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Dynamical clusters of infinite particle dynamics

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For any system $\{i\}$ of particles with the trajectories $x_i(t)$ in R^d on a finite time interval $[0,\tau]$ we define the interaction graph G. Vertices of G are the particles, there is an edge between two particles i and j iff for some $t\in[0,\tau]$ the distance between particles i and j is not greater than some constant. We undertake a detailed study of this graph for infinite particle dynamics and prove exponential estimates for its finite connected components. This also solves continuous percolation problem for complicated geometrical objects—the tubes around particle trajectories. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1955513]

I. INTRODUCTION

We undertake a detailed study of the interaction graph for infinite particle dynamics and prove exponential estimates for its finite clusters. Cluster properties of classical infinite-particle dynamics is a folklor notion now, cluster dynamics was discussed in Ref. 1. In more general sense, weak dependence of far away particle trajectories plays substantial role in many papers, see e.g., Refs. 2–5. For classical (deterministic or stochastic) dynamics, the cluster property reveals a clear geometric picture, where the system of infinite particles is subdivided into random finite subsets (clusters) which do not interact with each other on a fixed time interval. Thus the dynamics is reduced to finite particle dynamics. The new feature of this article is that we describe clusters in detail combinatorially and give combinatorial estimates of their probabilities. This solves continuous percolation problem for complicated geometrical objects—the tubes around particle trajectories.

Note that cluster property can have also different (but related) meaning: decay of correlations, existence of quasiparticles, especially for the quantum case, see Ref. 6, we do not pursue this issue here.

We assume that at time 0, for any cube Λ, Poisson point field with density ρ is given. Thus the number $N(\Lambda)$ of particles in Λ has Poisson distribution with $\langle N(\Lambda)\rangle/|\Lambda|=\rho$. The initial velocities $v_i(0)$ of the particles are assumed to be independently and identically distributed. Random initial configuration of coordinates $x_i(0)$ and velocities $v_i(0)$ is denoted by $\omega=\omega_\Lambda$.

We assume that for any cube Λ and any fixed number of particles $N(\Lambda)$ some finite particle dynamics in Λ is given. Here we mean by this that for (almost) any initial coordinates $x_i(0)$ and velocities $v_i(0)$ the trajectories $x_i(t)=x_i^{(\Lambda)}(t)$ are uniquely defined on the time interval $[0,\tau]$, they are assumed to be piecewise smooth (then the velocities $v_i(t)=\mathrm{d}x_i(t)/\mathrm{d}t$ are defined a.e.).

We further fix on some $r>0$. We call the tube (or r tube) $T_i(\tau)$ of the particle i the r-neighborhood of its trajectory $x_i(t,\omega)$, $0\leqslant t\leqslant\tau$. We say that two particles i, j interact at time t if

$$\mathrm{dist}(x_i(t),x_j(t))\leqslant 2r$$

that is if the closed r neighborhoods of $x_i(t)$ and $x_j(t)$ (the time t slices of the corresponding tubes) intersect. Then

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$$s_{ij} = \min\{t:\mathrm{dist}(x_i(t),x_j(t)) \leqslant 2r\}$$

is called the first interaction time of particles i and j.

To define dynamical clusters we consider (for any τ) the following finite random graphs $G^\Lambda = G^\Lambda(\tau) = G^\Lambda(\tau,\omega)$. Vertices of G^Λ are the particles. We can assume that they are labelled by the initial coordinates of the particles. Two vertices are connected by an edge if on the time interval $[0,\tau]$ these two particles interact at least at one time moment. The sets of vertices of connected components of $G^\Lambda = G^\Lambda(\tau)$ are called dynamical clusters in Λ (or simply clusters) if τ and Λ are fixed. Equivalently, the connected components are the same as the topological connected components of the union of all tubes $T = \cup T_i(\tau)$ in $\Lambda \times [0,\tau]$.

Within this general setting, sufficient for our purpose, we have to do three additional assumptions:

(1) For simplicity of presentation we assume translation invariance of the dynamics and periodic boundary conditions and that the particle i moves freely on the time interval $(t,t+s)$

$$x_i(t+s] = x_i(t) + v_i(t+0)s$$

if for any $t' \in (t,t+s)$ this particle does not interact with other particles.

(2) We assume that for any two particles i,j their first interaction times $s_{ij} = s_{ij}(\omega)$ are all different a.s. This holds trivially in many known dynamical models.

(3) Our main assumption is that the velocities are uniformly bounded, that is for some constant $v^0 > 0$ and any i and $t \in [0,\tau]$

$$|v_i(t)| \leqslant v^0.$$

This is a very simplifying assumption. However, even under such condition the combinatorics of clusters is not easy. I hope that the obtained estimates allow to weaken essentially this condition.

Remark 1.1: Note that our dynamics is very general—we do not even assume that the particle trajectories are related for different Λ. We could even take, with some precautions. any metric space instead of R^d with trajectories satisfying Lipschitz condition

$$\mathrm{dist}(x(t),x(t')) \leqslant v^0|t-t'|.$$

At the same time we could consider the infinite particle system directly in R^d with the same existence assumptions (1)–(3).

Physical intuition and percolation theory: If the velocities are uniformly bounded then on the time interval $[0,\tau]$ the tube of a particle belongs to the ball of radius $v^0\tau + r$ with the center in its initial coordinate. The results from continuous percolation theory, see Refs. 7–9, tell us that the infinite system of balls of radius $v^0\tau + r$ around Poisson points in R^d a.s. has no infinite clusters for ρ small. More exactly, simple space scaling shows that if ρ does not exceed $\alpha_0(v^0\tau + r)^{-d}$ for some fixed factor $\alpha_0 > 0$ sufficiently small, then all clusters are finite a.s. Moreover, exponential cluster estimates hold. However this result is too rough for our purpose.

Physics tells us that the correct answer is given by the Boltzman–Grad (BG) scaling. Note that the volume λ swept up by the particle, that is the volume of its tube, is of the order $v^0\tau r^{d-1}$, where $v^0\tau$ is the length of the trajectory and r^{d-1} is of the order of the circular section of the tube. Heuristically, for the densities smaller than $\alpha_0\lambda^{-1}$, the limiting dynamical clusters should be finite a.s. The reason is that $N(\Lambda)$ particles sweep up the volume $v^0\tau r^{d-1}N(\Lambda)$ which should be small compared to Λ.

We give the proof here. As far as I know, this does not follow from the existing results in continuous percolation theory. One of the reasons is that the particle tubes are not independent volumes around the Poisson points.

For obvious reasons we assume that $v^0\tau \gg r$, that is $v^0\tau > C_1 r$ for some C_1 sufficiently large. Otherwise, the percolation properties of the initial configuration of balls of radius r would prevail.

Moreover, exactly under this condition the BG volume $v_0\tau r^{d-1}+r^d$ is much smaller than the volume $(v_0\tau+r)^d$, and our result is stronger than the one obtained from continuous percolation theory.

Result: Denote $P_k^\Lambda(\tau|x)$ the conditional probability that, under the condition that there is a particle at the point x, the cluster, containing this particle, has exactly k particles. Then the following exponential estimate holds.

Theorem 1.2: *There are constants* C, $\alpha_0>0$ *such that for any* τ, v^0, r *and*

$$\rho=\frac{N(\Lambda)}{\Lambda}=\alpha(\tau v^0 r^{d-1})^{-1} \tag{1.1}$$

with $0<\alpha<\alpha_0$ *and any* k *we have, uniformly in* Λ *and in* x,

$$P_k^\Lambda(\tau|x)\leqslant(C\alpha)^{k-1}. \tag{1.2}$$

Corollary: If for any $t\in[0,\tau]$ the thermodynamic limit of the dynamics (that is $\lim_{\Lambda\to\infty}x^{(\Lambda)}\times(t)$ exists, then all clusters in R^d are finite and the exponential estimates (1.2) hold for $P_k(\tau)=\lim_{\Lambda\to\infty}P_k^\Lambda(\tau|x)$.

Under our general dynamics one cannot prove the existence of thermodynamic limit (it may not even exist). However, there are many results, obtained in a different way, concerning the existence of the thermodynamic limit.[2,4,10,11] In the last section we give an example of the (random) dynamics which satisfies our conditions.

To prove the theorem one has to decribe somehow the set of possible clusters. There are combinatorial and geometric aspects in the description of the set of clusters. The idea of the proof is to separate these two aspects. Combinatorial part consists of describing possible "collision schemes," that is the order in which particles interact at the first time. The central difficulty lies in the estimation of the number of such schemes.

II. TREES DESCRIBING THE DYNAMICS OF CLUSTERS

Dynamical clusters grow in time. Here we describe this growth in combinatorial terms.

A. Labeled trees

Assume that $N(\Lambda)$ particles together with their initial coordinates and velocities (that is a point ω in our probability space) are given in the volume Λ. Then by assumption 1 the trajectories of particles are uniquely defined. For given ω, any $t\in[0,\tau]$ and any set A of $N=N(A)\leqslant N(\Lambda)$ particles consider the subgraph $G_A(t)=G_A(t,\omega)$ of $G(t)=G(\tau)$. Then there is a finite number of moments t when the structure of this subgraph changes (the number of edges increases). We will be interested only in the moments when some connected components join together.

Denote $A_{-j}, j=1,2,\ldots,N_1\leqslant N$, the connected components of $G_A(0)$. Assume first, for convenience, that $N_1=N$, that is all these connected components are the one-point subsets of A.

Denote

$$0<t_1<\cdots<t_l<\cdots<t_{N-1}\leqslant\tau$$

all moments when the number of connected components decreases (by assumption 2 it decreases by 1). Denote A_k, $k>0$, the connected component which appeared at time t_k. It is a union of two nonintersecting connected components $A_{i(k)}$ and $A_{j(k)}$ for some $i(k)$, $j(k)<k$. One can say that t_k is the first collision time of the clusters $A_{i(k)}$ and $A_{j(k)}$. For example, $t_1=t_1(\omega)$ is the first moment when a pair of particles from A begin to interact. Thus the cluster A_1 consists of two vertices.

Now define the tree $T(A)=T(A,\omega)$ with $2N-1$ vertices. N vertices of this tree are labeled by the initial coordinates $x_1(0),\ldots,x_N(0)$, that is by clusters of $G_A(0)$. Denote this set of vertices V_0. Let us agree that to all vertices of V_0 time moment 0 is assigned. Other $N-1$ vertices are labeled by time moments t_i (or by the clusters A_i). Denote this set of vertices $V_1=\{1,\ldots,N-1\}$. If A_k is the union of connected components $A_{i(k)}$ and $A_{j(k)}$ then we draw a directed edge from $i(k)$ to k and

directed edge from $j(k)$ to k. Now V_0 can be defined as the set of vertices having no ingoing edges. The level of the vertex w is defined to be the maximal length of the (directed) paths from V_0 to w. Thus the vertices of level 0 are exactly the vertices of V_0.

Define the complete order R on V_1: $w<w'$ iff $t(w)<t(w')$. Note that this complete order is compatible with the natural partial order on the tree defined as: $w<w'$ iff $A_w \subset A_{w'}$.

At time t_k the connected component A_k appears because some particle from $A_{i(k)}$ and some particle from $A_{j(k)}$ interact at time t_k. Denote the pair of vertices from V_0, corresponding to these two particles, as $f(i(k)), f(j(k)) \in V_0$, where $f: V_1 \to V_0$ is the function, which to any $w \in V_1$ assigns an element of A_w.

Thus, each cluster with N points defines a tree T, some complete order relation R, defined by the time moments t_k, on the set V_1, and a function $f(w)$, which assigns to each vertex w a vertex $f(w)$ of level 0, lying under w.

Formally we did not assume that the particles from A do not interact with particles outside A on time interval $[0,\tau]$. However further we will consider the trees only for this case.

B. Unlabeled structured trees

There is a continuum of the labeled trees, introduced previously. Note that all trees $T(A,\omega)$ have the following property: each vertex (except those of level 0) has exactly two adjacent vertices of lower level. Denote the corresponding class of unlabeled trees $\mathcal{T}(N)$, N is thus the number of zero level vertices. Note that the number of trees in $\mathcal{T}(N)$ does not exceed C^N for some absolute constant $C>0$ and that for any (directed) unlabeled tree $T \in \mathcal{T}(N)$ the sets $V_0=V_0(T)$, $V_1=V_1(T)$ are uniquely defined.

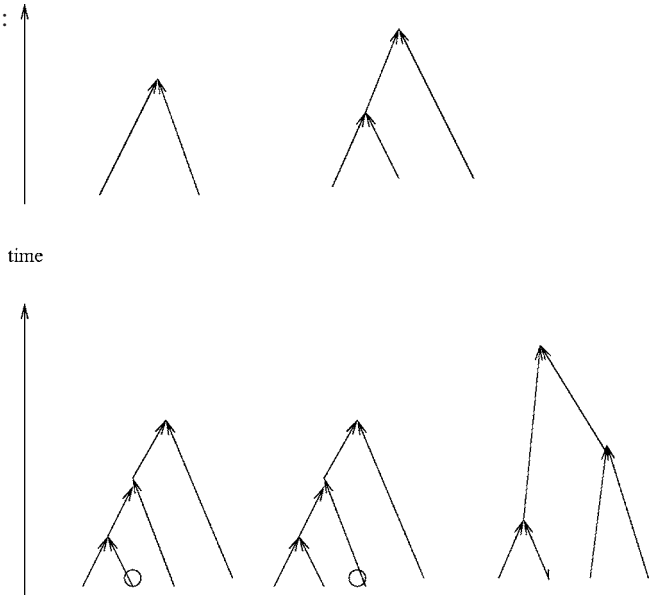
The triple

$$\mathbf{B} = (T,R,f)$$

will be our underlying combinatorial structure. Here $T \in \mathcal{T}(N)$, R is a complete order on $V_1(T)$ and function f on $V_1(T)$, where $f(w)$ is a directed path p from V_0 to w.

Now consider the set $\mathcal{B}(N)$ of equivalence classes of such triples under isomorphisms ψ of trees respecting the complete order R and such that $\psi f = f\psi$. Note that the number of elements in $\mathcal{B}(N)$ is finite for any N. The elements of $\mathcal{B}(N)$ can be called collision schemes (of cluster formation).

The following diagram shows unique collision schemes for $N=2,3$ and all 3 collision schemes for $N=4$:



On the first two isomorphic trees for $N=4$ the small circles show the particle of the left-most cluster which interacts at time t_3 with the right-most particle.

C. Main combinatorial estimate

Note first, that in the following sections one of the vertices of V_0 will be specified. This vertex will correspond to the particle situated in some fixed point x. One could agree that the isomorphisms, introduced previously, respect the specified vertices. Choice of the specified vertex gives factor N in combinatorial estimates. It is neglectable in the foregoing estimates, and we do not consider it in this section.

For any T we shall get an upper bound for $|\mathcal{B}(T,N)|$, the number of triples from $\mathcal{B}(N)$ with given T.

Let $K(T)\leqslant(N-1)!$ be the number of ordering R. For any vertex w let D_w be the number of elements in the subcluster A_w In total this gives the factor

$$D(T)=\prod_{w=1}^{N-1} D_w.$$

The main combinatorial estimate is the following result, which is interesting in its own.

Lemma 2.1: For any tree T

$$|\mathcal{B}(T,N)|\leqslant Q(T,N)=K(T)D(T)<C^N N! \tag{2.1}$$

for some absolute constant $C>0$.

Proof: The first inequality is evident. To prove the second we need the following notation. Let S_k, S_l be two completely ordered sets with k and l elements correspondingly. Denote $R(k,l)$ the number of complete orderings of the set $S_k\cup S_l$ which do not change the order inside S_k and inside S_l. Take, for example, $k\leqslant l$, then

$$R(k,l)=2\sum_{i=1}^{k} C_{k-1}^{i-1}C_{l-1}^{i-1}.$$

In fact, we can split each of S_k and S_l on $i=1,\ldots,k$ consecutive nonempty groups and arrange these groups in a sequence in alternative order. For example, S_k can be split on i consecutive parts by putting $i-1$ walls on $k-1$ empty places between consecutive elements of S_k, that gives the factor C_{k-1}^{i-1}.

For any tree T, $|T|=N$, we have the recurrent relation

$$Q(T,N)=NQ(k,T_1)Q(N-k,T_2)R(k,N-k) \tag{2.2}$$

if under the root vertex of T there are trees T_1 and T_2 with $|T_1|=k$, $|T_2|=N-k$ correspondingly. For $q=\log_2 Q$ we have

$$q(T,N)=\log N+q(k,T_1)+q(N-k,T_2)+\log R(k,N-k). \tag{2.3}$$

One can easily get uniform estimates separately for $K(T)\leqslant N!$ and $D(T)\leqslant N!$, but it is too rough, because there are cases where $K(T)=N!$, $D(T)=C^N$ and vice versa. This is seen from the following two examples. In the first one for the sequence of subclusters

$$\{1,2\},\{1,2,3\},\ldots,\{1,2,\ldots,N-1\}$$

we have $K(T)=1$, $D(T)=(N-1)!$.

From the second example one sees, moreover, that the estimate (2.1) cannot be improved. Put $N=2^n$ and consider the tree T with 2^{n-k} vertices on levels $k=0,\ldots,n$. Denote $Q(n)=K(T)D(T)$. We have the recurrent relation

$$Q(n) = 2^{2n}Q^2(n-1)R(2^{n-1},2^{n-1}).$$

It follows

$$Q(n) \leqslant 2^{2n}Q^2(n-1)a\frac{2^N}{\sqrt{N}}$$

for some constant $a>0$. For $q(n)=\log_2 Q(n)$ this gives for $n \geqslant 2$

$$q(n) \leqslant 2q(n-1) + 2^n + bn, q(1) = 1$$

for some constant $b>0$. The solution of this inequality is

$$q(n) \leqslant \sum_{k=1}^{n} 2^k(2^{n-k} + b(n-k)) = n2^n + b2^n\sum_{k=1}^{n}(n-k)2^{-n+k} = (n+c)2^n$$

for some $c>0$. This gives

$$Q(N) \leqslant 2^{N\log N + cN}.$$

Now we come to the general case. Put $Q(N)=P(N)N!$, then from (2.2) we get

$$P(N) \leqslant NP(k)P(N-k)r(k,N-k),$$

where

$$r(k,N-k) = \frac{2\sum_{i=1}^{k} C_{k-1}^{i-1}C_{N-k-1}^{i-1}}{C_N^k}$$

for some $k \leqslant [N/2]$. For $p(N)=\log P(N)$ we have

$$p(N) = p(k) + p(N-k) + a(v),$$

$$a(v) = a(k,N-k) = \log(Nr(k,N-k)).$$

This equation can be solved explicitely as

$$p(N) = \sum_{v} a(v).$$

To make estimation of this sum we need some notation and results. In the inductive procedure for a given tree T we will distinguish vertices of type A or B, where correspondingly $k \in [\alpha[N/2],[N/2]]$ and $k \in [0,\alpha[N/2]]$ where $\alpha=1-\varepsilon$ for some small $\varepsilon>0$. Denote their numbers N_A and N_B correspondingly. We have $N_A+N_B=N-1$.

Introduce the depth $m(v)$ of the vertex v of the tree—the distance from the root vertex. It is clear that $\log N \leqslant m(v) \leqslant N$. The A-depth $m_A(v)$ of the A-vertex v is the number of A vertices on the path from it to the root. We have

$$m_A(v) < \log_b N, \quad b = \tfrac{1}{2}(1+\epsilon). \tag{2.4}$$

The number

$$\sharp(v{:}m_A(v) = m) < c^m, \quad c = 2(1+\epsilon). \tag{2.5}$$

Denote $N(v)$ the number of level 0 vertices under v. We have

$$c_-^{m(v)} < N(v) < Nc_+^{m(v)}, \tag{2.6}$$

where

$$c_- = \tfrac{1}{2}(1-\epsilon), \quad c_+ = \tfrac{1}{2}(1+\epsilon).$$

We will need some inequalities.

(1) For any k,N

$$\frac{2\sum_{i=1}^{k} C_{k-1}^{i-1} C_{N-k-1}^{i-1}}{C_N^k} \leqslant 1. \tag{2.7}$$

This can be proved by simple combinatorial argument. Take four intervals $1,2,[3,k+1],[k+2;N]$. Then we choose k elements from these two intervals. The number $2C_{k-1}^{i-1}C_{N-k-1}^{i-1}$ gives only restricted choice: we choose one of the first two elements (factor of 2), $i-1$ elements from the last interval (factor C_{N-k-1}^{i-1}) and $k-i$ elements from the third interval (factor C_{k-1}^{i-1}).

(2) (Large deviation estimate) We will use the asymptotics

$$\log C_N^{\alpha N} \sim H(\alpha)N, H(\alpha) = -\alpha \log \alpha - (1-\alpha)\log(1-\alpha).$$

Then for $\gamma \leqslant \beta \leqslant \frac{1}{2}$ the maximum of

$$\log C_{\beta N}^{\gamma N} + \log C_{(1-\beta)N}^{\gamma N}$$

is attained for γ satisfying

$$\frac{1}{\beta}\log\frac{\beta-\gamma}{\gamma} + \frac{1}{1-\beta}\log\frac{1-\beta-\gamma}{\gamma} = 0.$$

In fact,

$$\log C_{\beta N}^{\gamma N} + \log C_{(1-\beta)N}^{\gamma N} \sim N\left(H\left(\frac{\gamma}{\beta}\right)\beta + H\left(\frac{\gamma}{1-\beta}\right)(1-\beta)\right),$$

$$\frac{d}{d\gamma}\left(H\left(\frac{\gamma}{\beta}\right)\beta + H\left(\frac{\gamma}{1-\beta}\right)(1-\beta)\right) = \frac{1}{\beta}\log\frac{\beta-\gamma}{\gamma} + \frac{1}{1-\beta}\log\frac{1-\beta-\gamma}{\gamma}.$$

Then by large deviation principle for any $k \sim \beta N$, $\beta < \frac{1}{2}$

$$\log r(k, N-k) < (1-\delta)^N. \tag{2.8}$$

We estimate $a_v < N$ for B vertices and $a_v < N(v)c_+^{m(v)}$ for A vertices. Using the bounds (2.4)–(2.7) we get the proof. □

III. GEOMETRY AND PROBABILITY OF CLUSTERS

In the previous section we assigned to any ω and any cluster A the triple $\mathbf{B}=\mathbf{B}(\omega,A)$. Here we follow inverse way: for any triple $\mathbf{B} \in \mathcal{B}(\mathbf{N})$ we describe the set of ω and the set of clusters A, containing given point x, such that $\mathbf{B}=\mathbf{B}(\omega,A)$.

Note that in finite volume Λ any cluster is finite, thus for any x

$$\sum_{k=1}^{\infty} P_k^{\Lambda}(\tau|x) = 1.$$

To prove the theorem we will describe the set of initial configurations containing k particles $X = (x_1(0), v_1(0), \ldots, x_k(0), v_k(0))$ which would give a cluster with exactly k particles, if other particles in Λ were not taken into account. We estimate by 1 the conditional probability (given X) that

there are no other particles, which could interact with the particles of the cluster on the time interval $[0,\tau]$. We get these estimates uniformly in Λ and x.

For any $\mathbf{B}=(T,R,f)$ and any sequence $0<t_1<\cdots<t_{N-1}\leqslant\tau$ define the set $I(\mathbf{B},t_1,\ldots,t_{N-1})$ of initial configurations ω of particles such that there exists a set $A,|A|=N$, among them (containing some particle at x) such that A defines a cluster, $N(A)=N$ and $\mathbf{B}(A,\omega)=\mathbf{B}$. The order of t_k, of course, should be compatible with the order R. Denote

$$I(\mathbf{B}) = \cup_{t_1,\ldots,t_{N-1}} I(\mathbf{B},t_1,\ldots,t_{N-1}).$$

We use the estimate

$$P_N^{\Lambda}(\tau|x) \leqslant \sum_{\mathbf{B}} P(I(\mathbf{B})|x).$$

In the previous section we have proved the factorial estimate of the number of terms in the sum $\Sigma_{\mathbf{B}}$. Thus from Lemma 1 and the following Lemma 2 the theorem follows.

Lemma 3.1: The following bound holds uniformly in $\mathbf{B}$

$$P(I(\mathbf{B})|x) \leqslant \frac{(C\alpha)^{N-1}}{N!}$$

for some absolute constant $C>0$.

The proof of this lemma is an easy (but unwieldy) matter and will be given in the next section. It is based on simple geometric considerations with piecewise smooth trajectories and simple facts about Poisson point fields.

IV. PROOF OF LEMMA 2

It is very instructive to understand first the proof for $N=2,3$. Anyway, this is the first step of the inductive procedure below.

A. Two particle cluster

For $N=2$ we have the only tree and the only $\mathbf{B}$. We have to describe geometrically all two particle clusters

Let two particles 1 and 2 have initial velocities v_1 and v_2 correspondingly. Assume that the initial coordinate of particle 1 is $x_1(0)=x$. For any r denote $S_r(x)$ the $(d-1)$-dimensional sphere of radius r and centre $x\in R^d$, let $B_r(x)$ be the corresponding d-dimensional ball. Put $S_r=S_r(0)$ and $B_r=B_r(0)$.

At the first collision time t_1 the particles 1 and 2 should be at the points $x_1(t_1)=x_1(0)+t_1v_1$ and $x_2(t_1)=x_1(0)+t_1v_1+y_1$ correspondingly, where y_1 is any point on the sphere S_{2r}, such that for any $0\leqslant t<t_1$ the spheres $S_r(x_1(0)+tv_1)$ and $S_r(x_2(0)+tv_2)$ do not intersect. This event $G_1(t_1,y_1)$ can also be defined as the event that initially at the point $x_2(0)=x+y_1+t_1(v_1-v_2)$ there is a particle. In other words the relative initial coordinate $x_2(0)-x_1(0)$ of the particle 2 is

$$z_{12}(0) = x_2(0) - x_1(0) = y_1 + t_1(v_1 - v_2).$$

Thus $x_2(0)$, for given $v_1,v_2,t_1,x_1(0)$, should be in a subset of the sphere

$$u_1(t_1) = S_{2r}(x + t_1(v_1 - v_2)) = S_{2r} + x + t_1(v_1 - v_2)$$

in R^d.

Consider the majorizing event

$$G_1(t_1) = \cup_{y_1\in S_{2r}} G_1(t_1,y_1)$$

that there is at least one particle in $u_1(t_1)$. The union

$$G = \cup_{t_1 \in [0,\tau]} G(t_1)$$

is the event that initially there is at least one particle in the union

$$U = \cup_{t_1 \in [0,\tau]} u_1(t_1)$$

which is a closed subset of Λ, parametrized (not one-to-one) by $[0,\tau] \times S_{2r}$. The volume of U does not exceed $C\tau v^0 r^{d-1}$, uniformly in v_1, v_2 (here the boundedness of velocities is used), where C_d is some absolute constant. That is obtained by integration in t_1. Thus the probability that initially there is a particle in this volume does not exceed

$$1 - \exp(-C_d \alpha).$$

This gives the bound $C_d \alpha$ if α is small.

Note that we estimated the conditional probability for given v_1 and v_2, we should now integrate over the velocity distributions

$$\int \int \mathrm{d}\mu(v_1)\mu(v_2),$$

where μ is the initial velocity distribution.

Given the initial coordinates of the two particles, we estimate from above the conditional probability that there are no more particles in their vicinity so that they could interact with them on the time interval $(0,\tau]$, by 1.

B. Three particle cluster

For general N, and in particular for $N=3$, we should only describe the set of relative coordinates of all particles. Then we choose one particle and put its coordinate equal x (this will give a nonessential factor N), thus shifting correspondingly all configuration.

For $N=3$ we also have the unique $\mathbf{B}=(T,R,f)$. Denote velocities of the particles (say 1 and 2) colliding at time t_1 correspondingly v_1, v_2. We can assume that at time t_2 the particle 3, having velocity v_3 collides with particle 1.

Introduce the events $G_1(y_1,t_1), G_1(t_1)$ exactly as in the case of two particles. Let $x_1(s) = x_1(s;y_1)$ be the trajectory of particle 1 for $s \in [t_1,t_2]$, it depends on y_1 (or on $x_2(0)$). Introduce the conditional (given t_1, y_1) event $G_2(t_2,y_2;t_1,y_1)$ that at the moment of collision of the particle 3 with particle 1

$$x_3(t_2) = x_1(t_2) + y_2, y_2 \in S_{2r}.$$

In other words

$$x_3(0) = x_1(t_2) + y_2 - v_3 t_2.$$

Consider the events

$$G_2(t_2;y_1,t_1) = \cup_{y_2 \in S_{2r}} G_2(t_2,y_2;t_1,y_1),$$

$$G_2([s,s'];t_1,y_1) = \cup_{t_2 \in [s,s']} G_2(t_2;t_1,y_1).$$

Using the independence property of Poisson point field, we can estimate, uniformly in y_1, the probability of the event $G_2([t_2,t_2+\mathrm{d}t_2];t_1,y_1)$ as

$$\rho C v^0 r^{d-1} \mathrm{d}t_2.$$

Thus the probability of the event

$$\cup_{s_1\in[t_1,t_1+dt_1]}\cup_{s_2\in[t_2,t_2+dt_2]}(G_1(y_1,s_1)\cap G_2(s_2;t_1,y_1))$$

can be estimated as

$$\rho C v^0 r^{d-1} dt_1 \rho C v^0 r^{d-1} dt_2.$$

Put also

$$G=\cup_{t_1\in[0,\tau]}(G_1(y_1,t_1)\cap G_2(t_1,y_1))=\cup_{t_1\in[0,\tau]}\cup_{t_2\in[t_1,\tau]}(G_1(y_1,t_1)\cap G_2(t_2;t_1,y_1)).$$

Then the probability of the latter event G is estimated by

$$\int_0^\tau C\rho v^0 r^{d-1} dt_1 \int_{t_1}^\tau C\rho v^0 r^{d-1} dt_2 = (C\rho v^0 r^{d-1})^2 \frac{\tau^2}{2!}.$$

C. General case

Assume now for any N that **B** and the velocities $v_1,\dots,v_N$ of the particles are fixed. We should describe the possible initial coordinates $x_1(0),\dots,x_N(0)$ which give rise to a cluster with N particles. The emergence of the cluster occurs as a sequence of $N-1$ conditional events

$$G_1(t_1,y_1),\dots,G_k(t_k,y_k;t_{k-1},y_{k-1};\dots;t_1,y_1),\dots,G_{N-1}(t_{N-1},y_{N-1};t_{N-2},y_{N-2};\dots;t_1,y_1)$$

at the moments $t_1,\dots,t_{N-1}$. We define these events inductively.

Our inductive assumption is that after step k the vectors $y_1,\dots,y_k\in S_{2r}$ and the events $G_1,\dots,G_k$ are defined. Moreover, there are $m(k)=N-k$ maximal (that is not contained in other clusters) clusters $A_{k,1},\dots,A_{k,m(k)}$ at time t_k, as on each step the number of clusters decreases by 1. For each $i=1,\dots,m(k)$, denote a vertex of the tree, corresponding to the cluster $A_{k,i}$, by i. Assume that at time moment t_{k+1} the clusters $A_{k,i(k+1)}$ and $A_{k,j(k+1)}$ (the choice of these clusters is unique as dictated by **B**) collide and the colliding particles be $f(i(k+1))$ and $f(j(k+1))$. Moreover, after k steps the initial relative coordinates inside all maximal clusters are fixed, they enter the definition of the events $G_1,\dots,G_k$.

Thus at time t_{k+1} the particle $f(i(k+1))$ will be at the point $x_{f(i(k+1))}(t_{k+1})$, the trajectory $x_{f(i(k+1))}(s)$ on the time interval $[t_k,t_{k+1}]$ depends in a rather complicated way on the relative coordinates inside the cluster $A_{i(k+1)}$ and in a simple way (translation invariance) on the initial coordinate $x_{f(i(k+1))}(0)$. Then the particle $f(j(k+1))$ will be at the point $x_{f(i(k+1))}(t_{k+1})+y_{k+1}$. Thus the initial coordinate of the particle $f(j(k+1))$ is

$$x_{f(i(k+1))}(t_{k+1})+y_{k+1}-x^{(0)}_{f(j(k+1))}(t_{k+1}),$$

where the upper index 0 means that the initial coordinate of the particle $f(j(k+1))$ is 0, and the relative coordinates inside this cluster were fixed in the inductive process. Here we use the translation invariance of the dynamics. It follows that we know now the initial relative coordinate of the particles $f(i(k+1))$ and $f(j(k+1))$, thus all relative initial coordinates inside the new cluster $A_{k+1}=A_{k,i(k+1)}\cup A_{k,j(k+1)}$.

We have, as in case $N=3$, the inductive estimates

$$\rho C v^0 r^{d-1} dt_k$$

for the probabilites of the events

$$G_k([t_k,t_k+dt_k])=\cup_{y_k\in S_{2r}}\cup_{s_k\in[t_k,t_k+dt_k]}G_k(s_k,y_k;t_{k-1},y_{k-1};\dots;t_1,y_1)$$

uniformly in $t_{k-1},y_{k-1},\dots,t_1,y_1$. Then for the product of events

$$\cap_{k=1}^{N-1} G_k([t_k, t_k + \mathrm{d}t_k])$$

we get the estimate

$$\prod_{k=1}^{N-1} \rho C v^0 r^{d-1} \mathrm{d}t_k.$$

Then, using the known formula

$$\int_0^\tau \cdots \int_{t_{N-3}}^\tau \int_{t_{N-2}}^\tau \mathrm{d}t_{N-1} \cdots \mathrm{d}t_1 = \int_0^\tau \cdots \int_0^{t_3} \int_0^{t_2} \mathrm{d}t_1 \cdots \mathrm{d}t_{N-1} = \frac{\tau^{N-2}}{(N-2)!}$$

we get the estimate of Lemma 2.

We assumed throughout the proof that the graph $G_A(0)$ has no clusters other than one-point clusters. If however there are some clusters at time zero, then the proof proceeds along the same lines. Moreover, it is easy to see, using our assumption $r \ll v^0 \tau$, that the cluster probability will be even smaller.

V. EXAMPLE OF DYNAMICS

Here we give an example of dynamics "with chemical reactions," satisfying the above general conditions. This dynamics however will be a random dynamics. In this dynamics randomness occurs because of random initial conditions and random interaction rules of dynamics.

Define first the finite volume dynamics. There are $N(\Lambda) < \infty$ particles in the cube Λ, each particle is characterized at time $t \in R_+$ by its coordinate $x(t) \in \Lambda \subset R^d$, velocity $v(t) \in R^d$ and type $q(t) \in \{1, \ldots, Q\}$. Initial coordinates $x_i(0)$ are distributed uniformly in Λ. The vectors $(q_i(0), v_i(0))$ are independently distributed with densities $p_q(v)$

$$\sum_q \int p_q(v) \mathrm{d}v = 1.$$

We will define a continuous time Markov process $(x_i(t), v_i(t), q_i(t) : i = 1, 2, \ldots, N)$ together with initial data $(x_i(0), v_i(0), q_i(0) : i = 1, 2, \ldots, N)$. It is a mixture of piecewise linear movement for the coordinates and random jumps for the velocities and types. We will not write down the generator of this process, but rather describe it more intuitively. The coordinates are defined as

$$x_i(t, \omega) = x_i(0, \omega) + \int_0^t v_i(t, \omega) \mathrm{d}t.$$

At the same time, any pair of particles i and j (independently of other pairs) at any time interval $(t, t+dt)$ can change their types and velocities with rates

$$\lambda(q_i(t) \,.\, v_i(t), q_j(t) \,.\, v_j(t), x_i(t) - x_j(t)) \chi_{2r}(x_i(t) - x_j(t)),$$

where $\chi_{2r}(x) = 1$ if $|x| \leqslant 2r$, and 0 otherwise. Functions λ are assumed to be bounded. As a result of this jump coordinates do not change, but types and velocities change

$$(q_i, v_i, q_j, v_j) = (q_i, v_i, q_j, v_j)(t) \to (q_i', v_i', q_j', v_j') = (q_i', v_i', q_j', v_j')(t + 0)$$

via conditional probability densities $P(q_i', v_i', q_j', v_j' \,|\, q_i, v_i, q_j, v_j)$ so that for any q_i, v_i, q_j, v_j we have

$$\sum_{q_i', q_j'} \int P(q_i', v_i', q_j', v_j' | q_i, v_i, q_j, v_j) dv_i' dv_j' = 1.$$

Our main assumption that velocities are uniformly bounded that is

$$P(q_i',v_i',q_j',v_j'|q_i,v_i,q_j,v_j)=0$$

if v_i' or v_j' exceed some constant $v^0>0$.

Thus, the velocities and types are piecewise constant on $[0,\infty]$, jumps occur at discrete time moments

$$0<t_1(\omega)<\cdots<t_k(\omega)<\cdots .$$

The process is well defined—$x_i(t,\omega)$ exist for any initial data and are piecewise linear a.s. This means that between the jumps the particles move freely with constant velocities $v_i(t)=\mathrm{d}x_i(t)/\mathrm{d}t$ defined a.e.

All previous results for the deterministic dynamics hold for this random dynamics with piecewise random trajectories as well. In this example, from this one can easily prove, in particular, the existence of the thermodynamic limit for small times. This limit coincides with the directly and similarly defined infinite particle dynamics.

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