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Markov Processes and Related Fields

Intrinsic Convergence Rate of Countable Markov Chains*

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Abstract. The exponential convergence rate to stationarity is very sensitive to perturbations of the transition probabilities. This motivates introducing a parameter that is invariant under perturbations within a finite domain, called "intrinsic rate". The intuitive interpretation of this invariant is the exponential rate at which the Markov chain converges back to a finite domain "from infinity".

For random walks in $\mathbf{Z}_+$ and $\mathbf{Z}_+^2$ we will study this invariant using probabilistic methods, in particular Large Deviations techniques, and analytic methods. Thus we connect convergence rates, action functionals, singularities of generating functions and spectral properties of transition matrices.

Keywords: countable Markov chains, exponential convergence, Large Deviations, essential spectrum

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Contents

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1. Introduction

The problem of determining the exact convergence rate of countable Markov chains is one of the central themes of many papers. Note, however, that the analogous problem for finite Markov chains is precisely the problem of finding the "second largest" eigenvalue of the transition matrix. Calculation of this eigenvalue is not a probabilistic problem: stochastic matrices are not simpler in this respect than arbitrary matrices. Nothing more exact can be said in general. So it only makes sense to look for this eigenvalue, or bounds for it, in interesting examples. For the countable case the situation appears to be similar with one exception: one should first find an appropriate space where the convergence rate coincides with the spectral gap (i.e. the distance of the spectrum, besides eigenvalue 1, from the unit circle). It could therefore be useful to have probabilistic insights in how to choose this space.

Unfortunately, convergence rates are very unstable with respect to changing the transition probabilities, even in only one point. From spectral theory we know that new discrete eigenvalues can appear after such a change. Probabilistic intuition tells us that the problem might be splitted into convergence "from infinity" and convergence within a "finite domain". This motivates looking for an invariant that is conserved under all finite number of changes of the transition probabilities. In Section 2 we define such an invariant called "intrinsic convergence rate" (α_{int}).

This intrinsic convergence rate can be expressed in probabilistic terms using Large Deviation theory. Also it can be guessed heuristically via pure probabilistic arguments. Unfortunately rigorous proofs demand more techniques using generating functions of first hitting probabilities. We find numerous relations between four different subjects:

i) convergence and intrinsic convergence rates;

ii) action functionals of Large Deviation theory;

iii) pole and non-pole singularities of generating functions of first hitting probabilities;

iv) spectrum and essential spectrum of the stochastic matrices in appropriate spaces.

Our main concern will be Markov Chains that do not satisfy the Dœblin condition, but application of these concepts to Dœblin chains appear to be interesting, in particular with respect to random perturbations of dynamical systems. Some connections with this work can be found in [1].

In Section 2 we derive our main results on the above connections for general Markov chains. In Section 3 we apply these results to random walks in $\mathbf{Z}_+^\nu$, $\nu = 1, 2$, and we give explicit expressions for $\alpha_{\rm int}$ and the appropriate spaces connecting $\alpha_{\rm int}$ to the essential spectrum of the transition matrix in this space. In the remaining part of the paper we prove the results formulated in Section 3.

We restrict to small dimensions $\nu \le 2$, since for dimensions $\nu > 2$ the required Large Deviation results on random walks in $\mathbf{Z}_+^\nu$ and the theory of Tœplitz operators do not exist yet.

2. Intrinsic rates

2.1. Main definitions

Let L_0 be an irreducible and aperiodic discrete time ($t = 0, 1, \ldots$) homogeneous Markov chain on the countable state space $\boldsymbol{S}$. Let $P = (P_{ij})_{i,j\in\boldsymbol{S}}$ denote the transition matrix of L_0. $P^{(t)} = (P_{ij}^{(t)})_{i,j\in\boldsymbol{S}}$ are the t-step transition probabilities of L_0, $\xi_t(i)$ denotes the position of the chain at time t when starting at i.

If L_0 is ergodic, then it is said to be exponentially ergodic if there exist constants $\alpha > 0$, $c(i, \alpha) > 0$ such that

$$\sum_j |P_{ij}^{(t)} - \pi_j| \le c(i, \alpha) \exp\{-\alpha t\}, \tag{2.1}$$

for all $i \in \boldsymbol{S}$ and t. If (2.1) holds for α, then we define the bounding constant $C(i, \alpha)$ by

$$C(i, \alpha) = \sup_{t\ge 0} \sum_j |P_{ij}^{(t)} - \pi_j| \exp\{\alpha t\}.$$

For the examples of random walks that we consider, $C(i,\alpha)$ tends to ∞ with i.

Introduce an equivalence relation on the set of irreducible and aperiodic Markov chains on $\boldsymbol{S}$: L_0 and L_0' are equivalent iff the corresponding transition matrices P and P' differ in a finite number of entries. Let $\mathcal{F}(\mathrm{L}_0)$ be the equivalence class of L_0. Denote elements of this class by L and by the superscript L the corresponding Markov chain, the transition matrices, etc., of this finite perturbation. L is said to be an *A-perturbation*, if this perturbation leaves the transitions leading from and to states $x \notin A$ invariant.

Ergodicity and exponential ergodicity are invariants for finite perturbations. This follows directly from the Lyapunov function criteria for these two properties.

Proposition 2.1. $\mathrm{L} \in \mathcal{F}(\mathrm{L}_0)$ *is ergodic (exponentially ergodic) iff* L_0 *is ergodic (exponentially ergodic).*

Assume that L_0 has the exponential convergence property with rate $\alpha(\mathrm{L}_0)$ defined by

$$\alpha(\mathrm{L}_0) = \sup\Big\{\alpha : \sum_{t\geq 0}\sum_j |P_{ij}^{(t)} - \pi_j| \exp\{\alpha t\} < \infty, \quad \text{for all } i \in \boldsymbol{S}\Big\}.$$

We note that $\alpha = \alpha(\mathrm{L}_0)$ need not satisfy (2.1).

Definition 2.1. $\alpha_{\mathrm{int}} = \sup_{\mathrm{L}\in\mathcal{F}(\mathrm{L}_0)} \alpha(\mathrm{L})$ is called the intrinsic rate for L_0.

One can spoil the convergence rate as much as one likes by changing a finite number of transition probabilities. By such changes the resulting rate will never improve the intrinsic rate (the intrinsic rate is an upper bound) but it can become as bad as one likes.

Proposition 2.2. $\inf_{\mathrm{L}\in\mathcal{F}(\mathrm{L}_0)} \alpha(\mathrm{L}) = 0.$

For the proof of Proposition 2.2 and for further use we need some notation. For $\mathrm{L} \in \mathcal{F}$ let $\tau_A^{\mathrm{L}} \geq 1$ denote the first hitting time of $A \subset \boldsymbol{S}$. We write $f_{iA}^{(t),\mathrm{L}} = \mathsf{P}\{\tau_A^{\mathrm{L}} = t \mid \xi_0 = i\}$ and

$$F_{iA}^{\mathrm{L}}(z) = \sum_{t\geq 0} f_{iA}^{(t),\mathrm{L}} z^t$$

stands for the corresponding generating function; similarly

$${}_A f_{ia}^{(t),\mathrm{L}} = \mathsf{P}\{\tau_A^{\mathrm{L}} = t, \xi_{\tau_A^{\mathrm{L}}} = a \mid \xi_0 = i\}, \quad a \in A,$$

with corresponding generating function ${}_A F_{ia}^{\mathrm{L}}(z)$. Since the function $F_{iA}(z)$ has positive coefficients, its first singularity occurs at a point, $r(F_{iA})$ say, in $\mathbf{R}_+$. The first hitting probabilities can also be expressed in terms of the taboo probability matrix ${}_AP^{\mathrm{L}}$, where the transitions to the set A are eliminated:

$${}_AP^{\mathrm{L}}_{ij} = \begin{cases} P^{\mathrm{L}}_{ij}, & j \notin A, \\ 0 & j \in A. \end{cases}$$

Then

$$f^{(t),\mathrm{L}}_{iA} = \sum_j {}_AP^{(t-1),\mathrm{L}}_{ij} P^{\mathrm{L}}_{jA}\,,$$

with ${}_AP^{(0),\mathrm{L}} = I$. To denote the generating functions of the basic Markov chain L_0 we shall omit the corresponding index.

The following lemma states the well-known connection between exponential convergence of L_0 and the domain of analyticity of the function $F_{00}(z)$ (cf. [3], [13]).

Lemma 2.3. i) L_0 *is exponentially convergent if and only if* $r(F_{00}) > 1$.

ii) *If* $F_{00}(z) = 1$ *for some* $z = z_0$ *with* $1 < |z_0| \le r(F_{00})$, *then the function* $\sum_t (P^{(t)}_{00} - \pi_0) z^t$ *has a singularity at the point* z_0 *and hence* $\alpha(\mathrm{L}_0) = \log|z_0|$.

Proof of Proposition 2.2. Suppose that

$$\inf_{\mathrm{L}\in\mathcal{F}(\mathrm{L}_0)} \alpha(\mathrm{L}) = \alpha_0 > 0\,.$$

Then L_0 is exponentially convergent at rate at least α_0. We will construct a finite perturbation L of L_0, such that $F^{\mathrm{L}}_{00}(z)$ takes the value 1 in some point z_0 with $\log|z_0| < \alpha_0$.

By Lemma 2.3 there is some constant $r > 1$ such that $F_{00}(z)$ is analytic for $|z| < r$. We can choose r such that $\log r < \alpha_0$. Let $r' < \min(2, r)$ and define the perturbation L (possibly not a finite perturbation) with P^{L}_{ij} differing from P_{ij} in entries $i = 0, 1$ only:

$$P^{\mathrm{L}}_{ij} = \begin{cases} 0, & i = j = 0; \\ 1, & i = 0, j = 1; \\ 1/r', & i = 1, j = 0; \\ 1 - 1/r', & i = j = 1. \end{cases}$$

Then

$$F^{\mathrm{L}}_{10}(z) = \frac{z}{r' - z(r'-1)}, \quad F^{\mathrm{L}}_{00}(z) = \frac{z^2}{r' - z(r'-1)}.$$

Furthermore, $F^{\mathrm{L}}_{i0}(z)$ is analytic for $|z| < r$ and $F^{\mathrm{L}}_{00}(-r') = 1$. This implies $\alpha(\mathrm{L}) < \alpha_0$.

Select a sequence $\mathrm{L}(n) \in \mathcal{F}(\mathrm{L}_0)$ with $P^{\mathrm{L}(n)}_{ij} = P_{ij}$, $i \ne 0, 1$, and $P^{\mathrm{L}(n)}_{ij} \to P^{\mathrm{L}}_{ij}$, $n \to \infty$, for $i = 0, 1$. $F^{\mathrm{L}(n)}_{00}(z)$ is analytic on a small disc $|z + r'| < \varepsilon$ and converges to $F^{\mathrm{L}}_{00}(z)$ uniformly in n on this disc. Choose the disc small enough not to contain any other zeros of $1 - F^{\mathrm{L}}_{00}(z)$, then by Hurwitz' theorem there exists a sequence $z_n \to -r'$, such that $F^{\mathrm{L}(n)}_{00}(z_n) = 1$, and a constant N such that $|z_n + r'| < \varepsilon$ for $n \geq N$. Consequently $\alpha_0 \leq \log|z_n| < \log r$, for $n \geq N$. □

2.2. Relationships with singularities of generating functions

The goal of this subsection is to understand when convergence rates can be obtained as the singularities of generating functions of first hitting times.

Let $0 \in \boldsymbol{S}$ be some fixed state. Define

$$r_0 = \sup_{\mathrm{L} \in \mathcal{F}(\mathrm{L}_0)} r(F^{\mathrm{L}}_{00}).$$

Our main claim is that

$$\alpha_{\mathrm{int}} = \log r_0, \tag{2.2}$$

and we shall first give an intuitive argument for this relation.

To this end, note that studying the rate at which the point 0 is reached for finite perturbations of L_0, can be reduced to the problem of studying the rate at which finite sets are reached for the basic chain L_0. Consequently

$$\log r_0 = \sup_{A \subset \boldsymbol{S}, |A| < \infty} \inf_x \rho_{A,x}, \tag{2.3}$$

with

$$\rho_{A,x} = \liminf_{N \to \infty} \Big(-\frac{1}{N} \log \mathsf{P}\big\{\xi_N(x) \in A,\ \xi_t(x) \notin A,\ t = 1, \ldots, N-1\big\}\Big).$$

This relation will be rigorously proved below. Consider a finite set A that has large probability mass under the stationary distribution. The convergence rate to stationarity for states $x \notin A$ is determined by two parameters. First, starting from x the process reaches A at rate $\rho_{A,x}$. After reaching A most time is spent in A and equilibrium is attained at some rate α_A. By perturbing the transitions within the set A, this can be made larger than $\rho_{A,x}$. Hence the parameters $\rho_{A,x}$ determine the intrinsic rate and so by relation (2.3) relation (2.2) should be valid.

These arguments do not motivate *why* the quickest convergence rate to stationarity is achieved by making the rate of convergence within a finite set as quick as possible. Therefore only one bound for α_{int} can be rigorously proved in general.

Theorem 2.4. *Let* L_0 *be an irreducible, aperiodic and ergodic Markov chain on the countable state space* $\boldsymbol{S}$*. Then* $\alpha_{\mathit{int}} \geq \log r_0$.

We shall give the proof below. For the other bound we will need additional assumptions using pole and non-pole singularities of $F^{\mathrm{L}}_{00}(z)$ for a Markov chain L.

Definition 2.2. $F^{\mathrm{L}}_{00}(z)$ is said to have a *pole singularity* in $z = z_0$, if $F^{\mathrm{L}}_{00}(z)$ is meromorphic in some disc $|z| < |z_0| + \varepsilon$ and has a pole in the point z_0. $F^{\mathrm{L}}_{00}(z)$ is said to have a (first) *non-pole singularity* in $r > 0$, if $F^{\mathrm{L}}_{00}(z)$ is meromorphic for $|z| < r$, but not for $|z| < r + \varepsilon$ for any $\varepsilon > 0$. A non-pole singularity is called *essential* if $F^{\mathrm{L}}_{00}(z)$ has no poles for $|z| < r$.

Theorem 2.5. *Let* L_0 *be an irreducible, aperiodic and ergodic Markov chain on a countable state space. Suppose that* r *is a non-pole singularity of* $F^{\mathrm{L}'}_{00}(z)$ *for some* $\mathrm{L}' \in \mathcal{F}(\mathrm{L}_0)$. *Then* $\alpha_{int} \leq \log r$.

In Lemma 2.12 we will give a simple sufficient criterion for the function $F^{\mathrm{L}}_{00}(z)$ to have a non-pole singularity at the point r_0 for some L. This will be applicable to one-dimensional random walks. For random walks in higher dimensions we need to consider also non-pole singularities associated with non-finite sets.

For any set $A \subset \boldsymbol{S}$ define $r_A = \inf_x r(F_{xA})$. The Markov chain L_0 is said to have property (P), if it has the three following properties:

(P1) $\{\#y \,|\, P_{xy} > 0\}$ is finite for any $x \in \boldsymbol{S}$ and $\{\#x \,|\, P_{xy} > 0\}$ is finite for any $y \in \boldsymbol{S}$;

(P2) there exists a, possibly non-finite, set $A \subset \boldsymbol{S}$, $A \neq \boldsymbol{S}$, such that the function $F_{xA}(z)$ has a non-pole singularity at r_A for some $x \in \boldsymbol{S}$, and the substochastic transition matrix $({}_AP_{xy})_{x,y\notin A}$ is irreducible and aperiodic;

(P3) there exist a sequence of finite sets $A(n) \subset A(n+1)$, $n = 1, 2, \ldots$, with $A(n) \subset A$, $|A(n+1) \setminus A(n)| \leq 1$ and $A(n) \to A$, $n \to \infty$, such that the substochastic transition matrix $({}_{A(n)}P_{xy})_{x,y\notin A(n)}$ is irreducible and aperiodic for $n = 1, 2, \ldots$

Theorem 2.6. *Let* L_0 *be an irreducible, aperiodic and ergodic Markov chain satisfying property* (P). *Let* A *be a set defined by* (P2) *and* (P3). *Then*

$$\log r_A \geq \alpha_{int} = \log r_0.$$

Remark 2.1. Generalisations are possible. It is for example not necessary to require the irreducibility conditions in (P2) and (P3). Consider the directed graph with vertices $x \notin A$ associated with $({}_AP_{xy})_{x,y\notin A}$, where A is some finite or infinite set. Our derivations allow this graph to contain a number of connected subsets, roughly speaking provided that for x, y in the same subset the first singularities of the functions $F_{xA}(z)$ and $F_{yA}(z)$ occur at the same point. If it is a pole singularity for both the above functions, then also the order of the poles is required be the same.

Remark 2.2. The restriction in (P1) to bounded jumps implies a stronger form of exponential convergence. The following relation can be shown to hold for $\mathrm{L} \in \mathcal{F}(\mathrm{L}_0)$:

$$\alpha(\mathrm{L}) = \sup\big\{\alpha \,:\, \sum_{t\geq 0} |P^{\mathrm{L},(t)}_{xy} - \pi^{\mathrm{L}}_y| \exp\{\alpha t\} < \infty, \ \text{for all } x, y, \in \boldsymbol{S}\big\}. \qquad (2.4)$$

Dœblin chains are examples where typically the conditions of Theorem 2.6 do *not* hold and where (2.4) is not valid. A simple example of such Dœblin chain is the Markov chain on $\mathbf{Z}_+$ with the following transition probabilities:

$$P_{xy} = \begin{cases} 1-\delta, & y = x+1, x > 0; \\ \delta, & y = 0, x > 0; \\ q_y, & y > 0, x = 0, \end{cases}$$

for some probability distribution $\{q_y\}_y$ with $\sum_{y>0} q_y = 1$ and some $\delta < 1/2$. In this case it is easily calculated that (2.2) is valid, that $\alpha_{\rm int} = \alpha(\mathrm{L}_0) = -\log(1-\delta)$ and that the functions $F^{\rm L}_{00}(z)$ all have a pole singularity at $z = 1/(1-\delta)$. However the right-hand side of (2.4) equals $-\log\delta$, which is greater than $\alpha(\mathrm{L}_0)$. Below we will give a very simple argument to show that $\alpha_{\rm int} = \log r_0$ by the use of generating functions.

Lemma 2.13 will give a simple sufficient condition for property (P) to hold. In the remainder of this subsection we will derive the proofs of the above mentioned results.

Consider the generating functions

$$P_{xy}(z) = \sum_{t\ge 0} P^{(t)}_{xy} z^t.$$

It is related to convergence rates in the following way. We can write

$$\sum_{t\ge 0} \big(P^{(t)}_{xy} - \pi_y\big) z^t = P_{xy}(z) - \frac{\pi_y}{1-z}. \tag{2.5}$$

Then the log radius of convergence of $(1-z)P_{00}(z)$ is equal to

$$\sup\big\{\alpha : \sum_{t\ge 0} |P^{(t)}_{00} - \pi_0| \exp\{\alpha t\} < \infty\big\}.$$

We will first study the convergence rate of L_0. Let $A \subset \boldsymbol{S}$ be finite. The first entrance-last exit decomposition formulae for generating functions are for any $A \subset \boldsymbol{S}$ given by

$$P_{xy}(z) = \sum_{a,a'\in A} {}_AF_{xa}(z)\Big(I(A) - F(z,A)\Big)^{-1}_{a,a'} {}_AP_{a'y}(z) + {}_AP_{xy}(z), \tag{2.6}$$

with $I(A)$ the identity operator on A and $F(z,A) : A \to A$ the matrix with elements ${}_AF_{aa'}(z)$.

Let the set A be fixed. The point r_A plays an important role in our analysis.

Proposition 2.7. *$r_A = \inf_{a\in A} r(F_{aA})$ for any set $A \subset \boldsymbol{S}$.*

Proof. Consider any $x \notin A$. By the irreducibility of the Markov chain there exists $a \in A$, such that x can be reached from $a \in A$ without passing through the states of A in between. Since the power series expansions of the functions $F_{aA}(z)$ and $F_{xA}(z)$ only have positive coefficients, this implies that $F_{aA}(z)$ has a singularity at $r(F_{xA})$ and hence $r(F_{aA}) \le r(F_{xA})$. □

Lemma 2.8. *Let $A \subset \boldsymbol{S}$ be a finite set.*

i) *Suppose that for some $a' \in A$ the function $F_{a'A}(z)$ has a non-pole singularity at the point r_A. Then $\alpha(\mathrm{L}_0) = \log r_A$, if $\det\big(I(A) - F(z,A)\big) \neq 0$, for $|z| < r_A$, $z \neq 1$. If $\det\big(I(A) - F(z,A)\big) = 0$ for some $z \neq 1$ with $|z| < r_A$, then we denote by z_A the smallest such point in absolute value. In this case $\alpha(\mathrm{L}_0) = \log |z_A|$.*

ii) *For any $\alpha < \min\{\log r_A, \alpha(\mathrm{L}_0)\}$ we have the following bounds. There exist a constant $k_1(\alpha) > 0$, such that*

$$\sum_t \sum_y |P^{(t)}_{xy} - \pi_y| \exp\{\alpha t\} \le k_1(\alpha) F_{xA}(\exp\{\alpha\}). \tag{2.7}$$

For the lower bound assume that there exists an increasing sequence of finite sets $\{A(n)\}$, with $A(1) = A$ and $A(n) \to \boldsymbol{S}$, $n \to \infty$, such that $F_{x(n)A(1)}(r) \to \infty$, $n \to \infty$, for any sequence $\{x(n)\}$, $x(n) \in A(n)\backslash A(n-1)$, and for any $r < r_A$. Then there exists a constant $k_2(\alpha) > 0$ and an integer N such that for any $x \notin A(N)$

$$\sum_t \sum_y |P^{(t)}_{xy} - \pi_y| \exp\{\alpha t\} \ge k_2(\alpha) F_{xA}(\exp\{\alpha\}). \tag{2.8}$$

If $F_{aA}(z)$ is finite at the point r_A for any $a \in A$, and there are no points $z \neq 1$, $|z| < r_A$ with $\det\big(I(A) - F(z,A)\big) = 0$, then the above assertions hold for $\alpha = \alpha(\mathrm{L}_0) = \log r_A$.

Proof. Note that by virtue of (2.5) and (2.6) the function $\sum_t (P^{(t)}_{xy} - \pi_y) z^t$ is analytic for $|z| < r_A$, if $\det(I(A) - F(z,A)) \neq 0$, and for $|z| < |z_A|$ otherwise. Note further that

$$\sum_y {}_A P_{ay}(z) = \frac{F_{aA}(z) - 1}{z - 1}. \tag{2.9}$$

Let $0 < \alpha < r_A$ be such that $\det\big(I(A) - F(z,A)\big) \neq 0$ for $|z| \le \exp\{\alpha\}$.

The upper bound (2.7) and i) follow immediately from (2.6) by using that

$$|P^{(t)}_{xy} - \pi_y| \le \sup_{|z| = \exp\{\alpha\}} \Big| P_{xy}(z) - \frac{\pi_y}{1 - z} \Big| \exp\{-\alpha t\}.$$

For the lower bound (2.8) note that for $r = \exp\{\alpha\}$ and $x \notin A$

$$\begin{aligned}
\sum_{t,y} \big| P^{(t)}_{xy} - \pi_y \big| r^t &\geq \sum_{y\in A} \Big| \sum_t \big(P^{(t)}_{xy} - \pi_y\big) r^t \Big| \\
&= \sum_{y\in A} \Big| \sum_{a\in A} F_{xa}(r) \big(I(A) - F(r,A)\big)^{-1}_{ay} - \frac{\pi_y}{1-r} \Big| \\
&\geq \Big| \sum_{a'\in A} \Big| \sum_{a\in A} {}_AF_{xa}(r) \big(I(A) - F(r,A)\big)^{-1}_{aa'} \Big| - \Big| \frac{\pi_A}{1-r} \Big| \Big| .
\end{aligned} \tag{2.10}$$

For $a \in A$, (2.9) implies that

$$\sum_b \big(I(A) - F(r,A)\big)^{-1}_{ab} \sum_y {}_AP_{by}(r) = -\frac{1}{r-1}.$$

Hence for $x \notin A$

$$\begin{aligned}
&\sum_{a'\in A} \Big| \sum_{a\in A} F_{xa}(r)(I(A) - F(r,A)\Big)^{-1}_{aa'} \Big| \\
&\geq \sum_{a'\in A} \Big| \sum_{a\in A} {}_AF_{xa}(r) \Big(I(A) - F(r,A)\Big)^{-1}_{aa'} \Big| \sum_y {}_AP_{a'y}(r) \frac{r-1}{\max_a F_{aA}(r) - 1} \\
&\geq \Big| \sum_{a\in A} {}_AF_{xa}(r) \big(I(A) - F(r,A)\big)^{-1}_{aa'} \sum_y {}_AP_{a'y}(r) \Big| \frac{r-1}{\max_a F_{aA}(r) - 1} \\
&= \frac{F_{xA}(r)}{\max_a F_{aA}(r) - 1}.
\end{aligned} \tag{2.11}$$

By assumption there exists a positive constant $\gamma < 1$ and an integer N such that for all $x \notin A(N)$

$$\frac{\pi_A}{r-1} \leq \gamma \frac{F_{xA}(r)}{\max_a F_{aA}(r) - 1}.$$

Combination with (2.10) and (2.11) shows (2.8). This completes the proof of ii) and hence of the Lemma. □

Proof of Theorem 2.4. Fix $\mathrm{L} \in \mathcal{F}(\mathrm{L}_0)$ and choose $\varepsilon > 0$ such that $r(F^{\mathrm{L}}_{00}) - \varepsilon > 1$. We will first show that we can delete zeros of $1 - F^{\mathrm{L}}_{00}(z)$, $z \neq 1$, from the disc $|z| < r(F^{\mathrm{L}}_{00}) - \varepsilon$ in the following way.

Define a set of perturbations $\mathrm{L}(\theta) \in \mathcal{F}(\mathrm{L}_0)$, $\theta \in [0,1]$, having transition probabilities $p_{xy}(\theta)$ with the following properties:

i) $p_{xy}(\theta) = p_{xy}$, $x \neq 0$ and all y;

ii) the mapping $\theta \to \{P_{0y}(\theta)\}_{y\in \boldsymbol{S}}$ is a continuous mapping from $[0,1]$ to ℓ^1, such that $P_{0y}(\theta)$ is non-increasing in θ for $y \neq 0$. In addition $p_{0y}(0) = p_{0y}$, $p_{0y}(1) = 1$ for $y = 0$ and $p_{0y}(1) = 0$ otherwise.

Then $\mathrm{L}(0) = \mathrm{L}$, $r(F_{00}^{\mathrm{L}(\theta)}) \geq r(F_{00}^{\mathrm{L}})$, $0 \leq \theta < 1$, and $F_{00}^{\mathrm{L}(1)}(z) = z$.

The function $F_{00}^{\mathrm{L}(\theta)}(z)$ is analytic on $|z| \leq r(F_{00}^{\mathrm{L}}) - \delta$ for any $0 < \delta < \varepsilon/2$ and any θ and it converges to $F_{00}^{\mathrm{L}(1)}(z)$, $\theta \to 1$, uniformly on this disc. Since the only zero of $1 - F_{00}^{\mathrm{L}(1)}(z)$ in this disc occurs at $z = 1$, we obtain by Hurwitz' theorem that there exists $\theta_0 < 1$, such that the only zero of $1 - F_{00}^{\mathrm{L}(\theta)}(z)$ on the disc $|z| < r(F_{00}^{\mathrm{L}}) - \varepsilon/2$ occurs at the point $z = 1$ for any $\theta > \theta_0$.

By virtue of Lemma 2.8 there exists $\mathrm{L}(\varepsilon) \in \mathcal{F}(\mathrm{L}_0)$ with $r(F_{00}^{\mathrm{L}(\varepsilon)}) = r(F_{00}^{\mathrm{L}})$ such, that $\alpha(\mathrm{L}(\varepsilon)) \geq \log r(F_{00}^{\mathrm{L}}) - \varepsilon$.

We can take L with $r(F_{00}^{\mathrm{L}})$ arbitrarily close to r_0 and $\varepsilon > 0$ arbitrarily small. This implies $\alpha_{\mathrm{int}} \geq \log r_0$. □

We next need a lemma connecting r_A to $r(F_{00}^{\mathrm{L}})$ for any A-perturbation L, thus connecting the rate of convergence to finite sets in the Markov chain L_0 to the rate of convergence to 0 for finite perturbations.

Lemma 2.9. *Let* L_0 *be an irreducible, aperiodic Markov chain on a countable state space* $\boldsymbol{S}$*. Let* $A \subset \boldsymbol{S}$ *be a finite set. Then the following holds:*

i) $r(F_{00}^{\mathrm{L}}) \leq r_A$ *for any* A*-perturbation* L*;*

ii) *for any* $\varepsilon > 0$ *there exists an* A*-perturbation* $\mathrm{L}(\varepsilon)$*, such that* $r(F_{00}^{\mathrm{L}(\varepsilon)}) \geq r_A - \varepsilon$*;*

iii) *if* $F_{aA}(r_A)$ *is finite for all* $a \in A$*, then there exists an* A*-perturbation* L *such that* $r(F_{00}^{\mathrm{L}}) = r_A$ *and* $F_{00}^{\mathrm{L}}(r_A)$ *is finite.*

Proof. To show i) let L be an A-perturbation and let $x \notin A$. It is clear that $r(F_{00}^{\mathrm{L}}) \leq r(F_{x0}^{\mathrm{L}})$. We have the following decomposition

$$F_{x0}^{\mathrm{L}}(z) = \sum_{a \in A} {}_AF_{xa}^{\mathrm{L}}(z) F_{a0}^{\mathrm{L}}(z) + {}_AF_{x0}^{\mathrm{L}}(z) = \sum_{a \in A} {}_AF_{xa}(z) F_{a0}^{\mathrm{L}}(z) + {}_AF_{x0}(z).$$

Because A is finite, the function ${}_AF_{xa}(z)$ has a singularity at r_A for at least one point $a \in A$. It follows immediately that $r(F_{x0}^{\mathrm{L}}) \leq r_A$. This proves i).

Next we prove ii). For any $\varepsilon > 0$ we need to construct a perturbation $\mathrm{L}(\varepsilon)$, such that $r(F_{a0}^{\mathrm{L}(\varepsilon)}) \geq r_A - \varepsilon$, $a \in A$. Let us assume that $0 \in A$. The proof for the case $0 \notin A$ is similar. Consider the following decomposition

$$F_{a0}^{\mathrm{L}}(z) = {}_AF_{a0}^{\mathrm{L}}(z) + \sum_{a' \in A'} {}_AF_{aa'}^{\mathrm{L}}(z) F_{a'0}^{\mathrm{L}}(z), \tag{2.12}$$

where $A' = A \setminus \{0\}$. The functions ${}_AF_{aa'}^{\mathrm{L}}(z)$ are analytic and bounded on $|z| < r_A - \varepsilon/2$ for any $a' \in A$. Denote by $I(A')$ the identity matrix on $A' \times A'$ and by $F^{\mathrm{L}}(z, A')$ the matrix function on $A' \times A'$ with elements ${}_AF_{aa'}^{\mathrm{L}}(z)$, $a, a' \in A'$.

Then (2.12) implies

$$F_{a0}^{\mathrm{L}}(z) = \sum_{a' \in A'} \big(I(A') - F^{\mathrm{L}}(z, A')\big)^{-1}_{aa'} {}_AF_{a'0}^{\mathrm{L}}(z).$$

It is sufficient to construct $\mathrm{L}(\varepsilon) \in \mathcal{F}(\mathrm{L}_0)$ such that $\det\big(I(A') - F^{\mathrm{L}}(z, A')\big)$ has no zeros for $|z| < r_A - \epsilon$, by changing only the transition probabilities from states $a \in A'$. Consider the sequence L^θ, $\theta \in [0,1]$, with transition probabilities $P_{xy}(\theta)$ defined as follows:

i) $P_{xy}(\theta) = P_{xy}$, $x \notin A'$ and all y;

ii) The mapping $\theta \to \{P_{xy}(\theta)\}_{y \in \boldsymbol{S}}$ is a continuous mapping from $[0,1]$ to ℓ^1 for $x \in A'$, such that $P_{xy}(\theta)$ is non-increasing in θ for $y \neq x$. Further $P_{xy}(0) = P_{xy}$, $P_{xy}(1) = 1$ for $y = x$ and $P_{xy}(1) = 0$, $x \in A'$.

The existence of θ_0 such that $\det\big(I(A') - F^{\mathrm{L}^\theta}(z, A')\big) \neq 0$, $\theta > \theta_0$, follows from Hurwitz' theorem in a similar manner as in the proof of Theorem 2.4.

We can iii) prove in a similar way. □

We have the following simple consequence.

Corollary 2.10. (2.3) *holds, or equivalently*

$$r_0 = \sup_{A \subset \boldsymbol{S}: |A| < \infty} \inf_{a \in A} r(F_{aA}).$$

To show Theorem 2.5 we need to show that non-pole singularities are invariant under finite perturbations.

Lemma 2.11. *Suppose that at the point $\hat{z}$ the functions $F^{\mathrm{L}}_{x0}(z)$, ${}_0P^{\mathrm{L}}_{xy}(z)$ do not have a non-pole singularity for any x, y, for some $\mathrm{L} \in \mathcal{F}(\mathrm{L}_0)$. Then this is true for any $\mathrm{L} \in \mathcal{F}(\mathrm{L}_0)$.*

Proof. Without loss of generality we may assume that these generating functions do not have any non-pole singularities at $\hat{z}$ for L_0. Let L be an A-perturbation and ν the number of elements in A. Write $f(z)$ for the vector with components $F_{x0}(z)$, $v(z)$ for the vector of differences $f_x(z) - F^{\mathrm{L}}_{x0}(z)$, $Q = P - P^{\mathrm{L}}$ and q^0 stands for the vector with elements q_{x0}. Then

$$\big(I + (I - z\,{}_0P)^{-1} z\,{}_0Q\big) v(z) = (I - z\,{}_0P)^{-1}\big(zq^0 + z\,{}_0Q f(z)\big). \tag{2.13}$$

The right-hand side of (2.13) we denote by the vector $w(z)$. It is a vector of the form

$$w_x(z) = \sum_{a \in A} {}_0P_{xa}(z) g_a(z),$$

with

$$g_a(z) = \sum_{a' \in A} \big({}_0q_{a0} + {}_0q_{aa'} F_{a'0}(z)\big)$$

having no non-pole singularities at $\hat{z}$. So $w_x(z)$ has no non-pole singularities at $\hat{z}$.

Only the $\nu = |A|$ columns of $(I - z_0P)^{-1} z\,{}_0Q$ corresponding to elements in A, are non-zero. Denote these columns by the vectors $k_a(z)$, $a = 1, \ldots, \nu$. Next denote by $K(z)$ the $\nu \times \nu$ matrix consisting of the ν components of the $k_a(z)$ that correspond to elements in A. Then the solution $v(z)$ of (2.13) is given by

$$v(z) = w(z) + \sum_{a,a' \in A} k_a(z)\big(I + K(z)\big)^{-1}_{aa'} w_a(z).$$

The components of $k_a(z)$ have no non-pole singularities at $\hat{z}$ by assumption. The matrix functions $(I - K(z))^{-1}$ have no elements with non-pole singularities, because they are all rational functions of ${}_0P_{xy}(z)$, $x, y \in A$, in z. Consequently $F^{\mathrm{L}}_{x0}(z)$ has no non-pole singularities.

Similar arguments are valid for the functions ${}_0P_{xy}(z)$. □

Proof of Theorem 2.5. For any $\mathrm{L} \in \mathcal{F}(\mathrm{L}_0)$ it follows from Lemma 2.11 that at least one of the functions $F^{\mathrm{L}}_{x0}(z)$, ${}_0P^{\mathrm{L}}_{xy}(z)$ has a non-pole singularity at r. Hence by using the decomposition (2.6) to state 0 at least one of the functions $P^{\mathrm{L}}_{xy}(z)$ has a non-pole singularity at r. (2.5) implies immediately that at least one of the functions $\sum_t \big(P^{(t),\mathrm{L}}_{xy} - \pi_y\big) z^t$ has a non-pole singularity. Consequently $\alpha_{\mathrm{int}} \leq \log r$. □

We finally give a simple sufficient condition for the existence of a perturbation L such that $F^{\mathrm{L}}_{00}(z)$ has a non-pole singularity at the point $r(F^{\mathrm{L}}_{00})$. It is applicable to one-dimensional random walks.

Lemma 2.12. *Let* L_0 *be an irreducible, aperiodic Markov chain on a countable state space* $\boldsymbol{S}$. *Suppose that there exist finite sets* $A \subset B \subset \boldsymbol{S}$, *with*

i) $P_{xA} = 0$ *for* $x \notin B$;

ii) $r(F_{yA}) = r(F_{xB}) = \hat{r}$, $y \notin A$, $x \notin B$;

iii) *for all* $x, y \notin A$ *it is possible to reach* y *from* x *with positive probability without passing through* A *and vice versa, i.e.* ${}_AF_{xy}(1)$, ${}_AF_{yx}(1) > 0$.

Then L *has an essential non-pole singularity at the point* $z = \hat{r}$ *for some* $\mathrm{L} \in \mathcal{F}(\mathrm{L}_0)$, *such that* $F^{\mathrm{L}}_{00}(\hat{r})$ *is finite.*

Proof. Clearly $F_{xA}(z)$ and $F_{xB}(z)$ have no singularities at any point z with $|z| < \hat{r}$. It is sufficient to prove that $F_{xB}(\hat{r})$ is finite. Since it is a singularity by assumption, it must be non-pole singularity.

Using Lemma 2.9 we can then construct a perturbation L having a non-pole singularity at $\hat{r}$, such that $F^{\mathrm{L}}_{00}(\hat{r})$ is finite. This will complete the proof.

Suppose that $F_{xB}(\hat{r}) = \infty$, then ${}_BF_{xb}(\hat{r}) = \infty$ for at least one point $b \in B \setminus A$, b_0 say. Hence there exists some sequence $r^{(n)} \uparrow \hat{r}$, such that ${}_BF_{xb_0}(r^{(n)}) \geq n$. Use the decomposition

$$F_{xA}(z) = \sum_{b \in B \setminus A} {}_BF_{xb}(z) F_{bA}(z),$$

for any $x \notin B$. Then $F_{xA}(r^{(n)}) \geq n$. There is a positive probability, p say, to reach x from b_0 without passing through A. Let n be such that $pn > 1$. Hence $F_{b_0 A}(r^{(n)}) \geq p \cdot n$, thus implying that

$$F_{xA}(r^{(n)}) \geq p \cdot n^2 \quad \text{and} \quad F_{b_0 A}(r^{(n)}) \geq p^2 \cdot n^2.$$

Continuing this process, it follows that $F_{xA}(r^n) = \infty$ for any n. This contradicts ii). □

Proof of Theorem 2.6. Theorem 2.5 and Lemma 2.12 imply the Theorem in case the set A defined by property (P2) is finite. Assume hence that this set A is an infinite set.

First we will show that $\log r_A \geq \alpha_{\text{int}}$.

Suppose that $\log r_A < \alpha(\mathrm{L})$ for some $\mathrm{L} \in \mathcal{F}(\mathrm{L}_0)$. By virtue of property (P) we may assume that $\mathrm{L} = \mathrm{L}_0$. Then the functions

$$\sum_{y \in C} P_{xy}(z), \quad F_{xs}(z), \quad {}_sF_{xs'}(z), \quad \sum_{y \in C} {}_sP_{xy}(z), \quad \sum_{y \in C} {}_{s,s'}P_{xy}(z)$$

are all meromorphic on $|z| < r_A + \varepsilon$, for some constant $\varepsilon > 0$, for any $x, s, s' \in \boldsymbol{S}$ and any subset $C \subset \boldsymbol{S}$. This easily follows by using decomposition in (2.6) to sets consisting of one state. By perturbing a finite number of transitions, the above functions for the perturbed chain are still meromorphic on $|z| < r_A + \varepsilon$. This follows in the same way as Lemma 2.12.

Let $a, b \in A$, $b \neq a$, be given. We denote by $\mathrm{L}(\varepsilon)$ the finite perturbation with transition probabilities

$$P_{xy}^{\mathrm{L}(\varepsilon)} = \begin{cases} P_{xy}, & x \neq a, b; \\ \varepsilon P_{ay}, & x = b, y \neq b; \\ 1 - \varepsilon \sum_{s \neq b} P_{as}, & x = y = b. \end{cases}$$

Further denote

$$\mathcal{D} = \{z \,:\, |z| < r_A + \varepsilon\}.$$

Since $P_{aa}^{\mathrm{L}}(z)$ and $\sum_{y \neq A} P_{ay}^{\mathrm{L}}(z)$ are meromorphic on $\mathcal{D}$ for any finite perturbation L, by virtue of (2.6) also

$$\sum_{y \notin A} \Big({}_AP_{ay}^{\mathrm{L}}(z) + \sum_{a' \in A, a' \neq a} {}_aP_{aa'}^{\mathrm{L}}(z) \, {}_AP_{a'y}^{\mathrm{L}}(z) \Big) = \frac{1}{P_{aa}(z)} \sum_{y \notin A} P_{ay}(z) \tag{2.14}$$

is meromorphic on $\mathcal{D}$, for any finite perturbation L. By using first entrance decomposition to the point b, we can rewrite the second expression in the left-hand side of (2.14) as follows

$$\begin{aligned} &\sum_{a' \in A, a' \neq a} {}_aP_{aa'}^{\mathrm{L}}(z) \, {}_AP_{a'y}^{\mathrm{L}}(z) = \\ &\quad \sum_{a' \in A, a' \neq a, b} {}_{a,b}P_{aa'}^{\mathrm{L}}(z) \, {}_AP_{a'y}^{\mathrm{L}}(z) + \sum_{a' \in A, a' \neq a} {}_aF_{ab}^{\mathrm{L}}(z) \, {}_aP_{ba'}^{\mathrm{L}}(z) \, {}_AP_{a'y}^{\mathrm{L}}(z). \end{aligned} \tag{2.15}$$

This yields that

$$\sum_{y\notin A}\Big({}_AP^{\mathrm{L}}_{ay}(z) + \sum_{a'\in A,a'\neq a,b} {}_{a,b}P^{\mathrm{L}}_{aa'}(z)\, {}_AP^{\mathrm{L}}_{a'y}(z) + \sum_{a'\in A,a'\neq a} {}_aF^{\mathrm{L}}_{ab}(z)\, {}_aP^{\mathrm{L}}_{ba'}(z)\, {}_AP^{\mathrm{L}}_{a'y}(z)\Big) \tag{2.16}$$

is meromorphic on $\mathcal{D}$, for any finite perturbation L and hence for $\mathrm{L} = \mathrm{L}(\varepsilon)$, for any ε. Only the third term in (2.16) depends on ε. For $\varepsilon = 0$

$$\sum_{a'\in A,a'\neq a} {}_aF^{\mathrm{L}(0)}_{ab}\, {}_aP^{\mathrm{L}(0)}_{ba'}(z)\, {}_AP^{\mathrm{L}(0)}_{a'y}(z) = 0$$

for $y \notin A$ and so (2.16) reduces to

$$\sum_{y\notin A}\Big({}_AP_{ay}(z) + \sum_{a'\in A,a'\neq a,b} {}_{a,b}P_{aa'}(z)\, {}_AP_{a'y}(z)\Big), \tag{2.17}$$

which must therefore be meromorphic on $\mathcal{D}$. By subtracting (2.17) from (2.16) it follows that for any ε

$$\sum_{y\notin A}\Big(\sum_{a'\in A,a'\neq a} {}_aF^{\mathrm{L}(\varepsilon)}_{ab}(z)\, {}_aP^{\mathrm{L}(\varepsilon)}_{ba'}(z)\, {}_AP^{\mathrm{L}(\varepsilon)}_{a'y}(z)\Big)$$

is meromorphic on $\mathcal{D}$. Since ${}_aF^{\mathrm{L}(\varepsilon)}_{ab}(z)$ is meromorphic, also

$$\sum_{y\notin A}\sum_{a'\in A,a'\neq a} {}_aP^{\mathrm{L}(\varepsilon)}_{ba'}(z)\, {}_AP^{\mathrm{L}(\varepsilon)}_{a'y}(z)$$

must be meromorphic on $\mathcal{D}$ for any ε. For $\varepsilon = 1$

$$\sum_{y\notin A}\sum_{a'\in A,a'\neq a} {}_aP^{\mathrm{L}(\varepsilon)}_{ba'}(z)\, {}_AP^{\mathrm{L}(\varepsilon)}_{a'y}(z) = \sum_{y\notin A}\sum_{a'\in A,a'\neq a} {}_aP_{aa'}(z)\, {}_AP_{a'y}(z),$$

and so it follows from (2.14) that $\sum_{y\notin A} {}_AP_{ay}(z)$ is meromorphic on $\mathcal{D}$. By virtue of (2.9), $F_{aA}(z)$ is therefore meromorphic on $\mathcal{D}$. However, (P1) together with (P3) imply that $F_{xA}(z)$ has a non-pole singularity at $z = r_A$ for any x with $\sum_{y\notin A} P_{xy} > 0$. We have derived a contradiction and thus $\alpha(\mathrm{L}_0) \le \log r_A$.

Similar arguments hold for any $\mathrm{L} \in \mathcal{F}(\mathrm{L}_0)$ by property (P) and thus we obtain $\alpha_{\mathrm{int}} \le \log r_A$.

Next we show that $\alpha_{\mathrm{int}} = \log r_0$. Consider the Markov chain L_0. Without loss of generality we may assume that $0 \in A$. Consider the increasing sequence of sets $A(n) \subset A$, $n = 1, \ldots$, from property (P3).

If $r_{A(n)} \to r_A$, $n \to \infty$, then

$$\log r_0 \ge \sup_n \log r_{A(n)} \ge \log r_A \ge \alpha_{\mathrm{int}}$$

and the result immediately follows from Theorem 2.4. Assume hence that $r_{A(n)} \to \hat{r}$, $n \to \infty$, with $\hat{r} < r_A$.

There are two cases. Either the sequence $r_{A(n)}$ is constant for all $n \geq N$ and some constant N, or it is strictly increasing for all n.

First consider the first case, i.e. $r_{A(n)} = \hat{r}$ for $n \geq N$.

Suppose that the function $F_{xA(n)}(z)$ has a pole singularity at $z = \hat{r}$ for some x with $\sum_{y \notin A(n)} P_{xy} > 0$ and for some $n > N$. By virtue of properties (P1) and (P3), $F_{xA(n)}(z)$ has a pole singularity at $z = \hat{r}$ for any x with $\sum_{y \notin A(n)} P_{xy} > 0$, and these poles are all of the same order. Consider the decomposition

$$F_{x'A(n)}(z) = \frac{{}_{A(n+1)}F_{x'A(n)}(z)}{1 - {}_{A(n+1)}F_{x'x'}(z)}, \tag{2.18}$$

with $x' = A(n+1) \setminus A(n)$. The functions ${}_{A(n+1)}F_{xx'}(z)$, ${}_{A(n+1)}F_{xA(n)}(z)$ and $F_{xA(n+1)}(z)$ all have a pole singularity at $\hat{r}$ for all x with $\sum_{y \notin A(n+1)} P_{xy} > 0$, and then they are of same order, or none of them has a pole singularity at this point; this is because of (P3). But they cannot have a pole singularity at $\hat{r}$, since this would imply by (2.18) that $F_{x'A(n)}(\hat{r})$ is finite, as the poles would cancel. Consequently $F_{xA(n+1)}(z)$ all have a non-pole singularity at $z = \hat{r}$ for x with $\sum_{y \notin A(n+1)} P_{xy} > 0$.

The conclusion is that for at least one value $n > N$ the functions $F_{xA(n)}(z)$ have a non-pole singularity at $z = \hat{r}$ for any x with $\sum_{y \notin A(n)} P_{xy} > 0$.

Fix n for which this is true and consider the $A(n+1)$-perturbation L with transition matrix

$$P^{\mathrm{L}}_{xy} = \begin{cases} P_{xy}, & x \notin A(n); \\ 1, & x \in A(n), x \neq 0, y = 0; \\ 1, & x = 0, y = x', \end{cases}$$

where $x' = A(n+1) \setminus A(n)$. It immediately follows that $F^{\mathrm{L}}_{00}(z)$ has a non-pole singularity at $z = \hat{r}$. By virtue of Theorems 2.4 and 2.5 it follows that

$$\log r_0 \geq \log \hat{r} \geq \alpha_{\mathrm{int}} \geq \log r_0\,,$$

thus completing the proof for the case that the sequence $r_{A(n)}$ is constant for all sufficiently large n.

Finally consider the case that $r_{A(n)}$ is strictly increasing for all n. Then $F_{xA(n)}(z)$ has a pole singularity of order 1 at $z = r_{A(n)}$ for any x for which $\sum_{y \notin A(n)} P_{xy} > 0$. Choose a reference state $x^* \notin A$. Then

$$\frac{F_{xA(n)}(r)}{F_{x^*A(n)}(r)} \to a^n_x, \quad r \uparrow r_{A(n)},$$

for some positive constant a^n_x, for any x with $\sum_{y \notin A(n)} P_{xy} > 0$. By virtue of (P2), for any $x \notin A$ there exist positive constants $c_1(x)$, $c_2(x)$, such that for all n

$$c_1(x) \leq a^n_x \leq c_2(x). \tag{2.19}$$

Indeed, for any x there exist constants m_1 and m_2, such that ${}_AP^{m_1}_{x^*x} > 0$, ${}_AP^{m_2}_{xx^*} > 0$ and so for any $r < r_{A(n)}$

$$F_{x^*A(n)}(r) \geq r^{m_1}\,{}_AP^{m_1}_{x^*x} F_{xA(n)}(r) \geq {}_AP^{m_1}_{x^*x} F_{xA(n)}(r),$$

and

$$F_{xA(n)}(r) \geq r^{m_2}\,{}_AP^{m_2}_{xx^*} F_{x^*A(n)}(r) \geq {}_AP^{m_2}_{xx^*} F_{x^*A(n)}(r).$$

For $r < r_{A(n)}$ consider the linear system

$$F_{xA(n)}(r) = r \sum_{y \notin A(n)} P_{xy} F_{yA(n)}(r) + r \sum_{y \in A(n)} P_{xy}.$$

Dividing both sides by $F_{x^*A(n)}(r)$, taking the limit $r \to r_{A(n)}$ and using (P1) we find

$$a^n_x = r_{A(n)} \sum_{y \notin A(n)} P_{xy} a^n_y. \tag{2.20}$$

Using (2.19) and a diagonalisation procedure we find the existence of a subsequence $\{n(k)\}$ and a set of positive numbers a_x, $x \notin A$, such that $a^{n(k)}_x \to a_x$, $k \to \infty$, for any $x \notin A$. By taking the limit $k \to \infty$ in (2.20) and using (P1) we get

$$a_x = \hat{r} \sum_{y \notin A} P_{xy} a_y, \tag{2.21}$$

for any $x \notin A$. This implies that the random process on $\boldsymbol{S}\setminus A$ with substochastic transition matrix $\{{}_AP_{xy}\}_{x,y\notin A}$ is $\hat{r}$-recurrent (cf. [13]). In particular, (2.21) implies

$$\liminf_{t\to\infty} \hat{r}^t\,{}_AP^{(t)}_{xy} > 0,$$

for any $x, y \notin A$, so that ${}_AP_{xy}(z)$ has a singularity at a point $r \leq \hat{r}$. This contradicts the assumption that $\hat{r} < r_A$. Hence, in the case of a strictly increasing sequence $r_{A(n)}$, we must have $r_{A(n)} \to r_A$, and the results follows from what has been said in the above. □

For checking property (P3) for random walks on $\mathbf{Z}^\nu_+$, we show that a slightly weaker version of Lemma 2.12 also holds in the case of non-finite sets.

Lemma 2.13. *Assume the conditions of Lemma* 2.12, *but now with A and B infinite sets. Further assume that $\#\{y : P_{xy} > 0\}$ is finite for any $x \in \boldsymbol{S}$. Then the functions $F_{xB}(z)$ have a non-pole singularity at $z = \hat{r}$ for any x with $\sum_{y\notin B} P_{xy} > 0$, such that $F_{xB}(\hat{r})$ is finite.*

Proof. Let $C = A$ or B. If $F_{xC}(\hat{r}) = \infty$ for some x with $\sum_{y\notin C} P_{xy} > 0$, then it follows that $F_{xC}(\hat{r}) = \infty$ for any such state x.

Let us assume that $F_{xB}(z)$ has a pole singularity at $z = \hat{r}$ for any x with $\sum_{y\notin B} P_{xy} > 0$. It follows that $F_{xA}(z)$ diverges for $z = \hat{r}$ for any x with $\sum_{y\notin A} P_{xy} > 0$.

It is sufficient to show that this implies that ${}_CP_{xy}(\hat{r})$ diverges for any x with $\sum_{y \notin C} P_{xy} > 0$, $y \notin C$ and C equal to A, B. The result then follows by using similar arguments to Lemma 2.12.

In the same way as in the proof of Theorem 2.6 we can show the existence of positive numbers a_x for x with $\sum_{y \notin A} P_{xy} > 0$, such that

$$a_x = \hat{r} \sum_{y \notin A} P_{xy} a_y.$$

Hence

$$\liminf_{t \to \infty} \hat{r}^t P_{xy}^{(t)} > 0$$

and ${}_AP_{xy}(z)$ diverges for $z = \hat{r}$, for any x with $\sum_{y \notin A} P_{xy} > 0$. The same arguments hold for the set B. □

Simple Dœblin chain. The proof that $\alpha_{\text{int}} = \log r_0$ for the Dœblin chain in Remark 2.2 is as follows. An easy computation gives

$$\begin{aligned} F_{00}(z) &= \frac{z^2 \delta}{1 - z(1-\delta)}, \\ (1-z) P_{00}(z) &= \frac{1 - z(1-\delta)}{z\delta + 1}, \\ {}_0P_{0x}(z) &= \sum_{y \leq x} q_y (1-\delta)^{x-y} z^{x-y+1}, \end{aligned}$$

and $r_0 = 1/(1-\delta)$.

Clearly for $1/\delta > r > 1/(1-\delta)$ we have that $(1-r)P_{00}(r)$ is negative. Since ${}_0P_{0x}(r) > 0$ for these values of r, it follows that $(1-r)P_{0x}(r)$ is also negative.

If $\alpha(\mathrm{L}_0) > 1/(1-\delta)$ then the functions $(1-z)\sum_x |P_{0x}(z)|$ are analytic for $|z| < 1/(1-\delta) + \varepsilon$ for some $\varepsilon > 0$ sufficiently small. This would imply that $(1-z)\sum_x P_{0x}(z) = 1$ for any $|z| < 1/(1-\delta) + \varepsilon$ (since this holds for $|z| < 1$). But this cannot be true, since all functions $(1-z)P_{0x}(z)$ are negative for $1/(1-\delta) < r < 1/(1-\delta) + \varepsilon$.

These arguments obviously apply for any finite perturbation and hence $\alpha_{\text{int}} = \log r_0$. □

2.3. Relationships with Large Deviations

Based on the connection of α_{int} to the rate at which finite sets are reached, we pose the following relation with action functionals in Large Deviation theory:

$$\alpha_{\text{int}} = \inf_{x \neq 0} \inf_{\tau > 0} \frac{1}{\tau} \mathcal{L}^0_{x,x,\tau}. \tag{2.22}$$

For the state space $\mathbf{Z}_+^\nu$ the action functional $\mathcal{L}^0_{x,x,\tau}$ is defined by

$$\mathcal{L}^0_{x,x,\tau} = \inf_{\varphi} \mathcal{L}_\tau(\varphi),$$

where $\mathcal{L}_\tau(\varphi)$ stands for the action functional of a path $\varphi : [0,\tau] \to \mathbf{R}_+^\nu$ with $\varphi(0) = \varphi(\tau) = x$, and the infimum is taken over all such paths φ from x to x that a.s. (i.e. on a set with Lebesgue measure equal to 0) do not hit the point 0.

An intuitive motivation is the following. We can perceive the probabilities in (2.3) via Large Deviations theory: we achieve A along the mean drift vector, but how is it possible to reach A for the first time only after a long time? The answer is that the decisive contribution is given by the paths that spend much time in the vicinity of x and then go to A along the mean drift.

This relation evidently cannot be proved under general conditions, as there are non-exponential situations, where Large Deviations theory does not work. This can even be the case in exponential situations.

For one- and two-dimensional random walks we will prove the characterisation (2.22) of α_{int} using probabilistic arguments combined with the generating function techniques from the previous subsection.

Theorem 2.14. *For the one- and two-dimensional random walks defined below*

$$\log r_0 = \inf_{x\neq 0} \liminf_{\tau\to\infty} \frac{1}{\tau}\mathcal{L}^0_{x,x,\tau}, \tag{2.23}$$

where

$$\mathcal{L}^0_{x,y,\tau} = \inf\big\{\mathcal{L}_\tau(\varphi) \,:\, \varphi(0) = x, \varphi(\tau) = y,\ \varphi \text{ is continuous and } \varphi(t) \neq 0 \text{ a.s.}\big\}.$$

In particular,

$$\alpha_{int} = \log r_0 = \inf_{x\neq 0} \liminf_{\tau\to\infty} \frac{1}{\tau}\mathcal{L}^0_{x,x,\tau}.$$

The proof will be given below.

Remark 2.3. As will appear in the course of the proof, (2.23) can be proved under more general conditions. We mainly used the fact that the one- and two-dimensional random walks satisfy the Large Deviation principle and the special structure of these walks was used marginally. Additional conditions will however be necessary. The exact form of such conditions is not clear, since Large Deviation theory with discontinuities in higher dimensions than 2 still does not exist. By virtue of Theorem 2.4 this proves one bound for α_{int} in (2.22). The upper bound holds for any Markov chain having property (P) including one- and two-dimensional random walks.

The calculation of the value

$$\inf_{x\neq 0} \liminf_{\tau\to\infty} \frac{1}{\tau}\mathcal{L}^0_{x,x,\tau}$$

is a classical variational problem. In Sections 4 and 5 we will explicitly calculate it for one- and two-dimensional random walks. In the remainder of this section we will introduce the Large Deviation principle and the one- and two-dimensional random walks that we study. Then we will prove Theorem 2.14.

One-dimensional random walks

We consider a one-dimensional random walk on $\mathbf{Z}_+$ with the following structure: there exist positive integers l, k, k' such that the following properties hold.

- *bounded jumps*: $P_{ij} = 0$ for $j < i - l$ or $j > i + k$;
- *spatial homogeneity*: $P_{ij} = p_{j-i}$, $i \geq k'$;
- *irreducibility and aperiodicity*;
- *ergodicity*: $\sum_{i=-l}^{k} ip_i < 0$.

The one-dimensional random walk satisfies the Large Deviation principle with the following action functionals $\mathcal{L}_\tau$ for paths $\varphi > 0$ (cf. [2]). Let

$$H_i(\alpha) = \log \sum_j P_{ij} \exp\{\alpha(j-i)\}$$

and

$$H(\alpha) = \log \sum_i p_i \exp\{\alpha i\}.$$

For $i \geq k'$, $H_i(\alpha) = H(\alpha)$. The Legendre transform of the function H is given by

$$\mathsf{L}(v) = \sup_\alpha \big\{ -H(\alpha) + v\alpha \big\}.$$

Let φ be a path that hits 0 on a set of Lebesgue measure 0 only. Then

$$\mathcal{L}_\tau(\varphi) = \tau \mathsf{L}(\dot{\varphi}),$$

if φ is linear and

$$\mathcal{L}_\tau(\varphi) = \int_0^\tau \mathsf{L}(\dot{\varphi}(t))\, dt\,,$$

if φ is absolutely continuous, provided that $-l < \dot{\varphi}(t) < k$. Otherwise we have $\mathcal{L}_\tau(\varphi) = \infty$.

Two-dimensional random walks

We study a homogeneous random walk L_0 on $\mathbf{Z}_+^2$ with the following structure. Let $\Lambda \subset \{1, 2\}$ and let B^Λ be the face

$$\{(x_1, x_2) \in \mathbf{Z}_+^2 \,|\, x_i > 0, i \in \Lambda;\ x_i = 0, i \notin \Lambda\}.$$

We will often use Λ to denote the face B^Λ. Then the following conditions are assumed to hold.

- *bounded jumps*: $P_{ij,kl} = 0$, if $\min\{k-i, l-j\} < -1$ or $\max\{k-i, l-j\} > d$ for some constant d;
- *spatial homogeneity on faces*: for any Λ there exists a probability distribution $a(\Lambda)$ on $\mathbf{Z}^2$, such that $P_{xy} = a(\Lambda, y-x)$, $x \in \Lambda$. Write p, p', p'' and p^0 to denote $a(\{1,2\})$, $a(\{1\})$, $a(\{2\})$ and $a(\{\emptyset\})$ respectively;
- *irreducibility and aperiodicity*;
- *irreducibility and aperiodicity of induced chains:* the induced chains on $\mathbf{Z}_+$ with transition matrices P_1 and P_2 defined by

$$P_{1,ij} = \sum_k P_{1i,kj}, \qquad P_{2,ij} = \sum_k P_{i1,jk}$$

  are irreducible and aperiodic;
- *ergodicity:* denote by M, M' and M'' the mean drift vectors on the faces $\{1,2\}$, $\{1\}$ and $\{2\}$ respectively, i.e.

$$\begin{aligned} M &= (M_x, M_y) &= \Big(\sum_{x\in Z} x p_{xy}, \sum_{y \in Z} y p_{xy}\Big), \\ M' &= (M'_x, M'_y) &= \Big(\sum_{x\in Z} x p'_{xy}, \sum_{y \in Z} y p'_{xy}\Big), \\ M'' &= (M''_x, M''_y) &= \Big(\sum_{x\in Z} x p''_{xy}, \sum_{y \in Z} y p''_{xy}\Big), \end{aligned}$$

  then one of the following conditions is assumed to hold:

  ND1: $M_x, M_y < 0$, $M_x M'_y - M_y M'_x < 0$, $M_y M''_x - M_x M''_y < 0$;

  ND2: $M_x \geq 0$, $M_y < 0$ and $M_x M'_y - M_y M'_x < 0$;

  ND3: $M_y \geq 0$, $M_x < 0$ and $M_y M''_x - M_x M''_y < 0$.

As in [6] define

$$\begin{aligned} H(\alpha,\beta) &= \log(\sum_{ij} p_{ij} \exp\{\alpha i + \beta j\}); \\ h_1(\alpha,\beta) &= \log(\sum_{ij} p'_{ij} \exp\{\alpha i + \beta j\}); \\ h_2(\alpha,\beta) &= \log(\sum_{ij} p''_{ij} \exp\{\alpha i + \beta j\}). \end{aligned}$$

Denote $H_\alpha(\alpha,\beta) = (\partial/\partial\alpha) H(\alpha,\beta)$, $H_\beta(\alpha,\beta) = (\partial/\partial\beta) H(\alpha,\beta)$ and let $\alpha'(\beta)$, $\beta'(\alpha)$ be solutions to

$$H_\beta(\alpha'(\beta), \beta) = 0, \qquad H_\alpha(\alpha, \beta'(\alpha)) = 0;$$

let $\beta''(\alpha)$, $\alpha''(\beta)$ be solutions to

$$\begin{cases} H(\alpha,\beta) &= h_1(\alpha,\beta); \\ H_\beta(\alpha,\beta) &< 0, \end{cases} \qquad \begin{cases} H(\alpha,\beta) &= h_2(\alpha,\beta); \\ H_\alpha(\alpha,\beta) &< 0. \end{cases}$$

Put

$$\beta(\alpha) = \begin{cases} \beta'(\alpha), & \text{if } H(\alpha,\beta'(\alpha)) \geq h_1(\alpha,\beta'(\alpha)); \\ \beta''(\alpha), & \text{otherwise,} \end{cases}$$

and

$$\alpha(\beta) = \begin{cases} \alpha'(\beta), & \text{if } H(\alpha'(\beta),\beta) \geq h_2(\alpha'(\beta),\beta); \\ \alpha''(\beta), & \text{otherwise,} \end{cases}$$

and define

$$\mathcal{H}_{\{1\}}(\beta) = H\big(\alpha,\beta(\alpha)\big), \qquad \mathcal{H}_{\{2\}}(\beta) = H\big(\alpha(\beta),\beta\big).$$

We recall some Large Deviations results from [6]. The action functionals are calculated from the Legendre transforms: for $v = (v_1, v_2) \in \mathbf{R}^2$

$$\begin{aligned} \mathsf{L}(v) &= \sup_{\alpha,\beta\in\mathrm{R}} \Big\{\alpha v_1 + \beta v_2 - H(\alpha,\beta)\Big\}, \\ \mathsf{L}_{\{1\}}(v_1) &= \sup_{\alpha\in\mathrm{R}} \Big\{\alpha v_1 - \mathcal{H}_{\{1\}}(\alpha)\Big\}, \\ \mathsf{L}_{\{2\}}(v_2) &= \sup_{\beta\in\mathrm{R}} \Big\{\beta v_2 - \mathcal{H}_{\{2\}}(\beta)\Big\}. \end{aligned}$$

Let for $x \in \mathbf{R}_+^2$

$$\mathsf{L}(x,v) = \begin{cases} \mathsf{L}(v), & \text{if } x_1, x_2 > 0; \\ \mathsf{L}_{\{1\}}(v_1), & \text{if } x_1 > 0, x_2 = 0; \\ \mathsf{L}_{\{2\}}(v_2), & \text{if } x_1 = 0, x_2 > 0, \end{cases}$$

then for any path $\varphi = (\varphi_1, \varphi_2) : [0,\tau] \to \mathbf{R}_+^2$ with $\varphi(t) \neq 0$ a.s. (with respect to the Lebesgue-measure)

$$\mathcal{L}_\tau(\varphi) = \begin{cases} \int_0^\tau \mathsf{L}(\varphi(t),\dot\varphi(t))dt, & \text{if } \varphi \text{ is absolutely continuous} \\ & \quad \text{with } -1 < \dot\varphi_k(t) < d, \\ \infty, & \text{otherwise.} \end{cases}$$

Large Deviation principle

Our derivations and notation are based on [6]. For $\tau \in \mathbf{R}_+$ let $C([0,\tau],\mathbf{R}_+^\nu)$ be the set of continuous functions $\varphi : [0,\tau] \to \mathbf{R}_+^\nu$. Let for any τ be given a functional $\mathcal{L}_\tau$ mapping $C([0,\tau],\mathbf{R}_+^\nu)$ into $[0,\infty]$. For any $s \geq 0$, $x \in \mathbf{R}_+^\nu$ let

$$\Phi_{x,\tau}(s) = \big\{\varphi \in C([0,\tau],\mathbf{R}_+^\nu) \,:\, \varphi(0) = x, \mathcal{L}_\tau(\varphi) \leq s\big\}.$$

Definition 2.3. The random walk $\mathrm{L}_0 = \{\xi_t\}$ in $\mathbf{Z}_+^\nu$ satisfies the Large Deviation principle with action functionals $\mathcal{L}_\tau$ if for all $\tau \geq 0$ and $x \in \mathbf{R}_+^\nu$ the following conditions hold:

i) *compactness*: $\Phi_{x,\tau}(s)$ is compact for any $s \geq 0$;

ii) *Large Deviation lower bound*: for any $\delta, \delta', s_0 > 0$ there exists N_0, such that for all $N \geq N_0$ and $\varphi \in \Phi_{x,\tau}(s_0)$

$$\mathsf{P}\Big\{ \sup_{t=0,\ldots,[N\tau]} \Big| \frac{1}{N}\xi_t([xN]) - \varphi(t/N) \Big| < \delta \Big\} \geq \exp\{-\delta' N - N\mathcal{L}_\tau(\varphi)\}; \tag{2.24}$$

iii) *Large Deviation upper bound*: for any $\delta, \delta', s_0 > 0$, there exists N_0, such that for all $N \geq N_0$ and $s \in (0, s_0)$

$$\mathsf{P}\Big\{ \sup_{t=0,\ldots,[N\tau]} \Big| \frac{1}{N}\xi_t([xN]) - \varphi(t/N) \Big| \geq \delta, \text{ for all } \varphi \in \Phi_{x,\tau}(s) \Big\} \leq \exp\{\delta' N - Ns\} \tag{2.25}$$

Denote

$$\begin{aligned} \Phi_{x,y,\tau} &= \big\{ \varphi \in C([0,\tau], \mathbf{R}_+^\nu) \,:\, \varphi(0) = x, \varphi(\tau) = y \big\}; \\ \Phi^0_{x,y,\tau} &= \big\{ \varphi \in C([0,\tau], \mathbf{R}_+^\nu) \,:\, \varphi(0) = x, \varphi(\tau) = y, \varphi(t) \neq 0 \text{ a.s.} \big\}. \end{aligned}$$

Definition 2.4. The path $\varphi \in \bigcup_{\tau \geq 0} \Phi_{x,y,\tau}$ with $\varphi : [0,\tau] \to \mathbf{R}_+^\nu$ is called an optimal path from x to y if for any τ' and any $\varphi' \in \Phi_{x,y,\tau'}$,

$$\mathcal{L}_\tau(\varphi) \leq \mathcal{L}_{\tau'}(\varphi').$$

Denote

$$\mathcal{L}_{x,y} = \inf_\tau \inf_{\varphi \in \Phi_{x,y,\tau}} \mathcal{L}_\tau(\varphi).$$

If the optimal path, $\varphi : [0,\tau] \to \mathbf{R}_+^\nu$ say, from x to y exists, then $\mathcal{L}_\tau(\varphi) = \mathcal{L}_{x,y}$. It is the path having highest probability amongst all paths from x to y.

Definition 2.5. The path $\varphi : [0,\tau] \to \mathbf{R}_+^\nu$ is called an optimal τ-path from x to y if for any $\varphi' \in \Phi_{x,y,\tau}$

$$\mathcal{L}_\tau(\varphi) \leq \mathcal{L}_\tau(\varphi').$$

Denote

$$\begin{aligned} \mathcal{L}_{x,y,\tau} &= \inf_{\varphi \in \Phi_{x,y,\tau}} \mathcal{L}_\tau(\varphi); \\ \mathcal{L}^0_{x,y,\tau} &= \inf_{\varphi \in \Phi^0_{x,y,\tau}} \mathcal{L}_\tau(\varphi). \end{aligned}$$

The subsection ends by proving Theorem 2.14. First of all we will show the following estimate

$$\log P^{([\tau N])}_{[xN][yN]} \sim -N\mathcal{L}_{x,y,\tau}, \quad N \to \infty. \tag{2.26}$$

Lemma 2.15. (2.26) *holds for the one- and two-dimensional random walks.*

Proof. First we show the upper bound. Let

$$\Phi_{x,y,\tau}(s) = \{\varphi \in C([0,\tau], \mathbf{R}^{\nu}_{+}) \,|\, \varphi(0) = x, \varphi(\tau) = y, \mathcal{L}_{\tau}(\varphi) \le s\}.$$

The set $\Phi_{x,y,\tau}(s)$ is compact, since it is closed in the compact set $\Phi_{x,\tau}(s)$. Let $\eta > 0$ be sufficiently small. Then there exists $\gamma > 0$, such that

$$\inf_{\varphi \in \Phi_{x,y,\tau}(\mathcal{L}_{x,y,\tau} - \eta)} |\varphi(\tau) - y| \ge \gamma.$$

Suppose that this is not true. Then there exists a sequence

$$\{\varphi_n\}_n \subset \Phi_{x,y,\tau}(\mathcal{L}_{x,y,\tau} - \eta)$$

with $\varphi_n(\tau) \to y$. By compactness of the set $\Phi_{x,y,\tau}(\mathcal{L}_{x,y,\tau} - \eta)$, it contains a path φ', with $\varphi'(\tau) \in \Phi_{x,y,\tau}$ and

$$\mathcal{L}_{\tau}(\varphi') \le \mathcal{L}_{x,y,\tau} - \eta.$$

This contradicts the minimality of the value $\mathcal{L}_{x,y,\tau}$.

Consequently

$$\{\xi_{[\tau N]}([xN]) = [yN]\} \quad \subset \quad \Big\{ \sup_{t=0,\dots,[\tau N]} \big|\xi_t([xN]) - N\varphi(t/N)\big| > \frac{\gamma}{2} N, \\ \text{for all } \varphi \in \Phi_{x,y,\tau}(\mathcal{L}_{x,y,\tau} - \eta) \Big\}, \tag{2.27}$$

for $N \ge N'$ for some N' and this choice of γ. Because of the validity of the Large Deviation upper bound, for any δ', η, γ there exists N_0, such that

$$\mathsf{P}\Big\{ \sup_{t=0,\dots,[\tau N]} \big|\xi_t([xN]) - N\varphi(t/N)\big| > \frac{\gamma}{2} N, \ \text{for all } \varphi \in \Phi_{x,y,\tau}(\mathcal{L}_{x,y,\tau} - \eta) \Big\}$$

$$\le \ \exp\{(\delta' + \eta) - N\mathcal{L}_{x,y,\tau}\}, \quad N \ge N_0, \tag{2.28}$$

and the desired upper bound follows by combining (2.27) and (2.28).

For notational convenience we will prove the lower bound for one-dimensional random walks only and for points $y \ne 0$. The generalisation is straightforward.

Let $\eta, \varepsilon > 0$ be sufficiently small and let $\varphi \in \Phi_{x,y,\tau-\varepsilon}$ with

$$\mathcal{L}_{\tau-\varepsilon}(\varphi) \le \mathcal{L}_{x,y,\tau} + \eta. \tag{2.29}$$

We lower bound $P^{([\tau N])}_{[xN][yN]}$ as follows. First we stay in a δN-neighbourhood of the path φ during time $[(\tau-\varepsilon)N]$, for δ sufficiently small. Then we reach $[yN]$ in time $[\varepsilon N]$.

Let $0<\delta\ll\varepsilon$. Then

$$\begin{aligned} P^{([\tau N])}_{[xN][yN]} \quad \geq \quad & \sum_{s:|s-[yN]|<2\delta N} \mathsf{P}\{\xi_{[\tau N]-[(\tau-\varepsilon)N]}=[yN]\,|\,\xi_0=s\} \\ & \times \mathsf{P}\{\xi_{[(\tau-\varepsilon)N]}=s\,|\,\xi_0=[xN]\}. \end{aligned}$$

Consider the homogeneous random walk $\tilde{\xi}_t$ on $\mathbf{Z}$ with jump probabilities p. By irreducibility there exists t', such that $\mathsf{P}\{\tilde{\xi}_t=s\,|\,\tilde{\xi}_0=0\}>0$, for $s=0,-1,1$ and any $t\geq t'$. Choose $t_0\geq t'$ and

$$\sigma=\log\Big(\min_{t=t_0,\ldots,2t_0-1}\ \min_{y=-1,0,1}\mathsf{P}\{\tilde{\xi}_t=y\,|\,\tilde{\xi}_0=0\}\Big).$$

Denote

$$M=\Big[\frac{[\tau N]-[(\tau-\varepsilon)N]}{t_0}\Big].$$

Let now $\varepsilon\gg t_0\delta$. Then for N_1 with $[yN_1]-\delta N_1>\max\{k,l\}t_0$

$$\mathsf{P}\{\xi_{[\tau N]-[(\tau-\varepsilon)N]}=[yN]\,|\,\xi_0=s\}\geq\exp\{M\sigma\}\geq\exp\Big\{\sigma\frac{\varepsilon N+1}{t_0}\Big\},\quad N\geq N_1,$$

so that

$$\begin{aligned} P^{(\tau N])}_{[xN][yN]} \quad \geq \quad & \exp\Big\{\sigma\frac{\varepsilon N+1}{t_0}\Big\} \qquad (2.30) \\ & \times \mathsf{P}\Big\{\sup_{t=0,\ldots,[(\tau-\varepsilon)N]}\big|\xi_t([xN])-N\varphi(t/N)\big|<\delta N\,|\,\xi_0=[xN]\Big\}. \end{aligned}$$

Combine (2.29) and (2.30). It easily follows that for any $\tau,\gamma>0$ there exists $N'=N'(\tau,\gamma)$, such that

$$P^{([\tau N])}_{[xN][yN]}\geq\exp\{-\gamma N+N\mathcal{L}_{x,y,\tau}\},\quad N\geq N'.$$

This proves the lower bound. □

Proof of Theorem 2.14. We shall first prove that

$$\log r_0=\inf_{x\neq 0}\ \liminf_{\tau\to\infty}\frac{1}{\tau}\mathcal{L}^0_{x,x,\tau}.$$

First we prove the lower bound for $\log r_0$. By virtue of (2.3) this means that we have to upper bound the logarithmic asymptotics of the probability of not hitting a finite set A before time T. For simplicity of notation we shall formulate the proof for dimension 1.

Fix $\tau > 0$. By Lemma 2.15, for any $\varepsilon > 0$ there exist $\delta > 0$, N', such that

$$\begin{aligned}\exp\{-N\mathcal{L}_{x,y,\tau} - \varepsilon N\} &\le \mathsf{P}\{\left|\xi_{[\tau N]}([x'N]) - [yN]\right| < \delta N\} \\ &\le \exp\{-N\mathcal{L}_{x,y,\tau} + \varepsilon N\},\end{aligned} \tag{2.31}$$

for all $|x' - x| < \delta$, $N \ge N'$, and all x, y. There exists $r = r(\tau)$, such that

$$\mathcal{L}_{x,y,\tau} = \mathcal{L}^0_{x,y,\tau},$$

$x, y \ge r$. This is because 0 cannot be reached from all states $[xN]$ with x sufficiently large, within time $[\tau N]$. This is due to the boundedness of jumps.

Let $N \ge N'$. Divide $\mathbf{Z}_+$ into intervals $\mathcal{B}_k(N) = \{[(k-1)\delta N]+1, \ldots, [k\delta N]\}$, and denote $b_k = (k - 1/2)\delta$. For N large we have $[b_k N] \in \mathcal{B}_k(N)$ and we will assume that this is the case.

Let $A(N) = \bigcup_{k=1,\ldots,[r/\delta]} \mathcal{B}_k(N)$. Consider first the embedded chain $\eta_t = \xi_{t[\tau N]}$. Its jump probabilities can be estimated by the Large Deviation probabilities. Let the initial position of η_t be $[xN] \notin A(N)$. Then

$$\mathsf{P}\{\eta_t \notin A(N), t \le T\} = \sum \mathsf{P}\{\eta_t \in \mathcal{B}_{k(t)}(N), t = 1, \ldots, T\},$$

where we take the summation over all sequences of T intervals not contained in $A(N)$. To such sequence of intervals $\mathcal{B}_{k(t)}(N)$ there corresponds a path $\varphi : [0, T\tau] \to \mathbf{R}_+$, with $\varphi(0) = x$, $\varphi(k\tau) = b_{k(t)}$ and by construction

$$\mathsf{P}\{\eta_t \in \mathcal{B}_{k(t)}(N), k = 1, \ldots, T\} \le \exp\bigl\{-N\mathcal{L}^0_{x,b_{k(T)},T\tau} + \varepsilon TN\bigr\}.$$

Consequently

$$\mathsf{P}\{\eta_t \notin A(N), t \le T\} \le \Bigl(\frac{2d\tau}{\delta}\Bigr)^T \exp\bigl\{-N \inf_{y \ge r} \mathcal{L}^0_{x,y,T\tau} + \varepsilon TN\bigr\}, \tag{2.32}$$

since from each interval it is possible to jump to at most $2d\tau/\delta$ intervals, where d denotes the maximum jump size

$$d = \max\{|y| : p_y > 0\}.$$

For the chain ξ_t

$$\mathsf{P}\bigl\{\xi_{[\tau N]}(y) \in \mathcal{B}_k(N), \xi_t \notin A(N), t \le [\tau N]\} \le \mathsf{P}\{\eta_1(y) \in \mathcal{B}_k(N)\bigr\},$$

for all $y \in \mathcal{B}_l(N)$. By combination with (2.32) we find

$$\begin{aligned}&\mathsf{P}\{\xi_t([xN]) \notin A(N), t \le T\} \\ &\qquad \le \mathsf{P}\Bigl\{\eta_t \notin A(N), t \le \Bigl\lfloor \frac{T}{[\tau N]} \Bigr\rfloor\Bigr\} \\ &\qquad \le \Bigl(\frac{2d\tau}{\delta}\Bigr)^{T/[\tau N]} \exp\Bigl\{ - N \inf_{y \ge r} \mathcal{L}^0_{x,y,\lfloor T\tau/[\tau N]\rfloor} + \varepsilon N \frac{T+1}{[\tau N]}\Bigr\}.\end{aligned}$$

And so

$$-\lim_{T\to\infty}\frac{1}{T}\log \mathsf{P}\{\xi_t([xN])\notin A(N), t\le T\}$$
$$\ge \quad -\frac{1}{[\tau N]}\log\Big(\frac{2d\tau}{\delta}\Big)+\liminf_{T\to\infty}\frac{1}{T}\inf_{y\ge r}\mathcal{L}^0_{x,y,T}-\varepsilon\frac{N}{[\tau N]}. \tag{2.33}$$

For N sufficiently large, the first term in the right-hand side of (2.33) can be made arbitrarily small.

Thus, for all N sufficiently large

$$\lim_{T\to\infty}\Big(-\frac{1}{T}\log \mathsf{P}\{\xi_t([xN])\notin A(N), t\le T\}\Big)\ge \inf_{x\ge r}\liminf_{T\to\infty}\frac{1}{T}\inf_{y\ge r}\mathcal{L}^0_{x,y,T}-\varepsilon.$$

Because of the irreducibility of the Markov chain

$$\log r_0 \;=\; \limsup_{N\to\infty}\;\lim_{T\to\infty}\Big(-\frac{1}{T}\log \mathsf{P}\{\xi_t([xN])\notin B(N), t\le T\}\Big),$$

for any increasing sequence of sets $B(N)$ with $\lim_{N\to\infty}B(N)=\boldsymbol{S}$ and any sequence of states $[xN]\notin B(N)$. As a consequence, for $x=r$

$$\log r_0\ge\liminf_{T\to\infty}\frac{1}{T}\inf_{y\ge r}\mathcal{L}^0_{x,y,T}.$$

To complete the proof for the lower bound we show for $|x|=r$

$$\liminf_{T\to\infty}\frac{1}{T}\inf_{y\ge r}\mathcal{L}^0_{x,y,T}=\liminf_{T\to\infty}\frac{1}{T}\mathcal{L}^0_{x,x,T}. \tag{2.34}$$

Let y and T be given. For any $\varphi\in\Phi_{x,y,T}$ we construct a path $\varphi'\in\Phi_{x,x,T'}$ for some $T'>T$ such that $\varphi(t)=\varphi'(t)$ for $t\le T$ and $\dot\varphi'(t)=\sum_i ip_i$, $T<t\le T'$. Then

$$\frac{1}{T'}\mathcal{L}_{T'}(\varphi')=\frac{1}{T'}\mathcal{L}_T(\varphi)\le\frac{1}{T}\mathcal{L}_T(\varphi).$$

This immediately implies (2.34) and so we have proved the lower bound. In the case of dimension 2 this construction is more complicated, since along the mean drift we do not necessarily come back to x. However, along the mean drift we hit a state z with $|z|=r$ (it is possible that along the mean drift first an ergodic 1-face is reached and then we move along the resulting drift for this 1-face). Since the set $\{z:|z|=r\}$ is compact, for any z with $|z|=r$ we can construct a path from z to x such that its contribution to the time-scaled action functionals $\mathcal{L}_{T'}(\varphi')/T'$ becomes negligible for T large.

For the upper bound we have to lower bound the logarithmic asymptotics of the probability of not hitting a finite set before time T.

Let $\gamma>0$ and choose x, τ such that

$$\frac{1}{\tau}\mathcal{L}^0_{x,x,\tau}\le\inf_{x'\ne 0}\;\liminf_{\tau'\to\infty}\frac{1}{\tau'}\mathcal{L}^0_{x',x',\tau'}+\gamma.$$

Next choose a path $\varphi \in \Phi_{x,x,\tau}$, such that $\inf_t \varphi(t) > 0$ and

$$\mathcal{L}_\tau(\varphi) \leq \mathcal{L}^0_{x,x,\tau} + \gamma\tau.$$

Denote $r = \inf_t \varphi(t)$. It can be shown that for any $\varepsilon > 0$, there exist $\delta > 0$ sufficiently small and N_0, such that

$$\mathsf{P}\Big\{ \sup_{t=0,\ldots,[\tau N]} \Big| \frac{1}{N}\xi_t([x'N]) - \varphi(t/N) \Big| < \delta \Big\} \geq \exp\{-\varepsilon N - N\mathcal{L}_\tau(\varphi)\},$$

for any x' with $|x - x'| < \delta$ and any $N \geq N_0$. For $\varepsilon > 0$ we can choose $\delta > 0$, such that $\delta \ll r$. Choose the set $B(N) = \{0, \ldots, [rN] - [\delta N] - 2\}$. Then it easily follows that

$$\mathsf{P}\{\xi_t([xN]) \notin B(N), t \leq T\} \geq \exp\Big\{ -N\Big(\Big[\frac{T}{[\tau N]}\Big] + 1\Big)\mathcal{L}_\tau(\varphi) - \varepsilon N\Big(\Big[\frac{T}{[\tau N]}\Big] + 1\Big)\Big\},$$

for N sufficiently large, and so

$$-\frac{1}{T}\log \mathsf{P}\{\xi_t([xN]) \notin B(N), t \leq T\} \leq \frac{N}{T}\Big(\Big[\frac{T}{[\tau N]}\Big] + 1\Big)\mathcal{L}_\tau(\varphi) + \varepsilon\frac{N}{T}\Big(\Big[\frac{T}{[\tau N]}\Big] + 1\Big).$$

Let N be sufficiently large such that

$$\frac{N}{T}\Big(\Big[\frac{T}{[\tau N]}\Big] + 1\Big)\mathcal{L}_\tau(\varphi) + \varepsilon\frac{N}{T}\Big(\Big[\frac{T}{[\tau N]}\Big] + 1\Big) \leq \frac{1}{\tau}\mathcal{L}_\tau(\varphi) + \frac{\varepsilon}{\tau} + \gamma,$$

for all sufficiently large values T. By taking the limit $T \to \infty$ we find that

$$\begin{aligned} \log r_0 &\leq \frac{1}{\tau}\mathcal{L}_\tau(\varphi) + \gamma + \frac{\varepsilon}{\tau} \\ &\leq \inf_{x' \neq 0} \liminf_{\tau' \to \infty} \frac{1}{\tau'}\mathcal{L}_{x',x',\tau'} + 3\gamma + \varepsilon\frac{1}{\tau} \end{aligned}$$

and the result follows, since we can take ε and γ arbitrarily small.

To complete the proof, we should show that $\alpha_{\mathrm{int}} = \log r_0$. The lower bound $\alpha_{\mathrm{int}} \geq \log r_0$ follows from Theorem 2.4. For the upper bound we will consider the one- and two-dimensional random walks separately.

For the one-dimensional random walk this is proved by two statements. The first is the general result of Theorem 2.5 that the existence of a non-pole singularity of $F^{\mathrm{L}}_{00}(z)$ for some L yields an upper bound on α_{int}. The second assertion is that some finite perturbation of the homogeneous random walk on $\mathbf{Z}_+$ has an essential non-pole singularity at the point r_0.

It is easy to see that $r(F_{xA}) = r(F_{yB})$, for any finite sets $A = \{i : i \leq i_0\}$, $B = \{i : i \leq i_1\}$ with i_0, $i_1 \geq k'$ and $i_0 + l < i_1$, and for any $x \notin A$, $y \notin B$. The conditions of Lemma 2.12 are therefore satisfied. The proof follows from Theorem 2.5 and Lemma 2.12.

For the two-dimensional random walk the same idea applies. For the sets $A = \{1\}$ and $B = \{x : x_2 \leq 2\}$ the conditions of Lemma 2.13 are satisfied and so this walk has property (P). The result then follows from Theorem 2.6. □

2.4. Relationships with the essential spectrum

The following problems are of interest:

1. Is there a space, such that both $\alpha(\mathrm{L}_0)$ and α_{int} can be obtained from analysing the spectrum of P in this space? If so, what space?

2. Suppose there is a space from which $\alpha(\mathrm{L}_0)$ can be obtained. Is this space the same for all $\mathrm{L} \in \mathcal{F}(\mathrm{L}_0)$: that is, can $\alpha(\mathrm{L})$ be obtained from analysing the spectrum of P^{L} in this space for *any* L?

We shall restrict the analysis to Banach spaces $\ell^1(\mu)$ of functions $f : \boldsymbol{S} \to \mathbf{R}$ with the norm $\|f\| = \sum_i |f(i)|\mu(i) < \infty$, for some positive real measure μ on $\boldsymbol{S}$. Assume that $P \in \mathcal{B}\big(\ell^1(\mu), \ell^1(\mu)\big)$. We will consider P as an operator acting from the left, i.e. $(Px)_i = \sum_j x_j P_{ji}$.

If the relation $\alpha_{\text{int}} = \log r_0$ is valid, then it follows that a suitable class of $\ell^1(\mu)$ spaces to consider is based on the generating functions of first hitting times: for $\mathrm{L} \in \mathcal{F}(\mathrm{L}_0)$ and $r \leq r(F^{\mathrm{L}}_{00})$ define

$$\mu^{\mathrm{L},r}(x) = \limsup_{r' \uparrow r} \frac{F^{\mathrm{L}}_{x0}(r')}{F^{\mathrm{L}}_{00}(r')}$$

and denote

$$F^{\mathrm{L},\prime}_{x0}(z) = \frac{d}{dz} F^{\mathrm{L}}_{x0}(z).$$

We could also define $\mu^{L,r}$ by $F^{\mathrm{L}}_{x0}(r)$: the above choice is only done to guarantee finiteness on the boundary $r(F^{\mathrm{L}}_{00})$.

Theorem 2.16. *Let* L_0 *be an irreducible, aperiodic and ergodic Markov chain on a countable state space* $\boldsymbol{S}$.

i) *If* $r < r(F^{\mathrm{L}}_{00})$ *we assume the existence of an infinite subsequence* $\{x(n)\}_n \subset \boldsymbol{S}$ *such that*

$$\frac{F^{\mathrm{L},\prime}_{x(n)0}(r)}{F^{\mathrm{L}}_{x(n)0}(r)} \to \infty, \quad n \to \infty. \tag{2.35}$$

If $r = r(F^{\mathrm{L}}_{00})$ *we assume the existence of infinitely many points* $\{x(n)\}_n$ *with*

$$\limsup_{r' \uparrow r} \frac{F^{\mathrm{L}}_{x(n)0}(r')}{F^{\mathrm{L}}_{00}(r')} > 0.$$

Then $\sup\big\{|\lambda| \,:\, \lambda \in \boldsymbol{\sigma}_{\mathbf{ess}}(P, \ell^1(\mu^{\mathrm{L},r}))\big\} = 1/r$.

ii) $\log \sup_{\mathrm{L} \in \mathcal{F}(\mathrm{L}_0)} r(F^{\mathrm{L}}_{00}) = \sup_{\mathrm{L},r} \Big(-\log \sup\{|\lambda| \,:\, \lambda \in \boldsymbol{\sigma}_{\mathbf{ess}}(P, \ell^1(\mu^{\mathrm{L},r})\}\Big).$

iii) *If* $\alpha_{int} = \log r_0$, *then* $\alpha_{int} = \sup_{\mathrm{L},r} \Big(-\log \sup\{|\lambda| \,:\, \lambda \in \boldsymbol{\sigma}_{\mathbf{ess}}(P, \ell^1(\mu^{\mathrm{L},r})\}\Big).$

iv) *If* $\alpha_{int} = \log r_0$ *and* $F_{00}^{\mathrm{L}'}(z)$ *has a singularity with a finite value at the point* r_0 *for some* $\mathrm{L}' \in \mathcal{F}(\mathrm{L}_0)$*, then* $\alpha_{int} = \log r(F_{00}^{\mathrm{L}'})$*; hence there exists an* $\ell^1(\mu)$ *space, such that* $\alpha(\mathrm{L})$ *can be obtained from the spectrum of* P^{L} *in this space, for any* $\mathrm{L} \in \mathcal{F}(\mathrm{L}_0)$.

The proof is given below. The conditions in the statement of the Theorem are satisfied in deflected random walks in $\mathbf{Z}_+^\nu$ with bounded jumps towards to origin: at $r \le r(F_{00}^{\mathrm{L}})$

$$F_{x0}^{\mathrm{L},\prime}(r) \ge \frac{|x|}{c} F_{x0}^{\mathrm{L}}(r)$$

for some norm $|\cdot|$ on $\mathbf{Z}_+^\nu$ and some fixed constant c. The condition for the case $r = r(F_{00}^{\mathrm{L}})$ is necessary to exclude that $\ell^1(\mu^{\mathrm{L},r})$ is a finite dimensional space.

In Sections 3, 4.2 and 5.2 we will explicitly calculate a measure μ for the one- as well as the two-dimensional random walks, such that the spectrum in $\ell^1(\mu)$ determines both α_{int} and $\alpha(\mathrm{L})$ for any $\mathrm{L} \in \mathcal{F}(\mathrm{L}_0)$. We will also explicitly calculate the essential spectrum of the transition matrix P for the corresponding random walk in this space.

Remark 2.4. Choosing $\mu \equiv 1$ is in general not possible. For Markov chains with bounded "downward" jumps we have $1 \in \boldsymbol{\sigma}_{\mathbf{ess}}(P, \ell^1)$: this follows from the fact that the Dœblin condition fails in this case, even though exponential ergodicity holds.

Remark 2.5. For non-reversible random walks on $\mathbf{Z}^+$ it follows from Section 3.1 that the measure space determining the convergence rates generally *cannot* be equal to $\ell^1(1/\sqrt{\pi})$, with $(1/\sqrt{\pi})(i) = 1/\sqrt{\pi_i}$, or equivalently to $\ell^2(1/\pi)$ (we have to take $1/\pi$ instead of π because we consider P as an operator acting from the left). For reversible Markov chains one can show that

$$r_0^{-1} = \sup\big\{|\lambda| \,:\, \lambda \in \boldsymbol{\sigma}_{\mathbf{ess}}\big(P, \ell^1\big(\frac{1}{\sqrt{\pi}}\big)\big)\big\},$$

by using that reversibility implies $\sum_i \pi_i (F_{i0}(z))^2 = (d/dz)F_{00}(z)$: this explains why $\ell^2(1/\pi)$ is the appropriate space to consider.

In the remainder of this subsection we will introduce the essential spectrum and we will prove Theorem 2.16.

A linear operator $P \in \mathcal{B}(X, X)$ for some Banach space X is Fredholm (or Nœther) if the range $\mathsf{R}(P)$ is closed in X, and if $\mathsf{def}(P)$ and $\mathsf{nul}(P)$ (i.e. the dimensions of $X/\overline{\mathsf{R}(P)}$ and the null-space of P respectively) are finite. The essential spectrum $\boldsymbol{\sigma}_{\mathbf{ess}}(P, X)$ of P in X is defined as the complement of the Fredholm domain

$$\boldsymbol{\sigma}_{\mathbf{ess}}(P, X) = \{\lambda \in \mathbf{C} \,:\, (\lambda - P) \text{ is not Fredholm}\};$$

the spectrum $\boldsymbol{\sigma}(P,X)$ is equal to

$$\boldsymbol{\sigma}(P) = \{\lambda \in \mathbf{C} : (\lambda - P)^{-1} \notin \mathcal{B}(X,X)\}.$$

Let us note first that isolated singularities are independent of the norm. To show this, we use the fact that the resolvent at an isolated singularity λ has a Laurent expansion (cf. [15], p. 228). Thus for at least for one pair of states (x,y) it holds that $P_{xy}(z)$ has a singularity at $1/\lambda$. Consequently $\lambda \in \boldsymbol{\sigma}(P,X)$ for any X.

An important question is the following.

Problem. Are $\exp\{-\alpha_{\rm int}\}$ and $1/r_0$ lower bounds for the largest value in the essential spectrum of P in $\ell^1(\mu)$?

Intuitively the answer should be yes: if it is not true, then there exists a perturbation that changes the rates within a finite set of states such that the largest value in the spectrum is smaller than $\exp\{-\alpha_{\rm int}\}$. This cannot be true if this perturbation corresponds to a Markov chain. The problem is to show that there exists a Markov chain perturbation that is arbitrarily close to the given finite perturbation.

Finally note the connection to solutions of Popov's criterion (cf. [10]) for exponential ergodicity. It states that L_0 is exponentially ergodic if and only if there exists $\gamma > 0$, a function $f \geq 1$ and a finite set A such that

$$\begin{aligned}\sum_y P_{xy} f_y &\leq \exp\{-\gamma\} f_x, && x \notin A;\\ \sum_y P_{xy} f_y &< \infty, && x \in A.\end{aligned}$$

First this implies that $F_{xA}(\exp\{\gamma\})$ is finite. Secondly, for any finite set A and $\gamma > 0$ for which $F_{xA}(\exp\{\gamma\})$ is finite, the function f given by

$$f_x = \begin{cases} F_{xA}(\exp\{\gamma\}), & x \notin A;\\ 1, & x \in A,\end{cases}$$

is a solution to Popov's criterion. Hence

$$\begin{aligned}\log r_0 &= \sup\{\gamma > 0 : \exists f \geq 1, A \subset \boldsymbol{S},\ \text{with } |A| < \infty,\\ &\qquad \text{that solve Popov's criterion for the rate } \gamma\},\end{aligned}$$

and so the "best" possible rate in Popov's criterion can only be equal to $\alpha_{\rm int}$ if $\alpha_{\rm int} = \log r_0$.

Proof of Theorem 2.16. We prove i). First let $r < r(F^{\rm L}_{00})$. Then there exists a finite set A, such that $F_{xA}(z)$ converges for $|z| \leq r + \varepsilon$, for some $\varepsilon > 0$. Hence for $r' \leq r + \varepsilon$ there exist constants $k_1(r')$ and $k_2(r')$, such that

$$k_1(r') F_{xA}(r') \leq \frac{F^L_{x0}(r')}{F^L_{00}(r')} \leq k_2(r') F_{xA}(r')$$

and ${}_AF_{xy}(z)$ and ${}_AP_{xy}(z)$ are analytic functions for $|z| < r + \varepsilon$.

For notational simplicity denote the norm of v in $\ell^1(\mu^{L,r})$ by $\|v\|$ and similarly for the corresponding operator norm. For $|z| \le r + \varepsilon$ the only singularities of $P_{xy}(z)$ can occur, if

$$\Delta(z) = \det\bigl(I(A) - F(z, A)\bigr) = 0.$$

Consider the decomposition (2.6). For $|z| < r$ and $v \in \ell^1(\mu^{\mathrm{L},r})$

$$\begin{aligned} \|v\, {}_AP(z)\| &= \sum_y \Bigl| \sum_x v_x\, {}_AP_{xy}(z) \Bigr| F_{yA}(r) \\ &\le \sum_x |v_x| \left| \frac{F_{xA}(r) - F_{xA}(z)}{r/z - 1} \right| < \infty, \end{aligned} \tag{2.36}$$

and if $\Delta(z) \ne 0$, then there is a constant k such that

$$\begin{aligned} &\sum_y \Bigl| \sum_x \sum_{a,a' \in A} v_x F_{xa}(z) (I - F(z, A))^{-1}_{aa'}\, {}_AP_{a'y}(z) \Bigr| F_{yA}(r) \\ &\qquad \le k \sum_x |v_x F_{xA}(z)| \sup_{a \in A} \left| \frac{F_{aA}(r) - F_{aA}(z)}{r/z - 1} \right| < \infty. \end{aligned} \tag{2.37}$$

For $|z| \le r + \varepsilon$ the number of zeros of the function $\Delta(z)$ can only be finite, since otherwise $\Delta(z)$ would be identically 0. Hence for $|\lambda| > 1/r$ the resolvent $(\lambda - P)^{-1}$ exists as an operator in $\mathcal{B}(\ell^1(\mu^{\mathrm{L},r}), \ell^1(\mu^{\mathrm{L},r}))$ with at most finitely many isolated singularities at points λ with $\Delta(1/\lambda) = 0$.

Let λ be a point with $\Delta(1/\lambda) = 0$, $|\lambda| > 1/r$. Then $(\lambda - P)^{-1}$ can be expanded into a Laurent series (cf. [7]). It has a pole of at most finite order, since $\Delta(1/\lambda)$ is a rational function. This implies that $(\lambda - P)$ is Fredholm for $|\lambda| > 1/r$.

To show that

$$1/r \in \boldsymbol{\sigma}_{\mathbf{ess}}(P, \ell^1(\mu^{\mathrm{L},r}))\,,$$

we will show that

$$1/r \in \boldsymbol{\sigma}_{\mathbf{ess}}(P^{\mathrm{L}}, \ell^1(\mu^{\mathrm{L},r}))\,.$$

If $F^{\mathrm{L}}_{00}(r) - 1 = 0$, this zero can be deleted. Clearly

$$\frac{F^L_{x0}(r) - F^L_{x0}(z)}{r/z - 1} \to r F^{L,\prime}_{x0}(r), \quad z \to r. \tag{2.38}$$

Inequality (2.36) easily implies that the first expression in the right-hand side of (2.6) defines an operator function $O(z) : \mathbf{C} \to \mathcal{B}(\ell^1(\mu^{\mathrm{L},r}), \ell^1(\mu^{\mathrm{L},r}))$ with a finite norm at $z = r$ for L and $A = \{0\}$. For the second expression we get for $v \ge 0$

$$\sum_y \Bigl| \sum_x v_{x0} P^{\mathrm{L}}_{xy}(r) \Bigr| F^{\mathrm{L}}_{y0}(r) \;=\; r \sum_x v_x F^{\mathrm{L},\prime}_{x0}(r),$$

by (2.37) and (2.38). It is possible to choose a Cauchy-sequence

$$\{v(n)\}_n \subset \ell^1(\mu^{\mathrm{L},r})$$

with $\|v(n)_0 P^{\mathrm{L}}(r)\| \to \infty$, for $n \to \infty$. Hence $\mathsf{R}(1/r - P^{\mathrm{L}})$ is not closed.

If $r = r(F^{\mathrm{L}}_{00})$ and if $F^{\mathrm{L}}_{x0}(z)$ has a pole at r, then $F^{\mathrm{L}}_{x0}(z)/F^{\mathrm{L}}_{00}(z)$ is analytic at r. Moreover, $F^{\mathrm{L},\prime}_{x0}(z)$ has a pole of higher order than $F^{\mathrm{L}}_{x0}(z)$ at r. Consequently (2.35) holds and we can apply the above arguments. Our condition that there exist infinitely many points $x(n)$, such that

$$\lim_{r' \to r} \frac{F_{x(n)0}(r')}{F_{00}(r')} > 0\,,$$

is necessary to show that $\mathsf{R}(1/r - P^{\mathrm{L}})$ is not closed.

Finally assume that $r = r(F^{\mathrm{L}}_{00})$ is a non-pole singularity. As in the foregoing $(\lambda - P)$ is Fredholm for $\lambda < 1/r$. The result follows, since (2.35) will also hold for $r = r(F^{\mathrm{L}}_{00})$.

Assertions ii), iii) and iv) immediately follow. □

3. Results for one- and two-dimensional random walks

Our main interest in this work is to study intrinsic rates for deflected random walks in $\mathbf{Z}^\nu_+$. This section presents a precise formulation of the results for $\nu = 1, 2$.

3.1. Results for one-dimensional random walks

Consider the function $H(\alpha)$. It is easily shown that $H(\alpha)$ is a convex function. The minimum is achieved at a *finite* point α_0:

$$H(\alpha_0) = \min_{\alpha \in \mathrm{R}} H(\alpha),$$

if $\sum_{i<0} p_i$, $\sum_{i>0} p_i$ are both positive. This is clearly true, since otherwise the Markov chain could not be irreducible.

Define

$$\varphi_{x,\tau}(t) = x, \quad t \le \tau.$$

Theorem 3.1. *The following assertions hold.*

i) $\alpha_{int} = \log r_0$;

ii) $\alpha_{int} = \inf_{x \neq 0} \inf_{\tau > 0} \dfrac{1}{\tau} \mathcal{L}^0_{x,x,\tau} = \lim\inf_{x \neq 0} \inf_{\tau > 0} \dfrac{1}{\tau} \mathcal{L}_\tau(\varphi_{x,\tau}) = -H(\alpha_0)$;

iii) $F^{\mathrm{L}}_{00}(z)$ *has an algebraic singularity at* $z = \exp\{-H(\alpha_0)\}$ *for any* $\mathrm{L} \in \mathcal{F}(\mathrm{L}_0)$.

Next we determine an appropriate space $\ell^1(\mu)$, such that the spectrum of P in this space determines the convergence rate and the intrinsic rate. We will consider measures μ from the following set

$$\mathcal{M} = \big\{\mu \,:\, \mu(i) = \exp\{\alpha i\}, \alpha \in \mathbf{R}\big\}$$

and we denote $\mu_\alpha(i) = \exp\{\alpha i\}$. Define $Q(z) = \sum_i p_i z^i$.

Theorem 3.2. i) $\boldsymbol{\sigma}_{\mathbf{ess}}(P, \ell^1(\mu_\alpha)) = \{\lambda \,:\, \lambda = Q(z\exp\{\alpha\}), |z| = 1\}$.

ii) *The space* $\ell^1(\mu_{\alpha_0})$ *determines both* α_{int} *and* $\alpha(\mathrm{L})$ *for any* $\mathrm{L} \in \mathcal{F}(\mathrm{L}_0)$, *more exactly*

$$\alpha_{int} = -\log\sup\{|\lambda| : \lambda \in \boldsymbol{\sigma}_{\mathbf{ess}}(P^{\mathrm{L}}, \ell^1(\mu_{\alpha_0}))\}$$

and

$$\alpha(\mathrm{L}) = -\log\sup\{|\lambda| : \lambda \in \boldsymbol{\sigma}(P^{\mathrm{L}}, \ell^1(\mu_{\alpha_0})), \lambda \neq 1\}.$$

3.2. Results for two-dimensional random walks

Consider the functions $H(\alpha,\beta)$, $\mathcal{H}_{\{1\}}$ and $\mathcal{H}_{\{2\}}$. It can be shown [6] that $\mathcal{H}_{\{1\}}$ and $\mathcal{H}_{\{2\}}$ are convex and that the equations $\mathcal{H}_{\{1\}}(\alpha) = 0$ and $\mathcal{H}_{\{2\}}(\beta) = 0$ have two real roots $\alpha_{\{1\}} \leq \alpha'_{\{1\}}$ and $\beta_{\{2\}} \leq \beta'_{\{2\}}$ respectively, with

$$\frac{\partial}{\partial\alpha}\mathcal{H}_{\{1\}}(\alpha_{\{1\}}) \leq 0\,, \quad \frac{\partial}{\partial\alpha}\mathcal{H}_{\{1\}}(\alpha'_{\{1\}}) \geq 0\,,$$

and

$$\frac{\partial}{\partial\beta}\mathcal{H}_{\{2\}}(\beta_{\{2\}}) \leq 0\,, \quad \frac{\partial}{\partial\beta}\mathcal{H}_{\{2\}}(\beta'_{\{2\}}) \geq 0\,.$$

Then there are unique points $\alpha^0_{\{1\}}$, $\beta^0_{\{2\}}$ with

$$\mathcal{H}_{\{1\}}(\alpha^0_{\{1\}}) = \min_{\alpha\in\mathrm{R}}\mathcal{H}_{\{1\}}(\alpha), \qquad \mathcal{H}_{\{2\}}(\beta^0_{\{2\}}) = \min_{\beta\in\mathrm{R}}\mathcal{H}_{\{2\}}(\beta).$$

There also exist α_0, β_0 with

$$H(\alpha_0,\beta_0) = \min_{\alpha,\beta\in\mathrm{R}} H(\alpha,\beta),$$

and the value $H(\alpha_0,\beta_0)$ is unique.

Theorem 3.3. *Assume that the Hessians*

$$\det\begin{pmatrix} \dfrac{\partial^2}{\partial\alpha^2}H(\alpha,\beta) & \dfrac{\partial^2}{\partial\beta\partial\alpha}H(\alpha,\beta) \\ \dfrac{\partial^2}{\partial\beta\partial\alpha}H(\alpha,\beta) & \dfrac{\partial^2}{\partial\beta^2}H(\alpha,\beta) \end{pmatrix} \neq 0.$$

Then

i) $\alpha_{int} = \log r_0$;

ii)
$$\begin{aligned}\alpha_{int} &= \inf_{x\neq 0} \liminf_{\tau>0} \frac{1}{\tau}\mathcal{L}^0_{x,x,\tau} = \inf_{x\neq 0}\inf_{\tau>0}\frac{1}{\tau}\mathcal{L}_\tau(\varphi_{x,\tau}) \\ &= -\max\{\mathcal{H}_{\{1\}}(\alpha_0), \mathcal{H}_{\{2\}}(\beta_0)\}.\end{aligned}$$

To determine a space $\ell^1(\mu)$ with an appropriate spectrum, we define a set $\mathcal{M}$ of measures in the following way. Denote $\gamma_x = x_2/x_1$.

We say that $\mu \in \mathcal{M}$ if and only if $\mu(x) = \exp\{\alpha(x)x_1 + \beta(x)x_2\}$ for a continuous vector field $(\alpha(x), \beta(x))$ on $\mathbf{R}^2_+$ satisfying the following conditions:

1. there exists a potential function for the vector field $(\alpha(x), \beta(x))$, i.e. there exists a function $F : \mathbf{R}^2_+ \to \mathbf{R}$ such that
$$\frac{\partial}{\partial x_1}F(x) = \alpha(x), \quad \frac{\partial}{\partial x_2}F(x) = \beta(x);$$
2. there exist $0 \le \gamma_1^\mu < \gamma_2^\mu \le \infty$, $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbf{R}$ such that
$$(\alpha(x), \beta(x)) = \begin{cases} (\alpha_1, \beta_1), & \gamma_x \le \gamma_1^\mu, \\ (\alpha_2, \beta_2), & \gamma_x \ge \gamma_2^\mu; \end{cases}$$
3. $(\alpha(x), \beta(x)) = (\alpha(y), \beta(y))$, if $\gamma_x = \gamma_y$.

Write $Q(z, u)$, $q'(z, u)$ and $q''(z, u)$ for the probability generating functions of p, p' and p''. The essential spectrum in $\ell^1(\mu)$, $\mu \in \mathcal{M}$, is characterised as follows.

Theorem 3.4. *Let $\mu \in \mathcal{M}$ and consider P as an operator in $\ell^1(\mu)$.*

i) *Let λ_0 be defined by*
$$\lambda_0 = \max_{x\neq 0}\Big\{Q(\exp\{\alpha(x)\}, \exp\{\beta(x)\})\Big\}.$$

Then $\lambda_0 = \sup\{|\lambda| : \lambda \in \boldsymbol{\sigma}_{\mathbf{ess}}(P, \ell^1(\mu))\}$ and $\lambda - P$ is Fredholm for $|\lambda| > \lambda_0$.

ii) *$\lambda \in \boldsymbol{\sigma}_{\mathbf{ess}}(P, \ell^1(\mu))$ if and only if one of the following conditions holds.*

a) *$\lambda - Q(z\exp\{\alpha(x)\}, u\exp\{\beta(x)\}) = 0$, for some $|z| = |u| = 1$ and for some $x \neq 0$.*

b) *$\lambda - q'(z\exp\{\alpha_1\}, u\exp\{\beta_1\}) = 0$, for some $|z| = |u| = 1$.*

c) *$\lambda - q''(z\exp\{\alpha_2\}, u\exp\{\beta_2\}) = 0$, for some $|z| = |u| = 1$.*

d) *$\mathsf{ind}_{|u|=1+0}\big(\lambda - Q(\exp\{\alpha_1\}, u\exp\{\beta_1\})\big) \neq 0$.*

e) *$\mathsf{ind}_{|z|=1+0}\big(\lambda - Q(z\exp\{\alpha_2\}, \exp\{\beta_2\})\big) \neq 0$.*

f) *For some* $|z| = 1$ *the equation* $\lambda - Q(z\exp\{\alpha_1\}, u\exp\{\beta_1\}) = 0$ *has a root* $|u(z)| < 1$ *with* $\lambda - q'(z\exp\{\alpha_1\}, u(z)\exp\{\beta_1\}) = 0$.

g) *For some* $|u| = 1$ *the equation* $\lambda - Q(z\exp\{\alpha_2\}, u\exp\{\beta_2\}) = 0$ *has a root* $|z(u)| < 1$ *with* $\lambda - q'(z(u)\exp\{\alpha_2\}, u\exp\{\beta_2\}) = 0$.

The next theorem shows the existence of a measure μ_0 for which α_{int} is determined by the essential spectrum of P in the space $\ell^1(\mu_0)$.

Theorem 3.5. *There exists* $\mu_0 \in \mathcal{M}$, *such that the space* $\ell^1(\mu_0)$ *is sufficient to determine both* α_{int} *and* $\alpha(\mathrm{L})$ *for any* $\mathrm{L} \in \mathcal{F}(\mathrm{L}_0)$, *more exactly*

$$\alpha_{int} = -\log\sup\{|\lambda| : \lambda \in \boldsymbol{\sigma}_{\mathbf{ess}}(P^{\mathrm{L}}, \ell^1(\mu_0))\}$$

and

$$\alpha(\mathrm{L}) = -\log\sup\{|\lambda| : \lambda \in \boldsymbol{\sigma}(P^{\mathrm{L}}, \ell^1(\mu_0)), \lambda \neq 1\}.$$

4. Proof of Theorem 3.1

4.1. Proofs

By virtue of Theorem 2.14 we only have to perform some explicit calculations, this means that we only have to show

$$\inf_{x\neq 0}\liminf_{\tau\to\infty}\frac{1}{\tau}\mathcal{L}^0_{x,x,\tau} = \inf_{x\neq 0}\inf_{\tau>0}\frac{1}{\tau}\mathcal{L}_\tau(\varphi_{x,\tau}) = -H(\alpha_0). \tag{4.1}$$

The second equality follows immediately from the explicit expressions for the action functionals. Indeed, for any $x \neq 0$ and any $\tau > 0$

$$\mathcal{L}_\tau(\varphi_{x,\tau}) = \tau\sup_\alpha\{-H(\alpha)\} = -\tau H(\alpha_0).$$

Theorem 4.1. (4.1) *holds.*

The proof is given in §4.2.

Proof of Theorem 3.1. This follows from Theorems 2.14 and 4.1. The fact that all functions $F^{\mathrm{L}}_{00}(z)$ do not only have a non-pole singularity at r_0, but in fact an algebraic singularity follows easily by using generating functions as in §6.2. □

4.2. Optimisation of the action functionals

The calculation of the action functionals is based on using a change of measure. As an illustration we will explicitly calculate the action functional $\mathcal{L}_\tau(\varphi_{x,\tau})$ for x sufficiently large, because the method in this simple case is typical for all similar calculations.

Introduce a family of random walks $\mathrm{L}^\alpha_0 = \{\xi^\alpha_t\}_t$, $\alpha \in \mathbf{R}$, with transition probabilities

$$P^\alpha_{ij} = \frac{P_{ij}\exp\{\alpha(j-i)\}}{\sum_k P_{ik}\exp\{\alpha(k-i)\}}$$

and denote the corresponding distributions and expectations by P_α and E_α respectively. Then

$$M_i(\alpha) = \frac{d}{d\alpha} H_i(\alpha)$$

is the mean drift from point i for $\mathcal{L}^\alpha$ and L_0 is ergodic if and only if $M_{k'}(0) < 0$. $\mathrm{L}_0 = \mathrm{L}_0^0$ for $\alpha = 0$ and we will suppress the dependence on α in our notation in this case.

The Markov chain L_0^α is a zero drift chain only for $\alpha = \alpha_0$: within the family L_0^α the probability that the scaled process is close to x during time τ is largest under this change of measure. Denote

$$B_{N,\tau,\delta,x} = \Big\{ \sup_{t=0,\dots,[N\tau]} \big|\xi_t - [xN]\big| < \delta N \Big\}$$

and let $I_{B_{N,\tau,\delta,x}}$ be the indicator of this event. Consider the random walk starting at $[xN]$. Then

$$\mathsf{P}\{B_{N,\tau,\delta,x}\} = \mathsf{E}_\alpha I_{B_{N,\tau,\delta,x}} \exp\Big\{ -\alpha\xi_{[N\tau]} + \alpha\xi_0 + \sum_{t=0}^{[N\tau]-1} H_{\xi_t}(\alpha)\Big\}.$$

This implies that

$$\mathsf{P}\{B_{N,\tau,\delta,x}\} \le \Big[\exp\{\alpha\delta N + [N\tau]H(\alpha)\}\, \mathsf{E}_\alpha I_{B_{N,\tau,\delta,x}}\Big],$$

for all α, and in particular for $\alpha = \alpha_0$. We thus obtain

$$\mathsf{P}\{B_{N,\tau,\delta,x}\} \le \exp\{\alpha_0\delta N + [N\tau]H(\alpha_0)\}, \tag{4.2}$$

provided that $\tau < x/l$. This condition on τ ensures that with probability 0 the set $\{0, \dots, k'-1\}$ is hit from $[xN]$ before time $[\tau N]$. For the lower bound, note that

$$\mathsf{E}_{\alpha_0}\{\xi_{t+1} - \xi_t \,|\, \xi_t = i\} = M(\alpha_0) = 0, \quad i > k'.$$

From the law of large numbers it follows that for some constant $c > 0$

$$\mathsf{E}_{\alpha_0} I_{B_{N,\tau,\delta,x}} \ge c\exp\{-\delta N\},$$

so that

$$\mathsf{P}\{B_{N,\tau,\delta,x}\} \ge c\exp\big\{ -\delta N - \delta|\alpha_0| + [N\tau]H(\alpha_0)\big\}.$$

This shows that

$$\log \mathsf{P}\{B_{N,\tau,\delta,x}\} \sim N\tau H(\alpha_0), \quad N \to \infty, \tag{4.3}$$

and we have proved that

$$\mathcal{L}_\tau(\varphi_{x,\tau}) = -\tau H(\alpha_0), \quad x > x(\tau),$$

for $x(\tau) = \tau \cdot l$.

For the proof of Theorem 4.1 we will prove a simple statement on the existence and structure of optimal τ-paths.

Lemma 4.2. *For any $x, y \neq 0$ the optimal τ-path φ from x to y is either linear*

$$\varphi(t) = x + \frac{t}{\tau}(y - x)$$

or it is piecewise linear, more exactly, there are $\tau_1 \leq \tau_2 \leq \tau$, such that

$$\varphi(t) = \begin{cases} x\dfrac{\tau_1 - t}{\tau_1}, & t \leq \tau_1; \\ 0, & \tau_1 \leq t \leq \tau_2; \\ y\dfrac{t - \tau_2}{\tau - \tau_2}, & t \geq \tau_2. \end{cases}$$

Proof. Let $x, y \neq 0$ and consider a path $\varphi' \in \Phi^0_{x,y,\tau}$. Let $\varphi \in \Phi^0_{x,y,\tau}$ be a linear path (with constant speed) and let its action functional be given by

$$\mathcal{L}_\tau(\varphi) = \tau \sup_\alpha \big\{ -H(\alpha) + \frac{y - x}{\tau}\alpha\big\} = -\tau H(\alpha') + (y - x)\alpha'.$$

Then

$$\begin{aligned} \mathcal{L}_\tau(\varphi') &= \int_0^\tau \mathsf{L}(\dot\varphi'(t))\, dt \\ &\geq \int_0^\tau \big(-H(\alpha') + \alpha'\dot\varphi'(t)\big)\, dt \\ &= -\tau H(\alpha') + (y - x)\alpha' \\ &= \mathcal{L}_\tau(\varphi), \end{aligned}$$

since the integral

$$\int_0^\tau \alpha'\dot\varphi'(t)\, dt$$

does not depend on the particular path, but only on its starting and end points. The assertion of the Lemma easily follows. □

Proof of Theorem 4.1. The previous lemma easily implies that amongst all paths $\varphi \in \Phi^0_{x,x,\tau}$ the path $\varphi_{x,\tau}$ identically x, $x \neq 0$, has the lowest action functional. □

Due to the convexity of $H(\alpha)$ the equation $H(\alpha) = 0$ has two real roots $\alpha_1 < 0 < \alpha_2$. The logarithmic asymptotics of $\pi_{[xN]}$ are well-known:

$$\log \pi_{[xN]} \sim -\alpha_2 x N, \quad N \to \infty.$$

Consequently the logarithmic asymptotics of the stationary probabilities are not related to α_{int}. This is not surprising: α_{int} is determined by the probability mass on the scaled path identically x, and the stationary probabilities by the probability mass on the scaled optimal path from 0 to the point x.

5. Proof of Theorem 3.3

5.1. Proofs

By virtue of Theorem 2.14

$$\alpha_{\text{int}} = \inf_{x\neq 0} \liminf_{\tau\to\infty} \frac{1}{\tau}\mathcal{L}^0_{x,x,\tau}. \tag{5.1}$$

So the only thing left, is to explicitly calculate the minimum time-scaled action functionals of closed paths. In particular we shall prove that this minimum is achieved by paths equal to a point, i.e.

$$\begin{aligned}\inf_{x\neq 0} \liminf_{\tau>0} \frac{1}{\tau}\mathcal{L}^0_{x,x,\tau} &= \inf_{x\neq 0}\inf_{\tau>0} \frac{1}{\tau}\mathcal{L}_\tau(\varphi_{x,\tau}) \\ &= -\max\{\mathcal{H}_{\{1\}}(\alpha^0_{\{1\}}), \mathcal{H}_{\{2\}}(\beta^0_{\{2\}})\}.\end{aligned} \tag{5.2}$$

The second equality immediately follows from the explicit expressions for the action functionals. To prove the first equality we have to solve a complicated variational problem.

Theorem 5.1. (5.2) *holds.*

The proof will be given in §5.2.

Proof of Theorem 3.3. This follows from Theorems 2.14 and 5.1. □

5.2. Optimisation of the action functionals

The action functionals are calculated by introducing the family of random walks $\mathrm{L}_0^{\alpha,\beta} = \{\xi_t^{\alpha,\beta}\}_t$, α, $\beta \in \mathbf{R}$ with transitions probabilities

$$P^{\alpha,\beta}_{ij,kl} = \frac{P_{ij,kl}\exp\{\alpha(k-i)+\beta(l-j)\}}{\sum_{k'l'} P_{ij,k'l'}\exp\{\alpha(k'-i)+\beta(l'-j)\}}.$$

The corresponding distributions and expectations denoted by $\mathsf{P}_{\alpha,\beta}$ and $\mathsf{E}_{\alpha,\beta}$.

There are only four different mean jump vectors, depending on the face. Only the drifts from points on the faces $\{1,2\}$, $\{1\}$ and $\{2\}$ are of interest:

$$\begin{aligned}M(\alpha,\beta) &= (M_x(\alpha,\beta), M_y(\alpha,\beta)) &= (\frac{\partial}{\partial\alpha}H_\alpha(\alpha,\beta), \frac{\partial}{\partial\beta}H_\beta(\alpha,\beta)),\\ M'(\alpha,\beta) &= (M'_x(\alpha,\beta), M'_y(\alpha,\beta)) &= (\frac{\partial}{\partial\alpha}h_{1,}(\alpha,\beta), \frac{\partial}{\partial\beta}h_1(\alpha,\beta)),\\ M''(\alpha,\beta) &= (M''_x(\alpha,\beta), M''_y(\alpha,\beta)) &= (\frac{\partial}{\partial\alpha}h_2(\alpha,\beta), \frac{\partial}{\partial\beta}h_2(\alpha,\beta)).\end{aligned}$$

Then $(\alpha,\beta) = (\alpha_0,\beta_0)$ is a point for which $\mathrm{L}_0^{\alpha,\beta}$ has zero drift $M(\alpha,\beta) = 0$ in the interior of the quarter plane. Further, $(\alpha,\beta) = (\alpha^0_{\{1\}}, \beta(\alpha^0_{\{1\}}))$ is the unique point for which $\mathrm{L}_0^{\alpha,\beta}$ has the property:

i) if $H(\alpha,\beta) > h_1(\alpha,\beta)$, then $M(\alpha,\beta) = 0$.

ii) if $H(\alpha,\beta) = h_1(\alpha,\beta)$, then the induced chain perpendicular to the face $\{1\}$ is ergodic. The long run drift $v^{\{1\}}$ from points on the face $\{1\}$ equals 0 (cf. [6]), i.e. for π_1 the stationary distribution of the induced chain on $\mathbf{Z}^+$ with transition matrix $P_1^{\alpha,\beta}$

$$v^{\{1\}} = \pi_{1,0} M'(\alpha,\beta) + (1-\pi_{1,0}) M(\alpha,\beta) \equiv 0.$$

For the point $(\alpha(\beta^0_{\{2\}}), \beta^0_{\{2\}})$ the analogous interpretation holds.

The action functionals for paths $\varphi_{x,\tau}$ identically x, $x \neq 0$, are easily calculated.

$$\lim_{N\to\infty} \frac{1}{N} \log \mathsf{P}\{B_{N,\tau,\delta,x} \,|\, \xi_0 = [xN]\} = \begin{cases} \tau H(\alpha_0,\beta_0), & x_1, x_2 > 0; \\ \tau \mathcal{H}_{\{1\}}(\alpha^0_{\{1\}}), & x_1 > 0, x_2 = 0; \\ \tau \mathcal{H}_{\{2\}}(\beta^0_{\{2\}}), & x_1 = 0, x_2 > 0, \end{cases} \tag{5.3}$$

where the sets $B_{N,\tau,\delta,x}$ were defined in §4.2. However, contrary to the case of one-dimensional random walks, the path $\varphi_{x,\tau}$ does not have largest probability amongst all paths $\varphi \in \Phi_{x,x,\tau}$ for *arbitrary* $x \neq 0$.

The proof of Theorem 5.1 requires a number of steps. First of all we shall prove that we need only consider τ-paths that are linear inside faces. To calculate the optimal τ-path from a point x to a point y is complicated and not necessary.

Lemma 5.2. *Let $\varphi \in \Phi_{x,y,\tau}$ and let τ_k be successive times that φ enters or leaves a 1-face or the $\emptyset$-face. Then $\mathcal{L}_\tau(\varphi) \geq \mathcal{L}_\tau(\varphi')$, where φ' is given by*

$$\varphi^k(t) = \varphi(\tau_k) + \frac{t-\tau_k}{\tau_{k+1}-\tau_k}\big(\varphi(\tau_{k+1}) - \varphi(\tau_k)\big), \quad \tau_k \leq t \leq \tau_{k+1}.$$

Proof. This follows in a similar way as the proof of Lemma 4.2, but for completeness we will give the details. Let $\varphi^k(t) = \varphi(\tau_k + t)$, $0 \leq t \leq \tau_{k+1} - \tau_k$ and suppose that $\varphi_k \subset \{1,2\}$ a.s. The other cases are similarly proved. Further denote by φ^* the linear path

$$\varphi^*(t) = \varphi(\tau_k) + \frac{t-\tau_k}{\tau_{k+1}-\tau_k}\big(\varphi(\tau_{k+1}) - \varphi(\tau_k)\big), \quad 0 \leq t \leq \tau_{k+1} - \tau_k,$$

and the parameters of the associated action functional by $(\alpha^*,\, \beta^*)$. Then

$$\begin{aligned} \mathcal{L}_{\tau_{k+1}-\tau_k}(\varphi^k) &= \int_0^{\tau_{k+1}-\tau_k} \mathsf{L}(\dot\varphi^k(t))\,dt \\ &\geq \int_0^{\tau_{k+1}-\tau_k} \big(-H(\alpha^*,\beta^*) + \alpha^* \dot\varphi_1^k(t) + \beta^* \dot\varphi_2^k(t)\big)\,dt \end{aligned}$$

$$\begin{aligned} &= -(\tau_{k+1}-\tau_k)H(\alpha^*,\beta^*) + \alpha^*(\varphi_1(\tau_{k+1}) - \varphi_1(\tau_k)) \\ &\quad + \beta^*(\varphi_2(\tau_{k+1}) - \varphi_2(\tau_k)) \\ &= \mathcal{L}_{\tau_{k+1}-\tau_k}(\varphi^*), \end{aligned}$$

since the integral

$$\int_0^{\tau_{k+1}-\tau_k} \big(\alpha^* \dot\varphi_1^k(t) + \beta^* \dot\varphi_2^k(t)\big)\, dt$$

does not depend on the particular path, but only on its starting and end points. □

Next we shall estimate the action functionals of closed paths. This is simple for closed paths crossing at most two faces with positive measure.

Lemma 5.3. *Let $\varphi \in \Phi^0_{x,x,\tau}$ be such that a.s. $\varphi \subset \{1,2\} \cup \{1\}$. Then $\mathcal{L}_\tau(\varphi) \geq -\tau \mathcal{H}_{\{1\}}(\alpha^0_{\{1\}})$. Similarly, if $\varphi \subset \{1,2\} \cup \{2\}$ a.s., then $\mathcal{L}_\tau(\varphi) \geq -\tau\mathcal{H}_{\{2\}}(\beta^0_{\{2\}})$.*

Proof. Let $\varphi \subset \{1,2\} \cup \{1\}$ a.s. For any $v \in \mathbf{R}^2$ and any $v_1 \in \mathbf{R}$

$$\begin{aligned} \mathsf{L}(v) &\geq -\mathcal{H}_{\{1\}}(\alpha^0_{\{1\}}) + \alpha^0_{\{1\}} v_1 + \beta(\alpha^0_{\{1\}}) v_2 \\ \mathsf{L}_{\{1\}}(v_1) &\geq -\mathcal{H}_{\{1\}}(\alpha^0_{\{1\}}) + \alpha^0_{\{1\}} v_1. \end{aligned}$$

The required estimate follows in the same way as the proof of Lemma 5.2. □

Next we consider the simplest closed paths that pass all three non-empty faces with positive measure, namely triangle paths. A triangle path is defined by five positive real parameters x, y, τ_k, $k = 1,2,3$ and its orientation. Paths with negative orientation are defined by

$$\varphi^-_{x,y,\tau_1,\tau_2,\tau_3}(t) = \begin{cases} \big(x\frac{\tau_1 - t}{\tau_1}, 0\big), & t \leq \tau_1; \\ \big(0, y\frac{t-\tau_1}{\tau_2}\big), & \tau_1 \leq t \leq \tau_1 + \tau_2; \\ \big(x\frac{t-\tau_1-\tau_2}{\tau_3}, y\frac{\tau_1+\tau_2+\tau_3-t}{\tau_3}\big), & \tau_1+\tau_2 \leq t \leq \tau_1+\tau_2+\tau_3. \end{cases}$$

Define triangle paths $\varphi^+_{x,y,\tau_1,\tau_2,\tau_3}$ with positive orientation in a similar way. For any positive constant $c > 0$

$$\frac{1}{\tau}\mathcal{L}_\tau(\varphi^-_{x,y,\tau_1,\tau_2,\tau_3}) = \frac{1}{2\tau}\mathcal{L}_{2\tau}(\varphi^-_{2x,2y,2\tau_1,2\tau_2,2\tau_3}),$$

where $\sum \tau_k = \tau$, and so it suffices to consider only triangle paths with $x+y=1$.

Lemma 5.4.

$$\frac{1}{\tau}\mathcal{L}_\tau(\varphi^{-,+}_{x,y,\tau_1,\tau_2,\tau_3}) \geq -\max\{\mathcal{H}_{\{1\}}(\alpha^0_{\{1\}}), \mathcal{H}_{\{2\}}(\beta^0_{\{2\}})\},$$

for any $x, y > 0$ with $x + y = 1$ and any $\tau_1, \tau_2, \tau_3 > 0$, where $\tau = \sum \tau_k$.

The next lemma shows that all non triangle paths crossing all three non-empty faces with positive measure, have bigger time-scaled action functionals than the minimal triangle path.

Lemma 5.5. *For any $\tau < \infty$ and any $x \neq 0$ it holds that for any non-triangle path $\varphi \in \Phi^0_{x,x,\tau}$ crossing all faces $\{1\}$, $\{2\}$ and $\{1,2\}$ with positive Lebesgue measure and $\{1,2\}$ at least twice*

$$\mathcal{L}_\tau(\varphi) \geq \inf_{x\neq 0} \inf_{\tau>0} \mathcal{L}^0_{x,x,\tau}.$$

For the proof of Lemmas 5.4 and 5.5 we will need a sequence of preparatory lemmas.

Introduce the family of random walks $\mathrm{L}^\mu_0 = \{\xi^\mu_t\}_t$, $\mu \in \mathcal{M}$, with transition probabilities

$$P^\mu_{xy} = \frac{P_{xy}\dfrac{\mu(y)}{\mu(x)}}{\displaystyle\sum_y P_{xy}\frac{\mu(y)}{\mu(x)}} \ ,$$

and denote the corresponding distributions and expectations by P_μ and E_μ respectively. Define

$$H_x(\mu) = \log \sum_y P_{xy} \frac{\mu(y)}{\mu(x)}.$$

Next define a subset $\mathcal{M}^0$ of measures in $\mathcal{M}$. We say that $\mu \in \mathcal{M}^0$ if and only if $\mu(x) = \exp\{\alpha(x) + \beta(x)\}$ for a continuous vector field $(\alpha(x), \beta(x))$ on $\mathbf{R}^2_+ \setminus \{0\}$ satisfying the following four conditions.

i) $(\alpha(x), \beta(x)) = (\alpha_1, \beta_1)$ for $x_2/x_1 \leq c_1$, for some $c_1 > 0$, and some α_1, β_1.

ii) $(\alpha(x), \beta(x)) = (\alpha_2, \beta_2)$ for $x_1/x_2 \leq c_2$, for some $c_2 > 0$ with $1/c_2 > c_1$, and some α_2, β_2.

iii) $(\alpha(x), \beta(x)) = (\alpha(y), \beta(y))$, $x_1 y_2 = x_2 y_1$.

iv) $\alpha(x)$, $\beta(x)$ are continuously differentiable with

$$\frac{\partial}{\partial x_2}\alpha(x) = \frac{\partial}{\partial x_1}\beta(x),$$

except for x with $x_2/x_1 = c_1$, $1/c_2$, and possibly for x with $x_2/x_1 = c_3$, for some constant c_3 with $1/c_2 > c_3 > c_1$. For such points x we assume that the "left" and "right" derivatives exist, i.e.

$$\frac{\partial^+}{\partial x_k}\alpha(x) = \lim_{\substack{y\to x,\\ y_2x_1>x_2y_1}} \frac{\alpha(y)-\alpha(x)}{y_k - x_k},$$

etc. are all finite and

$$\frac{\partial^{+(-)}}{\partial x_2}\alpha(x) = \frac{\partial^{+(-)}}{\partial x_1}\beta(x).$$

Lemma 5.6. *Suppose that there exists a measure $\mu \in \mathcal{M}^0$, such that*

$$H(\alpha(x), \beta(x)) \le \mathcal{H}\,,$$

for all x, and

$$h_1(\alpha_1, \beta_1), h_2(\alpha_2, \beta_2) \le \mathcal{H}$$

for some constant $\mathcal{H}$. Then the following assertions hold.

i) $\mathcal{L}^0_{x,y,\tau} \ge -\tau\mathcal{H} + \alpha(y)y_1 + \beta(y)y_2 - \alpha(x)x_1 - \beta(x)x_2$, *for any x, y and τ.*

ii) *For any $\varepsilon > 0$ there exist a finite set $A(\varepsilon)$ and a constant $N(\varepsilon)$, such that*

$$-\frac{1}{N}\log \mathsf{P}\{B_{A(\varepsilon),N,x,\tau}\} \ge -\tau\mathcal{H} - \alpha(x)x_1 - \beta(x)x_2 - \varepsilon, \quad N \ge N(\varepsilon),$$

with

$$B_{A,N,x,\tau} = \{\xi_{[\tau N]}([xN]) \in A, \xi_t([xN]) \notin A, t = 0, \dots, [\tau N] - 1\}.$$

Proof. The proof of i) is analogous to the proof of Lemma 5.2 by noting that the set of measures $\mathcal{M}_0$ is constructed in such a way that

$$\int\limits_0^\tau \big(\alpha(\varphi(t))\dot\varphi_1(t) + \beta(\varphi(t))\dot\varphi_2(t)\big)\,dt$$

only depends on the particular path though its starting and end points $\varphi(0)$ and $\varphi(\tau)$, and hence it is equal to

$$\alpha(\varphi(\tau))\,\varphi_1(\tau) + \beta(\varphi(\tau))\,\varphi_2(\tau) - \alpha(\varphi(0))\,\varphi_1(0) - \beta(\varphi(0))\,\varphi_2(0).$$

For the proof of ii) we use a change of measure. Rewrite

$$\mathsf{P}\{B_{A,N,x,\tau}\} = \mathsf{E}_\mu I_{B_{A,N,x,\tau}} \frac{\mu([xN])}{\mu\big(\xi_{[\tau N]}([xN])\big)} \exp\Big\{\sum_{t=0}^{N-1} H_{\xi_t}(\mu)\Big\}. \tag{5.4}$$

Next write $\gamma_x = x_2/x_1$. We consider $(\alpha(x), \beta(x))$ as functions of $\gamma \in [0, \infty]$ and we write $(\alpha(\gamma), \beta(\gamma))$. Then condition iv) in the definition of $\mathcal{M}^0$ is equivalent to

$$\alpha'(\gamma) = -\gamma\beta'(\gamma), \tag{5.5}$$

for $\gamma \neq 1/c_2$, c_1, c_3. At such points we have

$$\frac{d^{+(-)}}{d\gamma}\alpha(\gamma) = -\gamma \frac{d^{+(-)}}{d\gamma}\beta(\gamma).$$

This implies that $\alpha(\gamma)$ is increasing if $\beta(\gamma)$ is decreasing and vice versa. Then for $\gamma \neq 1/c_2$, c_1, c_3,

$$\begin{aligned}
&\lim_{N\to\infty} P_{[xN],[xN]+y} \exp\left\{\alpha(\gamma_{[xN]+y})([x_1N]+y_1) + \beta(\gamma_{[xN]+y})([x_2N]+y_2)\right. \\
&\qquad\qquad \left. - \alpha(\gamma_[xN])[x_1N] - \beta(\gamma_{[xN]}[x_2N]\right\} \\
&= \ p_y \exp\{\big(\alpha(\gamma) - \gamma\alpha'(\gamma) - \gamma^2\beta'(\gamma)\big)y_1 + \big(\beta(\gamma) + \alpha'(\gamma) + \gamma\beta'(\gamma)\big)y_2\} \\
&= \ p_y \exp\{\alpha(\gamma)y_1 + \beta(\gamma)y_2\},
\end{aligned}$$

for $x \in \{1,2\}$. For $x \in \{1\}$ or $x \in \{2\}$ the same result hold, as well as for $\gamma = 1/c_2$, c_1 or c_3. As a consequence

$$\lim_{N\to\infty} H_{[yN]}(\mu) = \begin{cases} H(\alpha(y), \beta(y)), & y \in \{1,2\}; \\ h_1(\alpha_1, \beta_1), & y \in \{1\}; \\ h_2(\alpha_2, \beta_2), & y \in \{2\}. \end{cases}$$

This implies for any $\varepsilon > 0$ the existence of a finite set $A(\varepsilon)$, such that $|H_y(\mu) - \mathcal{H}| < \varepsilon/2$ for $y \notin A(\varepsilon)$. Choose $N(\varepsilon)$ such that for any $y \in A(\varepsilon)$

$$\left|\frac{1}{N}\log\mu(y)\right| \leq \varepsilon/2, \quad N \geq N(\varepsilon).$$

Using (5.4) yields

$$-\frac{1}{N}\log \mathsf{P}\{B_{A(\varepsilon),N,x,\tau}\} \geq -\mathcal{H} - \alpha(x)x_1 - \beta(x)x_2 - \varepsilon, \quad N \geq N(\varepsilon).$$

□

The next lemma gives conditions under which measures satisfying the assumptions of Lemma 5.6 can be constructed.

Lemma 5.7. *Let* (α_1, β_1), (α_2, β_2) *and* $\mathcal{H}$ *be such that*

i) $h_1(\alpha_1, \beta_1)$, $H(\alpha_1, \beta_1) \leq \mathcal{H}$;

ii) $h_2(\alpha_2, \beta_2)$, $H(\alpha_2, \beta_2) \leq \mathcal{H}$;

iii) *one of the properties* a), b), c) *and* d) *holds:*

  a) $\alpha_1 < \alpha_2$, $\beta_1 > \beta_2$;

  b) $\alpha_1 > \alpha_2$, $\beta_1 < \beta_2$;

  c) $\alpha_1 < \alpha_2$, $\beta_1 < \beta_2$ *and* $H(\alpha_1, \beta_2) < \mathcal{H}$ *or* $H(\alpha_2, \beta_1) < \mathcal{H}$;

d) $\alpha_1 > \alpha_2$, $\beta_1 > \beta_2$ *and* $H(\alpha_1, \beta_2) < \mathcal{H}$ *or* $H(\alpha_2, \beta_1) < \mathcal{H}$.

Then there exists a measure $\mu \in \mathcal{M}^0$ *with*

$$(\alpha(x), \beta(x)) = (\alpha_1, \beta_1)$$

for $x \in \mathbf{R}_+^2$ *with* $x_2 = 0$,

$$(\alpha(x), \beta(x)) = (\alpha_2, \beta_2)$$

for $x \in \mathbf{R}_+^2$ *with* $x_1 = 0$ *and*

$$H(\alpha(x), \beta(x)) \leq \mathcal{H}$$

for any $x \in \mathbf{R}_+^2$.

Proof. Let us first assume that a) holds. Write $\gamma_x = x_2/x_1$ and consider $(\alpha(x), \beta(x))$ as functions of $\gamma \in [0, \infty]$. If $\alpha(\gamma)$ is strictly increasing in γ we can express $\beta(\gamma)$ locally as a function $\beta'(\alpha)$ of α. It follows that the following relation should hold:

$$\frac{d}{d\alpha}\beta'(\alpha(\gamma)) = -\frac{1}{\gamma}. \tag{5.6}$$

To achieve this, we need $\beta'(\alpha)$ to be a concave, strictly decreasing function of α. Note that the straight line connecting any two points contained in the set $\{(\alpha, \beta) : H(\alpha, \beta) \leq \mathcal{H}\}$ is also contained in this set by the structure of the function H. This applies in particular for the points (α_1, β_1) and (α_2, β_2). Clearly any linear, strictly decreasing function of α can be perturbed into a concave, strictly decreasing function of α.

Our construction is as follows. First construct a continuously differentiable, concave, strictly decreasing function $\beta'(\alpha)$ with $\beta'(\alpha_1) = \beta_1$, $\beta'(\alpha_2) = \beta_2$, $(d/d\alpha)\beta'(\alpha_1) < 0$ and $(d/d\alpha)\beta'(\alpha_2) > -\infty$. Then we put

$$(\alpha(\gamma), \beta(\gamma)) = \begin{cases} (\alpha_1, \beta_1), & \gamma^{-1} \geq -\dfrac{d}{d\alpha}\beta'(\alpha_1); \\ (\alpha, \beta'(\alpha)), & \gamma^{-1} = -\dfrac{d}{d\alpha}\beta'(\alpha), \alpha \in (\alpha_1, \alpha_2); \\ (\alpha_2, \beta_2), & \gamma^{-1} \leq -\dfrac{d}{d\alpha}\beta'(\alpha_2). \end{cases}$$

The constructed measure is in $\mathcal{M}^0$.

Next assume that c) holds and that $H(\alpha_1, \beta_2) < \mathcal{H}$. Then there exists (α', β') with $\alpha' < \alpha_1$, $\beta' > \beta_2$ and $H(\alpha', \beta') < \mathcal{H}$. Along the piecewise linear path in $\mathbf{R}^2$ determined by

$$(\alpha_1, \beta_1) \to (\alpha', \beta') \to (\alpha_2, \beta_2)$$

α increases if and only if β decreases. Define a continuously differentiable convex, decreasing function $\beta'(\alpha)$ on $[\alpha',\alpha_1]$ and a continuously differentiable concave, decreasing function $\beta''(\alpha)$ on $[\alpha',\alpha_2]$, such that $\beta'(\alpha')=\beta''(\alpha')=\beta'$, $\beta'(\alpha_1)=\beta_1$, $\beta''(\alpha_2)=\beta_2$, $(d/d\alpha)\beta'(\alpha')\le(d/d\alpha)\beta''(\alpha')$, $(d/d\alpha)\beta'(\alpha_1)>-\infty$ and $(d/d\alpha)\beta''(\alpha_2)<0$. Then put

$$(\alpha(\gamma),\beta(\gamma))=\begin{cases}(\alpha_1,\beta_1), & \gamma^{-1}\ge -\frac{d}{d\alpha}\beta'(\alpha_1);\\ (\alpha,\beta'(\alpha)), & \gamma^{-1}=-\frac{d}{d\alpha}\beta'(\alpha),\ \alpha\in(\alpha',\alpha_1);\\ (\alpha',\beta'), & -\frac{d}{d\alpha}\beta'(\alpha')\ge\gamma^{-1}\ge-\frac{d}{d\alpha}\beta''(\alpha');\\ (\alpha,\beta''(\alpha)), & \gamma^{-1}=-\frac{d}{d\alpha}\beta''(\alpha),\ \alpha\in(\alpha',\alpha_2);\\ (\alpha_2,\beta_2) & \gamma^{-1}\le-\frac{d}{d\alpha}\beta''(\alpha_2).\end{cases}$$

The constructed measure is an element of $\mathcal{M}^0$.

If b) or d) holds, then the construction is similar, only we should consider $(\alpha(x),\beta(x))$ as functions of $\gamma'_x=x_1/x_2$. □

Proof of Lemma 5.4. We will prove the assertion for triangle paths with negative orientation. Assume that

$$\inf_{\substack{x,y,\tau_1,\tau_2,\tau_3>0:\\ x+y=1}}\frac{1}{\tau}\mathcal{L}_\tau(\varphi^-_{x,y,\tau_1,\tau_2,\tau_3}) \ <\ -\max\{\mathcal{H}_{\{1\}}(\alpha^0_{\{1\}}),\mathcal{H}_{\{2\}}(\beta^0_{\{2\}})\}, \tag{5.7}$$

where $\tau=\sum\tau_k$. Further more, assume the existence of x', y', τ'_1, τ'_2, τ'_3 with $\sum\tau'_k<\infty$, such that

$$\frac{1}{\tau'}\mathcal{L}_{\tau'}(\varphi^-_{x',y',\tau'_1,\tau'_2,\tau'_3}) \ =\ \inf_{\substack{x,y,\tau_1,\tau_2,\tau_3>0:\\ x+y=1}}\frac{1}{\tau}\mathcal{L}_\tau(\varphi^-_{x,y,\tau_1,\tau_2,\tau_3}), \tag{5.8}$$

where $\tau'=\sum\tau'_k$. Write

$$\begin{aligned} f_1(x,\tau_1) &= \sup_\alpha\big(-\mathcal{H}_{\{1\}}(\alpha)-(x/\tau_1)\alpha\big),\\ f_2(y,\tau_2) &= \sup_\beta\big(-\mathcal{H}_{\{2\}}(\beta)+(y/\tau_2)\beta\big),\\ f_3(x,y,\tau_3) &= \sup_{\alpha,\beta}\big(-H(\alpha,\beta)+(x/\tau_3)\alpha-(y/\tau_3)\beta\big)\end{aligned} \tag{5.9}$$

and

$$f(x,y,\tau_1,\tau_2,\tau_3)=\frac{\tau_1}{\tau}f_1(x,\tau_1)+\frac{\tau_2}{\tau}f_2(y,\tau_2)+\frac{\tau_3}{\tau}f_3(x,y,\tau_3).$$

Then

$$\frac{1}{\tau}\mathcal{L}_\tau(\varphi^-_{x,y,\tau_1,\tau_2,\tau_3})=f(x,y,\tau_1,\tau_2,\tau_3).$$

Consider the function $f_1(x,\tau_1)$. The value $\alpha(x,\tau_1)$ for which the supremum in (5.9) is attained, is uniquely given by

$$-\frac{d}{d\alpha}\mathcal{H}_{\{1\}}(\alpha(x,\tau_1)) = \frac{x}{\tau_1}.$$

$\alpha(x,\tau_1)$ is continuously differentiable in x and $\tau_1 > 0$, and

$$f_1(x,\tau_1) = -\mathcal{H}_{\{1\}}(\alpha(x,\tau_1)) - \frac{x}{\tau_1}\alpha(x,\tau_1).$$

It easily follows that $f_1(x,\tau_1) \to \infty$ as $\tau_1 \to 0$ and

$$f_1(x,\tau_1) \to -\mathcal{H}_{\{1\}}(\alpha^0_{\{1\}})$$

as $\tau_1 \to \infty$. Similarly we have that $f_2(y,\tau_2) \to \infty$ as $\tau_2 \to 0$ and

$$f_2(y,\tau_2) \to -\mathcal{H}_{\{2\}}(\beta^0_{\{2\}})$$

as $\tau_2 \to \infty$. There is a unique value $\beta(y,\tau_2)$ for which the supremum in the second equation in (5.9) is attained. This defines a continuously differentiable function $\beta(y,\tau_2)$ for $\tau_2 > 0$, $y > 0$. Finally $f_3(x,y,\tau_3) \to \infty$ as $\tau_3 \to 0$ and

$$f_3(x,y,\tau_3) \to -H(\alpha_0,\beta_0)$$

as $\tau_3 \to \infty$ and there is a continuously differentiable vector function $(\alpha(x,y,\tau_3), \beta(x,y,\tau_3))$ for which the maximum in the third equation in (5.9) is attained.

Since by assumption the infimum is attained at a finite point $(x',y',\tau_1',\tau_2',\tau_3')$, we must have that the partial derivatives of f equal 0 (the boundary restriction on x,y play no role obviously). Denote $\partial_{\tau_1} f = (\partial/\partial\tau_1)f$, etc., and write

$$\alpha' = \alpha(x',\tau_1'), \quad \beta' = \beta(y',\tau_2'), \quad (\alpha'',\beta'') = (\alpha(x',y',\tau_3'), \beta(x',y',\tau_3')).$$

It follows that

$$\begin{aligned}
&\partial_{\tau_1} f(x,y,\tau_1,\tau_2,\tau_3)\\
&\qquad = \frac{\tau_2+\tau_3}{\tau^2}\Big\{-\mathcal{H}_{\{1\}}(\alpha(x,\tau_1) + \frac{x}{\tau_2+\tau_3}\alpha(x,\tau_1)\\
&\qquad\qquad - \frac{\tau_2}{\tau_2+\tau_3}f_2(y,\tau_2) - \frac{\tau_3}{\tau_2+\tau_3}f_3(x,y,\tau_3)\Big\}.
\end{aligned}$$

Thus $\partial_{\tau_1} f(x',y',\tau_1',\tau_2',\tau_3') = 0$ implies

$$f(x',y',\tau_1',\tau_2',\tau_3') = -\mathcal{H}_{\{1\}}(\alpha'), \tag{5.10}$$

and

$$\begin{aligned}
&-\mathcal{H}_{\{1\}}(\alpha') + \frac{x'}{\tau_2'+\tau_3'}\alpha'\\
&\quad = \frac{\tau_2'}{\tau_2'+\tau_3'}\Big(-\mathcal{H}_{\{2\}}(\beta') + \frac{y'}{\tau_2'}\beta'\Big) + \frac{\tau_3'}{\tau_2'+\tau_3'}\Big(-H(\alpha'',\beta'')\\
&\qquad + \frac{x'}{\tau_3'}\alpha'' - \frac{y'}{\tau_3'}\beta''\Big).
\end{aligned} \tag{5.11}$$

Similarly, $\partial_{\tau_2} f(x', y', \tau_1', \tau_2', \tau_3') = 0$ implies

$$f(x', y', \tau_1', \tau_2', \tau_3') = -\mathcal{H}_{\{2\}}(\beta'), \tag{5.12}$$

and essentially the same equation (5.11), and $\partial_{\tau_3} f(x', y', \tau_1', \tau_2', \tau_3') = 0$ implies

$$f(x', y', \tau_1', \tau_2', \tau_3') = -H(\alpha'', \beta''), \tag{5.13}$$

and also essentially the same equation (5.11). Next we compute

$$\partial_x f(x, y, \tau_1, \tau_2, \tau_3) = \frac{1}{\tau}\Big(- \alpha(x, \tau_1) + \alpha(x, y, \tau_3)\Big),$$

so that $\partial_x f(x', y', \tau_1', \tau_2', \tau_3') = 0$ implies

$$\alpha' = \alpha''. \tag{5.14}$$

Similarly $\partial_y f(x', y', \tau_1', \tau_2', \tau_3') = 0$ implies

$$\beta' = \beta''. \tag{5.15}$$

Combination of (5.10), (5.12) and (5.13) yields

$$\mathcal{H}_{\{1\}}(\alpha') = \mathcal{H}_{\{2\}}(\beta') = H(\alpha'', \beta''). \tag{5.16}$$

Since $\nabla H(\alpha'', \beta'') = (x'/\tau_3', -y'/\tau_3')$, $H_\alpha(\alpha(\beta'), \beta') \le 0$ and $H_\beta(\alpha', \beta(\alpha')) \le 0$, the following conclusions hold by virtue of (5.16):

i) $(\alpha'', \beta'') = (\alpha', \beta(\alpha'))$ and $h_1(\alpha'', \beta'') = H(\alpha'', \beta'')$;

ii) $\alpha(\beta') < \alpha''$;

iii) $H_\alpha(\alpha'', \beta'') > 0$.

We will show that

$$H_\beta(\alpha(\beta''), \beta'') \ge 0. \tag{5.17}$$

Suppose on the contrary that $H_\beta(\alpha(\beta''), \beta'') < 0$. Then there exists a pair α^*, β^* with the following properties:

i) $\mathcal{H}_{\{1\}}(\alpha^*) = \mathcal{H}_{\{2\}}(\beta^*) < H(\alpha'', \beta'')$;

ii) $H(\alpha(\beta^*), \beta(\alpha^*)) < H(\alpha'', \beta'')$.

Lemmas 5.7 and 5.6 yield for some $\varepsilon > 0$ that

$$-H(\alpha'', \beta'') = \frac{1}{\tau'}\mathcal{L}(\varphi_{x', y', \tau_1', \tau_2', \tau_3'}) > -H(\alpha'', \beta'') + \varepsilon.$$

Thus (5.17) must be true.

Next let $\delta > 0$ be sufficiently small and define the set

$$A(N) = \{x \in \mathbf{Z}_+^2 \mid x_1, x_2 \le \delta N\} \cup \{2\}.$$

Write $\tau_0 = \tau_1' + \tau_3'$ and let

$$\varphi(t) = \begin{cases} \Big(x' \dfrac{t}{\tau_3'}, y' \dfrac{\tau_3' - t}{\tau_3'}\Big), & t \le \tau_3'; \\ \Big(x' \dfrac{\tau_3' + \tau_1' - t}{\tau_1'}, 0\Big), & \tau_3' \le \tau_3' + \tau_1'. \end{cases}$$

Consider the Markov chain starting at $[(0, y')N]$. Using the above construction we can lower bound the probability of hitting the set $A(N)$ for the first time at $[\tau^0 N]$ by the probability of the path φ, and so

$$\limsup_{N\to\infty} -\frac{1}{N} \log \mathsf{P}\{B_{A(N),N,\tau^0}\} \le -\tau^0 H(\alpha'', \beta'') - \beta'' y, \tag{5.18}$$

with

$$B_{A(N),N,\tau^0} = \{\xi_{[\tau^0 N]} \in A(N), \xi_t \notin A(N), t = 0, \dots, [\tau^0 N] - 1\}.$$

Next we upper bound this same first hitting probability. We can choose a point (α^*, β^*) with the following properties:

i) $\alpha^* < \alpha''$, $\beta^* < \beta''$.

ii) $H(\alpha^*, \beta^*) = H(\alpha'', \beta'')$.

iii) $H(\alpha^*, \beta'') < H(\alpha'', \beta'')$.

Using similar arguments to the proof of Lemma 5.7 we can construct a measure $\mu \in \mathcal{M}^0$, such that the corresponding vector field $(\alpha(x), \beta(x))$ takes the values

$$(\alpha(x), \beta(x)) = \begin{cases} (\alpha^*, \beta^*), & x_1/x_2 \le c_2; \\ (\alpha^*, \beta''), & x_2/x_1 = c_3; \\ (\alpha'', \beta''), & x_2/x_1 \le c_1, \end{cases}$$

for some constants $1/c_2 > c_3 > c_1$. Using similar arguments as in the proof of Lemma 5.6 we can then show that

$$\liminf_{N\to\infty} \Big(-\frac{1}{N} \log \mathsf{P}\{B_{A(N),N,\tau^0}\}\Big) \ge -\tau^0 H(\alpha'', \beta'') - \beta^* y,$$

where we used that $\varphi_1(0) = \varphi_1(\tau_0) = 0$. This contradicts (5.18), since $\beta^* < \beta''$.

Consequently, the infimum cannot be attained in a point $(x', y', \tau_1', \tau_2', \tau_3')$ with all parameters τ_1', τ_2' and τ_3' finite. It follows that at least one of these must be infinite and so (5.8) cannot be true. From the limiting behaviour of the functions f_l as $\tau_l \to \infty$, $l = 1, 2, 3$, it follows that (5.7) cannot be true, whence the Lemma is proved. □

Proof of Lemma 5.5. Assume that the assertion of the Lemma is not true, i.e. there exist x', $\tau' < \infty$ and a non-triangle path $\varphi \in \Phi^0_{x',x',\tau'}$, crossing all non-empty faces with positive Lebesgue measure, such that

$$\frac{1}{\tau'}\mathcal{L}_{\tau'}(\varphi) = \inf_{x\neq 0}\inf_{\tau>0}\frac{1}{\tau}\mathcal{L}^0_{x,x,\tau}.$$

Let $x = (x_1, 0)$. It is sufficient to consider the case that φ is a path consisting of four linear pieces $\varphi_k : [0, \tau_k] \to \mathbf{R}^2_+$, $k = 1, \ldots, 4$. All other cases are similar. Let us write

$$\begin{array}{rcccl} \varphi_1(0) & = & \varphi_4(\tau_4) & = & x; \\ \varphi_1(\tau_1) & = & \varphi_2(0) & = & (x_2, 0); \\ \varphi_2(\tau_2) & = & \varphi_3(0) & = & (0, y_1); \\ \varphi_3(\tau_3) & = & \varphi_4(0) & = & (0, y_2) \end{array}$$

for some $0 < x_2 < x_1$, $0 < y_1 < y_2$. Then

$$\mathcal{L}_{\tau'}(\varphi) = \sum_k \mathcal{L}_{\tau_k}(\varphi_k),$$

and $\mathcal{L}_{\tau_k}(\varphi_k)$ is determined by some parameters, (α_k, β_k) say. In the same manner as in the proof of Lemma 5.4 we can show that $(\alpha_2, \beta_2) = (\alpha_4, \beta_4)$, i.e. the parameters determining the action functionals of the paths in the interior of the quarter plane are equal. This is not possible, since

$$\begin{pmatrix} -x_2 \\ y_1 \end{pmatrix} = \nabla H(\alpha_2, \beta_2) = \nabla H(\alpha_4, \beta_4) = \begin{pmatrix} x_1 - x_2 \\ -y_2 \end{pmatrix}.$$

□

Proof of Theorem 5.1. This follows from Lemmas 5.2, 5.3, 5.4 and 5.5 together with (5.3). □

Finally we note that similarly to the one dimensional case the logarithmic asymptotics of the stationary probabilities are not related to α_{int}:

$$\lim_{N\to\infty}\frac{1}{N}\pi_{[xN]} = -\mathcal{L}_{0,x},$$

whereas α_{int} is determined by the action functionals of the paths identically x. [6] calculates these action functionals (see [8] for the analytic derivation): let $\alpha^*(x)$, $\beta^*(x)$, $\tau^*(x)$ be the unique solution to

$$\begin{cases} H(\alpha, \beta) = 0; \\ \tau\nabla H(\alpha, \beta) = x; \\ \tau > 0. \end{cases}$$

For $\alpha'_{\{1\}}$, $\beta'_{\{2\}}$ as defined in §3.2 let $\beta'_{\{1\}}$, $\alpha'_{\{2\}}$ be defined by

$$\begin{cases} H(\alpha'_{\{1\}},\beta'_{\{1\}}) = 0, \\ \dfrac{\partial}{\partial\beta} H(\alpha'_{\{1\}},\beta'_{\{1\}}) \geq 0, \end{cases} \qquad \begin{cases} H(\alpha'_{\{2\}},\beta'_{\{2\}}) = 0, \\ \dfrac{\partial}{\partial\alpha} H(\alpha'_{\{2\}},\beta'_{\{2\}}) \geq 0, \end{cases}$$

γ_1, $\gamma_2 \in [0,\pi/2]$ be defined by

$$\tan\gamma_1 = \frac{(\partial/\partial\beta)H(\alpha'_{\{1\}},\beta'_{\{1\}})}{(\partial/\partial\alpha)H(\alpha'_{\{1\}},\beta'_{\{1\}})}, \qquad \tan\gamma_2 = \frac{(\partial/\partial\beta)H(\alpha'_{\{2\}},\beta'_{\{2\}})}{(\partial/\partial\alpha)H(\alpha'_{\{2\}},\beta'_{\{2\}})},$$

and $\gamma(x) \in [0,\pi/2]$ by

$$\gamma(x) = \arctan\left(\frac{x_1}{x_2}\right).$$

Then for $N \to \infty$

$$\frac{1}{N}\log\pi([xN]) \sim \begin{cases} -\alpha^*(x)x_1 - \beta^*(x)x_2, & \gamma_1 \leq \gamma(x) \leq \gamma_2; \\ -\alpha'_{\{1\}}x_1 - \beta'_{\{1\}}x_2, & \gamma(x) < \gamma_1, \gamma(x) \leq \gamma_2; \\ -\alpha'_{\{2\}}x_1 - \beta'_{\{2\}}x_2, & \gamma_1 \leq \gamma(x), \gamma_2 < \gamma(x); \\ -\min\{\alpha'_{\{1\}}x_1 + \beta'_{\{1\}}x_2, & \\ \qquad \alpha'_{\{2\}}x_1 + \beta'_{\{2\}}x_2\}, & \gamma_2 < \gamma(x) < \gamma_1. \end{cases}$$

6. Essential spectrum and spectrum of one-dimensional random walks

6.1. Proof of Theorem 3.2

The relation $\alpha_{\text{int}} = \log r_0$ is valid by virtue of Theorem 3.1 and hence by virtue of Theorem 2.16 it is possible to characterise α_{int} through the essential spectrum of some Banach space. In particular we can restrict to $\ell^1(\mu)$ spaces for the measures $\mu(x) = F^L_{x0}(r)$. We will first study the asymptotics of the functions $F^L_{x0}(r)$ for large x.

For $r \leq 1/Q(\exp\{\alpha_0\}) = \exp\{-H(\alpha_0)\}$ the equation $rQ(\exp\{x\}) = 1$ has two real roots: $\alpha_{(r)} < \alpha'_{(r)}$.

Lemma 6.1. *For any* $\mathrm{L} \in \mathcal{F}(\mathrm{L}_0)$ *and* $r \leq r(F^{\mathrm{L}}_{00})$, $r \geq 1$,

$$\log F^{\mathrm{L}}_{x0}(r) \sim \alpha_{(r)}x, \quad x \to \infty.$$

Proof. Let $A = \{0,\dots,a\}$, $a \geq k'$ and let L be an A-perturbation. It is sufficient to determine the logarithmic asymptotics of the function $F_{xA}(r)$. For $r \geq 1$ it is the minimal solution to

$$\begin{cases} \sum_j rP_{xj}f_j & \leq f_x, & x \notin A; \\ \sum_j P_{xj}f_j & < \infty, & x \in A; \\ f_x & \geq 1, & x \in \boldsymbol{S}. \end{cases} \tag{6.1}$$

The function f defined by $f_j = \exp\{\alpha_{(r)} j\}$ solves (6.1), and so $F_{xA}(r) \leq \exp\{\alpha_{(r)} x\}$. Consider the matrix P' differing from P in entries (x, j), $j \in A$, only:

$$P'_{xj} = p_{j-x} \cdot \exp\{\alpha_{(r)}(j - a)\}, \quad j \in A.$$

For the chain with this transition structure

$$F'_{xA}(r) = \exp\{\alpha_{(r)}(x - a)\}, x \notin A.$$

Since $\exp\{\alpha_{(r)}\} \geq 1$, it easily follows that $f^{(t)\prime}_{xA} \leq f^{(t)}_{xA}$. Consequently, $F_{xA}(r) \geq \exp\{\alpha_{(r)}(x - a)\}$. □

This implies that we can restrict to $\ell^1(\mu)$ spaces with μ a measure from the set $\mathcal{M}$ defined in §3.1. The connection of measures from this set $\mathcal{M}$ to the first hitting time generating functions explains why Large Deviation results can be obtained using changes of measure based on the set $\mathcal{M}$.

We will determine the complete essential spectrum of P in the space $\ell^1(\mu_\alpha)$.

Considering P as an operator acting on the space $\ell^1(\mu_\alpha)$, it is simpler (cf. [14]) to study the transformed operator T^α acting on ℓ^1 with the same spectrum, i.e. $T^\alpha = UPU^{-1}$ with $U : \ell^1(\mu_\alpha) \to \ell^1$ the operator with the only non-zero elements given by $U_{ii} = \exp\{i\alpha\}$, so that

$$T^\alpha_{ij} = \exp\{(j - i)\alpha\} P_{ij}.$$

In all but a finite number of states, T^α is the discrete convolution operator on the half-line given by

$$\eta_i = \sum_j p_{i-j} \exp\{\alpha(i - j)\} x_j, \quad i \in \mathbf{Z}. \tag{6.2}$$

By a "localisation principle" [9, 11] the essential spectrum of T^α can be determined from the analysis of this operator given by (6.2) at points at infinity. This can be done by applying a Fourier transform and using a Wiener-Hopf factorisation [5]. Later on we will give a complete characterisation of the spectrum.

Proof of Theorem 3.2. The essential spectrum is the set of values λ for which the convolution operator defined by

$$\eta_i = \lambda x_i - \sum_j p_{i-j} \exp\{\alpha(i - j)\} x_j$$

is not invertible. This is exactly the set of values λ for which $\lambda - Q(z \exp\{\alpha\}) = 0$ for some z with $|z| = 1$. □

We finally obtain a further characterisation of the constants $\alpha_{(r)}$: it is the rate of growth of the bounding constant $C^{\mathrm{L}}(x, \alpha_{(r)})$ as a function of x, for the perturbation L.

Theorem 6.2. *For any* $\mathrm{L} \in \mathcal{F}(\mathrm{L}_0)$ *the following holds. For* $\alpha < \alpha(\mathrm{L})$ *and for* $r = \exp\{\alpha\}$

$$\log C^{\mathrm{L}}(x, \alpha) \sim \alpha_{(r)} x, \quad x \to \infty.$$

If $\alpha(\mathrm{L}) = \alpha_{int}$, *then*

$$\log C^{\mathrm{L}}(x, \alpha_{int}) \sim \alpha_0 x, \quad x \to \infty.$$

Proof. It is sufficient to give the proof for $\mathrm{L} = \mathrm{L}_0$.

By Lemma 2.8, $C(x, \alpha) \le cF_{xA}(r)$. Let $\delta > 0$ be sufficiently small with $\log(r + \delta) < \alpha(\mathrm{L}_0)$ and let $\varepsilon > 0$. Then for

$$T_{x,r} \; > \; \frac{x\alpha_{(r+\delta)} - \log\varepsilon + \log(r+\delta) - \log\delta}{\log(r+\delta) - \log r}$$

it follows that

$$\sum_{t \ge T_{x,r}} r^t \sum_j |P_{xj}^{(t)} - \pi_j| < \varepsilon.$$

For any $\delta > 0$ there exists I_0, such that for $x \ge I_0$

$$\begin{aligned} C(x, \alpha) \;&\ge\; \frac{1}{T_{x,r}} \sum_{t < T_{x,r}} r^t \sum_j |P_{xj}^{(t)} - \pi_j| \\ &\ge\; \frac{1}{T_{x,r}} \Big\{ \sum_{t \ge 0} r^t \sum_j |P_{xj}^{(n)} - \pi_j| - \varepsilon \Big\} \\ &\ge\; c' \cdot \frac{1}{T_{x,r}} F_{xA}(r) - \frac{\varepsilon}{T_{x,r}}, \end{aligned}$$

for some constant $c' > 0$ by virtue of Lemma 2.8. The result follows from Lemma 6.1 by choosing $T_{x,r}$ appropriately. □

6.2. Spectrum

We will calculate the spectrum of P in $\ell^1(\mu_\alpha)$ by calculating the spectrum of T^α in ℓ^1. First we need to introduce the following parts of the spectrum.

Let X be a Banach space. The point spectrum $\boldsymbol{\sigma}_{\mathbf{p}}(P, X)$, the continuous spectrum $\boldsymbol{\sigma}_{\mathbf{c}}(P, X)$ and the residual spectrum $\boldsymbol{\sigma}_{\mathbf{r}}(P, X)$ are given by

$$\begin{aligned} \boldsymbol{\sigma}_{\mathbf{p}}(P, X) \;&=\; \{\lambda \in \mathbf{C} : (\lambda - P) \text{ is not one-to-one}\}; \\ \boldsymbol{\sigma}_{\mathbf{c}}(P, X) \;&=\; \{\lambda \in \mathbf{C} : (\lambda - P) \text{ is one-to-one, } \mathsf{R}(\lambda - P) \text{ is dense in } X, \\ &\qquad\qquad \text{but not equal to } X\}; \\ \boldsymbol{\sigma}_{\mathbf{r}}(P, X) \;&=\; \{\lambda \in \mathbf{C} : (\lambda - P) \text{ is one-to-one, } \mathsf{def}(\lambda - P) > 0\}. \end{aligned}$$

For $x \in \ell^1$ define

$$\begin{aligned} x(z) &= \sum_{i \geq k'} x_i \, z^i, \\ T^\alpha x(z) &= \sum_i \sum_{j \geq k'} x_i \, T^\alpha_{ij} \, z^j. \end{aligned}$$

Then $y = (\lambda - T^\alpha)x$ if and only if

$$y_i = \lambda x_i - \sum_{j \leq k'+l-1} x_j T^\alpha_{ji}, \qquad i = 0, \ldots, k'-1; \tag{6.3}$$

$$y(z) = \big(\lambda - Q\big(z \exp\{\alpha\}\big)\big)x(z) - D(z), \tag{6.4}$$

with

$$D(z) = -\sum_{i=0}^{l-1} x_{k'+i} \sum_{j=i+1}^{l} p_{-j} \exp\{-\alpha j\} z^{k'+i-j} + \sum_{i=0}^{k'-1} x_i \sum_{j=k'}^{k'+k-1} T^\alpha_{ij} z^j. \tag{6.5}$$

It is clear that $T^\alpha \in \mathcal{B}(\ell^1, \ell^1)$.

Let λ be given. By the triple $(\kappa_1, \kappa_2, \kappa_3)$ we will denote the number of zeros of $\lambda - Q(z \exp\{\alpha\})$ with modulus smaller than 1, equal to 1 and bigger than 1 respectively. Write

$$Q^\alpha(z) = Q(z \exp\{\alpha\}).$$

First we characterize the spectrum through the triples $(\kappa_1, \kappa_2, \kappa_3)$.

1. $(\kappa_1, \kappa_2, \kappa_3) = (l, 0, k)$.

Given $y \in \ell^1$, (6.3) and (6.5) are a system of $k' + l$ linear equations in the $k' + l$ unknowns $x_0, \ldots, x_{k'+l-1}$ by projecting $y(z) = D(z)$ unto the zeros $z_1(\lambda), \ldots, z_l(\lambda)$ of $\lambda - Q^\alpha(z)$ in $|z| < 1$:

$$y\big(z_j(\lambda)\big) = D\big(z_j(\lambda)\big), \quad j = 1, \ldots, l. \tag{6.6}$$

More specificly, (6.3) and (6.6) defines a linear system $O(\lambda)\tilde{x} = \tilde{y}$ with $\tilde{x}$ and $\tilde{y}$ the vector restrictions of x and y to their first $k + l'$ components. Since $\det O(\lambda)$ is a symmetric function of the zeros $z_1(\lambda), \ldots, z_l(\lambda)$, it follows that $\det O(\lambda)$ is analytic in λ and so there are at most $k' + l$ values of λ for which (6.3) and (6.6) have a solution for $y \equiv 0$.

Given a solution $\tilde{x}$, $x(z)$ can then be solved uniquely by putting

$$x(z) = \frac{y(z) - D(z)}{\lambda - Q^\alpha(z)}. \tag{6.7}$$

The conclusion is that if $\det O(\lambda') = 0$ then $\lambda \in \boldsymbol{\sigma}_{\mathbf{p}}(T^\alpha)$ and $\lambda' - T^\alpha$ has a finite dimensional null-space. Otherwise $\lambda - T^\alpha$ is invertible, i.e. $\lambda \notin \boldsymbol{\sigma}(T^\alpha)$.

2. $(\kappa_1, \kappa_2, \kappa_3) = (l-1, 1, k)$.

As in the previous case we have to solve a system of $k'+l$ linear equations in $k'+l$ unknowns $x_0, \ldots, x_{k'+l-1}$, and so there are at most $k'+l$ values λ for which there is a solution to the homogeneous set of equations. For these values $\lambda \in \boldsymbol{\sigma}_{\mathbf{p}}(T^\alpha)$.

Note that $\mathsf{R}(\lambda - T^\alpha)$ is dense in ℓ^1, because there is at least one solution $x \in \ell^1$ to (6.3), (6.6) and (6.7) for each y with finitely many non-zero components (because in this case $y(z)$ has an isolated pole at the zero with modulus 1). By construction we will show that $\mathsf{R}(\lambda - T^\alpha)$ is not closed. Denote by $\hat{z}$ the zero of $Q^\alpha(z)$ with modulus 1.

Choose $y' \in \ell^1$ with $y'_i \geq 0$ and

$$\sum_i \sum_{j>i} y_j = \infty\,.$$

Let y'' be given by $y''_i = y'_i/\hat{z}^i$, $i \geq k'$, and $y''_i = 0$ otherwise. Consider the finite truncations y^n of y'', i.e. $y^n_i = y''_i$, $i \leq n$, and $y^n_i = 0$, $i > n$. $\{y^n\}_n$ is a Cauchy-sequence in ℓ^1 with limit y''.

This implies that $\hat{y}^n$ defined by

$$\hat{y}^n(z) = \prod_{j: z_j(\lambda) \neq \hat{z}} \big(z - z_j(\lambda)\big)\big(y^n(z) - y^n(\hat{z})\big)$$

is a vector in ℓ^1, and the sequence $\{\hat{y}^n\}_n$ is a Cauchy-sequence in $\mathsf{R}(\lambda - T^\alpha)$. However, the limit $\hat{y}$, defined by

$$\hat{y}(z) = \prod_{j: z_j(\lambda) \neq \hat{z}} \big(z - z_j(\lambda)\big)\big(y''(z) - y''(\hat{z})\big)$$

in a vector in $\mathsf{R}(\lambda - T^\alpha)$ only if $x'' \in \ell^1$ for x'' given by

$$x''(z) = \frac{y''(z) - y''(\hat{z})}{z - \hat{z}}.$$

But

$$x''_{k'+i} \;=\; \hat{z}^{-k'-i-1} \sum_{l \geq k'+i+1} y'_l, \quad i \geq 0,$$

so that $x'' \notin \ell^1$.

Hence $\mathsf{R}(\lambda - T^\alpha)$ is not closed in ℓ^1 and so $\lambda \in \boldsymbol{\sigma}_{\mathbf{ess}}(T^\alpha)$. For all λ for which $\det O(\lambda) \neq 0$, it follows that $\lambda \in \boldsymbol{\sigma}_{\mathbf{c}}(T^\alpha)$.

3. $(\kappa_1, \kappa_2, \kappa_3) = (l, 1, k-1)$.

We have to solve a system of $k'+l+1$ linear equations in the $k'+l$ unknown variables $x_0, \ldots, x_{k'+l-1}$: the zero of $\lambda - Q^\alpha(z)$ with absolute value 1 also has to be divided out to get a vector $x \in \ell^1$.

By construction it is easy to show that $\mathsf{R}(\lambda - T^\alpha)$ is not closed. To this end choose any numbers y_i, $i < k'$, and $\tilde{y}(z_j(\lambda))$, $j \leq l+1$ for which (6.3) and (6.6) have a solution. Let $y' \in \ell^1$ be arbitrary. Then for y determined by y_i, $i < k'$ and by

$$y(z) \;=\; \sum_{i=1}^{l+1} \frac{y'(z)}{y'(z_i(\lambda)} \prod_{j\neq i} \frac{z - z_j(\lambda)}{z_i(\lambda) - z_j(\lambda)} \tilde{y}(z_i(\lambda)),$$

there is a solution x to (6.3), (6.6) and (6.7). The non-closedness of the range follows by setting $\tilde{y}$ equal to y^n and y'' from case 2 respectively. Non-closedness can also be concluded from the fact that the linear subspace

$$\{y \,:\, y_1 \neq 0, y_i = 0, i > 0\} \not\in \mathsf{R}(\lambda - T^\alpha).$$

Hence $\lambda \in \boldsymbol{\sigma}_{\mathbf{ess}}(T^\alpha)$.

If the rank of the $k' + l + 1$ homogeneous equations is smaller than $k' + l$ then $\lambda \in \boldsymbol{\sigma}_{\mathbf{p}}(T^\alpha)$: this can occur for at most finitely many values of λ as has been argued before. If the rank is equal to $k' + l$ then $\lambda \in \boldsymbol{\sigma}_{\mathbf{r}}(T^\alpha)$.

4. $(\kappa_1, \kappa_2, \kappa_3) = (l + 1, 0, k - 1)$. We have to solve a system of $k' + l + 1$ linear equations in $k' + l$ unknowns.

Since in this case $y(z)$ is analytic in the zeros $z_i(\lambda)$ for all $y \in \ell^1$, it follows that $\mathsf{R}(\lambda - T^\alpha)$ is closed. $\mathsf{def}(\lambda - T^\alpha)$ and $\mathsf{nul}(\lambda - T^\alpha)$ are finite, since they are determined by solutions to a finite system of linear equations.

Further this case is the same as case 3.

5. $(\kappa_1, \kappa_2, \kappa_3) = (l - 1, 0, k + 1)$.

We have to solve a system of $k' + l - 1$ linear equations in $k' + l$ unknowns. This has a solution for $y = 0$ and so $\lambda \in \boldsymbol{\sigma}_{\mathbf{p}}(T^\alpha)$.

In the same way as before, it can be shown that $\lambda \not\in \boldsymbol{\sigma}_{\mathbf{ess}}(T^\alpha)$. □

All other possibilities can be derived similarly. The following lemma will be used to characterise the second largest value in the spectrum.

Lemma 6.3. i) $Q'(x) \geq 0$ *implies* $\mathsf{ind}_{|z|=1+0}\Big\{Q(x) - Q(xz)\Big\} = 1$.

ii) $Q'(x) < 0$ *implies* $\mathsf{ind}_{|z|=1+0}\Big\{Q(x) - Q(xz)\Big\} = 0$.

This lemma has the following consequence.

Corollary 6.4. *If* $|\lambda| > Q^\alpha(1)$, *then* $\lambda - Q^\alpha(z)$ *has* l *roots with modulus smaller than 1 and* k *roots with modulus greater than 1. If* $|\lambda| < Q^\alpha(1)$, *and* λ *is sufficiently close to* $Q^\alpha(1)$, *then* $\lambda - Q^\alpha(z)$ *has* $l+1$ *roots with modulus smaller than 1 and* $k-1$ *roots with modulus greater than 1.*

Define

$$\Gamma^\alpha = \{Q^\alpha(z) \,:\, |z| = 1\}.$$

Γ^α divides $\mathbf{C}$ into one unbounded region, say C_0^α and a number of bounded regions. Corollary 6.4 implies that

$$\lambda \in C_0^\alpha \iff (\kappa_1, \kappa_2, \kappa_3) = (l, 0, k).$$

Note further that

$$\lambda \in \Gamma^\alpha \iff \lambda \in \boldsymbol{\sigma}_{\mathbf{ess}}(T^\alpha).$$

Let $S^\alpha \subset \mathbf{C}$ be the set of points with

$$\lambda \in S^\alpha \iff \begin{cases} \lambda \in C_0^\alpha \quad \text{and there exists a solution} \\ \text{to the homogeneous equations (6.3), (6.6).} \end{cases}$$

Lemma 6.5. *Let*

$$\lambda(\alpha) = \sup\{|\lambda| : \lambda \in S^\alpha \cup \Gamma^\alpha, \lambda \neq 1\}.$$

Then $\alpha(\mathrm{L}_0) = -\log \lambda(\alpha)$, *if* $S^\alpha \neq \emptyset$.

Remark 6.1. It seems that all listed possibilities for the triples $(\kappa_1, \kappa_2, \kappa_3)$ can occur. This follows from Lemma 6.3 and the following argument. If the sequence $p_{-l}, \ldots, p_0, \ldots, p_k$ is periodic with period n say, then Γ^α has n self-intersections. A slight perturbation of the p_i gives at least $n-1$ self-intersections.

Remark 6.2. Explicit expressions for the functions $F_{xA}(z)$ and $P_{xy}(z)$ can be obtained by solving a similar system of equations to (6.3), (6.4) and (6.5).

Example. Consider the homogeneous random walk with downward jumps of size 1, i.e. $l = 1$, and $k' = 1$ and transitions from state 0 given by

$$P_{0y} = \begin{cases} p_y, & y > 0; \\ p_0 + p_1, & y = 0. \end{cases}$$

It easily follows that

$$F_{x0}(z) = \big(F_{10}(z)\big)^x$$

and so $F_{00}(z) = 1$ if and only if $z = 1$. This implies that $S^\alpha = \emptyset$ and so

$$\alpha_{\mathrm{int}} = \alpha(\mathrm{L}_0) = -\log Q(\alpha_0) = -H(\alpha_0).$$

7. Essential spectrum of two-dimensional random walks

7.1. Composite convolutions on a half-space

This subsection derives the required results on the Fredholm (Nœtherian) property for composite convolutions on a half-space in $\mathbf{Z}^2$, that we need for Theorem 3.4. Our notation and derivations are based on [12].

Let A be the convolution operator given by

$$(Ax)_{ij} = \sum_{(kl)\in\mathrm{Z}^2} a_{i-k,j-l} x_{kl}, \quad a \in \ell^1(\mathbf{Z}^2). \tag{7.1}$$

Let $a(z,u) = \sum_{ij} a_{ij} z^i z^u$ be the kernel of A and $\ell(A)$ the vector with integer components

$$\begin{aligned} \ell_1(A) &= \operatorname*{ind}_{|z|=1+0} a(z,1) = \frac{1}{2\pi} \int\limits_{|z|=1} d\arg a(z,1); \\ \ell_2(A) &= \operatorname*{ind}_{|u|=1+0} a(1,u) = \frac{1}{2\pi} \int\limits_{|u|=1} d\arg a(1,u). \end{aligned}$$

We write $\mathcal{P}_U : \ell^p(\mathbf{Z}^2) \to \ell^p(U)$, $U \subset \mathbf{Z}^2$, for the projection operator on U, i.e.

$$x = \{x_{ij}\}_{(ij)\in \mathrm{Z}^2} \;\to\; \{x_{ij}\}_{(ij)\in U}.$$

$\mathcal{P}_+$ ($\mathcal{P}_-$) stands for the projection on $\mathbf{Z} \times \{1,2,\ldots\}$ ($\mathbf{Z} \times \{-1,-2,\ldots\}$), and $\Gamma = \mathbf{Z} \times \{0\}$.

Introduce the convolution $(*)$ as the multiplication of kernels in $\ell^1(\mathbf{Z}^2)$. There is a one-one correspondence between convolution operators of type (7.1) and kernels a, and so A is invertible if and only if a is invertible, i.e. when the symbol $a(z,u) \neq 0$ on the torus $T_2 : |z| = |u| = 1$.

Write $\ell = \ell(A)$ and $a_\ell(z,u) = z^{\ell_1} u^{\ell_2}$. $\arg(a * a_\ell^{-1})(z,u)$ and $\ln(a * a_\ell^{-1})(z,u)$ are one-valued on T_2, if the symbol $a(z,u)$ does not vanish on T_2. Put

$$b_{+,-,\Gamma} = \mathcal{P}_{+,-,\Gamma} \ln(a * a_\ell^{-1}), \quad a_{+,-,\Gamma} = e^{b_{+,-,\Gamma}}$$

respectively. Then $a = a_+ * a_- * a_\Gamma * a_\ell$. Denote the corresponding convolution operators by $A_{+,-,\Gamma,\ell}$, then we have obtained the decomposition

$$A = A_+ A_- A_\Gamma A_\ell, \tag{7.2}$$

where $A_{+,-,\Gamma,\ell}$ are clearly commutative, invertible, and the first three leave invariant the respective spaces $\mathcal{P}_{+,-,\Gamma}\, \ell^p(\mathbf{Z}^2)$.

The above arguments follow from a generalised Wiener theorem on locally analytic functions on the maximal ideal space of the ring of functions $a(z,u)$, $a \in \ell^p(\mathbf{Z}^2)$ ([4], §35, p.266).

Next consider the composite convolution C on the half-space $\mathbf{Z} \times \mathbf{Z}_+$:

$$C = AP_+ + BP_\Gamma, \tag{7.3}$$

where A, B are of type (7.1) with kernels a, b of the form

$$\begin{aligned} a_{ij} &= 0, \qquad j < -1; \\ b_{ij} &= 0, \qquad j < 0, \end{aligned}$$

so that C is closed in $\mathcal{P}_{+\Gamma}\ell^p$.

For a clearer exposition we assume that the kernel b has finite support in the second variable. This will only be explicitly used to treat the case that b vanishes on T_2.

Theorem 7.1. *Let $b(z,u)$ have finite degree m as a function of u.*

i) *C is invertible if and only if the following conditions hold.*

a) *$a(z,u), b(z,u) \neq 0$, for $(z,u) \in T_2$.*

b) *$\ell_2(A) = 0$.*

c) *Let $u_0(z,A)$ be the unique single-valued branch of $a(z,u) = 0$, $z \in T_1$, with $|u_0(z,A)| < 1$. Then $b(z, u_0(z,A)) \neq 0$ for $z \in T_1$.*

ii) *C is not Nœtherian in all other cases. This is because of the following properties: $\mathsf{nul}(C) = \infty$ for $\ell_2(A) < 0$, $\mathsf{def}(C) = \infty$ for $\ell_2(A) > 0$, $\mathsf{R}(C)$ is not closed in the remaining cases.*

Proof. We consider solutions $f_+ \in \mathcal{P}_+\ell^p$, $f_\Gamma \in \mathcal{P}_\Gamma\ell^p$ of

$$Cf = Af_+ + Bf_\Gamma = g, \tag{7.4}$$

for $g \in \mathcal{P}_{+\Gamma}\ell^p$.

The necessity of

$$a(z,u) \neq 0, \quad (z,u) \in T_2, \tag{7.5}$$

for the Fredholm property follows from the localisation principle [11], [12] for points at infinity along rays $\varphi \in (0,\pi)$ emanating from the origin. It also follows easily by construction, that $\mathsf{R}(C)$ is not closed if (7.5) is violated.

Assume hence (7.5) for the remainder of the proof. It requires a number of steps.

a°) Let $\ell_2 < 0$, i.e. $\ell_2 = -1$ ($\ell = \ell(A)$). We show that $\mathsf{nul}(C) = \infty$.

Choose f_Γ arbitrarily. Write $\delta(r)$ for the characteristic function of the point $r \in \mathbf{Z}^2$. Form the decomposition $A = \tilde{A}_+ A_- A_\ell$, with $\tilde{A}_+ = A_+ A_\Gamma$ and consider f_+ of the form

$$f_+ = \sum_r c_r \tilde{A}_+^{-1} \delta(r) + h_+,$$

where the constants c_r and the function $h_+ \in \mathcal{P}_+\ell^p$ will be determined below. Then

$$Af_+ = \sum_r c_r A_- \delta(r+\ell) + Ah_+.$$

$r + \ell \in \Gamma$ for $r = (r_1, 1)$ and then $A_-\delta(r+\ell) \in \mathcal{P}_{-\Gamma}\ell^p$. Solve h_+ from

$$\mathcal{P}_+ A h_+ = -\mathcal{P}_+ B f_\Gamma$$

and use this to solve c_r from

$$\mathcal{P}_\Gamma \sum_r c_r A_- \delta(r - \ell) = -\mathcal{P}_\Gamma A h_+ - \mathcal{P}_\Gamma B f_\Gamma,$$

then for this choice $\mathcal{P}_{+\Gamma} Cf = 0$. Since we can take $\mathcal{P}_\Gamma f = \delta(r)$, $r = (r_1, 0)$ arbitrarily, $\mathsf{nul}(C) = \infty$.

b°). Next let $\ell_2 > 0$. We show that $\mathsf{def}(C) = \infty$.

Denote the projection on $j = j'$ by $\mathcal{P}_{j'}$, the projections on $0 \le j \le j'$ and $0 < j \le j'$ by $\mathcal{P}_{\le j'}$ and $\mathcal{P}_{+\le j'}$, etc. Then there is a solution (f_+, f_Γ) to (7.4) if and only if there is a solution $(h_{>\ell}, f_\Gamma)$, $h_{>\ell} \in \mathcal{P}_{>\ell}\ell^p$ to

$$A_- h_{>\ell} + B f_\Gamma = g \tag{7.6}$$

and the relation between $h_{>\ell}$ and f_+ is one-to-one:

$$h_{>\ell} = \tilde{A}_+ A_\ell f_+.$$

Since A_- is invertible and leaves $\mathcal{P}_{\le j}\ell^p$ invariant, (7.6) is equivalent to

$$h_{>\ell} + A_-^{-1} B f_\Gamma = A_-^{-1} g,$$

and so

$$\mathcal{P}_\ell\, A_-\, B\, f_\Gamma = \mathcal{P}_\ell\, A_-^{-1} g. \tag{7.7}$$

If $\mathcal{P}_\ell\, A_-\, B\, \mathcal{P}_\Gamma \equiv 0$ then $\mathsf{def}(C) = \infty$, otherwise (7.7) has at most a finite dimensional set of solutions. Since $\mathcal{P}_{\ge \ell-1} A_-^{-1} g$ is only determined by $\mathcal{P}_{\ge \ell-1} g$ and $\mathcal{P}_{l-1} A_-^{-1} \mathcal{P}_{l-1} g = \mathcal{P}_{l-1} g$, there is an infinite dimensional extension of $\mathcal{P}_{\ge \ell} g$ in $\mathcal{P}_{\ge \ell-1}\ell^p$ that cannot be attained.

c°) Let $\ell_2 = 0$ and $b(z,u) \ne 0$ on T_2. Write $\ell' = \ell(B)$.

Let $f = (f_\Gamma, f_+)$ be a solution to (7.4) for $g \in \mathcal{P}_{+\Gamma}\ell^p$. Form the decomposition $B = \tilde{B}_+ B_- B_{\ell'}$, $\tilde{B}_+ = B_+ B_\Gamma$. (7.4) is equivalent to

$$\tilde{B}_+^{-1} A f_+ + B_- B_{\ell'} f_\Gamma = \tilde{B}_+^{-1} g.$$

Let $h_+ = A_\ell A_+ \tilde{B}_+^{-1} f_+$, then $h_+ \in \mathcal{P}_+\ell^p$ and

$$A_- h_+ + B_- B_{\ell'} f_\Gamma = \tilde{B}_+^{-1} g. \tag{7.8}$$

Clearly, (7.8) consists of two different sets of equations on $\mathcal{P}_{>\ell'}\ell^p$ and $\mathcal{P}_{\le \ell'}\ell^p$ respectively, the first of which is simply solved.

1) On $\mathcal{P}_{>\ell'}\ell^p$ (7.8) is equal to

$$A_- \mathcal{P}_{>\ell'} h_+ = \tilde{B}_+^{-1} g, \tag{7.9}$$

which has the unique solution

$$\mathcal{P}_{>\ell'} h_+ = \mathcal{P}_{>\ell'} A_-^{-1} \mathcal{P}_{>\ell'} \tilde{B}_+^{-1} g = \mathcal{P}_{>\ell'} A^{-1} \tilde{B}_+^{-1} g, \tag{7.10}$$

since A_- leaves $\mathcal{P}_{\le \ell'}\ell^p$ invariant. The value $\mathcal{P}_{+,\le \ell'} A_- \mathcal{P}_{>\ell'} h_+$ cannot be controlled.

2) On $\mathcal{P}_{\le \ell'}\ell^p$ we have

$$A_- \mathcal{P}_{\le \ell'} h_+ + B_- B_{\ell'} f_\Gamma = \tilde{B}_+^{-1} g - A_- \mathcal{P}_{>\ell'} h_+. \tag{7.11}$$

If $\ell_2' = 0$, (7.11) can be uniquely solved given $\mathcal{P}_\Gamma A_- h_+$, determined in i), since $\mathcal{P}_\Gamma B_- \mathcal{P}_\Gamma$ is the identity. In formula:

$$\begin{cases} f_+ &= A_{-\ell}\tilde{A}_+^{-1}\tilde{B}_+\mathcal{P}_+A_-^{-1}\tilde{B}_+^{-1}g, \\ f_\Gamma &= B_{-\ell'}\tilde{B}_+^{-1}g - B_{\ell'}\mathcal{P}_\Gamma A_-\mathcal{P}_+A_-^{-1}\tilde{B}_+^{-1}g. \end{cases} \tag{7.12}$$

Let $\ell_2' > 0$. (7.11) are a system of $l'+1$ linear functional equations in the unknowns $h_1, \ldots, h_{\ell'}$, f_Γ. It can be solved in ℓ^p provided

$$\Delta = \det \begin{pmatrix} a_{-,0} & & & & b_{-,0} \\ a_{-,-1} & a_{-,0} & & & b_{-,-1} \\ & \ddots & \ddots & & \vdots \\ & & & a_{-,0} & b_{-,\ell'-1} \\ & & & a_{-,-1} & b_{-,\ell'} \end{pmatrix} \neq 0, \quad |z| = 1,$$

where $a_{-,j}$, $b_{-,j}$ are equal to $\mathcal{P}_j a_-$, $\mathcal{P}_j b_-$, for a_- and b_- the kernels of A_- and B_-. To see this, let $\Delta = 0$ for some $|z| = 1$ and denote

$$\tilde{g} = \tilde{B}_+^{-1}g - A_-\mathcal{P}_{>\ell'}h_+.$$

Then as $a_{-,0} = 1$

$$\Delta f_\Gamma = a_{-,-1}^{\ell'}\,\tilde{g}_{\ell'} - a_{-,-1}^{\ell'-1}\,\tilde{g}_{\ell'-1} + a_{-,-1}^{\ell'-2}\,\tilde{g}_{\ell'-2} + \cdots + (-1)^{\ell'}\tilde{g}_0,$$

and it is possible to select a value $\mathcal{P}_\Gamma g$, such that f_Γ does not represent a vector in $\ell^p(\mathbf{Z})$.

Let $\Delta \neq 0$, $|z| = 1$. Note that $\Delta \in \ell^1(\mathbf{Z})$. By Wiener's theorem, it has an inverse in $\ell^1(\mathbf{Z})$.

d^o) We study Δ.

By assumption $a(z,u) = 0$ has one branch $u_0(z,A)$, with modulus smaller than 1, $|z| = 1$; $b(z,u) = 0$ has ℓ_2' branches $u_0(z,B)$, $\ldots, u_{\ell_2'-1}(z,B)$, with moduli smaller than 1.

By uniqueness of the decomposition it follows that

$$\begin{aligned} a_-(z,u) &= 1 - \frac{u_0(z,A)}{u}, \\ b_-(z,u) &= \prod_{k=0}^{\ell_2'-1}\big(1 - \frac{u_k(z,B)}{u}\big), \end{aligned}$$

so that $a_{-,0} = 1$, $a_{-,-1}(z) = -u_0(A,z)$, $b_{-,0}(z) = 1$, $b_{-,-1}(z) = -\sum_k u_k(z,B)$, etc. A simple calculation shows, that

$$\Delta = \prod_{k=0}^{\ell_2'-1}\big(u_0(z,A) - u_k(z,B)\big).$$

e^0). Let $b(z,u)=0$ for some $(z,u)\in T_2$. Here we will use that $b(z,u)$ has finite degree in u.

For $|z|=1$, $b(z,u)=0$ has m roots $u_k(B,z)$, and we can write

$$b(z,u)=b_m(z)\prod_{k=0}^{m-1}\big(u-u_k(z,B)\big).$$

In the previous derivations, replace $\tilde{B}_+$ by the identity, $B_-B_{\ell'}$ by B and ℓ' by $-m$: then $\Delta=0$ on $|z|=1$ and the theorem is proved. □

7.2. Proofs of Theorems 3.4 and 3.5

We study the essential spectrum of P in $\ell^1(\mu)$ for $\mu\in\mathcal{M}$ (defined in §3.2). This is equivalent to studying the essential spectrum of T^μ given by

$$T^\mu_{xy}=P_{xy}\exp\Big\{\alpha(\gamma_y)y_1+\beta(\gamma_y)-\alpha(\gamma_x)x_1-\beta(\gamma_x)x_2\Big\}$$

in ℓ^1, with $\gamma_x=x_2/x_1$. To this end we consider the compactification $\ell^1_*(\mathbf{Z}^2)$ of $\ell^1(\mu)$ defined by adding to each ray emanating from 0 in $\mathbf{R}^2$ the corresponding point at infinity. Write $\mathcal{N}=\ell^1_*(\mu)\setminus\ell^1(\mu)$. Extend the basis of the topology on $\ell^1(\mu)$ to a topology on the compactification by adding the open cones with point at 0. B is called an operator of *local type* if for any two disjoint sets F_1, F_2 the operator $\mathcal{P}_{F_1}B\mathcal{P}_{F_2}$ is compact. Then T^μ is a locally continuous operator of local type [11], [9] Chapter XV, for $\mu\in\mathcal{M}$.

To determine whether $\lambda\in\boldsymbol{\sigma}_{\mathbf{ess}}(P,\ell^1(\mu))$ it is sufficient to show that $\lambda-T^\mu$ is locally Fredholm. For the latter it is sufficient to show that $\lambda-T^\mu$ is locally Fredholm at points at infinity along rays emanating from the origin.

Proof of Theorem 3.4. T^μ is locally Fredholm at points at infinity determined by the rays $\gamma,1/\gamma\neq 0$ if and only if the symbol

$$\lambda-\sum_y p_y\exp\{\alpha(\gamma)y_1+\beta(\gamma)y_2\}z^{y_1}u^{y_2}$$

does not vanish for $|z|$, $|u|=1$. Along rays $\gamma=0$, $1/\gamma=0$ the operator T^μ is locally Fredholm if the corresponding composite convolution operator on the half-space in $\mathbf{Z}^2$ is Fredholm. The Fredholm property for such composite convolutions has been characterised in Theorem 7.1. The result follows from Theorem 7.1, [12], [9] Theorem XV.4.1 and Corollary 6.4. □

Proof of Theorem 3.5. Denote

$$\mathcal{H}=\max\{\mathcal{H}_{\{1\}}(\alpha^0_{\{1\}}),\mathcal{H}_{\{2\}}(\beta^0_{\{2\}})\}.$$

Consider the sets of points

$$S_1=\{(\alpha,\beta(\alpha)):H(\alpha,\beta(\alpha))\le\mathcal{H}\}$$

and

$$S_2 = \{(\alpha(\beta), \beta) : H(\alpha(\beta), \beta) \leq \mathcal{H}\}\,.$$

For the proof of the Theorem it is sufficient to construct a measure $\mu \in \mathcal{M}^0$, such that

i) $(\alpha(x), \beta(x)) = (\alpha, \beta(\alpha)) \in S_1$ for x with $x_2 = 0$;

ii) $(\alpha(x), \beta(x)) = (\alpha(\beta), \beta) \in S_2$ for x with $x_1 = 0$;

iii) $H(\alpha(x), \beta(x)) \leq \mathcal{H}$.

The existence of such measure follows immediately from Lemma 5.7 if the conditions of this Lemma apply to one pair of points

$$(\alpha, \beta(\alpha)) \in S_1, \quad (\alpha(\beta), \beta) \in S_2.$$

Let us assume that there does not exist such pair of points. It follows that either $\alpha \geq \alpha(\beta)$, $\beta(\alpha) \geq \beta$ for all $(\alpha, \beta(\alpha)) \in S_1$, $(\alpha(\beta), \beta) \in S_2$ or vice versa. Consider the first case. This can occur, if S_1 and S_2 consist of one point each, $(\alpha', \beta(\alpha'))$ and $(\alpha(\beta'), \beta')$ say, such that $\beta(\alpha') = \beta'$, $H_\beta(\alpha', \beta(\alpha')) = 0$ and $H_\beta(\alpha(\beta'), \beta') = 0$. Analysis of the functions $\mathcal{H}_{\{1\}}$ and $\mathcal{H}_{\{2\}}$ yields that in that case also $H_\alpha(\alpha', \beta(\alpha')) = 0$ and so $(\alpha', \beta(\alpha')) = (\alpha(\beta'), \beta') = (\alpha_0, \beta_0)$. As the measure μ we can take $\mu(x) = \exp\{\alpha_0 x_1 + \beta_0 x_2\}$.

Suppose that at least one of S_1 and S_2 contains more than one point. It can be shown that in this case there exists a pair of points $(\alpha', \beta(\alpha'))$, $(\alpha(\beta'), \beta')$, such that

i) $H(\alpha', \beta')$, $H(\alpha(\beta'), \beta(\alpha')) > \mathcal{H}$;

ii) $\partial_\alpha H(\alpha', \beta(\alpha')) > 0$, $\partial_\alpha H(\alpha(\beta'), \beta') < 0$;

iii) $(d/d\alpha)\mathcal{H}_{\{1\}}(\alpha') < 0$, $(d/d\beta)\mathcal{H}_{\{2\}}(\beta') > 0$;

iv) $\beta' = \beta(\alpha')$.

We can then construct a triangle path $\varphi^-_{x,y,\tau_1,\tau_2,\tau_3}$, with $\tau_3 = 1$, x, y such that

$$\nabla H(\alpha', \beta(\alpha')) = (x, -y)$$

and the parameters τ_1 and τ_2, such that

$$-\frac{d}{d\alpha}\mathcal{H}_{\{1\}}(\alpha') = x/\tau_1 \quad \text{and} \quad \frac{d}{d\beta}\mathcal{H}_{\{2\}}(\beta') = y/\tau_2\,.$$

Then for $\tau = \sum_k \tau_k$

$$\mathcal{L}_\tau(\varphi^-_{x,y,\tau_1,\tau_2,\tau_3}) = -(\tau_1 + 1)H(\alpha', \beta(\alpha')) - \tau_2 H(\alpha(\beta'), \beta') < -\tau\mathcal{H}.$$

But this is not possible by virtue of Lemma 5.4. □

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