_0(t)$ (it is our first approximation) and seek $g(t)$ as the solution of Eq. (5), which we thus we regard as an equation for g with a given f. We note that this equation is nonlinear because $x(t)$ depends on g (and is uniquely determined by it). If there exists a solution of functional equation (5), then there exists a quasistationary solution of Eq. (1).

3.2. The generating function. We consider the equation

$$\frac{d^2x}{dt^2} = G(t) - A\frac{dx}{dt} \tag{7}$$

on the interval $t \in [0, T_L]$ with the initial conditions

$$x(0) = 0, \qquad \frac{dx}{dt}(0) = v_0 = A^{-1}w, \tag{8}$$

where

$$G(t) = w + \nu t + g(t)$$

for some constant $w > 0$, which has the meaning of the basic constant force whose action on each particle is directed to the right. The seed function νt with the small parameter $\nu = \nu(N)$ determines the weak increase of this force with its movement to the right during some short time interval, but nevertheless of the order $O(1)$.

The solution of (7) has the form

$$v(t) = \frac{dx}{dt} = A^{-1}w + \nu e^{-At}\int_0^t ue^{Au}\,du + e^{-At}\int_0^t g(u)e^{Au}\,du. \tag{9}$$

Then

$$e^{-At}\int_0^t ue^{Au}\,du = A^{-1}t - A^{-2} + A^{-2}e^{-At}, \tag{10}$$

which gives

$$v(t) = \frac{dx}{dt} = A^{-1}w + \nu\biggl(\frac{t^2}{2} + O(t^3)\biggr) + e^{-At}\int_0^t g(u)e^{Au}\,du$$

for small t. From (9) and (10), we then have

$$\begin{aligned} x(t) = \int_0^t v(t)\,dt = A^{-1}wt + \nu A^{-1}\frac{t^2}{2} - \nu A^{-2}t - \nu A^{-3}(e^{-At} - 1) + \\ + \int_0^t \biggl(\int_0^p g(u)e^{-A(p-u)}\,du\biggr)\,dp. \end{aligned} \tag{11}$$

3.3. Scales and asymptotic values. We have three leading scales. The main scale has the order $O(1)$; such are L, $T(L)$, the particle velocity, and the force G acting on each particle. The second scale has the order of the inverse density N^{-1}; in this scale, we have the period $s = N^{-1}$ between the entries of neighboring particles at the point 0 and the distances

$$r(t) = x(t+s) - x(t) = r + \epsilon_0 + \epsilon_g$$

between neighboring particles, where

$$\begin{aligned} r &= A^{-1}ws, \\ \epsilon_0 &= \nu A^{-1}\biggl(ts + \frac{1}{2}s^2\biggr) - \nu A^{-2}s - \nu A^{-3}e^{-At}(e^{-As} - 1), \\ \epsilon_g &= \int_t^{t+s}\biggl(\int_0^p g_1(u)e^{-A(p-u)}\,du\biggr)\,dp. \end{aligned} \tag{12}$$

The third scale measures the difference between consecutive distances

$$z(t) = x(t+s) - 2x(t) + x(t-s) = \nu s^3 + \delta_0 + \delta_g,$$

where

$$\delta_0 = -\nu A^{-3}(e^{-As} - 1)\big(e^{-A(t+s)} - 2e^{-At} + e^{-A(t-s)}\big) - \nu s^3 = \nu s^3(e^{-At} - 1) + O(\nu s^4),$$
$$\delta_g = \int_t^{t+s} \left(\int_0^p g(u)e^{-A(p-u)}\,du\right) dp - \int_{t-s}^{t} \left(\int_0^p g(u)e^{-A(p-u)}\,du\right) dp. \tag{13}$$

3.4. Solution of the functional equation.

Theorem 2. *There exists $T_0 > 0$ such that for any $0 < T_L < T_0$ and any sufficiently large N, the quasistationary solution defined by formulas* (4) *and* (11) *exists and is defined on $(-\infty, \infty)$. It is periodic with the period s if the enumeration of particles is neglected. Moreover, the statements in Theorem* 1 *hold.*

In fact, we rewrite functional equation (5) using the asymptotic value of the interaction potential at zero. Taking $c_1 = 1$, for $z = o(r)$, we have the formal series, whose convergence is proved below,

$$f\big(r(t)\big) - f\big(r(t) + z(t)\big) = r(t)^{-a} - \big(r(t) + z(t)\big)^{-a} = r(t)^{-a} \sum_{k=1}^{\infty} (-1)^{k-1} d_k(a) \left(\frac{z(t)}{r(t)}\right)^k, \tag{14}$$

where $d_k(a) = C_{a+k-1}^{a-1}$. We rewrite the expansion as

$$(r + \epsilon_0 + \epsilon_g)^{-a} \sum_{k=1}^{\infty} (-1)^{k-1}\, d_k(a) \left(\frac{\nu s^3 + \delta_0 + \delta_g}{r + \epsilon_0 + \epsilon_g}\right)^k =$$
$$= r^{-a}\left(1 + \frac{\epsilon_0}{r} + \frac{\epsilon_g}{r}\right)^{-a} \sum_{k=1}^{\infty} (-1)^{k-1}\, d_k(a) \left(\frac{\nu s^3/r + \delta_0/r + \delta_g/r}{1 + \epsilon_0/r + \epsilon_g/r}\right)^k =$$
$$= r^{-a} \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} f_{i,j,k,l,m} \left(\frac{\nu s^3}{r}\right)^i \left(\frac{\epsilon_0}{r}\right)^j \left(\frac{\epsilon_g}{r}\right)^k \left(\frac{\delta_0}{r}\right)^l \left(\frac{\delta_g}{r}\right)^m, \tag{15}$$

where

$$f_{i,j,k,l,m} = (-1)^{i+l+m-1} d_{i+l+m}(a).$$

We note that the summation over i begins with $i = 1$. We subdivide this series into two parts. The first part, denoted by η, includes all terms with $k = m = 0$. The second part, denoted by Bg, includes the remaining terms. Thus, η is a constant, and B can be considered a nonlinear operator.

We are interested with the leading term of η with $i = 1$ and $j = k = l = m = 0$. We set it equal to w,

$$w = \frac{a\nu s^3}{r^{a+1}} = \frac{a\nu s^3}{(A^{-1}ws)^{a+1}},$$

which gives the value of the parameter

$$\nu = s^{a-2} a^{-1} A^{-a-1} w^{a+2}.$$

The basic series with the leading term subtracted can be rewritten as

$$Bg + (\eta - w) = w \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} f_{i,j,k,l,m} \left(\frac{\nu s^3}{r} \right)^{i-1} \left(\frac{\epsilon_0}{r} \right)^{j} \left(\frac{\epsilon_g}{r} \right)^{k} \left(\frac{\delta_0}{r} \right)^{l} \left(\frac{\delta_g}{r} \right)^{m} - w. \tag{16}$$

We use the estimates

$$\left| \frac{\epsilon_0}{r} \right| = O(\nu), \qquad \left| \frac{\delta_0}{r} \right| = o(\nu), \qquad \left| \frac{\epsilon_g}{r} \right| \leq C T_L \|g\|, \qquad \left| \frac{\delta_0}{r} \right| \leq C T_L \|g\|$$

for some constant $C > 0$, where the norm of a function $\|g\| = \sup |g|$ is understood as its supremum on this interval. The estimates from above easily follow from formulas (12) and (13).

We rewrite the equation in the form

$$g = Bg - \nu t + \eta, \tag{17}$$

where $\eta = -\nu t$. We note that the next-order g-independent term is the term with $i = j = 1$ and $k = l = m = 0$. It is equal to

$$w r^{-1} \left(\nu A^{-1} \left(ts + \frac{1}{2} s^2 \right) - \nu A^{-2} s - \nu A^{-3} e^{-At} (e^{-As} - 1) \right).$$

This gives a correction to our seed term νt. The remaining terms have a lower order either in N^{-1} or in T_L.

For sufficiently small $T_L > 0$, the operator B maps the ball $\|g\| \leq 1$ into itself. Moreover, we have $\|B\| < \beta$ for small $\beta = \beta(T_L)$ on the ball $\|g\| \leq 1$. Also, $\|\eta\| = O(\nu)$. It follows that the solution of Eq. (17) exists, is unique, and can be written as

$$g = (1 - B)^{-1} (\eta - \nu t) = \sum_{k=0}^{\infty} B^k (\eta - \nu t).$$

We note that our formulas define $g(t)$ on the interval $[-\epsilon, T_L + \epsilon]$, which gives the quasistationary solution.

4. Remarks

Smallness of T_L. On one hand, the condition that T_L is small is technical. On the other hand, some condition seems necessary: current cannot flow along a conductor that is too long. Moreover, this condition can be transformed (with scaling) to conditions on other parameters in the main equation: the mass, the potential (the charge in particular), the friction coefficient A, and also, of course, x and t.

Transient regime. In the original Drude model, the dynamics of one particle tend to the equilibrium motion exponentially fast. It would be interesting to show that exponential convergence also holds for our model with the boundary conditions fixed as above and for some initial configuration of N_L particles close to the uniform configuration of the particles on the interval.

If the boundary conditions are empty and the N particles are initially concentrated on a small subinterval, then we have a discharge problem of the rate of convergence to the uniform configuration at the points $0, L/N, 2L/N, 3L/N, \ldots$.

Continuous charge. Our result shows that the classical model of the movement of charges without an external force is based on a quite delicate mechanism. A natural question therefore arises of whether a similar mechanism can be found in a rougher continuous-charge approximation. We present two possible candidates.

The first rough approximation is to use the Boltzman equation. In this approximation, a smooth density $\rho = \rho(t, x, v)$ is considered instead of the point particles. Formally, the Boltzman equation for the system of noninteracting particles with the external force $F = F(x, v)$ is

$$\frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial x} - \frac{F}{m}\frac{\partial f}{\partial v}.$$

If we assume that the difference of forces from right and left neighbors under some scaling yields a force proportional to the density gradient,

$$F = c\int \frac{\partial f}{\partial x}\,dv,$$

then we obtain the integral-differential equation

$$\frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial x} - \frac{1}{m}\frac{\partial f}{\partial v}\left(c\int \frac{\partial f}{\partial x}\,dv - Av\right).$$

The rougher hydrodynamic approximation assumes that all the mass $\rho(x)$ at the point x has the same velocity $v(x)$. The Navier–Stokes equation for a viscous compressible fluid is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{1}{\rho}\left(\mu\frac{\partial^2 v}{\partial x^2} - \frac{d}{dx}p\right).$$

Together with the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0$$

and the isentropy condition $p = c\rho$, this gives a closed system of equations for $\rho(x, t)$ and $v(x, t)$. The isentropy condition is a substitute for the nearest-neighbor interaction.

Quantum models. Various analogues of (1) are possible in the quantum case. Moreover, the problem is complicated by the question of the extent to which the dispersion of wave packets should be taken into account.

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