Markov Processes Relat. Fields **24**, 337–365 (2018)

Markov Processes and Related Fields

Classical Microscopic Electrodynamics: short introduction

V.A. Malyshev

Faculty of Mechanics and Mathematics, Lomonosov Moscow State University. Leninskie Gory 1, Moscow, 119991 Russia. E-mail: 2malyshev@mail.ru

Received March 12, 2018

Abstract. I tried to do this introduction both short and self-contained but I hope that it could serve at least as a list of what is necessary to know. For more details there are many books in References. Also I wanted to emphasize what can be considered now as absolutely rigorous in this science.

KEYWORDS: electrodynamics, wave equation, Maxwell, Lorentz, short introduction

AMS SUBJECT CLASSIFICATION: 81S99

Contents

1. Elementary vector analysis

We shall consider scalar and vector functions (and will call them fields) on $R^3 = \{x = (x_1, x_2, x_3)\}$. We assume them as smooth as necessary and hope that the reader will be able to formulate it properly whenever necessary. It is useful sometimes to compare smooth objects with the corresponding objects on the lattice $Z^3 = \{v = (v_1, v_2, v_3)\}$ – these will be scalar functions on the set of vertices v and edges $l = l_{v,i} = (v, v + e_i)$, where $e_i, i = 1, 2, 3$, are the unit coordinate vectors. For the lattice the proofs are often essentially simpler and even obvious.

Main differential operators **The gradient** transforms the scalar function $\varphi(x)$, $x \in R^3$, (potential) to the vector function

$$grad\varphi(x) = (\bigtriangledown\varphi)(x) = \Big(\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \frac{\partial\varphi}{\partial x_3}\Big), \quad \bigtriangledown = \Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\Big),$$

the gradient is a measure of steepness in coordinate directions. Minus *grad* defines the direction of the force.

On the lattice the potential is a scalar function $\varphi(v)$ on the set of vertices, and its gradient is the function

$$(grad\varphi)(l_{v,i}) = \varphi(v + e_i) - \varphi(v)$$

on the set of edges, or the vector at point v

$$(grad\varphi)(v) = (\varphi(v + e_1) - \varphi(v), \varphi(v + e_2) - \varphi(v), \varphi(v + e_3) - \varphi(v))$$

Divergence of vector field, on the contrary, transforms the vector function $A(x) = (A_1(x), A_2(x), A_3(x))$ to the scalar function

$$divA(x) = (\bigtriangledown, A)(x) = \sum_{i=1}^{3} \frac{\partial A}{\partial x_i}, \quad (divA)(v) = \sum_{i=1}^{3}(A(l_{v,i}) - A(l_{v-e_i,i}))$$

If the vector field corresponds to some flow $A_i = \rho u_i$ (for example the flow of liquid, where ρ is the density (mass or charge density), and $\{u_i\}$ is the velocity vector, then the derivative $\partial A_i/\partial x_i$ measures flow in (or flow out) to x in the direction i.

Rotor (curl) $rot\, A(x)$ of the vector field measures "rotation" of the field A around point x. This is the vector field, which components can be considered as vectors along coordinate axes perpendicular to the corresponding coordinate plane. It is defined as

$$rot\, A = \nabla \times A,$$

where the vector product of two vectors A and B is defined as

$$A \times B = (A_2 B_3 - A_3 B_2, A_3 B_1 - A_1 B_3, A_1 B_2 - A_2 B_1)$$

For the lattice the rotor of vector field A can be considered as the function on the set of two-dimensional cells $\Delta_{v,i,j}$, spanned by two edges $l_{v,i}, l_{v,j}, i \neq j$. For example,

$$(rotA)(l_{v,3}) = (rotA)(\Delta_{v,1,2}) = A(l_{v,1}) + A(l_{v+e_1,2}) - A(l_{v+e_1+e_2,1}) - A(l_{v+e_2,2})$$

that is the sum along the boundary of the square, where the signs take into account the direction of the circumvention. Alternatively, it can be considered as the function on the set of the edges of the dual lattice $Z^d + (\frac{1}{2}, \ldots, \frac{1}{2})$, which pass through the center of the squares in perpendicular direction.

The connection between the definitions on R^3 and on the lattice Z^3_ε with lattice step ε becomes more clear if, for example, the component $(\nabla \times A)_1(x)$ is imagined as the limit of the integral along the contour of the contour of the square on the (x_2, x_3)-plane with center in x and having side lengths ε

$$\lim \frac{\oint (A(s), t(s)) ds}{\varepsilon^2}$$

where (A, t) is the scalar product and $t(s)$ are the unit vectors tangent to given contour.

Laplacian of the scalar function φ on R^3 or on Z^3 is also a scalar function and is defined correspondingly as follows

$$(\Delta\varphi)(x) = \sum_{i=1}^{3} \frac{\partial^2 \varphi}{\partial x_i^2},$$

$$(\Delta\varphi)(v) = \sum_{i=1}^{3} (\varphi(v + e_i) - 2\varphi(v) + \varphi(v - e_i))$$

Local identities Below we will need the following trivially checked identities on R^3

$$rotgrad = \nabla \times \nabla = 0, \quad divgrad = \Delta \tag{1.1}$$

$$divrot = 0, \quad rotrot = graddiv - \Delta \tag{1.2}$$

Moreover, the first two hold also on the lattice.

Remark 1.1. Note that the formulas (1.2) for the lattice should be more exactly defined. In fact, rotor can be defined as vector function both on the dual lattice and on the initial Z^3. However, on Z^3_ε with ε small the difference is of the order $O(\varepsilon^2)$, but could play role in some new discrete models of classical electrodynamics.

The following formulas for functions of $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ (at the points where all derivatives exist) are useful

$$grad(\varphi(r)) = \varphi'(r) r_0, r_0 = \frac{x}{|x|} \tag{1.3}$$

$$div(\varphi(r)x) = (x, grad\varphi(r)) + \varphi(r) divx = r\varphi'(r) + 3\varphi(r) \tag{1.4}$$

$$rot(\varphi(r)x) = -[x, grad\varphi(r)] + \varphi(r) rotx = 0 \tag{1.5}$$

$$\Delta\varphi(r) = div(\frac{\varphi'(r)}{r}x) = \varphi''(r) + 2\frac{\varphi'(r)}{r}, \tag{1.6}$$

and for example,

$$\Delta(r^{-1+\varepsilon}) = (-\varepsilon + \varepsilon^2) r^{-3+\varepsilon}$$

To include the singular point $r = 0$, one should (for smooth function $f(r)$) find the limit of the integral

$$\int \Big(\Delta\frac{1}{r}\Big) f(r) dr = \lim_{\varepsilon\to 0} \int \Delta(r^{-1+\varepsilon}) f(r) dr =$$

$$= \lim_{\varepsilon\to 0} (-\varepsilon + \varepsilon^2) \int r^{-3+\varepsilon} f(r) dx =$$

$$= 4\pi \lim_{\varepsilon\to 0} (-\varepsilon + \varepsilon^2) \int r^{-1+\varepsilon} f(r) dr =$$

$$= -4\pi \lim_{,\delta\to 0} \lim_{\varepsilon\to 0} (\varepsilon \int_0^\delta r^{-1+\varepsilon} f(0) dx + O(\delta))$$

$$= -4\pi f(0) \lim_{,\delta\to 0} \lim_{\varepsilon\to 0} (\delta^\varepsilon + O(\delta)) =$$

$$= -4\pi f(0)$$

whence

$$\Delta\frac{1}{r} = -4\pi\delta(x) \tag{1.7}$$

Three integral theorems To better understand the following assertions it is useful to imagine the integrals as the sums over the vertices of a conveniently chosen graph (or lattice), where the proof is a simple cancellations.

Integral along the curve L with ends x_1, x_2 for the scalar function φ

$$\int\limits_L (grad\,\varphi, dL) = \varphi(x_2) - \varphi(x_1) \tag{1.8}$$

On the lattice it is the sum over the edges of the path where all terms are cancelled except two extreme vertices.

Integral over the surface, the boundary of which is the closed curve L (Stokes theorem about the flow of the rotor)

$$\int\limits_S (rot\,A, n)dS = \int\limits_L (A, dL) \tag{1.9}$$

On the lattice it is evident already for the rectangle on one of coordinate planes, where the terms on any internal edge gives zero, and only terms along the boundary survive.

Integral over the volume Λ having boundary surface S (Gauss – Ostrogradskij theorem)

$$\int\limits_\Lambda div\,A d\Lambda = \int\limits_S (A(s), n)dS \tag{1.10}$$

where $n = n(s)$ is the unit vector at the point $s \in S$, perpendicular to S and directed to outside S. It is also easy to understand on the lattice one can think, for example, about a cube Λ, Then after cancellations we are left with the sum over edges of Z^3_ε which have exactly one vertex of which belongs to the boundary of the cube. Note that the edges of Λ should not be taken into account because, while $\varepsilon \to 0$, their measure with respect to the measure of S becomes infinitely small.

Helmholtz decomposition Further on we assume that the considered functions on R^3 are sufficiently smooth in some (simply connected) domain G. Moreover, if G is unbounded then all necessary derivatives of these functions decay at infinity.

Note first that the equation $grad\varphi = 0$, on R^3 and on Z^3, evidently has only constant solutions. This is wrong for the equations $divA = 0$ and $rotA = 0$. Their solutions are called solenoidal and irrotational vector fields correspondingly.

Theorem 1.1. *Any sufficiently smooth vector field $A(x)$ in G can be uniquely decomposed (up to additive constants) on twp vector functions: irrotational A_{curl} and solenoidal A_{sol}*

$$A = A_{irrot} + A_{sol}$$

Proof. Using (1.7), that is

$$\delta(x-y)=-\frac{1}{4\pi}\Delta_x\frac{1}{|x-y|}$$

and the second formula of (1.2) we have

$$A(x)=\int\delta(x-y)A(y)dy=-\frac{1}{4\pi}\int(\Delta_x\frac{1}{|x-y|})A(y)dy=$$

$$=-\frac{1}{4\pi}\int(graddiv_x-rotrot_x)\frac{1}{|x-y|})A(y)dy=A_{irrot}+A_{sol}$$

where

$$A_{sol}=rot\mathrm{B(x)},\quad \mathrm{B(x)}{=}\frac{1}{4\pi}rot_x\int\frac{1}{|x-y|}A(y)dy, \tag{1.11}$$

$$A_{irrot}=-grad\varphi,\ \ \varphi=\frac{1}{4\pi}div\int\frac{1}{|x-y|}A(y)dy \tag{1.12}$$

It remains to check that

$$divA_{sol}=rotA_{irrot}=0$$

using formulas (1.1) and (1.2). □

Note that if the vector field A satisfies both equations $divA=rotA=0$ then also $\Delta A=0$. More generally, we have the following theorem.

Theorem 1.2. *The solution of the system of equations*

$$\nabla\times A=C(x),\mathrm{div}A=D(x),$$

for any (generalized) functions C,D, decaying at infinity sufficiently fast and such that $divC\equiv 0$, exists and is unique up to additive constant.

Proof. See [1, 9].

2. Wave equation

Below we shall often encounter wave equations. Moreover, Maxwell equations are, in some sense, a couple of wave equations. That is why it is helpful to look at wave equations in more detail.

Wave equation with unknown function $u(x,t),x\in R^d,t\in R$, is defined as

$$Lu=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}-\Delta u=f(x,t) \tag{2.1}$$

with initial conditions (Cauchy problem for $x \in R^d, t \geq 0$)

$$u(x,0) = f_0(x), \quad \frac{\partial u}{\partial t}(x,0) = f_1(x) \tag{2.2}$$

The solution is then the sum of some solution $u_f(x,t)$ of inhomogeneous equation (2.1) and the solution of homogeneous equation (when $f(x,t) = 0$) with initial conditions

$$u(x,0) = f_0(x) - u_f(x,0), \quad \frac{\partial u}{\partial t}(x,0) = f_1(x) - \frac{\partial}{\partial t} u_f(x,0)$$

Often one can find solution u_f with zero initial conditions.

Due to linearity all functions here are allowed to be generalized functions (distributions).

2.1. One-dimensional wave equation

D'Alembert solution of the homogeneous equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \tag{2.3}$$

Note that all derivatives of u satisfy the same wave equation.

Simple remark is that for any function $F(x)$ both functions $F(x \pm ct)$ are solutions of the homogeneous equation. Examples are two "photon" solutions

$$u = \delta(x \pm ct) \tag{2.4}$$

Remark 2.1. The solution (2.4) gives a hint that wave equation is tightly connected with classical point particle mechanics. This corresponds also to what physicists said long ago: Pauli assumed that light energy is radiated by quanta $\varepsilon = h\omega$ where ω is the frequency, Einstein assumed that this quantum of energy has momentum $p = hk$, where $k = \frac{2\pi}{\lambda}$ and λ is the wave length; we shall not consider this connection here.

More generally, any solution of the Cauchy problem can be written as

$$u = F_+(x + ct) + F_-(x - ct) \tag{2.5}$$

for some functions $F_\pm(x)$. In fact, assume, for example, that f_0 and f_1 have compact support. Then the unique solution of the homogeneous equation (2.3) with initial conditions (2.2) is

$$u(x,t) = \frac{1}{2}(f_0(x+ct) + f_0(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} f_1(y)dy \tag{2.6}$$

We are looking for the solution in the form (2.5) and get two equations for the initial conditions

$$F_+(x) + F_-(x) = f_0(x), F'_+(x) - F'_-(x) = \frac{1}{c} f_1(x)$$

Differentiating the first equation and solving the obtained system we get

$$F'_\pm(x) = \frac{1}{2}(f'_0 \pm \frac{1}{c} f_1) \Longrightarrow F_\pm(x) = \frac{1}{2}\left(f_0 \pm \frac{1}{c} \int\limits_0^x f_1 dx + C_\pm \right) \Longrightarrow$$

$$\Longrightarrow F_+ + F_- = f_0 + C_+ + C_-$$

This gives $C_+ + C_- = 0$, and then (2.5) implies (2.6).

Escape to infinity It is easy to see that for any bounded interval $I \subset R$ the solution will be zero or constant (that is force field will be zero) starting from some $t(I)$.

For example, for any smooth $F(x)$ define the set $S(F)$ as the closure of the set $\{x : F'(x) \neq 0\}$. Consider two cases:

1) $f_1 = 0$ and $supp(f_0) = S(f_0) = [a, b]$. Then $supp(u(t)) = S(f_0(t))$ starting from some t_1 the support $supp(u(t)) = S(f_0(t))$ will consist of two segments $[a - ct, b - ct]$ and $[a + ct, b + ct]$. In particular, L_2-norm becomes equals to the initial L_2-norm;

2) $f_0 = 0$ and $S(f_1) = [a, b]$. Then for any x starting from some $t = t(x)$ and $S(f_1(t))$ will again consist of the same two segments.

Inhomogeneous equation In equation (2.1) the right hand term is called forcing term. The solution of equation (2.1) with $f = \delta(x)$ is called the fundamental solution or the Green's function of this equation.

The following assertion states that the solution $u(x, t)$ of the inhomogeneous equation (2.1) with zero initial conditions can be obtained as

$$u(x, t) = \int\limits_0^t w(x, t, t_1) dt_1 \tag{2.7}$$

where $w(x, t.t_1), 0 \leq t_1 \leq t$, is the solution of the homogeneous equation $Lu = 0$ with initial conditions

$$w(x, t_1, t_1) = 0, \quad \frac{d}{dt} w(x, t, t_1)|_{t=t_1} = f(x, t_1) \tag{2.8}$$

where t_1 is considered as a parameter. Proof is just by substitution

$$Lu = L \int\limits_0^t w(x, t, t_1) dt_1 = \int\limits_0^t Lw(x, t, t_1) dt_1 + \frac{d}{dt} w(x, t, t) = f(x, t)$$

This trick however has deeper interpretation: the forced term at time t_1 provides the velocity change like in Newtonian mechanics. And linearity of the equation allows to take superposition (integral) of these infinitesimally small changes.

If f does not depend on t then for any x the solution $u(x,t)$ stabilizes for t greater than some $t(x)$.

2.2. Three dimensional wave equation

First of all, we consider 3 special and very simple cases.

Plane waves For any functions F_+, F_- and any vector $a = (a_1, a_2, a_3)$ such that $a_1^2 + a_2^2 + a_3^2 = 1$ the following linear combination is a solutions of the homogeneous equation

$$u(x,t) = F_+((a,x) + ct) + F_-((a,x) - ct)$$

Series expansion Equation (2.1) with $f = 0$ and initial conditions (2.2) can be solved as the series

$$u(x,t) = f_0(x) + \sum_{n=1}^{\infty} \frac{(ct)^{2n}}{(2n)!} \Delta^n f_0(x) + t f_1(x) + t \sum_{n=1}^{\infty} \frac{(ct)^{2n}}{(2n+1)!} \Delta^n f_1(x)$$

if this series and its second derivatives are absolutely convergent. The proof is just a substitution.

Spherical symmetry First of all, it is tempting to consider spherically symmetric solutions. For $d = 3$ and any two functions $F_\pm$ there is such solution

$$u(r,t) = \frac{1}{r}(F_+(r + ct) + F_-(r - ct))$$

of the homogeneous wave equation, where $r = \sqrt{x_1^2 + x_2^2 + x_3^2} > 0, x_i \in R$, To see this we will need the following simple facts:

1) Consider spherically symmetric function $w = w(r,t)$ on R^3. By formula (1.6) the function $w(r,t)$ satisfies homogeneous wave equation (2.1) iff it satisfies the so called spherical wave equation

$$\frac{1}{c^2} w_{tt} = \left(w_{rr} + \frac{2}{r} w_r \right) \tag{2.9}$$

2) Under these conditions $u(r,t) = rw$ satisfies one-dimensional wave equation (2.3) with $x = r > 0$. and boundary condition $u(0,t) = u_t(0,t) = 0$. In fact,

$$u_{tt} = r w_{tt} = r c^2 (w_{rr} + \frac{2}{r} w_r) = c^2 (rw)_{rr} = c^2 u_{rr}$$

3) To get

$$w(r,t) = \frac{1}{r}u(r,t)$$

we consider one-dimensional equation for $x = r \in R$ with odd initial conditions

$$f_i(x) = -f_i(-x), i = 1,2$$

Then the solution $u(x,t)$ is given by

$$u(x,t) = \frac{1}{2}(f_0(x+ct) + f_0(x-ct)) + \frac{1}{2c}\int\limits_{x-ct}^{x+ct} f_1(y)dy, \quad x > ct,$$

$$u(x,t) = \frac{1}{2}(f_0(x+ct) - f_0(ct-x)) + \frac{1}{2c}\int\limits_{ct-x}^{x+ct} f_1(y)dy, \quad x < ct,$$

4) The solution for the case with spherically symmetric force term can be obtained similarly.

Fourier transform In the general case the most familiar method could be the Fourier transform. We use

$$w(k,t) = \int\limits_{R^3} u(x,t)e^{-i(x,k)}dx,$$

$$\psi(k,t) = \int\limits_{R^3} f(x,t)e^{-i(x,k)}dx, \qquad \psi_j(k) = \int\limits_{R^3} f_j(x)e^{-i(x,k)}dx, \ j = 0,1$$

This gives the initial value problem for ordinary differential equation:

$$w_{tt} = -c^2|k|^2 w + \psi(k,t), \quad w(k,0) = \psi_0(k), \quad w_t(k,0) = \psi_1(k)$$

Then the Fourier transformed homogeneous equation has evident solution

$$w(k,t) = \psi_0(k)\cos(c|k|t) + \psi_1(k)c^{-1}|k|^{-1}\sin(c|k|t)$$

Denote by $A_i = A_i(x,t), i = 0,1$, the inverse Fourier transforms of $a_1(k,t) = c^{-1}|k|^{-1}\sin(c|k|t)$ and $a_1(k,t) = \frac{\partial}{\partial t}a_1 = \cos(c|k|t)$ correspondingly. Then we get the solution as the sum of two convolutions

$$u(x,t) = A_0 \star f_0 + A_1 \star f_1 = \frac{\partial}{\partial t}(A_1 \star f_0) + A_1 \star f_1 \tag{2.10}$$

as

$$A_o(x,t) = \frac{\partial A_1}{\partial t}$$

Thus to find the solution we have only to find $A_1(x,t)$. But instead of this standard calculation, more interesting is the following method.

Spherical means Denote $B(x,r)$ the ball of radius r with the centre at $x \in R^3$ (its volume is $\frac{4\pi}{3}r^3$), $\partial B(x,r) = S(x,r)$ the boundary of this ball – the sphere with the area $4\pi r^2$. Let $d\sigma = d\sigma^S = d\sigma$ be the uniform measure on $S = S(x,r)$ normalized to the area. and $d\sigma_1 = d\sigma_1^S = \frac{d\sigma}{4\pi r^2}$ be the uniform measure on S normalized to 1.

Define spherical means U, F_j of functions $u, f_j, j = 0, 1$, correspondingly. Namely, for any $r > 0$ put

$$\begin{aligned} U(x,r,t) &= \int\limits_{S(x,r)} u(s,t)d\sigma_1(s), \\ F_j(x,r) &= \int\limits_{S(x,r)} f_j(s)d\sigma_1(s) = \frac{1}{4\pi}\int\limits_{|y|=1} f_j(x+ry)d\sigma(y) \end{aligned} \tag{2.11}$$

The latter expression is defined also for $r \le 0$, then $F_j(x,r)$ are even functions of $r \in R$.

We present 4 formulas, often used in the literature, for the solution of the wave equation (2.1) in case $f = 0$:

$$u(x,t) = \frac{\partial}{\partial t}(tF_0(x,ct)) + tF_1(x,ct) = \tag{2.12}$$

$$= \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\int\limits_{|y-x|=ct} f_0(y)d\sigma(y)\right] + \frac{1}{4\pi c^2 t}\int\limits_{|y-x|=ct} f_1(y)d\sigma(y) = \tag{2.13}$$

$$= \frac{\partial}{\partial t}\left[t\int\limits_{S:|y|=1} f_0(x+cty)d\sigma_1(y)\right] + t\int\limits_{S:|y|=1} f_1(x+cty)d\sigma_1(y) = \tag{2.14}$$

$$= \frac{1}{4\pi c^2 t^2}\int\limits_{\partial B(x,ct)} (f_0(y) + (\nabla f_0(y), y-x) + tf_1(y))d\sigma(y) \tag{2.15}$$

Proof. Enumerate these four terms correspondingly by 1, 2, 3, 4.

1) It is easy to see that term 1 satisfies the initial conditions. In fact, integration over the sphere with radius $r \to 0$ gives

$$F_j(x,0) = \lim_{r\to 0} F_j(x,r) = f_j(x)$$

Then

$$u(x,0) = F_0(x,0) = \lim_{r\to 0} F_0(x,r) = f_0(x),$$

$$\frac{\partial}{\partial t}u(x,o) = 2c\frac{\partial}{\partial r}F_0(x,0) + F_1(x,0) = f_1(x,0)$$

where we used the reflection symmetry

$$F_j(x,r) = F_j(x,-r) \Longrightarrow \frac{\partial}{\partial r} F_j(x,0) = 0$$

Now, for example, we want to prove that $tF_1(x,ct)$ satisfies the wave equation. Note first that for $r - ct$

$$\frac{\partial^2}{\partial t^2}(tF_1(x,ct)) = 2\frac{\partial}{\partial t}F(x,ct) + t\frac{\partial^2}{\partial t^2}F(x,ct) =$$

$$= 2c\frac{\partial}{\partial (ct)}F(x,ct) + c^2 t \frac{\partial^2}{\partial (ct)^2}F(x,ct) =$$

$$= cr\Big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\Big)F_1(x,r) = cr\Delta_x(F_1(x,r)) = c^2\Delta_x(tF_1(x,ct))$$

For the latter equality we used **Darboux formula**

$$\Big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\Big)F_j(x,r) = \Delta_x F_j(x,r) \tag{2.16}$$

that can be proved by direct calculation

$$\frac{\partial}{\partial r}F(x,r) = \frac{1}{4\pi}\int\limits_{|y|=1}(\nabla f_j(x+ry),y)d\sigma(y) = \frac{r}{4\pi}\int\limits_{|y|\le 1}\Delta_x f_j(x+ry)dy$$

2) The transition $1 \to 2$ is the definition of the spherical mean, taking into account that the surface of the sphere is $4\pi c^2 t^2$.

3) 3 is obtained from 2 by change of variables.

4) Term 4 can be obtained from the second one by differentiating. □

Any of these formulas (especially the fourth one) is often called **Kirchhoff formula** (for $f = 0$).

The corollary is the **Huygens principle**: the solution at point x at time t depends on the initial data only at the points at the distance ct from x. And moreover, from all these points equally.

Forcing term As we saw above, wave equation uses manipulations with change of variables. That is why it could be more transparent to get rid of some constants. For example, to consider equation

$$\frac{\partial^2 v}{\partial t^2} - \Delta v = \varphi(x,t) \tag{2.17}$$

instead of (2.1), then

$$u(x,t) = v(x,ct), \quad f(x,t) = \varphi(x,ct)$$

satisfies equation (2.1).

We shall solve non homogeneous equation (2.1) with zero initial conditions exactly as in the one dimensional case, and in the same notation. Thus, as in (2.7)

$$v(x,t) = \int_0^t w(x,t,t_1)dt_1$$

and by Kirchhoff formula (2.15) for $t > t_1$

$$w(x,t,t_1) = \frac{1}{4\pi(t-t_1)} \int_{|y-x|=t-t_1} \varphi(y,t_1)d\sigma(y)$$

and then

$$v(x,t) = \int_0^t w(x,t,t_1)dt_1 = \frac{1}{4\pi}\int_0^t \frac{1}{t-t_1} \int_{|x-y|=t-t_1} \varphi(y,t_1)d\sigma(y)dt_1 =$$

$$= \frac{1}{4\pi}\int_0^t \frac{1}{r}\int_{|x-y|=r} \varphi(y,t-r)d\sigma(y)dr = \frac{1}{4\pi}\int_{|x-y|\le t} \frac{\varphi(y,t-|x-y|)}{|x-y|}dy \quad (2.18)$$

where dy is the Lebesgue measure on R^3. The latter expression is often also called **Kirchhoff formula**. Then

$$u(x,t) = \frac{1}{4\pi}\int_{|x-y|\le ct} \frac{\varphi(y,ct-|x-y|)}{|x-y|}dy = \frac{1}{4\pi}\int_{|x-y|\le ct} \frac{f(y,\frac{ct-|x-y|}{c})}{|x-y|}dy$$

3. Maxwell – Lorentz equations in vacuum

Physical equations Maxwell equations concern two time dependent vector fields on R^3: electric $E = E(t,x)$ and magnetic $H = H(t,x)$ field strengths. Also they concern two types of material objects: point charges and charge densities. The first are defined by the charges q_i (there is no mass in Maxwell equations), coordinates $x_i(t)$ and velocities $v_i(t)$ of the point particles $i = 1,\ldots,N$. The second one is defined by the charge density $\rho(t,x)$ and velocity field $v(t,x)$. However, instead of velocity field the current density $j(t,x) = \rho(t,x)v(t,x)$ is used.

We write down the Maxwell equations in two systems of units: International system (SI) with fundamental units: meter m, kilogram kg, second sec, coulomb C, and also Gaussian unit system with corresponding units cm, gr, sec and electrostatic unit without fixed name. We use also the following three constants:

$c = 3.10^9 m\, sec^{-1}$ – velocity of light in vacuum, $\varepsilon_0 = 10^{-11} C^2 kg\, m^{-2} sec^2$ – dielectric constant of vacuum (vacuum permittivity), $\mu_0 = \frac{1}{\varepsilon_0 c^2}$ – (magnetic) vacuum permeability.

Maxwell equations in SI and Gaussian units

$$\nabla E = \frac{1}{\varepsilon_0}\rho \quad (SI), \qquad \nabla E = 4\pi\rho \qquad (Gaussian) \tag{3.1}$$

$$\nabla H = 0 \quad (SI), \qquad \nabla H = 0 \quad (Gaussian) \tag{3.2}$$

$$\nabla \times E = -\frac{\partial H}{\partial t} \quad (SI), \qquad \nabla \times E = -\frac{1}{c}\frac{\partial H}{\partial t} \qquad (Gaussian) \tag{3.3}$$

$$\nabla \times H = \mu_0 J + \varepsilon_0\mu_0 \frac{\partial E}{\partial t} \quad (SI), \quad \nabla \times H = \frac{4\pi}{c}J + \frac{1}{c}\frac{\partial E}{\partial t} \quad (Gaussian) \tag{3.4}$$

Lorentz force

$$F = q(E + v \times H) \quad (SI), \qquad F = q\Big(E + \frac{v}{c} \times H\Big) \quad (Gaussian) \tag{3.5}$$

acts on the point particle of charge q and mass m, and forces it to move by Newton's law

$$m\frac{d^2x}{dt^2} = F \tag{3.6}$$

Equations (3.1)–(3.4) are often called the laws of: Coulomb – Gauss, absence of magnetic charge (monopoles), Faraday and Ampere correspondingly. We do not concern here the history of these equations, but normally these four equations are called Maxwell equations, however in Maxwell's paper [6] there are 8 equations, or 6 equations for microscopic (in vacuum) version.

Below we use mainly Gaussian system. The correspondence between Gaussian and SI units is given by the following formulas (for example between E_G and E_{SI})

$$E_G\sqrt{4\pi\varepsilon_0} = E_{SI}\sqrt{4\pi\varepsilon_0}, \;\; V_G \Longleftrightarrow V_{SI}\sqrt{4\pi\varepsilon_0}, \;\; q_G \Longleftrightarrow \frac{q_{SI}}{\sqrt{4\pi\varepsilon_0}}, \;\; \rho_G \Longleftrightarrow \frac{\rho_{SI}}{\sqrt{4\pi\varepsilon_0}},$$

$$J_G \Longleftrightarrow \frac{J_{SI}}{\sqrt{4\pi\varepsilon_0}}, \;\; H_G \Longleftrightarrow H_{SI}\sqrt{\frac{4\pi}{\mu_0}}, \;\; A_G \Longleftrightarrow A_{SI}\sqrt{\frac{4\pi}{\mu_0}},$$

where V, A are potentials (see below).

Mathematical problems The Maxwell equations (3.1)–(3.4), from mathematical point of view, look quite formal. It is not clear what variables are given and which are unknown: there are 10 scalar functions and only 8 scalar equations. With Lorentz equations there will be 11 scalar equations. Also the problems are not formulated exactly. There are the following possibilities:

1) Forget about Lorentz equation and consider only Maxwell equations (3.1)–(3.4), where we assume that charges and currents are given functions and the fields E, H are unknown. The corresponding theory is closed, consistent, and sufficiently simple. It is discussed in the next section. As the equations (3.1)–(3.4) are linear, then ρ and J can be considered as generalized functions. In particular, if the charges are N point particles with given parameters $q_i, x_i(t)$, we can write down the Maxwell equations with point particles. For this one should change in (3.1) the charge density, and in (3.4) the current density as follows

$$\rho(t,x) \to \sum_{i=1}^{N} q_i \delta(x - x_i(t)), j(t,x) \to \sum_{i=1}^{N} q_i \frac{dx_i(t)}{dt} \delta(x_i(t))$$

2) Forget about Maxwell equations (3.1), (3.4) and, assuming that the fields E, H are given, find the trajectories of the point particle according to the Newton equation with Lorentz force (3.5); This is also consistent theory.

3) Concerning complete system (3.1)–(3.5), it does not have still consistent theory (existence theorems). Now we give the list of problems.

a) First of all, the problem with the notion of density. How Newton's law can be applied to the density ? One needs additional axioms, for example additional forces of unknown nature which keep the charges in small closed balls, see for example. [8].

b) Concerning charged point particles the most important is the self-interaction problem – one of the main consistency problem in classical mathematical physics. The simplest case is when there is only one particle and the field of the particle acts on this particle itself. One particular case was recently considered in [11], see also Project 5.

c) In case of two and more particles, one can forget about fields and self-interaction, and consider two-particle **retarded interaction** defined by Lienard – Wiechert potentials (see below) of these particles. There are many papers in this direction (for example, one by Feynman – Wheeler [17]). I am not ready to comment this.

4. Solving Maxwell equations – particles create fields

Linear differential equations (ODE and PDE) with constant coefficients, can be explicitly solved using Fourier transform. This is the case for Maxwell equations if charges and currents are given. General problems concerning existence and uniqueness of solution are well studied, see for example [7]. We describe three methods of solution: direct by Fourier transform, wave equations for E and H, introducing potentials.

Note first that smooth densities ρ and j can depend on time, but only under the condition that the charge conservation law

$$\frac{\partial \rho}{\partial t} + divj = 0 \tag{4.1}$$

holds. It formally follows from the Maxwell equations if one applies divergence to the Ampere law and after this use the first equation of (1.2). For point particles this condition holds as well.

4.1. Direct solution in general case

Two first Maxwell equations do not contain time. Thus if we consider Cauchy problem, these are not evolution equations but the restrictions on initial conditions. Applying *div* to the third and fourth equations, and using the first equation of (1.2), we see that if the first and second equations hold at time $t = 0$, they hold at any time.

Remind first that vector operators of differentiation and translation

$$\Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\Big), \quad (S_a f)(x) = f(x+a),$$

under vector Fourier transform (here f, x, a, k are 3-dimensional vectors)

$$\hat{f}(k) = \int f(x) e^{-i(k,x)} dx$$

become operators of multiplication on ik_1, ik_2, ik_3 and $e^{i(a,k)}$ respectively. Moreover,

$$divE \to i(k, \hat{E}), \;\; rotE \to ik \times \hat{E}$$

Thus, together with equations on initial conditions

$$i(k, \hat{E}) = 4\pi \hat{\rho}, \;\; (k, \hat{H}) = 0$$

we get also the system of 6 scalar first order ODE

$$\frac{1}{c}\frac{dF}{dt} = AF - 4\pi J(k,t)$$

for column-vectors

$$F(k,t) = (\hat{E}_1, \hat{E}_2, \hat{E}_3, \hat{H}_1, \hat{H}_2, \hat{H}_3)'$$

$$J(k,t) = (\hat{j}_1, \hat{j}_2, \hat{j}_3, 0, 0, 0)'$$

with (6×6)-matrix

$$A = i\begin{pmatrix} 0 & -B \\ B & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{pmatrix}$$

General solution of the Cauchy problem is the sum of solution of homogeneous equation with given initial conditions $F(k, 0)$ and non-homogeneous equation with zero initial conditions

$$\hat{F}(k,t) = e^{cAt}\hat{F}(k,0) - 4\pi c e^{At}\int\limits_0^t e^{-At_1}\hat{J}(k,t_1)dt_1$$

The initial conditions for homogeneous equations should satisfy of course two first Maxwell equations. Then the obtained solution is unique.

Now consider the limiting procedure from continuous charge density to point particles and vice-versa. So, if $\rho_{i,\varepsilon}$ is a symmetric charge density with total charge q_i in the sphere of radius ε around point x_i, and $j_{i,\varepsilon} = v_i\rho_{i,\varepsilon}$, then as $\varepsilon \to 0$ the solution for the densities converges to the solution for point charges. Vice-versa, any smooth density $\rho(x,t)$ can be approximated by point charge system on the lattice Z^3_ε with step ε.

4.2. Wave equations for $\boldsymbol{E}$ and $\boldsymbol{H}$

Applying rot to equation (3.4), then substituting $rotE$ from (3.3), and using formula (1.2), from $divH = 0$ we get wave equation for H (and similarly for E)

$$\frac{1}{c^2}\frac{\partial^2 H}{\partial t^2} = \Delta H + \frac{4\pi}{c} rotJ \tag{4.2}$$

$$\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \Delta E + 4\pi\Big(\frac{1}{c^2}\frac{\partial}{\partial t} rotJ - grad\rho\Big) \tag{4.3}$$

Note that we would not get wave equations if in the third and fourth Maxwell equations were the same signs. We consider these wave equations not only with initial conditions for $E(0), H(0)$ satisfying two first Maxwell equations, but also with initial conditions for time derivatives

$$\frac{\partial H}{\partial t}(0) = -rotE(0), \quad \frac{\partial E}{\partial t}(0) = c^2 rotH(0) - \frac{1}{\varepsilon_0}J(x,0)$$

which follow from the Maxwell equations. As we have shown above, Maxwell equations with these initial conditions have the unique solution. And we saw that this solution satisfies wave equations (4.2,4.3). But the solution of this wave equations with our initial conditions is also unique. Thus these solutions coincide. See also [10].

4.3. Potentials and gauges

Potentials are very useful in point particle mechanics, the same is for field theories. As $divH = 0$, that is H is solenoidal, then formula (1.11) of theorem 1.1 shows that there exists vector function A such that

$$H = rotA \tag{4.4}$$

Substituting this to Faraday law we get

$$\nabla \times \Big(E + \frac{1}{c}\frac{\partial A}{\partial t}\Big) = 0$$

Then formula (1.12) of theorem 1.1 shows that for this irrotational field there exists scalar field $\varphi(x,t)$ such that

$$E + \frac{1}{c}\frac{\partial A}{\partial t} = -\nabla\varphi \Longleftrightarrow E = -\frac{1}{c}\frac{\partial A}{\partial t} - \nabla\varphi \tag{4.5}$$

The functions φ and A are called scalar and vector potentials correspondingly. Substituting (4.4) and (4.5) to Maxwell equations we see that two homogeneous equations give identities, but both inhomogeneous equations look quite ugly

$$\nabla^2\varphi + \frac{1}{c}\frac{\partial divA}{\partial t} = -4\pi\rho$$

$$\nabla^2 A - \frac{1}{c}\frac{\partial^2 A}{\partial t^2} - grad\Big(divA + \frac{1}{c}\frac{\partial\varphi}{\partial t}\Big) = -\frac{4\pi}{c}j$$

To make them simpler one can use the fact that the potentials are not uniquely defined by formulas (4.4) and (4.5). It is common to say that their choice is the choice of gauge.

Lorentz gauge However if the Lorentz condition

$$divA + \frac{1}{c}\frac{\partial\varphi}{\partial t} = 0 \tag{4.6}$$

holds then we get two beautiful wave equations for potentials

$$\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \Delta A = \frac{4\pi}{c}j \tag{4.7}$$

$$\frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} - \Delta\varphi = 4\pi\rho \tag{4.8}$$

But can we justify the Lorentz condition ? The answer is yes if we use the following **gauge transformation** with arbitrary scalar function $\lambda(x,t)$

$$A \to A + grad\lambda, \quad \varphi \to \varphi - \frac{1}{c}\frac{\partial\lambda}{\partial t} \tag{4.9}$$

It is easy to see that this does not change the fields E and H. Now we should choose λ so that Lorentz condition hold. To get such λ it is sufficient to find it from the following wave equation

$$divA + \frac{1}{c}\frac{\partial \varphi}{\partial t} = -\nabla^2 \lambda + \frac{1}{c}\frac{\partial^2 \lambda}{\partial t^2} \tag{4.10}$$

The fields defined by formulas (4.4) and (4.5) satisfy Maxwell equations. Vice-versa, assume that we need to solve Cauchy problem for Maxwell equations with given initial conditions $E(0), H(0)$. To do this we can proceed as follows. Let us first solve independently equations (4.7) and (4.8) with zero initial conditions, then also solve (4.10). We get potentials satisfying Lorentz condition. Note that to get back to E and H one should not know λ, as the gauge transform (4.9) does not change them.

Coulomb gauge Here again we define the potential by (4.4) and (4.5), but also assume the Coulomb gauge condition

$$divA = 0 \tag{4.11}$$

Note that this condition automatically holds for A defined by (1.11). Then the potentials satisfy the equations

$$\Delta \varphi = -4\pi\rho \tag{4.12}$$

$$\Delta A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\frac{4\pi}{c} j + \frac{1}{c}\frac{\partial (grad\varphi)}{\partial t} \tag{4.13}$$

This gauge is useful with the absence of sources (charges). Then

$$\varphi = 0, \quad \Delta A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = 0$$

and the fields are defined from potentials by

$$H = rotA, \quad E = -\frac{1}{c}\frac{\partial A}{\partial t}$$

4.4. Lienard – Wiechert potential

Assume that the point charge q has the trajectory $x_0(t), t \in [-\infty, T)$, with velocity $v(t)$ such that $|v(t)|$ is less than the speed of light c. Then for any given x and any time $T > 0$ there exists at most one time moment $t_{ret} = t_{ret}(x, T) < T$ such that

$$|x - x_0(t_{ret})| = c(T - t_{ret})$$

In fact, for any $t > 0$ let $S(x, t)$ be the sphere in R^3 with centre at x and radius ct. Assume that $x_0(t_{ret}) \in S(x, t = T - t_{ret})$ for some $t_{ret} < T$. Then the trajectory cannot be on the sphere $S(x,t\pm\varepsilon)$ at time $t\pm\varepsilon$ because it would take time more than ε. The uniqueness of t_{ret} is thus proved.

We use the Kirchhoff formula (2.18) with

$$f(x,t)=4\pi q\rho(x-x_0(t))$$

$$f(x,t)=\frac{4\pi}{c}J(x,t)=\frac{4\pi q}{c}v(t)\rho(x-x_0(t))$$

for scalar and vector potentials correspondingly. Thus

$$\varphi(x,t)=\frac{1}{4\pi}\int\limits_{|x-y|\le ct}\frac{f(y,t-\frac{|x-y|}{c})}{|x-y|}dy=q\int\limits_{|x-y|\le ct}\frac{\rho(y,t-\frac{|x-y|}{c})}{|x-y|}dy=$$

$$=q\int\limits_{|x-y|\le ct}\frac{\delta(y-x_0(t_{ret}))}{|x-y|}dy$$

where we put

$$t'=t_{ret}=t-\frac{|x-x_0(t')|}{c}$$

Then we can introduce the variable under the integral

$$\varphi(x,t)=q\int\limits_{|x-y|\le ct}\int dt'\delta\Big(t'-\Big(t-\frac{|x-x_0(t')|}{c}\Big)\Big)\frac{\delta(y-x_0(t'))}{|x-y|}dy$$

Integrating in dy we get

$$\varphi(x,t)=q\int dt'\delta\Big(t'-\Big(t-\frac{|x-x_0(t')|}{c}\Big)\Big)\frac{1}{|x-x_0(t')|}$$

Again we need new variables

$$s=t'-\Big(t-\frac{|x-x_0(t')|}{c}\Big),\quad n(t')=\frac{x-x_0(t')}{|x-x_0(t')|}$$

Then

$$\frac{ds}{dt'}=1-\frac{(n(t'),v(t'))}{c}$$

and change of variables gives

$$\varphi(x,t)=q\int ds\delta(s)\Big[|x-x_0(t')|\Big(1-\frac{(n(t'),v(t'))}{c}\Big)\Big]^{-1}=$$

$$=\frac{q}{|x-x_0(t')|\big(1-\frac{(n(t'),v(t'))}{c}\big)} \tag{4.14}$$

if t_{ret} exists and $\varphi(x,t) = 0$ otherwise.

Note now that to obtain this formula we used only Kirchhoff formula and wave equation (4.8). For the vector potential A we should use now wave equation (4.7) instead. But the difference between these wave equations is that in the right hand side of (4.7) there is $\frac{4\pi}{c}J$ instead of $4\pi\rho$. Thus we get

$$A(x,t) = \frac{v(t_{ret})}{c}\varphi(x,t) \tag{4.15}$$

The corresponding electric and magnetic field strengths are

$$E(x,t) = \frac{1}{4\pi\varepsilon_0}\frac{|r|}{(r,u)^3}[(c^2 - v^2)u + r \times (u \times a)]$$

$$H(x,t) = \frac{1}{c} n \times E(x,t)$$

where $u = cn - v$ and $a = \frac{dv}{dv}$ is the particle acceleration at time $t = t_{ret}$.

5. Lorentz equation – fields move particles

Assume now that the fields $E(t,x)$ and $H(t,x)$ are given. And consider the Newtonian dynamics of one particle of charge q with Lorentz force F

$$m\frac{dv}{dt} = F = q\Big(E + \frac{1}{c}[v,H]\Big) \tag{5.1}$$

Here we will mainly consider only the second (magnetic) force in (5.1).

Example For example, if $E = 0$ and H is constant and parallel to the x_1-axis, then the particle will (dependently of initial conditions) move with uniform velocity v_1 in the x_1-direction (that is along the lines of magnetic force) and simultaneously rotates parallel to the (x_2, x_3)-plane. In fact, let $H = (B_1, 0, 0)$, then

$$\frac{dv_1}{dt} = 0, \quad \frac{dv_2}{dt} = v_3 B_1, \quad \frac{dv_3}{dt} = -v_2 B_1,$$

Then $v_1(t) = v_1(0)$ and for $i = 2, 3$ (if $\frac{dv_i}{dt}(0) = 0$)

$$\frac{d^2 v_i}{d^2 t} = -v_i B_1^2 \Longrightarrow v_i(t) = C_1 \sin|B_1|t + C_2 \cos|B_1|t = v_i(0)\cos|B_1|t,$$

$$x_i(t) = x_i(0) + \int_0^t v_i(0)\cos|B_1|t dt = x_I(0) + \frac{v_I(0)}{|B_1|}\sin|B_1|t$$

Remark 5.1. **About forces depending on velocity of the particle.** Examples of such forces:

1. Hamiltonians $\mathbf{H}(p)$ are used in geometric optics;

2. Forces $F = -\alpha v$ or other functions $f(v)$ are often used as friction forces; Systems with this force do not respect the energy conservation law. Moreover, such forces are not fundamental forces and should be deduced from the laws of non-equilibrium statistical physics;

3. Force $F = F(v)$ is called **gyroscopic** if it depend on v and does not do work (does not produce energy) that is at any time t the scalar product

$$Fv = 0$$

Examples are magnetic force and Coriolis force, that is the force $F = -2m[\omega, v]$ on the particle having velocity v relatively to the reference system, having angular velocity ω,

One can imagine even more general Hamiltonian systems

In particle mechanics any function $\mathbf{H}(q, p)$ can be considered as the Hamiltonian. which defines time evolution of q and p via Hamiltonian equations

$$\frac{dq}{dt} = \frac{\partial \mathbf{H}}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial \mathbf{H}}{\partial q} \tag{5.2}$$

and moreover it is an integral for these equations. However, it does not mean that this is directly related to Newton equations by

$$q = x, \quad v = \frac{dq}{dt}, \quad p = mv,$$

and even less directly to the following identification for some function $U(x)$

$$T = T(v) = \frac{mv^2}{2}, \quad \mathbf{H} = T(v) + U(x), \quad F(x) - -\nabla U(x)$$

Lorentz force does not prevent energy conservation law, and one could try to write down the corresponding Hamiltonian form using magnetic vector potential A,. Why do we need this – one of the explanation is that we are looking for the conserved quantity for Lorentz dynamics.

We use general approach when also $q = x$ but $p \neq mv$. Our example is: for the vector function $A = A(q)$ the Hamiltonian

$$\mathbf{H} = \frac{1}{2m}(p - A)^2 = \frac{p^2}{2m} - \frac{1}{m}(p \cdot A) + \frac{1}{2m}A^2 \tag{5.3}$$

To simplify notation we write here A instead of $\frac{e}{c}A$. If we define the momentum as $p = mv + A$ then

$$p = mv + A \Longrightarrow v = \frac{p}{m} - \frac{A}{m} \Longrightarrow \mathbf{H} = \frac{mv^2}{2} \tag{5.4}$$

then our Hamiltonian and also the kinetic energy in the sense of Newton mechanics are conserved. This means also that magnetic force does not produce work.

Now direct calculations show that (magnetic part of) the Lorentz force follows from this. In fact, the first Hamilton equation in (5.2) is the first equation in (5.4)

$$v_i = \frac{1}{m}(p_i - A_i) = \frac{dq_i(t)}{dt} = \frac{\partial \mathbf{H}}{\partial p_i}$$

The second Hamilton equation gives for example for $i = 1$

$$\begin{aligned} m\frac{d^2 q_1}{dt^2} = m\frac{dv_1}{dt} = \frac{dp_1}{dt} - \frac{dA_1}{dt} = -\frac{\partial \mathbf{H}}{\partial q_1} - \frac{dA_1}{dt} = \\ = \frac{1}{m}\sum_j p_j \frac{\partial A_j}{\partial q_1} - \frac{1}{m}\sum_j A_j \frac{\partial A_j}{\partial q_1} - \sum_j v_j \frac{\partial A_1}{\partial q_j} = \\ = \frac{1}{m}\sum_j (mv_j + A_j)\frac{\partial A_j}{\partial q_1} - \frac{1}{m}\sum_j A_j \frac{\partial A_j}{\partial q_1} - \sum_j v_j \frac{\partial A_1}{\partial q_j} = \\ = \sum_j v_j \frac{\partial A_j}{\partial q_1} - \sum_j v_j \frac{\partial A_1}{\partial q_j} = \sum_j v_j \Big(-\frac{\partial A_1}{\partial q_j} + \frac{\partial A_j}{\partial q_1} \Big) = \\ = v_2 \Big(\frac{\partial A_2}{\partial q_1} - \frac{\partial A_1}{\partial q_2} \Big) - v_3 \Big(\frac{\partial A_1}{\partial q_3} - \frac{\partial A_3}{\partial q_1} \Big) = \\ = v_2 (\nabla \times A)_3 - v_3 (\nabla \times A)_2 = ([v, rotA])_1 \end{aligned}$$

The Lorentz force (5.1) follows.

Remark 5.2. Note that the Lorentz force is an axiom. For example, changing the sign of A in (5.3) and in (5.4), we get minus sign in the Lorentz force.

6. Particular solutions

Microscopic electrodynamics presents some global view on the basic axioms concerning electricity. In despite of the fact that its consistency is not yet proved there are very important parts of it which are absolutely consistent. Examples of such theories:

1) algebraic theories related to Maxwell equations. There is a lot: symmetries (including special relativity), language of differential forms and other geometric algebra, etc.,

2) simplified cases, for example time independent, or when some parameters are scaled as necessary,

3) situations with large number of particles and external forces of not yet understood nature (stochastic influence).

We shortly describe here only some examples of 2) and 3). Many practical applications we do not even mention: optics, transmission lines etc.

6.1. Elementary electrostatics

If $\rho(x)$ and $j(x)$ do not change in time then there can be static (not depending on time) solutions $E(x), H(x)$. If so, each of them independently satisfies the corresponding pair of equations

$$divE = 4\pi\rho, \quad rotE = 0 \tag{6.1}$$

$$divH = 0, \quad rotH = \frac{4\pi}{c}J \tag{6.2}$$

The first pair defines the science called **electrostatics**. By equation (1.12) there exists scalar function φ called **potential** such that

$$E = -\nabla\varphi$$

Applying divergence to it, we get the following **Poisson equation**

$$(\Delta\varphi)(x) = -4\pi\rho(x), x \in R^3 \tag{6.3}$$

If $\rho(x) = q_0\delta(x)$ then it has the following solution (unique up to an additive constant)

$$\varphi(x) = \frac{q_0}{r} \Longrightarrow E(x) = -\nabla\varphi = \frac{q_0}{r^2}\frac{x}{r} \tag{6.4}$$

where $r = |x|$. This follows from formula (1.7). This follows also from L-W potential formula (4.14) if the charge stands still all the time (starting from $-\infty$). If the charge is introduced at point 0 at some finite time, then (6.4) holds starting from some time moment.

For smooth density $\rho(x)$

$$\varphi(x) = \int \frac{\rho(y)}{|x-y|}dy$$

Connection of E with the force gives the Lorentz force axiom: we get the famous Coulomb law: *electric charge q_0 at point 0 acts on the charge q at the point x with the force*

$$F = \frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r^2}\frac{x}{r} \quad (SI) \quad = \frac{qq_0}{r^2}\frac{x}{r} \quad (Gaussian) \tag{6.5}$$

Thus the charges of the same sign repel each other, and charges of different signs attract to each other. Note that the vector function $F = F(x)$ is absolutely integrable at zero and thus it can be considered as generalized function.

Finally, for system of N charged particles one can define (as in Newton's mechanics) the two particle interaction potential $U(|x_i - x_j|)$ and (electrostatic) potential energy of the particle system

$$U = \sum_{i<j} U(|x_i - x_j|) = \sum_{i<j} \frac{q_i q_j}{|x_i - x_j|} = \frac{1}{2}\sum_{i\neq j} \frac{q_i q_j}{|x_i - x_j|} \tag{6.6}$$

Then Newtonian equations for this particle system will be

$$m_i \frac{d^2 x_i}{dt^2} = -\frac{\partial U}{\partial x_i}$$

This is Coulomb mechanics which still does not exist, see Project 4.

Similarly we can define the field created by the charge density $\rho(y)$ (say with compact support)

$$V_\rho(x) = \int \frac{\rho(y)}{|x-y|} dy, \quad E_\rho(x) = -\nabla V_\rho$$

and the potential energy of continuous charge system

$$U_{cont} = \frac{1}{2} \int \frac{\rho(x)\rho(y)}{|x-y|} dxdy = \frac{1}{2} \int \rho(x) V_\rho(x) dx =$$

$$= -\frac{1}{8\pi} \int (\Delta V_\rho) V_\rho(x) dx = \frac{1}{8\pi} \int (grad V_\rho)^2(x) dx = \frac{1}{8\pi} \int E_\rho^2 dx = U_{field}$$

At the end we rewrote this potential energy in terms of E and called this the energy of electrostatic field (produced by this charge system). We see that V_ρ and U_{field} are finite if ρ is smooth and has compact support. We see also that $U_{field,cont}$ is always positive. On the contrary that is not true for the similar expression for the energy of the field of discrete charges

$$E^2(x)dx = \left(\sum_i \frac{e_i}{|x-x_i|^2} \right)^2 dx = \left(\sum_i \frac{e_i^2}{|x-x_i|^4} + \sum_{i \neq j} \frac{e_i}{|x-x_i|^2} \frac{e_j}{|x-x_j|^2} \right) dx$$

The first sum is moreover always infinite. It is related to self interaction of the point charge. If we discard this sum then we get finite expression. This is the simplest example of **renormalization** (elimination of divergencies).

6.2. Elementary magnetostatics

This is defined by the second pair of equations (6.2). First of all, it follows from (4.1) that the current density $J(x)$ satisfies

$$\nabla J = 0$$

This means that there are only stationary currents where the charge density is conserved at any point. We assume also that the support of $J(x)$ is bounded. As $H = \nabla \times A$, the second formula (6.2) gives Poisson equation for the vector potential A:

$$\frac{4\pi}{c} J = rot H = rot rot A = -\Delta A \tag{6.7}$$

similar to equation (6.3) in electrostatics. Then the only solution of equations (6.2) is given by

$$A(x) = \frac{1}{c} \int\limits_{\Lambda} \frac{J(y)}{|x-y|} dy$$

Direct calculation gives (in Gauss system)

$$H(x) = \frac{1}{c} \int\limits_{\Lambda} \frac{j(y) \times (x-y)}{|y-x|} dy,$$

where Λ is the volume such that $J(x) = 0$ if $x \neq \Lambda$. This is the **Biot − Savart law:** stationary current creates stationary magnetic field.

6.3. Elementary radiation

Consider now the case when the radiating particle with charge q moves with constant velocity $0 \leq v < c$ along x-axis R as $x_0(t) = vt < x \in R, t \geq 0$

$$f(x,t) = 4\pi q \rho(x,t) = 4\pi q \delta(x - vt)$$

In this case

$$t'v + c(t-t') = x \Longleftrightarrow t' = \frac{ct - x}{c - v} = a^{-1}(t - \frac{x}{c}), \quad a = 1 - \frac{v}{c}.$$

Then

$$x - x_0(t') = x - vt' = x - a^{-1}v\Big(t - \frac{x}{c}\Big) = a^{-1}\Big(ax - vt + x\frac{v}{c}\Big) = a^{-1}(x - vt)$$

and substituting to (6.4) we get

$$\varphi(x,t) = q\frac{1}{x - vt}$$

This looks like the static Coulomb law: the particle at time t at point $x - vt$ created Coulomb field at the point x at the same moment t. With more bulky calculations (or see [5], page 434) one can sow that if x does not belong to the x-axis then the answer is

$$\varphi(x,t) = q\frac{1}{|x - vt|\sqrt{1 - \frac{v^2 \sin^2 \varphi}{c^2}}}$$

where φ is the angle between vectors x and v. If the case $\varphi = 0$ could seem exceptional, but the formula above shows that the result seems to have sense for small velocities $v^2 \ll c^2$. And one could suspect that self interaction could play role here for large v (see Project 5).

Remark 6.1. Now we give a wrong calculation which shows how one can make mistake with multidimensional δ-functions

$$\varphi(x,t) = q \int\limits_{|x-y|\leq ct} \frac{\delta(y - t'v)}{|x-y|} dy =$$

$$= q\frac{1}{x - a^{-1}(t - \frac{x}{c})v} = q\frac{a}{ax - (t - \frac{x}{c})} = q\frac{a}{x - vt} = q\frac{1 - \frac{v}{c}}{x - vt}$$

6.4. Energy of free fields

Consider now Maxwell equations without sources (we call this free fields), that is when $\rho = 0, J = 0$. Then the simplest solution, not dependent on x, y, is

$$E(z,t) = E_0 f(kz - \omega t), \quad H(z,t) = H_0 f(kz - \omega t) \tag{6.8}$$

with constant vectors E_0, H_0 and function f of one real variable z. In particular, these can be monochromatic **plane waves**, that is

$$E(z,t) = E_0 e^{i(kz - \omega t)}, \quad H(z,t) = H_0 e^{i(kz - \omega t)} \tag{6.9}$$

or “point packets” with $f(\cdot) = \delta(\cdot)$, that is each component fluctuates with the same frequency and spread with the speed $c = \frac{\omega}{k}$ along one direction (z axis). At any time moment the field is constant along xy plane.

From $divE = divH = 0$ follows transversality of the vectors E_0, H_0, that is their z-components are zero. Moreover,

$$H_{0,x} = -E_{0,y}, \quad H_{0,y} = E_{0,x}$$

that is electric and magnetic fields are orthogonal and their real amplitudes are equal.

There always existed beautiful algebraic activity concerning (for example Hamilton – Lagrange approach) Maxwell – Lorentz equations.

For example, energy density of the electromagnetic field should be taken as

$$u = \frac{1}{8\pi}(E^2 + H^2)$$

Then (see for example [15])

$$\frac{\partial}{\partial t}u = \Big(\frac{1}{8\pi}2E\frac{\partial E}{\partial t} + 2H\frac{\partial H}{\partial t}\Big) = \frac{1}{4\pi}(E(c\nabla \times H - 4\pi J) + H(-c\nabla \times E)) =$$

$$= -(J,E) + \frac{c}{4\pi}((E, \nabla \times H) - (H, \nabla \times E)) =$$

$$= -(J,E) - \frac{c}{4\pi}(\nabla, E \times H) = -(J,E) - (\nabla, S) \tag{6.10}$$

where we introduced the Poynting vector

$$S = \frac{c}{4\pi}(E \times H) \tag{6.11}$$

Assume now that $J(x) = 0$ and that initial data $E(x, 0)$, $H(x, 0)$ are smooth and have compact support. Then at any time t fields have compact support. Moreover, for any t the energy

$$\int_{\Lambda} u(x,t)dx \tag{6.12}$$

where $\Lambda = R^3$ is finite and conserved. One can prove this taking, for given t, in (6.12) sufficiently large volume $\Lambda = \Lambda(t)$ so that E and H are zero outside this volume. Then

$$\frac{\partial}{\partial t}\int_{\Lambda} udx = \int_{\Lambda} \frac{\partial}{\partial t}udx = -\int_{\Lambda} (\nabla, S)dx = 0$$

by Gauss – Ostrogradskij formula (1.10).

Besides its conservation property, there are other indications to call $u(x, t)$ the energy (see, for example, section concerning electrostatics).

Remark 6.2. However, until there is no consistency proof, all examples are interesting where these algebraic techniques can be justified with consistency proof.

6.5. First steps towards rigorous Macroscopic Electrodynamics

Coulomb mechanics It could be great mathematical science but does not still exist. See Project 3 for details.

Electric current This is one-dimensional Coulomb mechanics, see Project 4 where it is explained in detail.

References

[1] G.E. Shilov (1954) *Lectures on vector analysis,* Moscow.

[2] J. Jackson (1962) *Classical Electrodynamics*, Wiley.

[3] W. Greiner (1991) *Classical Electrodynamics*, Springer.

[4] D. Ivanenko, A. Sokolov (1951) *Classical field theory* (in Russian).

[5] D. Griffiths (1999) *Introduction to Electrodynamics*, Prentice Hall.

[6] J. Maxwell (1865) *A dynamical theory of the electromagnetic field*, Phil. Transactions Royal Soc. of London, Vol. 155, pp. 459–512.

[7] V. Palamodov (2002) *Topics in mathematical physics.*

[8] H. Spohn (2004) *Dynamics of charged particles and their radiation field*, Cambridge Univ. Press.

[9] Y. Gui, W. Dou (2007) *A rigorous and completed statement of Helmholtz theorem*, Progress in Electromagnetic research, PIER, Vol. 69, pp. 287–304.

[10] C.E. Gutierrez (2014) *Wave and Maxwell's equations.*

[11] V.A. Malyshev, S.A. Pirogov (2014) *Self-Interaction Model of Classical Point Particle in One Dimension*, Doklady Mathematics, Vol. 90, No. 3, pp. 769–772.

[12] J. Lebowitz, E. Lieb (1972) *The constitution of matter: existence of thermodynamnics for systems composed of electrons and nuclei.* Adv. in Math., Vol. 9, pp. 316–398.

[13] H. Oymak, S. Erkoc (2000) *Distribution of point charges on a thin conducting disk.* International Journal of Modern Physics C, Vol. 11, Issue 5, pp. 891–900.

[14] S. Erkoc, H. Oymak (2000 Sep) *Rules for the distribution of point charges on a conducting disk*, Phys. Rev. E Stat. Phys. Plasmas Fluids Relat. Interdiscip. Topics., 62(3 Pt A):R3075-6.

[15] F. Melia (2001) *Electrodynamics.*

[16] J. Arthur (2011) *Understanding Geometric Algebra for Electromagnetic Theory,* Wiley.

[17] R.P. Feynman, J.A. Wheeler (1949) *Classical Electrodynamics in Terms of Direct Interparticle Action*, Rev. Mod. Phys., Vol. 21, pp. 425–433.

[18] L. Evans (1998) *Partial differential equations*, AMS.

[19] W. Strauss (1992) *Partial differential equations. An Introduction*, Wiley.