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ЛИНЕЙНЫЕ ОПЕРАТОРЫ В БЕСКОНЕЧНОЧАСТИЧНЫХ СИСТЕМАХ

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Главная задача, возникающая при изучении любой физической системы—построение ее эволюции во времени (динамики) и, в частности, описание возможных асимптотических режимов этой эволюции (картина "рассеяния"). В случае квантовой системы эта задача тесно связана с исследованием спектра оператора энергии (гамильтониана) такой системы.

Основная тема книги—изложение современных математических методов изучения спектра и рассеяния гамильтонианов бесконечночастичных систем со слабым взаимодействием между частицами (модели квантовой теории поля, статистической физики и теории твердого тела). Большинство изложенных в книге результатов принадлежит ее авторам, их сотрудникам и ученикам. Кроме указанной основной темы в первых главах книги содержится обширный иллюстративный и учебный материал (примеры физических и теоретико-вероятностных моделей, для которых удается построить и описать их динамику, основные способы описания физических систем и их динамик: C^*-алгебры, фоковские пространства и вторичное квантование, основные и температурные состояния, т. н. "эвклидов подход" и т. д.).

Для специалистов по математической физике, функциональному анализу и теории вероятностей, а также для студентов и аспирантов, специализирующихся в этих областях.

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CONTENTS

INTRODUCTION.
WHAT THIS BOOK IS ABOUT

The subject of this book can be looked at from various points of view. From the standpoint of functional analysis, we study the spectral properties of a certain class of linear operators; from the viewpoint of probability theory, we are concerned with the analysis of singular Markov processes; and finally, from the viewpoint of mathematical physics, we study the dynamics of equilibrium systems in quantum statistical physics and quantum field theory.

The area in which this book concentrates is still under active development, as can be seen from the vast number of publications. However, the main concepts have already been worked out, and the principal aim has been formulated; namely, for any particular physical system, we want to describe the complete "corpuscular" picture that corresponds to it that is, find all the collective oscillations ("quasiparticles") of the system and their "bound states", and also describe their "scattering". In other words, in the language of modern scattering theory, one seeks to establish the "asymptotic completeness" of the physical system. Of course, it is quite possible that unexpected anomalies and surprises may be encountered in such a program.

A single technique (so far, unique) serves as the centerpiece of all the methods and problems considered in this book. This is the cluster expansion method, which can be applied in various forms to obtain information about the spectral properties of the operators studied here. This technique was described quite fully in our previous book [26], to which we shall frequently refer.

There are today three main approaches to determining the spectral properties of infinite physical Hamiltonians:

1. The method of Bethe-Salpeter kernels, which has been developed by J. Glimm and A. Jaffe and their coworkers and followers (see the bibliographic notes at the end of this book).

2. The Moscow method, which we are currently developing together with our students.

3. The method based on direct cluster expansion in "real" time, which has its roots in the earlier "Hamiltonian approach" in the work of K. Friedrichs, K. Hepp, J. Glimm, and A. Jaffe.

At present, the first two approaches enable one to study only the "lower" spectral branches of the Hamiltonians, whereas the third method can be used to prove that the Hamiltonians are asymptotically complete "in the large" (for the entire spectrum).

Our book is devoted to an exposition of the second and third approaches, which have not previously been treated in monograph form; we also discuss one of the ingredients of the Bethe-Salpeter theory, the so-called Dyson equation. In addition, we present some necessary background material along with some supplementary information, and there are a few digressions.

In our view, a complete and exhaustive treatise on this subject would be premature because, as already noted, the theory is far from being complete and the principal results expected from it have not been obtained. The present book is therefore intended as an introduction to the subject which, however, covers all the main ideas. We have tried to include in this book some new material not present in other monographs; however, to facilitate the task of the reader (and also for purely pedagogical reasons) we have also included some well-known material in the form in which it will be needed. We have relied on the following list of monographs. For probability theory, see [8, 11, 16]; functional analysis and the spectral theory of operators are nicely discussed in [36], where a good treatment of quantum mechanics and scattering theory may also be found. Finally, the theory of C^*-algebras and its applications to statistical physics are discussed in detail in the two-volume work [7, 49]. The material we need on second quantization is contained in the books [5, 36] and [44]. Finally, [12] is also very helpful (among other things, the method of Bethe-Salpeter kernels is briefly discussed there).

Formulas will be referred to as follows: (1) indicates equation (1) in the current section; (2.1) means equation 2 in §1 of the current chapter; (3.2.1) denotes equation 3 in §2 of Chapter 1. A similar system will be used to refer to subsections, theorems, and lemmas.

CHAPTER

EXTENDED INTRODUCTION

The purpose of the present introduction is to acquaint the reader with the general theme of the book. We also discuss here the main concepts and give an intuitive explanation of their meaning.

§1. General outline

1. How linear operators arise in mathematical physics. In this book we study the dynamics (or in other words, the time evolution) of various physical systems. The dynamics is describable in terms of (infinite-dimensional) linear spaces and linear operators that act on them.

In the case of quantum systems, such a description is predetermined by the very language of quantum mechanics: for any quantum physical system, the states of the system coincide with the vectors in a suitable complete Hilbert space $\mathcal{H}$ (more precisely, with the rays in $\mathcal{H}$), and the dynamics is given by a unitary one-parameter group of operators $\{U_t, t \in R^1\}$ acting on $\mathcal{H}$ (t is the time). Sometimes one also introduces the reduced dynamics in the algebra $\mathfrak{B}(\mathcal{H})$ of bounded operators acting on $\mathcal{H}$ by the formula

$$\alpha_t A = U_t A U_t^{-1}, \qquad t \in R^1, \qquad A \in \mathfrak{B}(\mathcal{H}). \tag{1}$$

It is easily seen that $\{\alpha_t, t \in R^1\}$ is a group of $*$-automorphisms of the algebra $\mathfrak{B}(\mathcal{H})$, i.e., it preserves the adjoint operation: $\alpha_t(A^*) = (\alpha_t(A))^*$.

The dynamics $\{U_t, t \in R^1\}$ in the space $\mathcal{H}$ is usually called the *Schrödinger dynamics,* and the dynamics α_t generated by it in the algebra $\mathfrak{B}(\mathcal{H})$ (or in some subalgebra invariant under α_t) is called the *Heisenberg dynamics.*

In the case of a classical mechanical system, the dynamics is initially given by a group $\{T_t, t \in R^1\}$ of invertible transformations of the state space Ω (phase space) of the physical system into itself. However, this dynamics can also be described in terms of the following group of linear operators U_t or U_t^* acting respectively on the space of functions defined on Ω:

$$(U_t f)(\omega) = f(T_t^{-1}\omega), \qquad \omega \in \Omega, \tag{2}$$

and on the space of measures (charges) defined on Ω:

$$(U_t^* \mu)(A) = \mu(T_t^{-1} A), \qquad A \subseteq \Omega. \tag{3}$$

This approach to the dynamics on Ω is frequently simpler and more fruitful than a direct study of the rather complicated group of transformations $\{T_t, t \in R^1\}$. In particular, in problems in nonequilibrium statistical mechanics, the main object of

study is the dynamics (3) given by U_t^* in the space of measures on phase space; the fundamental concepts used by physicists (decay of correlation functions, spectral modes, hierarchies of moment equations, etc.) pertain specifically to the dynamics of measures. Naturally, the study of the evolution of U_t or U_t^* does not fully obviate the need to analyze the initial dynamics T_t in Ω, and many of its properties (the existence of attractors, the randomness or hyperbolic behavior of the system, etc.) cannot be expressed well in terms of the dynamics of measures or functions on Ω.

On the other hand, one often replaces the description of the full dynamics U_t^* of measures by a reduced and simplified description, which, however, may still lead to nonlinear equations (the Boltzmann equation, equations of hydrodynamics, etc., see [59]).

2. General scheme for describing a dynamics (noncommutative probability theory). The above two examples, the Heisenberg dynamics (1) and the dynamics (2), are special cases of a more general setup, to which we shall often have recourse in this book. Namely, suppose we are given a triple $(\mathfrak{A}, \langle\cdot\rangle, \alpha_t)$, where $\mathfrak{A}$ is an algebra with unit 1 and involution $A \mapsto A^*$; $\langle\cdot\rangle$ is a *state* on $\mathfrak{A}$, i.e., a linear functional such that $\langle 1\rangle = 1$ and $\langle A^*A\rangle \geq 0$ for every $A \in \mathfrak{A}$; and finally, $\{\alpha_t, t \in R^1\}$ is a one-parameter group of $*$-automorphisms of the algebra $\mathfrak{A}$.

For quantum systems, where as we have seen, $\mathfrak{A} = \mathfrak{B}(\mathcal{H})$ (or some subalgebra of $\mathfrak{B}(\mathcal{H})$), and when α_t is the Heisenberg dynamics (1), a state $\langle\cdot\rangle$ is usually specified in the form

$$\langle A\rangle = \operatorname{Tr}(\rho A), \tag{4}$$

where ρ is a positive trace-class operator acting in $\mathcal{H}$ such that $\operatorname{Tr}\rho = 1$ (ρ is called the *density matrix* of the state (4); for more details, see [46]).

In the case of classical systems, the algebra $\mathfrak{A}$ is a (commutative) algebra of bounded functions defined on the phase space Ω; the dynamics $\alpha_t = U_t$ is given by (2), and a state is given by an integral

$$\langle f\rangle = \int_\Omega f(\omega)d\mu(\omega), \tag{5}$$

where μ is a probability measure on Ω.

A number of concepts in probability theory can be generalized to apply to states $\langle\cdot\rangle$: $\langle A\rangle$ is the mean value of the element $A \in \mathfrak{A}$, $\langle(A - \langle A\rangle 1)^2\rangle$ is its *variance*, $\langle A^n\rangle$ is the nth moment of A, $\langle\exp\{itA\}\rangle$ is the *characteristic function* of A, and so on.

The general theory of states on algebras is sometimes called noncommutative probability theory.

A state $\langle\cdot\rangle$ on $\mathfrak{A}$ is said to be *invariant* (or *equilibrium*) with respect to the dynamics α_t if $\langle\alpha_t A\rangle = \langle A\rangle$ for all $A \in \mathfrak{A}$.

§2. Basic examples of physical systems

To orient the reader, and also to introduce the terminology and concepts that will be needed, we list here the principal examples of the physical systems that are encountered in the literature on mathematical physics.

We start by noting that in this book we focus mainly on the study of infinite systems (consisting of infinitely many particles occupying all of the space). However, such systems are conveniently described and studied as limits of systems with finitely many particles. We will therefore begin with a description of finite systems.

1. System of classical particles (classical gas). For a system of N identical pointlike particles contained in a bounded region $\Lambda \subset R^\nu$, the phase space $\Omega_{\Lambda,N}$ consists of all sequences of pairwise distinct pairs

$$\omega = \{(q_1, v_1), \dots, (q_N, v_N)\}, \quad (q_i, v_i) \in \Lambda \times R^\nu, \qquad i = 1, \dots, N. \tag{1}$$

Here q_i is the position of the ith particle and v_i is its velocity.

The *energy* (or *Hamiltonian function*) of the system is generally taken to be

$$H_{\Lambda,N}(\omega) = \frac{m}{2}\sum_{i=1}^{N} v_i^2 + \sum_{1\le i<j\le N} U(q_i, q_j), \tag{2}$$

where m is the mass of each particle, and $U(q_1, q_2)$ is the *potential energy* for two interacting particles located at the points q_1 and q_2, respectively. One generally assumes that $U(q_1, q_2) = U(|q_1 - q_2|)$, i.e., that the interaction energy depends only on the distance between the particles.

The dynamics

$$\begin{aligned} T_t : \Omega_{\Lambda,N} \to \Omega_{\Lambda,N} :& \omega^0 = \{(q_1^0, v_1^0), \dots, (q_N^0, v_N^0)\} \\ &\mapsto \omega^t = \{(q_1^t, v_1^t), \dots, (q_N^t, v_N^t)\} \end{aligned} \tag{3}$$

is determined by solving the Newtonian differential equations

$$\frac{dq_i}{dt} = v_i, \quad m\frac{dv_i}{dt} = \sum_{j\ne i} F(q_i, q_j) \tag{4}$$

with initial condition $\omega^0 = \{(q_1^0, v_1^0), \dots, (q_N^0, v_N^0)\}$, where the function $F(q_1, q_2) = -(\nabla_{q_1} U)(q_1, q_2)$ is the force exerted on the first particle by the second. The system of equations (4) must be supplemented by a "boundary condition", i.e., one must specify how a particle moves once it has reached the boundary $\partial\Lambda$ of the region Λ. If Λ has a smooth boundary $\partial\Lambda$, one usually imposes the "elastic reflection" condition: the velocity component of a particle normal to $\partial\Lambda$ (at the point where the particle hits the boundary) changes sign. Sometimes, if Λ is a ν-dimensional cube, "periodic boundary conditions" are imposed, i.e., instead of the cube one considers the torus obtained by "gluing" together the opposite faces and replacing $U(q_1, q_2)$ by the corresponding periodic potential. If the potential $U(q_1, q_2)$ is a smooth function for all $q_1 \ne q_2$, with possibly a singularity on the hyperplane $q_1 = q_2$, in a neighborhood of which it increases monotonically to $+\infty$ as $q_1 - q_2 \to 0$, then the existence (and uniqueness) of the solution of (4) for all initial conditions $\{(q_1^0, v_1^0), \dots, (q_N^0, v_N^0)\} \in \Omega_{\Lambda,N}$ follows from general theorems in the theory of differential equations, and also from the energy conservation law:

$$H(\omega^t) = H(\omega^0). \tag{5}$$

This equality is easily derived from the equation of motion (4). For a potential $U(q_1, q_2)$ with a singularity of the above type on the hyperplane $q_1 = q_2$, equation (5) implies that no two particles can occupy the same point. One sometimes considers a system of solid elastic rods, i.e., a system of particles that cannot approach one another closer than the distance δ, where δ is the diameter of a ball. The phase

space $\Omega^\delta_{\Lambda,N}$ for such a system consists of sequences of the form (1) with the additional condition

$$|q_1 - q_2| \geq \delta, \qquad i \neq j. \tag{6}$$

Here the potential $U(q_1, q_2)$ of the system is defined for all pairs (q_1, q_2) such that $|q_1 - q_2| > \delta$. Two cases are possible:

1) The potential $U(q_1, q_2) \to +\infty$ as $|q_1 - q_2| \to \delta$. Then the dynamics $T_t \colon \Omega^\delta_{\Lambda,N} \to \Omega^\delta_{\Lambda,N}$ is determined as before by the system of differential equations (4) (with elastic reflection as the boundary condition), because no two balls can collide.

2) The potential $U(q_1, q_2)$ is a smooth, bounded function in the entire region $|q_1 - q_2| > \delta$. Here the balls can collide and the dynamics must be supplemented by a prescription of how the particles must move after a collision. One usually first considers the "elastic collision" condition for two balls, which means that the colliding balls "exchange" the normal components of their velocities (i.e., the projections of their velocities on the straight line joining their centers at the instant of collision). When three or more balls collide simultaneously, no reasonable prescription can be given (for $\nu > 1$). We therefore remove from the space $\Omega^\delta_{\Lambda,N}$ all the initial states $\{(q_1^0, v_1^0), \dots, (q_N^0, v_N^0)\}$, whose motion would lead to triple, quadruple, and higher-order collisions. It turns out that the $2N\nu$-dimensional Lebesgue measure of the set of such initial states is equal to zero (Alexander's theorem). On the remaining set $\widetilde{\Omega}^\delta_{\Lambda,N}$, the dynamics T_t is given as described above. Note, however, that when $\nu = 1$, the dynamics T_t can be defined on the entire space $\Omega^\delta_{\Lambda,N}$. In a simultaneous collision of k rods centered at the points

$$q_1 < q_2 < \dots < q_k, \qquad q_{i+1} - q_i = \delta, \qquad i = 1, \dots, k-1,$$

which is possible only if $v_1 > v_2 > \dots > v_k$, the order of these velocities must become reversed as the first ball acquires the velocity v_k, the second v_{k-1}, and so on, and the last ball acquires the velocity v_1; after this, the balls move apart freely.

In all of the above examples, the dynamics T_t obeys *Liouville's theorem*: *The Lebesgue measure $\prod dq_i\, dv_i$ of any subset $A \subset \Omega_{\Lambda,N}$ (or $\Omega^\delta_{\Lambda,N}$) of the phase space of the system is preserved by the motion.* Thus, the transformations U_t in the space of functions $f \in L_2(\Omega_{\Lambda,N}, \prod_i dq_i\, dv_i)$ defined by equation (2.1) generate a unitary group of operators in this Hilbert space.

We note at once that of the measures on $\Omega_{\Lambda,N}$ that are invariant under the dynamics T_t, the most important are the Gibbs (equilibrium) measures $\mu_{\beta,\Lambda,N}$. Their density with respect to the Lebesgue measure $\prod_i dq_i\, dv_i$ is given by the formula

$$d\mu_{\beta,\Lambda,N} = p_{\beta,\Lambda,N}(\{(q_i, v_i), \quad i = 1, \dots, N\}) \prod_i dq_i\, dv_i,$$

where

$$\begin{aligned} &p_{\beta,\Lambda,N}(\{(q_1, v_1), \dots, (q_N, v_N)\}) \\ &\qquad = \frac{1}{Z_{\beta,\Lambda,N}} \exp\{-\beta H_{\Lambda,N}(\{(q_1, v_1), \dots, (q_N, v_N)\})\}. \end{aligned} \tag{7}$$

Here $\beta > 0$ is a parameter and $Z_{\beta,\Lambda,N}$ a normalization factor (the *partition function*):

$$Z_{\beta,\Lambda,N} = \int_{\Omega_{\Lambda,N}} \exp\{-\beta H_{\Lambda,N}(\{(q_1, v_1), \dots, (q_N, v_N)\})\} \prod_i dq_i\, dv_i. \tag{8}$$

A discussion of the pair potentials $U(q_1, q_2)$ in (2) for which the Gibbs measure density (7) is well defined (i.e., $Z_{\beta,\Lambda,N} \neq 0, \infty$) can be found in [37]. Since we deal with identical particles, it is convenient to regard them as indistinguishable, i.e., we take the phase space to be $\Omega^{\text{indist}}_{\Lambda,N}$, defined as the quotient space

$$\Omega^{\text{indist}}_{\Lambda,N} = \Omega_{\Lambda,N}/S_N, \tag{9}$$

where S_N is the group of all permutations (relabelings) of the particles. Here we take the Lebesgue measure of a set $A \subset \Omega^{\text{indist}}_{\Lambda,N}$ to be $\text{mes}(\pi^{-1}(A))/N!$, where $\pi^{-1}(A)$ is the complete inverse image of the set $A \subseteq \Omega^{\text{indist}}_{\Lambda,N}$ under the natural map $\pi\colon \Omega_{\Lambda,N} \to \Omega^{\text{indist}}_{\Lambda,N}$, and $\text{mes}\, B$ is the $2\nu N$-dimensional Lebesgue measure of $B \subset \Omega_{\Lambda,N}$. A state $\omega \in \Omega^{\text{indist}}_{\Lambda,N}$ for a system of indistinguishable particles can be regarded as an N-point subset of the one-particle space $\Lambda \times R^\nu$. The dynamics T^{indist}_t in $\Omega^{\text{indist}}_{\Lambda,N}$ and the Gibbs measure on this space are naturally reduced by the dynamics T_t and Gibbs measure (7) on $\Omega_{\Lambda,N}$.

In statistical physics, for convenience, one often also considers systems with a variable number of particles. For the case of the classical gas described in this section, the state space is the set

$$\Omega^{\text{indist}}_{\Lambda} = \bigcup_{N\geq 0} \Omega^{\text{indist}}_{\Lambda,N}, \tag{10}$$

i.e., the collection of all finite subsets $c \subset \Lambda \times R^\nu$. The dynamics T_t on $\Omega^{\text{indist}}_{\Lambda}$ acts independently on each stratum $\Omega^{\text{indist}}_{\Lambda,N}$, and the Lebesgue measure dc on $\Omega^{\text{indist}}_{\Lambda}$ is also defined as the measure $(1/N!)\prod_i dq_i\, dv_i$ on the stratum $\Omega^{\text{indist}}_{\Lambda,N}$, while the Gibbs measure $\mu_{\beta,\mu,\Lambda}$ on $\Omega^{\text{indist}}_{\Lambda,N}$ is given by the density (with respect to the Lebesgue measure on $\Omega^{\text{indist}}_{\Lambda}$)

$$d\mu_{\beta,\mu,\Lambda} = P_{\beta,\mu,\Lambda}(c)\, dc,$$

where

$$P_{\beta,\mu,\Lambda}(c) = \frac{1}{\Xi_{\beta,\mu,\Lambda}} \exp\{-\beta(H_\Lambda(c) + \mu N(c))\}. \tag{11}$$

Here $c = \{(q_i, v_i),\ i = 1, 2, \ldots, N\}$, $H_\Lambda(c) = H_{\Lambda,N}(\{(q_1, v_1), \ldots, (q_N, v_N)\})$, and $N(c) = N$, $\beta > 0$, and μ are parameters; the normalization factor $\Xi_{\beta,\mu,\Lambda}$ is equal to

$$\Xi_{\beta,\mu,\Lambda} = \sum_{N=0}^{\infty} \frac{e^{-\beta\mu N}}{N!} Z_{\beta,\Lambda,N}. \tag{12}$$

The distribution (11) is often called the *Gibbs grand canonical ensemble,* while (7) is the canonical ensemble. It is easily verified that the conditional distribution generated by the grand canonical ensemble on the stratum $\Omega^{\text{indist}}_{\Lambda,N}$ coincides with the canonical ensemble on this stratum. More details on all the above can be found in [37] and [30].

2. Classical lattice system. In the previous section we have described a gas of identical particles moving in a bounded region $\Lambda \subset R^\nu$. In this section we will describe a system of finitely many particles, each oscillating about its equilibrium position. Let Z^ν be a ν-dimensional lattice, $\Lambda \subset Z^\nu$ a finite subset, and suppose that to each $x \in \Lambda$ there is assigned a particle whose displacement from the point x is q_x

and whose velocity is v_x. The state space of the system is $\Omega_\Lambda = (R^\nu \times R^\nu)^\Lambda$; the energy (Hamiltonian function) is taken to be

$$H_\Lambda = \frac{m}{2}\sum_{x\in\Lambda} v_x^2 + \sum_{\substack{x,y\subset\Lambda\\ x\neq y}} \Phi_{x,y}(q_x, q_y) + \sum_{x\in\Lambda} U(q_x). \tag{13}$$

Here $U(\cdot)$ is the potential energy for the interaction of particle q_x with a force which is centered at the point $x \in \Lambda$ and tends to restore it to the equilibrium position $q_x = 0$, and $\Phi_{x,y}(q_x, q_y)$ is the interaction potential energy for two particles assigned to the points x and y and displaced by q_x and q_y, respectively. If $U(q)$ and $\Phi_{x,y}(q_x, q_y)$ are quadratic forms in their variables, then the system with Hamiltonian (13) is called a system of *linear interacting oscillators.*

The dynamics $T_t : \Omega_\Lambda \to \Omega_\Lambda$, $\omega^0 \to \omega^t$ of the above system is again obtained by solving the Newtonian differential equations

$$\frac{dq_x}{dt} = v_x, \qquad m\frac{dv_x}{dt} = F^{(1)}(q_x) + \sum_{y\neq x} F^{(2)}_{x,y}(q_x, q_y) \tag{14}$$

with initial condition $\omega^0 = \{\{q_x^0, v_x^0\}, x \in \Lambda\}$. Here $F^{(1)}(q_x) = -(\nabla U)(q_x)$, $F^{(2)}_{x,y}(q_x, q_y) = -(\nabla_{q_x}\Phi)(q_x, q_y)$ are the forces acting on the particle at x exerted by the attracting center (at x) and by the other particles.

Again, one verifies easily that the Hamiltonian function (13) and the Lebesgue measure $\prod_{x\in\Lambda} dq_x\, dv_x$ are preserved under the motion. The Gibbs distribution $\mu_{\beta,\mu,\Lambda}$ in Ω_Λ is again given by a formula analogous to equation (11).

3. Quantum systems. Consider a system of N particles contained in a bounded region $\Lambda \subset R^\nu$ of the space R^ν. We will describe the three most important cases.

A. *Distinguishable particles.* The Hilbert space of the states for such a system is the space $\mathcal{H}_{\Lambda,N} = L_2(\Lambda^N, \prod_i dq_i)$ of functions $f(q_1, \ldots, q_N)$, $q_i \in \Lambda$. The energy operator is given by

$$H_{\Lambda,N} f = -\sum \frac{1}{2m_i}\Delta_{q_i} f + U(q_1, \ldots, q_N) f, \tag{15}$$

where m_i is the mass of the ith particle, the potential energy $U(q_1, \ldots, q_N)$ is bounded from below and depends on the N variables $q_1, \ldots, q_N$, and Δ_{q_i} is the Laplace operator (with respect to the variable q_i), subject to a suitable selfadjoint boundary condition on the boundary $\partial\Lambda$ of the region Λ. One usually takes Dirichlet boundary conditions, i.e., the functions in the domain of definition of the operator $D_{H_{\Lambda,N}}$ $H_{\Lambda,N}$ vanish if one or more of the arguments q_i is in $\partial\Lambda$. In the case when $\Lambda \subset R^\nu$ is a cube, one also considers "periodic boundary conditions": the values of f and its first derivatives $\partial f/\partial q_i$ coincide for N-tuples $(q_1, \ldots, q_i, \ldots, q_N)$, $(q_1, \ldots, q_i', \ldots, q_N)$ whenever (for any i) q_i and q_i' lie on two opposite faces of the cube. One usually considers a pairwise interaction of particles, possibly in an external field; i.e., we take

$$U(q_1, \ldots, q_N) = \sum_{i<j} \Phi_{ij}(q_i, q_j) + \sum_i \Phi_i(q_i),$$

where Φ_{ij} is the interaction potential for particles i and j, and Φ_i is the potential of the external field acting on the ith particle.

Conditions on the potentials Φ_{ij} and Φ_i for which the operator (15) is selfadjoint are analyzed in detail in [36]. As usual, the dynamics $U_t^{\Lambda,N}\colon \mathcal{H}_{\Lambda,N} \to \mathcal{H}_{\Lambda,N}$ is given in the form

$$U_t^{\Lambda,N} f = \exp\{itH_{\Lambda,N}\}f, \qquad f \in \mathcal{H}_{\Lambda,N}, \quad t \in R^1. \tag{16}$$

Of the states on the algebra $\mathfrak{B}(\mathcal{H}_{\Lambda,N})$ of bounded operators acting on $\mathcal{H}_{\Lambda,N}$ and invariant under the dynamics (16), the most important is the Gibbs state, defined by the formula

$$\langle A\rangle_\beta = \operatorname{tr}(\rho A), \qquad A \in \mathfrak{A}(\mathcal{H}_{\Lambda,N}), \tag{17}$$

where

$$\rho = \frac{1}{Z_{\beta,\Lambda,N}} \exp\{-\beta H_{\Lambda,N}\},$$

$\beta > 0$ is a parameter, and $Z_{\beta,\Lambda,N}$ is the normalization factor,

$$Z_{\beta,\Lambda,N} = \operatorname{Tr} \exp\{-\beta H_{\Lambda,N}\}. \tag{18}$$

For a large class of potentials Φ_{ij} and Φ_i, the operator ρ (the density matrix for the state (17)) is of trace class for all N, all bounded regions Λ, and every $\beta > 0$, i.e., (17) and (18) are well defined. For more on this we refer to [37].

States of the form (17) are sometimes called *temperature states* (the parameter $\beta = T^{-1}$, where T is the temperature of the system). In addition to these, one also studies the *ground states* on the algebra $\mathfrak{B}(\mathcal{H}_{\Lambda,N})$, which are also invariant under the dynamics (16). Let $\Psi_{\mathrm{gr}} \in \mathcal{H}_{\Lambda,N}$ be a normalized eigenvector (*ground-state vector*) of the operator $H_{\Lambda,N}$ corresponding to an eigenvalue which is simple and as small as possible. Then the ground state $\langle\cdot\rangle_{\mathrm{gr}}$ on $\mathfrak{B}(\mathcal{H}_{\Lambda,N})$ is defined by the formula

$$\langle A\rangle_{\mathrm{gr}} = (A\Psi_{\mathrm{gr}}, \Psi_{\mathrm{gr}}), \qquad A \in \mathfrak{B}(\mathcal{H}_{\Lambda,N}). \tag{19}$$

B. *Indistinguishable particles* (*bosons and fermions*). Let us suppose that all the particle have the same mass: $m_i = m$, and also that the interaction potentials are identical:

$$\Phi_{ij}(q_i, q_j) = \Phi(q_i, q_j), \qquad \Phi_i(q_i) = \Phi(q_i).$$

Then the operator $H_{\Lambda,N}$ on $\mathcal{H}_{\Lambda,N}$ commutes with all permutations of the particles and the following two subspaces of $\mathcal{H}_{\Lambda,N}$ are invariant under $H_{\Lambda,N}$: the subspace $\mathcal{H}_{\Lambda,N}^{\mathrm{sym}}$ of functions $f(q_1, \dots, q_N)$ symmetric in their arguments, and the subspace $\mathcal{H}_{\Lambda,N}^{\mathrm{asym}}$ of functions antisymmetric in $(q_1, \dots, q_N)$ (i.e., which change sign when any pair of particles is interchanged). The components of $H_{\Lambda,N}$ acting on $\mathcal{H}_{\Lambda,N}^{\mathrm{sym}}$ and $\mathcal{H}_{\Lambda,N}^{\mathrm{asym}}$ will be denoted by $H_{\Lambda,N}^{\mathrm{sym}}$ and $H_{\Lambda,N}^{\mathrm{asym}}$, respectively. Particles described by vectors in $\mathcal{H}_{\Lambda,N}^{\mathrm{sym}}$ and by the Hamiltonian $H_{\Lambda,N}^{\mathrm{sym}}$ are called *Bose* particles (or *bosons*), while those described by vectors in $\mathcal{H}_{\Lambda,N}^{\mathrm{asym}}$ are called *Fermi* particles (*fermions*). The dynamics U_t^a on the spaces $\mathcal{H}_{\Lambda,N}^a$, $a = \mathrm{sym}, \mathrm{asym}$, is given by restricting the dynamics U_t (16) on $\mathcal{H}_{\Lambda,N}$ to the invariant subspaces $\mathcal{H}_{\Lambda,N}^a$, $a = \mathrm{sym}, \mathrm{asym}$; the temperature and ground states are given by equations (17) and (19) with $H_{\Lambda,N}$ replaced by $H_{\Lambda,N}^{\mathrm{sym}}$ and $H_{\Lambda,N}^{\mathrm{asym}}$, respectively, and with the vector Ψ_{gr} replaced by $\Psi_{\mathrm{gr}}^{\mathrm{sym}}$ $(= \Psi_{\mathrm{gr}})$ or $\Psi_{\mathrm{gr}}^{\mathrm{asym}}$ $(\neq \Psi_{\mathrm{gr}})$.

One often considers boson and fermion systems with a variable number of particles. The Hilbert spaces in this case are taken to be

$$\mathcal{F}_\Lambda^{\mathrm{sym}} \equiv \mathcal{H}_\Lambda^{\mathrm{sym}} = \mathbb{C}^1 \oplus \left(\bigoplus_{N=1}^{\infty} \mathcal{H}_{\Lambda,N}^{\mathrm{sym}} \right) \tag{20}$$

and

$$\mathcal{F}_\Lambda^{\mathrm{asym}} \equiv \mathcal{H}_\Lambda^{\mathrm{asym}} = \mathbb{C}^1 \oplus \left(\bigoplus_{N=1}^{\infty} \mathcal{H}_{\Lambda,N}^{\mathrm{asym}} \right) \tag{21}$$

which consist of infinite sequences

$$F = \{f_0, f_1(q_1), \dots, f_n(q_1, \dots, q_N), \dots\} \tag{22}$$

of symmetric (respectively, antisymmetric) functions in increasingly many arguments. The norm of a sequence (22) is defined to be

$$\|F\|^2 = |f_0|^2 + \sum_{N=1}^{\infty} \|f_N\|^2_{\mathcal{H}_{\Lambda,N}}. \tag{23}$$

Here $\mathcal{H}_\Lambda^a = \mathcal{F}_\Lambda^a$, $a = \mathrm{sym}, \mathrm{asym}$ are called *Fock spaces*. The Hamiltonian H_Λ^a acting on the space $\mathcal{F}_\Lambda^a$ is given by

$$H_\Lambda^a = \bigoplus_{N=0}^{\infty} H_{\Lambda,N}^a, \quad a = \mathrm{sym}, \mathrm{asym}, \qquad (H_{\Lambda,0}^a = 0), \tag{24}$$

and the dynamics is once again given by equation (16); it coincides on each subspace $\mathcal{H}_{\Lambda,N}^a$ with the previous dynamics $U_t^{\Lambda,N}$. The Gibbs temperature state is given by the density matrix

$$\begin{aligned} \rho_\Lambda^a &= \frac{1}{\Xi^a} \exp\{-\beta(H_\Lambda^a + \mu\widehat{N})\}, \\ \Xi^a &= \mathrm{Tr} \exp\{-\beta(H_\Lambda^a + \mu\widehat{N})\}, \end{aligned} \tag{25}$$

where $\beta > 0$ and μ are parameters, and $\widehat{N}$ is the particle-number operator, acting on each subspace $\mathcal{H}_{\Lambda,N}^a$ as multiplication by N. The ground state is given by equation (19), where Ψ_{gr}^a is the ground-state vector for the operator $H_\Lambda^a + \mu N$ on $\mathcal{H}_\Lambda^a$.

A more general construction of the Fock spaces and associated operator formalism (the second-quantization method) will be described in §3.1. There we will also give examples of Hamiltonians (in particular, a Hamiltonian of the form (15)) expressed in "second-quantization" form.

4. Quantum spin (lattice) systems. Let Z^ν be a ν-dimensional lattice, each point x of which corresponds to a finite-dimensional Hilbert space $\mathcal{H}_x$ isomorphic to $\mathbb{C}^n$. For each finite set $\Lambda \subset Z^\nu$ we write $\mathcal{H}_\Lambda$ for the tensor product of the spaces $\mathcal{H}_x$, $x \in \Lambda$:

$$\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x. \tag{26}$$

The vectors in $\mathcal{H}_\Lambda$ describe the states of a system of "particles", where one particle is located at each point $x \in \Lambda$; their "internal degrees of freedom" ("spin") are specified by the elements of the space $\mathcal{H}_x$. The operator algebra $\mathfrak{A}_x = \mathfrak{B}(\mathcal{H}_x)$ is clearly isomorphic to the algebra $\mathfrak{M}$ of $n \times n$ matrices, and the algebra $\mathfrak{B}(\mathcal{H}_\Lambda) = \bigoplus_{x\in\Lambda} \mathfrak{A}_x$. Given two sets $\Lambda_1 \subset \Lambda_2$ we have an (isometric) inclusion homomorphism

$$\mathfrak{A}_{\Lambda_1} \to \mathfrak{A}_{\Lambda_1} \otimes 1_{\Lambda_2\setminus\Lambda_1} \subset \mathfrak{A}_{\Lambda_2}, \tag{27}$$

where $1_{\Lambda_2\setminus\Lambda_1} \in \mathfrak{A}_{\Lambda_2\setminus\Lambda_1}$ is the unit element of the algebra. Now suppose that for each finite set $A \subset Z^\nu$ there corresponds a selfadjoint element $\Phi_A \in \mathfrak{A}_\Lambda$, which by (27) can be regarded as an element of $\mathfrak{A}_\Lambda$ with $A \subseteq \Lambda$. We now define the selfadjoint operator $H_\Lambda \in \mathfrak{A}_\Lambda$

$$H_\Lambda = \sum_{A\subseteq\Lambda} \Phi_A \tag{28}$$

as the Hamiltonian of our system. The dynamics is as usual given by equation (16). The temperature and ground states are given by formulas similar to equations (17)–(19).

5. Systems with stochastic dynamics. Models of systems in which the dynamics is not uniquely determined as in the previous examples, but is given instead by stochastic mechanisms, have recently become popular in mathematical physics. Different situations can arise here either the evolution equations of the system are themselves random (Markov chains), or else one can consider a family of deterministic equations that depend on random parameters ("random medium"). One then studies the "typical" properties of the family of dynamics that arises. Finally, an intermediate case (random walks in a "random" medium) is also possible. We will discuss the first situation here and consider the case of Markov chains with *local interaction* in a finite region Λ (see [24]). The simplest examples are those for which the transition distributions are conditionally independent. In more detail, let $L \subset Z^\nu$ be a finite set and let the configuration (state of the system) be given by a function $\{s(x), x \in \Lambda\}$ with values in some finite set S. For a given initial state $s_0 \in S^\Lambda$ of the system at $t = 0$, the probability that the system will be in a state s at time $t = 1$ is equal to

$$p(s/s_0) = \prod p_x(s(x)/s_0), \tag{29}$$

where $p_x(\cdot/s_0)$ is a family of distributions on S depending on the configuration s_0. Formula (29) expresses the conditional independence of the distributions for the configurations s at time $t = 1$ for a fixed configuration s_0. The family $p_x(\cdot/s_0)$ depends on s_0 as follows:

$$p_x(\cdot/s_0) = p_x(\cdot/s_0|_{Q_x}), \tag{30}$$

where $Q_x = Q+x$ is a finite fixed neighborhood of the point x (obtained by translating a fixed neighborhood Q of 0 to the point x), and $s_0|_{Q_x}$ is the restriction of the configuration s_0 to Q_x. Formula (30) expresses the "locality of the interaction" for a Markov chain.

§3. Infinite systems and the thermodynamic limit

Macroscopic physical systems consist of a large number of particles and fill a region of R^ν (or of the lattice Z^ν) that is large compared with the characteristic dimensions of the interaction. It is therefore convenient to pass to an idealized infinite system infinitely many particles moving in all of space whose attributes (in particular, the dynamics and equilibrium states) are close to those for the corresponding finite systems. Three purely mathematical arguments can also be advanced in favor of infinite systems:

a) it is only for them that the concept of phase transition for equilibrium systems can be formulated;

b) the dynamics has a continuous spectrum only for an infinite system, and only for such systems are the "corpuscular" ("quasiparticle") picture and scattering theory meaningful (see §6 for more details);

c) finally, it is only for the limit equilibrium states that the equivalence principle for ensembles is rigorously valid. According to this principle, the limit states obtained from the canonical and grand canonical Gibbs ensembles (and also from the microcanonical ensemble, see [37]) coincide in "single-phase" regions of the space of parameters on which they depend, provided the parameters are suitably related (these are the temperature and density of the particles for the canonical ensemble, the temperature and chemical potential for the grand canonical ensemble, and finally, the energy and particle densities for the microcanonical ensemble).

Here we will briefly describe the infinite systems corresponding to each of the finite systems enumerated in the preceding section, their dynamics and their equilibrium (Gibbs) states.

1. Classical gas (of indistinguishable particles). The phase space Ω for this system is the collection of all locally finite subsets $c \subset R^\nu \times R^\nu$ of the space of states $(q, v) \in R^\nu \times R^\nu$ of a single particle, where q is the position and ν the velocity of the particle. A subset $c \subset R^\nu \times R^\nu$ consisting of pairs (q, v) is said to be locally finite if every bounded subset $\Lambda \subset R^\nu$ contains only finitely many particles in c.

For a system with Hamiltonian of the form (2.2), two families of limit Gibbs distributions can be introduced in the space Ω. The first is obtained from the canonical ensemble $\mu_{\beta,\Lambda,N}$ (7.2) by taking the thermodynamic limit:

$$\mu_{\beta,\rho} = \lim_{\substack{\Lambda \uparrow R^\nu \\ \frac{N}{|\Lambda|} \to \rho}} \mu_{\beta,\Lambda,N}, \tag{1}$$

where ρ is the limiting density of the particles and the limit in (1) is in the sense of weak convergence of finite-dimensional distributions (see [26] for more details). The second family of limit distributions on Ω is obtained by taking the limit of the grand canonical Gibbs ensemble $\mu_{\beta,\mu,\Lambda}$ (11.2),

$$\mu_{\beta,\mu} = \lim_{\Lambda \uparrow R^\nu} \mu_{\beta,\mu,\Lambda}. \tag{2}$$

It turns out that for regular regions of the parameters (β, ρ) and (β, μ), in which all the characteristics of the corresponding family of distributions — the correlation functions, free energy, and so on (see [37]) — depend analytically on the parameters, the two families coincide:

$$\mu_{\beta,\rho} = \mu_{\beta,\mu} \tag{3}$$

provided μ is chosen so that the mean limit density of the particles in the large ensemble

$$\rho_{\beta,\mu} = \lim_{\Lambda \uparrow R^\nu} \frac{\langle N \rangle_{\beta,\mu,\Lambda}}{|\Lambda|}$$

is equal to ρ:

$$\rho_{\beta,\mu} = \rho \tag{4}$$

(here, $\langle \cdot \rangle_{\beta,\mu,\Lambda}$ denotes an average over $\mu_{\beta,\mu,\Lambda}$). Assertion (3) together with equation (2.2) is called the equivalence principle for ensembles.

For systems with a pair interaction potential $U(q_1, q_2)$ in equation (2.2), the dynamics in the phase space Ω is given by a infinite system of Newtonian differential equations, which is obtained by formally taking the limit of system (4.2). Let $c = \{(q, v)\} \in \Omega$; then for a pair $(q, v) \in c$ we have

$$\begin{aligned} \dot{q} &= v, \\ \dot{v} &= \frac{1}{m} \sum_{q' \neq q} F(q, q'), \end{aligned} \tag{5}$$

where the sum is over all particles in the configuration c not coinciding with the given particles. In the simplest case when the potential $U(q_1, q_2)$ is finite[1], the sum $\sum_{q'} F(q, q')$ is finite and the right-hand side of (5) is well defined.

Even the simplest examples show that for certain initial configurations $c^0 = \{(q^0, v^0)\} \in \Omega$, equations (5) may fail to have a solution even for arbitrarily short time intervals (the system may collapse instantaneously, so that a bounded region contains an arbitrarily large number of particles). Therefore, a fundamental difficulty in using equations (5) to construct a well-defined dynamics is to choose a sufficiently large subset $\widetilde{\Omega} \subset \Omega$ of initial configurations c^0 for which (5) has a solution $c(t)$ for all $t \in R^1$ which does not leave the set $\widetilde{\Omega}$: $c(t) \in \widetilde{\Omega}$.

There are two approaches in the literature to constructing the set $\widetilde{\Omega}$; either it is described explicitly (the associated dynamics in $\widetilde{\Omega}$ is said to be *absolute*), or else one implicitly takes a set with the property that it has full measure relative to every equilibrium (Gibbs) distribution in Ω. The result here is called an *equilibrium* dynamics, because every equilibrium distribution is invariant under the dynamics on $\widetilde{\Omega}$ (and also, by the way, invariant under every relatively absolute dynamics, if the set $\widetilde{\Omega}$ is big and has full measure with respect to the equilibrium distribution). The first (absolute) approach is convenient in leading automatically (via equations (3.1)) to a dynamics on the space of measures on $\widetilde{\Omega}$, whereas the second approach gives such a dynamics only on the class of measures that are absolutely continuous with respect to some equilibrium measure.

However, we note that regardless of how the dynamics T_t on the phase space Ω is constructed, one can define and study the dynamics U_t in the space $C_0^{\mathrm{loc}}(\Omega)$ of smooth local functions in Ω, defined by the Liouville equation

$$\frac{\partial f}{\partial t} = \sum_q \left(v \frac{\partial f}{\partial q} \right)(c) + \frac{1}{m} \sum_{q,q'} \left(\frac{\partial f}{\partial v}, F(q, q') \right), \qquad c = \{(q, v)\} \in \Omega. \tag{6}$$

[1]Translator's note: This means that $U(q_1, q_2)$ is zero for all but finitely many q_1, q_2.

In the first sum the summation is over all particles in c, while in the second it is over all unordered pairs of particles in c. Since the function f depends only on a finite part of the configuration c (as follows from the definition of a local function f), the sums in (6) are finite.

One can also directly define (again, regardless of how the dynamics T_t on Ω is constructed) a dynamics U_t^* on the space of measures on Ω. The easiest way to do this is to use the evolution equations for the correlation functions of the measures. This leads to an infinite system of coupled equations which has been studied by many authors, as is reflected in its name, the BBGKY hierarchy, after Bogolyubov, Born, Green, Kirkwood, and Yvon (for more details, see [34] and also [35]).

We note that the dynamics enumerated above (in the configuration, function, and measure spaces) have been fully constructed only in a few very simple cases. In all these cases, the dynamics are obtained as limits (in a natural sense) of associated finite dynamics, that is, by taking the thermodynamic limit (see [50] for more details).

2. Classical lattice systems. The state space Ω for such a system is the space $(R^\nu \times R^\nu)^{Z^\nu}$ of infinite configurations $\{(q_x, v_x),\ x \in Z^\nu\}$. The dynamics is given by an infinite system of equations obtained from system (14.2) by formally taking the limit $\Lambda \uparrow Z^\nu$:

$$\dot{q}_x = v_x, \qquad m\dot{v}_x = F^{(1)}(q_x) + \sum_{y \neq x} F^{(2)}(q_x, q_y). \tag{7}$$

Here again the problem arises of choosing a large enough set of initial configurations $\widetilde{\Omega} \subset \Omega$, for which (7) is solvable and which is invariant under the resulting dynamics. This problem has been solved only for some special cases (e.g., for a system of linear oscillators). The same also holds for the dynamics U_t and U_t^* on the spaces of functions and measures on Ω which, as in the previous case, can be studied independently of how the dynamics T_t has been constructed on the configuration space.

A family of limit Gibbs measures $\mu_{\beta,\mu}$ on the space Ω can be defined as the thermodynamic limits as $\Lambda \uparrow Z^\nu$ of the finite Gibbs distributions $\mu_{\beta,\mu,\Lambda}$. These measures are invariant under the dynamics U_t^*.

3. Quantum systems. In the case of an infinite quantum system, the construction of the Hilbert space of states (together with a Schrödinger dynamics) is less transparent than for infinite classical systems.

Here two approaches are used the Euclidean approach, which we will briefly describe in one of the following sections, and also an approach that directly constructs the limiting Heisenberg dynamics starting from the finite Heisenberg dynamics on the algebras $\mathfrak{A}_\Lambda = \mathfrak{B}(\mathcal{H}_\Lambda)$, where $\mathcal{H}_\Lambda$ is the Hilbert space of states for a finite system. This dynamics is given by a group of $*$-automorphisms of a suitable limit algebra $\mathfrak{A}$, called the algebra of pseudolocal observables, to be described shortly. In most cases the equilibrium (temperature or Gibbs) states on the algebras $\mathfrak{A}_\Lambda$ (as well as their ground states) generate limit states (temperature or ground states) on the quasilocal algebra $\mathfrak{A}$ upon taking the thermodynamic limit, and the limit states are invariant under the limit Heisenberg dynamics on $\mathfrak{A}$. The limit Hilbert space and associated Schrödinger dynamics are then constructed by the standard Gelfand-Naimark-Segal construction (GNS construction), which will be described in §2.1.

We now turn to the description of the quasilocal algebra $\mathfrak{A}$. We will define it for the case of continuous quantum systems consisting of identical indistinguishable particles (bosons, say). We note that, as follows easily from the definition (20.2), when

$\Lambda = \Lambda_1 \cup \Lambda_2$, where $\Lambda_1 \cap \Lambda_2 = \emptyset$, the Fock space $\mathcal{F}^{\mathrm{sym}}$ admits a natural expansion as the tensor product

$$\mathcal{F}^{\mathrm{sym}}_{\Lambda} = \mathcal{F}^{\mathrm{sym}}_{\Lambda_1} \otimes \mathcal{F}^{\mathrm{sym}}_{\Lambda_2}. \tag{8}$$

It follows that for $\Lambda_1 \subset \Lambda_2$ the algebra $\mathfrak{A}_{\Lambda_1} = \mathfrak{B}(\mathcal{F}^{\mathrm{sym}}_{\Lambda_1})$ imbeds homeomorphically and isometrically in the algebra $\mathfrak{A}_{\Lambda_2} = \mathfrak{B}(\mathcal{F}^{\mathrm{sym}}_{\Lambda_2})$

$$\mathfrak{A}_{\Lambda_1} \to \mathfrak{A}_{\Lambda_1} \otimes 1_{\Lambda_2 \setminus \Lambda_1} \subset \mathfrak{A}_{\Lambda_2}, \tag{9}$$

where $1_{\Lambda_2 \setminus \Lambda_1}$ is the identity operator on the space $\mathcal{F}^{\mathrm{sym}}_{\Lambda_2 \setminus \Lambda_1}$ (cf. equation (27.2), which gives an analogous expression for the case of quantum lattice systems).

Thus, we may consider the inductive limit $\overline{\mathfrak{A}} = \cup \mathfrak{A}_{\Lambda}$ of the algebras $\mathfrak{A}_{\Lambda}$, $\Lambda \subset R^{\nu}$, equipped with the norm $\| \cdot \|$. We call $\mathfrak{A}$ the algebra of *local observables,* and its completion with respect to $\| \cdot \|$ is the algebra of *quasilocal observables.* Since all the $\mathfrak{A}_{\Lambda}$ are C^*-algebras (see §2.1 below) with respect to the adjoint operation, the same is true of the limit algebra $\overline{\mathfrak{A}}$. Assume further that for all the algebras $\mathfrak{A}_{\Lambda}$, groups of isometric automorphisms $\alpha^{\Lambda}_t \colon \mathfrak{A}_{\Lambda} \to \mathfrak{A}_{\Lambda}$ are defined, and that for every local element $A \in \mathfrak{A}$ the limit

$$\lim_{\Lambda \uparrow R^{\nu}} \alpha^{\Lambda}_t A = \alpha_t A \in \overline{\mathfrak{A}} \tag{10}$$

exists. Since the mapping $\alpha_t \colon \mathfrak{A} \to \overline{\mathfrak{A}}$ preserves the norm, it extends to an isometric automorphism

$$\alpha_t : \overline{\mathfrak{A}} \to \overline{\mathfrak{A}} \tag{11}$$

and the set of all these forms a one-parameter group that gives the limit dynamics of the quasilocal algebra.

In exactly the same way, one can define the quasilocal algebra and its dynamics for a system of Fermi particles, as well as for lattice quantum systems.

As limits of Gibbs states on the "finite" algebras $\mathfrak{A}_{\Lambda}$, the Gibbs (and ground) states on the quasilocal algebra are (as already mentioned) invariant under the limit dynamics α_t on $\overline{\mathfrak{A}}$.

The investigation of the limiting dynamics α_t together with the Gibbs (and ground) states on $\overline{\mathfrak{A}}$ is in fact a principal aim of this book. As we have already noted, this can be done either using the Euclidean approach, or else by directly analyzing the Dyson Schwinger series representing the dynamics on the algebra $\overline{\mathfrak{A}}$. Both approaches, which will be briefly described in the next sections, rely on the cluster expansion technique (see [26]).

4. Limiting stochastic dynamics. The properties of stochastic dynamics for infinite systems are much more diverse than for finite systems, and their study requires subtler methods. They are often analyzed (and even constructed) by taking limits of finite stochastic dynamics. To be sure, Markov chains with local interaction and conditionally independent transition distributions (see the example at the end of the previous section) can be defined directly in an infinite volume just as well as in a finite volume, but in the more general case when conditional independence is not required, their construction again requires taking a thermodynamic limit. Some examples will be considered in §6.1.

§4. Expansion of Heisenberg dynamics

In the previous sections we have briefly sketched the main topics of this book. We will now briefly touch upon the methods to be used. There are, in fact, just two: direct power series expansion of the dynamics and study of the resulting series (the subject of the present section), and the Euclidean approach, to be discussed in the next section.

Let $\mathfrak{A}$ be a Banach algebra with involution equipped with a one-parameter group of inner $*$-automorphisms,

$$\alpha_t A = \exp\{itH\} A \exp\{itH\} \equiv F_A(t), \tag{1}$$

where H is a selfadjoint element of $\mathfrak{A}$. Clearly, the family of elements $F_A(t)$ satisfies the equation

$$\begin{gathered} \frac{dF_A}{dt} = i[H, F_A(t)] \equiv i(HF_A - F_A H) \stackrel{\text{def}}{=} \mathcal{H} F_A, \\ F_A(0) = A, \end{gathered} \tag{2}$$

where

$$\mathcal{H} : \mathfrak{A} \to \mathfrak{A} : F \to i[H, F], \qquad F \in \mathfrak{A}, \tag{3}$$

is a bounded linear operator on $\mathfrak{A}$ with norm $\|\mathcal{H}\| \leq 2\|H\|$. The solution of (2) can be expressed in the form

$$\begin{aligned} F_A(t) = \exp\{t\mathcal{H}\} A &= A + \sum_{n=1}^{\infty} \frac{t^n}{n!} \mathcal{H}^n A \\ &= A + \sum_{n=1}^{\infty} \frac{(it)^n}{n!} [\underbrace{H, [H, \dots, [H, A]}_{n \text{ times}} \dots]], \end{aligned} \tag{4}$$

where the series converges with respect to the norm of the algebra $\mathfrak{A}$.

Another series turns out to be more convenient in practice. Let $H = H_0 + V$, where H_0 and V are selfadjoint elements of $\mathfrak{A}$. Then

$$\begin{aligned} F_A(t) = \alpha_t^0(A) + \sum_{n=1}^{\infty} i^n \int\limits_{0<t_1<t_2<\dots<t_n<t} \cdots \int dt_1 \dots dt_n \\ \times \underbrace{[\alpha_{t_1}^0(V), [\alpha_{t_2}^0(V), \dots, [\alpha_{t_n}^0(V), \alpha_t^0(A)}_{n \text{ times}} \dots]], \end{aligned} \tag{5}$$

where $\alpha_t^0(B) = \exp\{itH_0\} B \exp\{-itH_0\}$ is the automorphism group generated by the element $H_0 \in \mathfrak{A}$.

To prove (5), we write

$$B_t = \alpha_{-t}^0(F_A(t)) = \exp\{-itH_0\} F_A(t) \exp\{itH_0\}. \tag{6}$$

Hence

$$\frac{dB_t}{dt} = i[V_t, B_t], \tag{7}$$

where

$$V_t = \alpha_t^0(V) = \exp\{itH_0\}V\exp\{-itH_0\}$$

and consequently,

$$\begin{aligned} B_t &= B_0 + i\int_0^t [V_s, B_s]\,ds \\ &= B_0 + \sum i^n \int\limits_{0<s_n<s_{n-1}<\cdots<s_1<t} \cdots \int ds_1 \ldots ds_n [V_{s_1}, [V_{s_2}, \ldots, [V_{s_n}, B_0]\ldots]]. \end{aligned} \tag{8}$$

Applying the automorphism α_t^0 to the last formula and making the substitution $s_i = t - t_i$, $i = 1, \ldots, n$, we get (5) since $B_0 = A$.

We note that our formulas remain valid for any Banach Lie algebra with multiplication $[\cdot, \cdot]$. In this case, $\alpha_t^0(B)$ is understood to be the solution of the equation

$$\frac{dB_t}{dt} = i[H_0, B_t], \qquad B_{t=0} = B. \tag{9}$$

In particular, in the study of classical systems one takes $\mathfrak{A}$ to be the Lie algebra of smooth functions on a symplectic manifold, in which the multiplication $[\cdot, \cdot]$ is given by Poisson brackets. This operation is given in local coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ by

$$i[H, F] \equiv \{H, F\} = \sum_{i=1}^{n} \left(-\frac{\partial H}{\partial q_i}\frac{\partial F}{\partial p_i} + \frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q_i} \right).$$

The series in equations (4), (5) for expanding the dynamics are also formally applicable to the case of a group of $*$-automorphisms of generated by a symmetric derivation on the algebra $\mathfrak{A}$. Recall that a derivation is an operator δ, defined on a dense $*$-subalgebra D_δ, $\delta\colon D_\delta \to \mathfrak{A}$, such that

1) $\delta(AB) = \delta(A)B + A\delta(B)$, $A, B \in D_\delta$,

2) $\delta(A^*) = (\delta(A))^*$.

In particular, the operator $\mathcal{H}$ introduced above is a symmetric derivation.

The action of the group α_t^δ of $*$-automorphisms on $\mathfrak{A}$ corresponding to the derivation δ,

$$A \to \alpha_t^\delta(A) \equiv F_A(t)$$

is given as the solution of the equation

$$\frac{dF_A(t)}{dt} = \delta(F_A(t)), \qquad F_A(t=0) = A$$

(see [7] for more details).

The formal series for the automorphism α_t^δ can sometimes also be applied to the case $H \in \mathfrak{A}$, when the derivation δ on $\mathfrak{A}$ is of a form more general than $[H, \cdot]$. In this case, it may be necessary to consider the series on some dense subset of elements of $\mathfrak{A}$, and to interpret their convergence in a suitable weak sense.

§5. Euclidean approach

The spectral analysis of the operators $U_t = \exp\{itH\}$ for the Schrödinger dynamics is equivalent to the same problem for the operator H, or for the semigroup of operators $\exp\{-tH\}$. In many cases, this semigroup allows a very simple probabilistic representation known as the Feynman-Kac formula. This representation permits one to employ cluster expansions to study the spectrum of the operator $\exp\{-tH\}$; Chapter 3 is devoted to this. Here we will describe the probabilistic representation $\exp\{-tH\}$ itself for some very simple examples. More general cases are considered in Chapter 3.

1. The original Feynman-Kac formula. The general scheme for deriving the Feynman-Kac formula is as follows. Let (S, ν) be a space equipped with a probability measure ν, and let a selfadjoint operator H acting on the space $L_2(S, \nu)$ have the form

$$H = H_0 + \widehat{V},$$

where $\widehat{V}$ is the operator given by multiplication by the bounded function $V(x)$, $x \in S$, and H_0 is a selfadjoint operator for which $\exp\{-tH_0\}$ is a stochastic semigroup. More precisely, assume that the action of $\exp\{-tH_0\}$ is given by

$$(\exp\{-tH_0\}f)(x) = \int_S P_t(x,y)f(y)d\nu(y), \qquad x \in S, \qquad f \in L_2(S, d\nu), \tag{1}$$

where $P_t(x, y)$ is the density (with respect to the measure ν) of the transition probability (for a particle to go from x to y) during the time t for some Markov process $(\xi_t,\ t > 0)$ with values in the set S. Let $Q_t(x, y)$ denote the kernel of the operator $\exp\{-tH_0 + \widehat{V}\}$,

$$(\exp\{-tH_0 + \widehat{V}\}f)(x) = \int_S Q_t(x,y)f(y)d\nu(y). \tag{2}$$

The Feynman-Kac formula expresses $Q_t(x, y)$ as an average over the process $\{\xi_t,\ t > 0\}$.

For simplicity we will limit ourselves here to the case of a finite set S. Note that the requirement for the semigroup $\exp\{-tH_0\}$ to be stochastic leads to the following conditions on the matrix $H_0 = \{h^0_{x,y}\}_{x,y\in S}$:

$$h^0_{x,y} \le 0 \quad \text{for } x \ne y, \qquad \sum_{y\in S} h^0_{x,y} = 0 \quad \text{for all } x. \tag{3}$$

We now use Trotter's formula (see [36])

$$\exp\{-t(H_0 + \widehat{V})\} = \lim_{n\to\infty} \left(\exp\left\{ -\frac{t}{n}H_0 \right\} \exp\left\{ -\frac{t}{n}\widehat{V} \right\} \right)^n \tag{4}$$

and denote by $\{Q^{(n)}_t(x, y)\}_{x,y\in S}$ the matrix of the operator $(\exp\{-\frac{t}{n}H_0\}\times\exp\{-\frac{t}{n}\widehat{V}\})^n$.

It is clear that for $x = x_0$ and $y = x_n$

$$
\begin{aligned}
Q_t^{(n)}(x,y) = \underbrace{\int\cdots\int}_{(n-1)\text{ times}} \exp\Big\{-\frac{t}{n}\sum_{i=1}^{n} V_{x_i}\Big\} \\
\times P_{t/n}(x_0,x_1)\dots P_{t/n}(x_{n-1},x_n)\,d\nu(x_1)\dots d\nu(x_{n-1}) \\
= \Big\langle \exp\Big\{-\frac{t}{n}\sum_{i=1}^{n} V(x_i)\Big\}\Big\rangle_{t,x,y}^{(n)}.
\end{aligned} \tag{5}
$$

The mean $\langle\cdot\rangle_{t,x,y}^{n}$ has the following interpretation. Let $x_k = \xi_{tk/n}^{x}$, $k = 0,1,\dots,n$, denote the Markov chain starting at the point x: $x_0 = x$, with discrete time $0, t/n, \dots, tk/n, \dots, t$ and transition probabilities $P_{t/n}(x,y)d\nu$. Then $\langle\cdot\rangle_{t,x,y}^{n}$ is the mean of the trajectories $x_0, x_1, \dots, x_n$ of our chain for which $x_n = y$, with respect to the conditional (unnormalized!) measure on the set $\Omega_{t,x,y}^{(n)}$.

By virtue of (5), in the limit $n \to \infty$ we obtain the following representation for the matrix $Q_t(x,y)$ of the operator $\exp\{-t(H+\widehat{V})\}$

$$
Q_t(x,y) = \Big\langle \exp\Big\{-\int_0^t V(\xi_\tau)\,d\tau\Big\}\Big\rangle_{t,x,y}, \tag{6}
$$

this is the Feynman-Kac formula. The average $\langle\cdot\rangle_{t,x,y}$ is taken over the conditional (as before, unnormalized) measure on the set $\Omega_{t,x,y}$ of those trajectories $\{x(\tau),\, 0<\tau<t\}$ of the Markov process $\{\xi^x,\, 0<\tau<t\}$, starting at x ($\xi_0^x \equiv x$) and reaching y at time $\tau = t$: $x(t) = y$.

Formula (6) implies the following representation for the action of the operator $\exp\{-t(H_0+\widehat{V})\}$:

$$
\begin{aligned}
(\exp\{-t(H_0+\widehat{V})\}f)(x) &= \int_S f(y)d\nu(y)\Big\langle \exp\Big\{-\int_0^t V(\xi_\tau)\,d\tau\Big\}\Big\rangle_{t,x,y} \\
&= \Big\langle \exp\Big\{-\int_0^t V(\xi_\tau)d\tau\Big\} f(\xi_t)\Big\rangle_{t,x},
\end{aligned} \tag{7}
$$

where $\langle\cdot\rangle_{t,x}$ denotes the mean over all trajectories of the Markov process $\{\xi_\tau^x,\, 0<\tau<t\}$. Similarly, the matrix elements of the operator $\exp\{-t(H_0+\widehat{V})\}$ are given by

$$
(\exp\{-t(H_0+V)\}f, g) = \int_S \overline{g}(x)\Big\langle \exp\Big\{-\int_0^t V(\xi_\tau)\,d\tau\Big\} f(\xi_t)\Big\rangle_{t,x} d\nu(x). \tag{8}
$$

2. Renormalized Feynman-Kac-Nelson formula. As x, y, t vary in equation (6), so does the set $\Omega_{x,y,t}$ over which the averaging is performed. In the Feynman-Kac-Nelson formula, the mean is taken over trajectories of a single stationary Markov process. Indeed, let us suppose that the Markov process ξ_t introduced above with transition probability density $P_t(x,y)$ (from x to y in time t), and with the infinitesimal operator H_0, is a stationary Markov process defined for all $t \in R^1$, and let the measure ν on S be its invariant distribution.

Then from equation (8) we find that

$$
(1, \exp\{-2t(H_0+V)\}1) = \Big\langle \exp\Big\{-\int_{-t}^{t} V(\xi_\tau)\,d\tau\Big\}\Big\rangle_0 \stackrel{\text{def}}{=} Z_{2T}, \tag{9}
$$

where $(\cdot,\cdot)$ is the inner product on $L_2(S,\nu)$ and the mean $\langle\cdot\rangle_0$ is taken over the distribution μ_0 of the stationary process $\{\xi_t,\, t\in R^1\}$. Define the new measure μ_T by

$$\frac{d\mu_T}{d\mu_0} = Z_{2T}^{-1}\exp\Big\{-\int_{-T}^{T} V(\xi_\tau)\,d\tau\Big\}. \tag{10}$$

Then for any functions F, G defined on the set S and for any time t_0, $0\le t_0<T$, the mean is given by

$$\begin{aligned}
\langle F(\xi_0)G(\xi_{t_0})\rangle_{\mu_T} &\stackrel{\text{def}}{=} Z_{2T}^{-1}\Big\langle F(\xi_0)G(\xi_{t_0})\exp\Big\{-\int_{-T}^{T}V(\xi_\tau)d\tau\Big\}\Big\rangle_0\\
&= Z_{2T}^{-1}\int\Big\langle\exp\Big\{-\int_{-T}^{0}V(\xi_\tau)d\tau\Big\}\Big\rangle_{x_{-T},x_0,T}F(x_0)\\
&\quad\times\Big\langle\exp\Big\{-\int_{0}^{t_0}V(\xi_\tau)d\tau\Big\}\Big\rangle_{x_0,x_{t_0},t_0}G(x_{t_0})\\
&\quad\times\Big\langle\exp\Big\{-\int_{t_0}^{T}V(\xi_\tau)d\tau\Big\}\Big\rangle_{x_{t_0},x_T,T-t_0}d\nu(x_{-T})\,d\nu(x_0)\,d\nu(x_{t_0})\,d\nu(x_T)\\
&= Z_{2T}^{-1}(1,\exp\{-T(H_0+\widehat V)\}\widehat F\exp\{-t_0(H_0+\widehat V)\}\widehat G\\
&\quad\times\exp\{-(T-t_0)(H_0+\widehat V)\}1)\\
&= Z_{2T}^{-1}(\exp\{-T(H_0+\widehat V)\}1,\widehat F\exp\{-t_0(H_0+\widehat V)\}\widehat G\\
&\quad\times\exp\{-(T-t_0)(H_0+\widehat V)\}1),
\end{aligned} \tag{11}$$

where $\widehat F$ and $\widehat G$ are given by multiplication by the functions $F(x)$ and $G(x)$, respectively, and $(\cdot,\cdot)$ is the inner product on $L_2(S,\nu)$. Let $\Omega_0(x)$ be a normalized eigenvector of the operator H_0+V which corresponds to an isolated eigenvalue λ_0 that is simple and as small as possible. Then it is obvious that when $T\to\infty$ we have the asymptotic relation

$$\exp\{-T(H_0+\widehat V)\}1 = \exp\{-\lambda_0 T\}(\Omega_0(1,\Omega_0)+o(1)), \tag{12}$$

and the expression in (11) has the asymptotic form

$$\begin{aligned}
&(\exp\{-T(H_0+\widehat V)\}1,\widehat F\exp\{-t_0(H_0+\widehat V)\}\widehat G\exp\{-(T-t_0)(H_0+\widehat V)\}1)\\
&\quad=\exp\{-\lambda_0(2T-t_0)\}\times[(\Omega_0,\widehat F\exp\{-t_0(H_0+\widehat V)\}\widehat G\Omega_0)|(1,\Omega_0)|^2+o(1)].
\end{aligned} \tag{13}$$

On the other hand, with $F=G=1$ we obtain that

$$\begin{aligned}
Z_{2T} &= \exp\{-2\lambda_0 T\}((\Omega_0,\Omega_0)|(1,\Omega_0)|^2+o(1))\\
&= \exp\{-2\lambda_0 T\}(|(1,\Omega_0)|^2+o(1)).
\end{aligned} \tag{14}$$

Thus, the limit

$$\begin{aligned}
\lim_{T\to\infty}\langle F(\xi_0)G(\xi_{t_0})\rangle_{\mu_T} &= \langle F(\xi_0)G(\xi_{t_0})\rangle_\mu\\
&= (\widehat F\Omega_0,\exp\{-t_0(H_0+\widehat V-\lambda_0 E)\}\widehat G\Omega_0),
\end{aligned} \tag{15}$$

exists, where $\langle\cdot\rangle_\mu$ is the mean with respect to the limiting Gibbs measure

$$\mu = \lim_T \mu_T. \tag{16}$$

The second equality in (15) is called the renormalized Feynman-Kac-Nelson formula.

If $\Omega_0(x) > 0$ for all x, it follows from the above formulas that the measure μ is the distribution of a stationary Markov process $\{\eta_t, -\infty < t < \infty\}$ with an invariant measure $\widehat{\nu}$ defined by the formula

$$d\widehat{\nu} = \Omega_0^2(x)d\nu \tag{17}$$

and with transition probability density $\widehat{P}_t(x,y)$ (relative to the invariant measure $\widehat{\nu}$)

$$\widehat{P}_t(x,y) = \Omega_0^{-1}(x)\exp\{\lambda_0 t\}Q_t(x,y)\Omega_0^{-1}(y),$$

where $Q_t(x,y)$ is the kernel of the operator $\exp\{-t(H_0+\widehat{V})\}$ (see (2)). The kernel $\widehat{P}_t(x,y)$ defines a semigroup

$$(\mathcal{J}_t f)(x) = \int \widehat{P}_t(x,y)f(y)d\widehat{\nu}(y), \quad t>0, \tag{18}$$

on the space $L_2(S,\widehat{\nu})$ with generator $\widehat{H}$: $\mathcal{J}_t = \exp\{-t\widehat{H}\}$. Formula (17) implies that the mapping

$$U : L_2(S,\widehat{\nu}) \to L_2(S,\nu) : f \to \Omega_0 f \tag{19}$$

is unitary. Then from equation (15), rewritten in the form

$$(F, \exp\{-t\widehat{H}\}G)_{L_2(S,\widehat{\nu})} = (UF, \exp\{-t(H_0+\widehat{V}-\lambda_0 E)\}UG)_{L_2(S,\nu)}$$

we obtain that

$$\widehat{H} = U^{-1}(H_0+\widehat{V}-\lambda_0 E)U,$$

i.e., the operators $\widehat{H}$ and $(H_0+\widehat{V}-\lambda_0 E)$ are unitarily equivalent. Thus the spectral analysis of $H_0+\widehat{V}$ reduces to that for $\widehat{H}$.

We note that the Feynman-Kac formula is valid quite generally when ν is an arbitrary σ-finite measure on S. As we have seen, the validity of the renormalized Feynman-Kac-Nelson formula requires that ν be an invariant probability measure for a stationary process ξ_t, and also the existence of a ground state Ω_0 for the operator $H_0+\widehat{V}$ with an isolated simple eigenvalue

$$\lambda_0 = \inf\sigma(H_0+\widehat{V}).$$

The use of the Euclidean approach in the study of more complicated systems (such as, say, quantum field theory) will be described in 5.2. Here we consider some simple examples that will be used later.

I. We consider first the Hamiltonian

$$Hf = -\frac{1}{2}\frac{d^2 f}{dx^2} + \frac{\omega^2 x^2}{2}f, \qquad f \in L_2(R^1, dx), \tag{20}$$

for a linear quantum oscillator. This operator acts on $L_2(R^1, dx)$ and is well known to have a ground state given by the eigenvector

$$\text{const} \exp\{-\omega x^2/2\} \tag{21}$$

with eigenvalue $\omega/2$.

We now introduce the Gaussian distribution with density

$$p(x) = \sqrt{\omega/\pi} \exp\{-\omega x^2\} \tag{22}$$

and make the unitary transformation

$$U : L_2(R^1, dx) \to L_2(R^1, p\,dx), \qquad f \to (p(x))^{-1/2} f.$$

One computes readily that this takes the operator $\widetilde{H} = H - \omega E/2$ into the selfadjoint operator

$$\widetilde{H} f = UHU^{-1} f = -\frac{1}{2}\frac{d^2 f}{dx^2} + \omega x \frac{df}{dx} \tag{23}$$

on $L_2(R^1, dx)$; $\widetilde{H}$ is the infinitesimal generator of a stationary Markov process $\{\xi_t, t \in R^1\}$ with mean zero, covariance

$$\langle \xi_{t_1}, \xi_{t_2} \rangle = \frac{1}{2\omega} \exp\{-\omega |t_1 - t_2|\} \tag{24}$$

and stationary distribution density equal to (22) (an Ornstein-Uhlenbeck process).

II. We next consider a system of n interacting one-dimensional quantum harmonic oscillators with the Hamiltonian

$$Hf = -\frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} + \frac{1}{2}\Big(\sum a_{ij} x_i x_j\Big) f, \tag{25}$$

acting on $L_2(R^n, d^n x)$. The quadratic form $\sum a_{ij} x_i x_j$ with matrix $A = \{a_{ij}\}$ is assumed to be strictly positive.

Again, one can show that the operator $H - \lambda_0 E$ (where λ_0 is the smallest eigenvalue of H) is unitarily equivalent to the infinitesimal generator $\widetilde{H}$ of a stationary Markov Gaussian process $\{\xi_t^{(i)}, i = 1, \ldots, n, t \in R^1\}$ with mean zero and covariance matrix

$$\{\langle \xi_{t_1}^{(i)} \xi_{t_2}^{(j)} \rangle\} = \frac{1}{2} A^{-1/2} \exp\{-A^{-1/2} |t_1 - t_2|\} \tag{26}$$

(a multidimensional Ornstein-Uhlenbeck process). The density for the invariant measure of this process is given by

$$p(x) = \frac{(\text{Det } A)^{1/4}}{\pi^{1/2}} \exp\{-A^{1/2} x, x\}, \tag{27}$$

where $(\cdot, \cdot)$ is the inner product on R^n.

The operator $\widetilde{H}$ acts on the space $L_2(R^n, p(x)d^nx)$ by the formula

$$\widetilde{H}f = -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 f}{\partial x_i^2} + \left(A^{1/2}x, \frac{\partial}{\partial x}\right)f, \tag{28}$$

where

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right).$$

III. In the case of a system of anharmonic oscillators with Hamiltonian

$$H = -\frac{1}{2}\sum_i \frac{\partial^2 f}{\partial x_i^2} + \frac{1}{2}\left(\sum a_{ij}x_i x_j\right)f + V(x_1, \ldots, x_n)f, \tag{29}$$

where $V(x_1, \ldots, x_n)$ is a function of the variables $x_1, \ldots, x_n$ bounded from below, the operator $H - \lambda_0 E$ (where λ_0 is the smallest eigenvalue of H) is unitarily equivalent to the generator of a process $\xi_t = \{\xi_t^{(i)},\ i = 1, \ldots, n,\ t \in R^n\}$, whose distribution μ is obtained as the Gibbs limit (see (10) and (16)) of the Gaussian measure μ_0 (the distribution of the Ornstein-Uhlenbeck process) via the "interaction"

$$\int_{-T}^{T} V(x(\tau))d\tau.$$

REMARK. All the results of the foregoing analysis can be obtained by a formal heuristic procedure, as is usually done by physicists and which leads more quickly to the final answer. The action functional for a classical one-dimensional linear oscillator is of the form

$$S(x(\tau)) = \frac{1}{2}\int_{-\infty}^{\infty}[\dot{x}^2(\tau) - \omega^2 x^2(\tau)]d\tau, \tag{30}$$

where $x(\tau)$ is a smooth path tending to zero at infinity. Passing formally to imaginary time $\tau \to i\tau$, we find that the action becomes

$$S(x(\tau)) = -\frac{i}{2}\int_{-\infty}^{\infty}[\dot{x}^2(\tau) + \omega^2 x^2(\tau)]d\tau = -iS_{\text{Eucl}}(x(\tau)) = -\frac{i}{2}(Bx, x), \tag{31}$$

where $(\cdot, \cdot)$ is the inner product on $L_2(R^2, dx)$ and B is the selfadjoint operator

$$Bx = -\frac{d^2x}{d\tau^2} + \omega^2 x. \tag{32}$$

One computes without difficulty that the kernel of the operator B^{-1} is

$$B^{-1}(\tau, \sigma) = \frac{1}{2\omega}\exp\{-\omega|\tau - \sigma|\}, \tag{33}$$

i.e., it coincides with the covariance function (24) for the Ornstein-Uhlenbeck process. Thus, heuristically at least, the functional

$$\text{const}\exp\{-S_{\text{Eucl}}(x(\tau))\} = \text{const}\exp\left\{-\frac{1}{2}(Bx, x)\right\} \tag{34}$$

can be regarded formally, in analogy with the case of a finite-dimensional Gaussian distribution, as the "distribution density" for the values of the process $\{\xi_t,\ t \in R^1\}$ relative to the "measure $\prod_\tau dx(\tau)$." The same considerations apply to the case of a multidimensional Ornstein-Uhlenbeck process, for which the Euclidean action $S_{\text{Eucl}}^{\text{harm}}$ has the form

$$S_{\text{Eucl}}^{\text{harm}}(x(\tau)) = \frac{1}{2}(Bx, x), \tag{35}$$

where $x(\tau) = \{x_i(\tau),\ i = 1, \dots, n\}$ and the operator B is given by

$$Bx = -\frac{d^2x}{d\tau^2} + Ax, \tag{36}$$

with $A = \{a_{ij}\}$ the matrix of the quadratic form in (25).

For a system of anharmonic oscillators the Euclidean action $S_{\text{Eucl}}^{\text{anharm}}$ is given by

$$S_{\text{Eucl}}^{\text{anharm}} = S_{\text{Eucl}}^{\text{harm}} + \int_{-\infty}^{\infty} V(x(\tau))d\tau, \tag{37}$$

where $V(x_1, \dots, x_n)$ is the function appearing in (29). Regarding the functional $\exp\{-S_{\text{Eucl}}^{\text{anharm}}\}$ as the distribution "density" μ of a Markov process, corresponding to anharmonic oscillators, with respect to the "measure $\prod_\tau dx(\tau)$," and recalling that $\exp\{-S_{\text{Eucl}}^{\text{harm}}\}$ gives the "density" for the Ornstein-Uhlenbeck process, we obtain formally that

$$d\mu = \text{const} \exp\left\{-\int_{-\infty}^{\infty} V(x(\tau))d\tau\right\} d\mu_0, \tag{38}$$

where μ_0 is the distribution for the Ornstein-Uhlenbeck process. This heuristic formula requires only a slight correction: one must replace $\int_{-\infty}^{\infty} V(x(\tau))\, d\tau$ by the integral $\int_{-T}^{T} V(x(\tau))\, d\tau$ and then take the limit $T \to \infty$ in order to get an exact formula.

The formal procedure described here can be employed directly to obtain the final formulas (10) and (16) using only the form of the (Euclidean) action for the corresponding classical system.

§6. Corpuscular picture (quasiparticles and scattering theory for infinite quantum systems)

In this concluding section we will briefly outline the main program and goals in the area of mathematical physics with which our book is concerned.

For a quantum system consisting of n identical free (i.e., noninteracting) particles, the Hilbert space $\mathcal{H}_n$ of states is a tensor product (symmetric or antisymmetric in the case of indistinguishable particles) of copies of the one-particle space $\mathcal{H}_1 = f$ describing the states of a single particle:

$$\mathcal{H}_n = \mathfrak{f}^{\otimes n} \text{ (or } \mathcal{H}_n = \mathfrak{f}_a^{\otimes n}, \quad a = \text{sym}, \text{asym}), \tag{1}$$

and the energy operator $\mathcal{H}_n$ is given as the tensor sum

$$H_n = H_1 + \cdots + H_n \tag{2}$$

of the operators $H_k = \underbrace{1 \otimes \cdots \otimes h}_{k\text{th place}} \otimes 1 \otimes \cdots \otimes 1$, where h is the energy operator for a single particle acting on f.

If one considers a system with an arbitrary finite number of identical particles, the state space $\mathcal{H}$ and energy operator H are given as direct sums

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n \quad (\mathcal{H}_0 = \mathbb{C}^1) \tag{3}$$

and

$$H = \bigoplus_{n=0}^{\infty} H_n \quad (H_0 = 0) \tag{4}$$

of the n-dimensional spaces $\mathcal{H}_n$ and operators H_n, respectively.

If an individual particle possesses a symmetry group G, i.e., there is a unitary representation $\{g \to u_g,\ g \in G\}$ of G acting on the space f and commuting with the operator h, then the tensor product of the representation generates a unitary representation $u_g^{(n)}$ of G on $\mathcal{H}_n$, and the direct sum of these representations $u_g^{(n)}$ gives a full representation $\{u_g,\ g \in G\}$ in $\mathcal{H}$ commuting with the operator H. It turns out that one can regard many spatially homogeneous infinite physical systems, described by vectors in a Hilbert space $\widehat{H}$, by an energy operator $\widehat{H}$ acting on $\widehat{\mathcal{H}}$, and by a representation $\{\widehat{U}_x,\ x \in R^\nu\}$ of the group of spatial translations of R^ν (or of the subgroup Z^ν) on $\mathcal{H}$, as a system of noninteracting particles ("quasiparticles" or "elementary excitations"). More precisely, this means that one can find a one-particle Hilbert space f, a one-particle energy operator h acting on f, and a representation of the group $R^\nu\colon x \to u_x,\ x \in R^\nu$ on f such that the triple $(\mathcal{H}, \widehat{H}, \{\widehat{U}_x\})$ is unitarily equivalent to the space $\mathcal{H}$, operator H, and representation $\{u_x,\ x \in R^\nu\}$ on $\mathcal{H}$ constructed from the one-particle triple $(f, h, \{U_x\})$ using equations (1)–(4).

In general, such a prescription is clearly wrong. First, "quasiparticles" of various types can occur, and second, bound states can exist, i.e., clusters of quasiparticles that move as a unit. Formally, these complexes can be regarded as new quasiparticles. The above construction can thus be generalized to the case when several types of particles are present. Indeed, suppose we are given a finite set of triples $\{f, h_i, \{u_x^{(i)}\}\}$, $i = 1, \ldots, s$, each describing a specific type of "quasiparticle." We then construct the triples

$$\begin{aligned}
\mathcal{H}^{(i)} &= \bigotimes_{n=0}^{\infty} \mathfrak{f}_i^{\otimes n}, \\
H^{(i)} &= \bigotimes_{n=0}^{\infty} (\underbrace{h^{(i)} + \cdots + h^{(i)}}_{n\text{-fold tensor sum of } h^{(i)}}), \\
u_x^{(i)} &= \bigotimes_{n=0}^{\infty} (u_x^{(i)})^{\otimes n}, \quad x \in R^\nu,
\end{aligned}$$

which describe a "gas" of quasiparticles of the same type. Finally, we construct the

total space, total Hamiltonian, and total representation by

$$\mathcal{H} = \bigotimes_{i=1}^{s} \mathcal{H}^{(i)},$$
$$H = (\underbrace{H^{(1)} + \cdots + H^{(s)}}_{\text{tensor sum}}),$$
$$u_x = \bigotimes_{i=1}^{s} (u_x^{(i)}), \quad x \in R^{\nu}.$$

It seems plausible that every spatially uniform system should be such a mixture of "gases" of several types of quasiparticle in the above sense. However, this conjecture has so far been verified only for a very small class of models. We observe that scattering theory is the main tool for verifying this conjecture. The Hamiltonian $\widehat{H}$ of the system is usually of the form

$$\widehat{H} = H_0 + V,$$

where the operator H_0 describes a free system of noninteracting particles, and the perturbation V is small in a suitable sense. Then in many cases the wave (Møller) operators

$$W_{\pm} = \operatorname*{s-lim}_{t \to \pm} e^{-iH_0 t} e^{iHt},$$
$$W_{\pm}^{*} = \operatorname*{s-lim}_{t \to \pm} e^{-iHt} e^{iH_0 t}$$

exist (here s-lim denotes the strong operator limit, see [36]). It follows that $\widehat{H}$ and H_0 are unitarily equivalent under either of the operators $W_{\pm}$ (or $W_{\pm}^{*}$).

In the more general case when $\widehat{H}$ is not unitarily equivalent to H_0 because bound states are present, the bound states must be used to construct a larger space $\widehat{\mathcal{H}}$ and Hamiltonian $\overline{H}_0$, which would describe both the "gas" of original free particles and the "gas" of bound states (in the sense indicated above). The proof of unitary equivalence for the Hamiltonian $\overline{H}_0$ is usually called the asymptotic completeness problem for $\overline{H}$; it again reduces to constructing certain wave operators that now act in different spaces (see [36]).

The contents of this introductory chapter can be summarized by saying that the study of an infinite system proceeds in the following steps.

1. Construct the dynamics of the infinite system (either by the Euclidean approach, or by directly constructing a limiting Heisenberg dynamics).

2. Find one-particle subspaces (including any "bound states") for the Hamiltonian of the limiting dynamics.

3. Construct the wave operators and prove asymptotic completeness.

As we have already observed, this program has been fully implemented only for a very few models. However, portions of it have been completed for a wide class of examples, and these form the content of our book.

CHAPTER I

CONSTRUCTION OF A NONEQUILIBRIUM DYNAMICS

Nonequilibrium dynamics is the established, not very apt term for the evolution of a system "by itself", the evolution being determined for all initial states of the system (or for a sufficiently large subset of states). It has no direct relation with the "equilibrium distributions" on the set of these states.

We will analyze here three types of nonequilibrium dynamics: 1) the dynamics of an infinite (nonideal) classical gas; 2) the Heisenberg dynamics on C^*-algebras associated with infinite quantum systems; 3) random dynamics for fields on an infinite lattice. Along the way we will present some expository material on C^*-algebras, Fock space, the second-quantization formalism, etc.

§1. The dynamics of an infinite one-dimensional classical gas of interacting solid rods

In the previous chapter we touched on the dynamics of an infinite system of interacting particles in the space R^ν. Here we will describe this dynamics in more detail for a one-dimensional gas of particles with a "hard core" (system of "hard rods").

In the one-dimensional case, the state of such a system is specified by an infinite sequence of pairs $\{q_i, v_i\}$ enumerated so that

$$\cdots < q_{-1} < q_0 < q_1 < \dots, \tag{1}$$

where for definiteness we assume that

$$q_0 \geq 0, \quad q_{-1} < 0 \tag{2}$$

(recall that q_i is the position of the center of the ith rod and v_i is its velocity). In addition, the conditions

$$\delta \leq q_{i+1} - q_i, \quad i = 0, \pm 1, \pm 2, \dots \tag{3}$$

are satisfied. Condition (2) will be assumed to hold only for the initial configurations, because in general it is violated as the particles evolve. Let Ω denote the space of all the states of the infinite system.

As we have already said, the dynamics in Ω is given formally by an infinite system of differential equations

$$\frac{dq_i}{dt} = v_i, \quad m\frac{dv_i}{dt} = \sum_{j \neq i} F(q_i - q_j), \tag{4}$$

where $F(\xi) = -\partial U/\partial \xi$ and $U(\xi) = U(-\xi)$ is the interaction potential, defined for all $\xi \geq \delta$. We assume that

1) the potential $U(\xi)$ has compact support:

$$U(\xi) = 0 \quad \text{for } |\xi| > R, \tag{5}$$

where $R > \delta$ is the radius of the interaction;

2) the function $U(\xi)$ is continuously differentiable on the interval $\delta \le \xi \le R$.

As we have already discussed in 2.0, the equations (4) must be supplemented by an "elastic reflection" condition for collisions of finitely many rods. As we explained in 3.0, to construct a dynamics using equations (4) one must choose a subset $\widetilde{\Omega} \subseteq \Omega$ of initial conditions

$$\omega^0 = \{(q_i, v_i), \quad i = 0, \pm 1, \pm 2, \dots\},$$

for which (4) has a solution that is well defined for all values $0 \le t < \infty$ and remains in $\widetilde{\Omega}$:

$$\omega^t = \{(q_i^t, v_i^t)\} \in \widetilde{\Omega}.$$

We now introduce the set of states $M(C_1, C_2) \subset \Omega$, where $C_1 > 0$, $C_2 > 0$ are arbitrary constants determined by the following conditions: for each $\omega^0 = \{(q_i^0, v_i^0)\} \in \widetilde{\Omega}$, there exists a $\overline{k} = \overline{k}(\omega^0)$ such that for every $k > \overline{k}(\omega^0)$

1) we have

$$\max_{|i|<2^k} |v_i^0| < C_1\sqrt{k}, \tag{6}$$

2) there exist integers i_k^+ and i_k^- satisfying

$$2^k \le i_k^+ < 2^{k+1}, \qquad -2^{k+1} < i_k^- \le -2^k,$$

such that

$$|q_{i_{k+1}^\pm} - q_{i_k^\pm}| > C_2 k. \tag{7}$$

Let us show that for every $\omega^0 \in M(C_1, C_2)$, equations (4) have a solution ω^t which is well defined for all t, $0 \le t < \infty$. Let $T > 0$ and $\overline{\overline{k}}(\omega^0) > \overline{k}(\omega^0)$ be an integer, to be determined below. We divide all the particles in the state ω^0 into finite groups η_s^0 of particles (clusters) that arrive in succession, as follows.

The zeroth cluster η_0^0 consists of all particles with indices

$$i_{\overline{\overline{k}}}^- \le i < i_{\overline{\overline{k}}}^+.$$

The cluster η_s^0 with index $s > 0$ consists of the particles with indices

$$i_{\overline{\overline{k}}+s-1}^+ \le i < i_{\overline{\overline{k}}+s}^+,$$

while the clusters η_s^0 with index $s < 0$ contain the particles for which

$$i_{\overline{\overline{k}}+|s|}^- \le i < i_{\overline{\overline{k}}+|s|-1}^-.$$

Condition (7) ensures that the gap between clusters s and $s+1$ (or $s-1$) is at least

$$C_2(|s|+\overline{\overline{k}}). \tag{8}$$

We now determine the solution of equations (4) with initial condition $\omega^0 \in M(C_1, C_2)$ on the interval $t \in [0, T]$, as follows. Assume that for $0 \le t \le T$, the motion of each cluster $\eta_s^0 \subset \omega^0$ entering the state ω^0 is governed by the finite system of equations

$$\frac{dq_i}{dt} = v_i, \qquad m\frac{dv_i}{dt} = \sum_j F(q_i - q_j), \tag{9}$$
$$\{q_i, v_i\} \in \eta_s^t, \quad \eta_s^{t=0} = \eta_s$$

(together with the elastic reflection conditions), all the other clusters being completely ignored. *A priori,* for such a motion it would be possible for particles in different clusters to enter a zone of interaction, and then the joint motion of all the clusters would not be described by equations (4). However, the next lemma shows that this situation cannot occur.

LEMMA 1. *Let $\overline{\overline{k}} > \overline{k}(\omega^0)$ be sufficiently large, and assume that the motion of the clusters is governed by equations* (9). *Then the distance between them always remains greater than R.*

PROOF. The initial velocity of any particle in the sth cluster is at most

$$C_1\sqrt{\overline{\overline{k}} + |s| + 1}.$$

Since the potential satisfies conditions 1) and 2), the absolute magnitude of the force acting on any particle is less than some constant $D = D(U, \delta)$, and thus the velocity of a particle in the sth cluster is at most

$$C_1\sqrt{\overline{\overline{k}} + |s| + 1} + DT$$

for all times in $[0, T]$ (when colliding rods exchange velocities, the maximum velocity of the particles in a cluster does not change). Thus, particles in adjacent clusters η_s^0 and η_{s+1}^0 can travel toward one another by a distance of at most

$$\left(C_1\sqrt{\overline{\overline{k}} + |s| + 1} + DT\right)T + \left(C_1\sqrt{\overline{\overline{k}} + s + 2} + DT\right)T.$$

However, for $\overline{\overline{k}}$ sufficiently large and all s, we have

$$\left(C_1\sqrt{\overline{\overline{k}} + |s| + 1} + DT\right)T + \left(C_1\sqrt{\overline{\overline{k}} + |s| + 2} + DT\right)T < C_2(\overline{\overline{k}} + |s|) - R,$$

which in view of (8) gives the assertion of the lemma.

Thus, the above-defined motion of all the particles for times $0 \le t \le T$ with initial condition $\omega^0 \in M(C_1, C_2)$ satisfies an infinite system of equations. Since T is arbitrary, we get a solution ω^t for all $0 \le t < \infty$. However, we have still not constructed the required set $\widetilde{\Omega}$, because in general the set $M(C_1, C_2)$ is not invariant under the motion. Let $M^t(C_1, C_2) \subset \Omega$ be the image of the set $M(C_1, C_2)$ after the particles have moved for a time t. Evidently, the set

$$\widetilde{\Omega}(C_1, C_2) = \bigcap_{t \ge 0} M^t(C_1, C_2) \subset M(C_1, C_2) \tag{10}$$

is invariant under the dynamics defined above.

Although $\widetilde{\Omega}(C_1, C_2)$ cannot be described explicitly, for suitably chosen constants C_1 and C_2, it is quite large. Indeed, let $\mu_{\beta,\mu}$ be the limiting Gibbs distribution on the space Ω defined using the finite Hamiltonians

$$H_{\Lambda,N} = \sum \frac{mv_i^2}{2} + \sum_{\substack{i\neq j \\ q_i\in\Lambda}} U(q_i - q_j)$$

($\Lambda \subset R^1$ is a finite interval, N an integer) and the parameters β and μ defining the grand canonical ensemble (see 2.0 and 3.0).

LEMMA 2. *For $C_1 > \overline{C}_1(\beta,\mu)$ sufficiently large and $C_2 < C_2(\beta,\mu)$ sufficiently small, the set $\widetilde{\Omega}(C_1, C_2)$ has full Gibbs measure*:

$$\mu_{\beta,\mu}(\widetilde{\Omega}(C_1, C_2)) = 1. \tag{11}$$

For a proof of this lemma we refer to [39].

The final result is stated in the next theorem.

THEOREM 3. 1) *A dynamics $T_t\colon \widetilde{\Omega}(C_1, C_2) \to \widetilde{\Omega}(C_1, C_2)$ taking ω^0 to ω^t is defined for initial conditions $\omega^0 \in \widetilde{\Omega}(C_1, C_2)$. It is a cluster dynamics in the sense that for any fixed $t_0 > 0$, the initial state ω^0 can be split into clusters $\{\eta_s^0 = \eta_s^0(t_0),\ s = 0, \pm1, \pm2, \dots\}$, which for times $0 < t < t_0$ move independently without interacting with one another.*

2) *For sufficiently small C_2 and large C_1, the set $\widetilde{\Omega}(C_1, C_2)$ has full Gibbs measure $\mu_{\beta,\mu}$, and this measure on $\widetilde{\Omega}(C_1, C_2)$ is invariant under the dynamics*:

$$\mu_{\beta,\mu}(T_t A) = \mu_{\beta,\mu}(A), \quad A \subset \widetilde{\Omega}(C_1, C_2). \tag{12}$$

The last assertion (12) is also proved in [39]. To give the reader an idea of the arguments involved here, we will prove a simpler statement than in Lemma 2.

LEMMA 4. *For sufficiently large C_1 and small C_2, the set $M(C_1, C_2)$ has full Gibbs measure,*

$$\mu_{\beta,\mu}(M(C_1, C_2)) = 1.$$

PROOF. Let $\widehat{N}_k \subset \Omega$ be the set of states $\omega^0 \in \Omega$ for which

$$\max_{|i|<2^k} |v_i^0| > C_1\sqrt{k}, \tag{13}$$

and let $\overline{N}_k \subset \Omega$ be the set of states for which condition (2) fails for the given k:

$$q_{i+1} - q_i < C_2 k$$

for all i in the intervals $[-2^{k+1}, -2^k]$ and $[2^k, 2^{k+1}]$. We will show that

$$\sum_k \mu_{\beta,\mu}(\widehat{N}_k) < \infty \quad \text{and} \quad \sum_k \mu_{\beta,\mu}(\overline{N}_k) < \infty \tag{14}$$

for suitable C_1 and C_2. The lemma will then follow from (14) by the well-known Borel-Cantelli lemma (see [11]). To estimate $\mu_{\beta,\mu}$ we fix all the positions $X = \{q_i\}$ of the particles and consider the conditional distribution $\mu_{\beta,\mu}(\cdot \mid X)$ of the velocities. By (7.2), this distribution is a product of identical Gaussian distributions with densities

$$\text{const} \exp\left\{ -\frac{\beta m v_i^2}{2} \right\}. \tag{15}$$

Then

$$\begin{aligned} \mu_{\beta,\mu}(\widehat{N}_k|X) &= \mu_{\beta,\mu}(\max_{|i|<2^k} |v_i^0| > C_1\sqrt{k}|X) \\ &< \sum_{|i|<2^k} \mu_{\beta,\mu}(|v_i^0| > C_1\sqrt{k}|X) \\ &< (2^{k+1}+1)\,\text{const} \int\limits_{|v|>C_1\sqrt{k}} \exp\left\{ -\frac{\beta m v^2}{2} \right\} dv \\ &< (2^{k+1}+1)\,\text{const}\exp\{-\beta m C_1^2 k\} < \text{const}\exp\{-\alpha k\}, \end{aligned} \tag{16}$$

where $\alpha = \beta m C_1^2 - \ln 2 > 0$ for sufficiently large C_1. We now use the simple estimate

$$\int_{|x|>A} e^{-cx^2}\,dx < \text{const}\exp\{-cA^2\}$$

for a Gaussian integral for large A. Estimate (16) implies a similar bound for the unconditional probability $\mu_{\beta,\mu}(\widehat{N}_k)$, and hence also that the first of the series in (14) converges. To prove the convergence of the second series in (14), we can pass to the Gibbs distribution $\widetilde{\mu}_{\beta,\mu}$ in the particle configuration space $\{q_i\}$ obtained from $\mu_{\beta,\mu}$ by averaging over all velocities. Then

$$\widetilde{\mu}_{\beta,\mu}(\overline{N}_k) = \mu_{\beta,\mu}(\overline{N}_k). \tag{17}$$

We now fix all the positions $\{\overline{q}_i\}$ of the particles with index $i < n$ and consider the density $p(q_{n+1} \mid \overline{q}_n, \overline{q}_{n-1}, \dots)$ of the conditional distribution for the position q_{n+1} of the next, $(n+1)$th particle. Under the condition that $q_{n+1} - \overline{q}_n > R$, this density is equal to

$$p(q_{n+1}|\overline{q}_n, \overline{q}_{n-1}, \dots) = \text{const}\exp\{-\alpha(q_{n+1} - \overline{q}_n)\}, \tag{18}$$

(see [43]), where the constant depends on the configuration $\overline{q}_n, \overline{q}_{n-1}, \dots$, and the argument of the exponential depends only on the parameters β and μ of the Gibbs distribution: $\alpha = \alpha(\beta, \mu)$.

We now consider the conditional probability of the event

$$\overline{N}'_k = \{q_{i+1} - q_i < C_2 k, \quad i = -2^{k+1}, \dots, -2^k\}$$

under the condition that the positions $\{\overline{q}_j\}$ of the particles with index $j < -2^{k+1}$ are fixed. This probability can be expressed in the form

$$\begin{aligned} &\widetilde{\mu}_{\beta,\mu}(\overline{N'}_k|\overline{q}_j, j < -2^{k+1}) \\ &\qquad = \int_{|q_{i+1}-q_i|<C_2 k} \prod_{i=-2^{k+1}}^{i=-2^k} p(q_{i+1} \mid q_i, q_{i-1}, \dots, \overline{q}_j, \dots) \prod dq_i. \end{aligned} \tag{19}$$

By (18), for large enough k we have

$$\int\limits_{|q_{i+1}-q_i|>C_2k} p(q_{i+1}|q_i, q_{i-1}, \dots)\, dq_{i+1} < \text{const} \exp\{-C_2\alpha k\}$$

and hence the integral (19) is at most

$$(1 - \exp\{-C_2\alpha k\})^{2^k} < \exp\{-2^k \exp\{-C_2\alpha k\}\} < \exp\{-2^{k/2}\}$$

if $C_2\alpha < 1/2 \ln 2$. A similar estimate holds for the unconditional probability of the set $\overline{N}'_k$, and also for the probability of the set

$$\overline{N}''_k = \{q_{i+1} - q_i < C_2 k, \quad i = 1^k, \dots, 2^{k+1}\}.$$

Since $\overline{N}_k = \overline{N}'_k \cap \overline{N}''_k$, the above estimates imply the convergence of the second series in (14), proving Lemma 4.

There are also other more explicit ways of describing the set $\widetilde{\Omega}$ of initial states for which system (4) has a solution such that the corresponding dynamics leaves $\widetilde{\Omega}$ invariant. We present here such a description, taken from [50]. Namely, for each particle $\{q_i, v_i\}$ in the initial state ω^0, consider the energy density of the particles in ω^0 within a c-neighborhood of the point q_i,

$$\frac{1}{2c}\Big[\sum_{j:|q_j-q_i|<c} \frac{mv_j^2}{2} + \sum_{j,j':|q_j-q_i|<c,|q_{j'}-q_i|<c} U(q_j - q_{j'})\Big] = e_i(c).$$

We introduce the set $\widetilde{\Omega}$ of initial states ω^0 such that

$$\sup_i \sup_{c>B+|q_i|} e_i(c) < \infty,$$

where $B = B(U)$ is some fixed constant.

THEOREM 5. 1) *For every initial state $\omega^0 \in \widetilde{\Omega}$ there exists a solution ω^t of system (4) such that $\omega^t \in \widetilde{\Omega}$ for all t (and the solution is unique).*

2) *For every probability distribution ν on Ω with sufficiently good ergodicity properties relative to translations along the line (in particular, for the Gibbs distributions $\mu_{\beta,\mu}$, see [50] for more details), the set $\widetilde{\Omega}$ has full ν-measure,*

$$\nu(\widetilde{\Omega}) = 1.$$

3) *The Gibbs distribution on $\widetilde{\Omega}$ is invariant under the dynamics constructed on $\widetilde{\Omega}$.*

A proof of this theorem can be found in [50].

One checks easily that $\widetilde{\Omega}(C_1, C_2) \subset \widetilde{\Omega}$, and for the elements $\omega^0 \in \widetilde{\Omega}(C_1, C_2)$ the dynamics in Theorem 5 coincides with the cluster dynamics in $\widetilde{\Omega}(C_1, C_2)$ constructed above.

§2. A quick review of C^*-algebras

In the construction and analysis of the Heisenberg dynamics, i.e., the time evolution of the "observables" (operators), for quantum systems it is helpful to define the dynamics on a suitable algebra of "observables". In most cases, this will be a C^*-algebra. In terms of their properties, these algebras resemble the algebra of continuous functions on the state space of a classical infinite system. Here we will briefly state a number of facts concerning C^*-algebras; more detailed information can be found in the monographs [7, 13].

We note at once that C^*-algebras are for the most part employed in the study of quantum spin systems or fermion systems; they are not well suited to the analysis of boson systems.

An *involution* of a Banach algebra $\mathfrak{A}$ is a transformation $\mathfrak{A} \to \mathfrak{A}\colon A \to A^*$ of the algebra into itself such that

$$\begin{aligned} (\lambda_1 A_1 + \lambda_2 A_2)^* &= \overline{\lambda}_1 A_1^* + \overline{\lambda}_2 A_2^*, \\ (A_1 A_2)^* = A_2^* A_1^* \quad &\text{and} \quad (A^*)^* = A. \end{aligned} \tag{1}$$

Definition 1. A Banach algebra with an involution $*\colon \mathfrak{A} \to \mathfrak{A}$ satisfying the condition

$$\|A\|^2 = \|AA^*\|, \qquad A \in \mathfrak{A} \tag{2}$$

is called a C^*-algebra.

An example of a C^*-algebra is the algebra $\mathfrak{B}(\mathcal{H})$ of all bounded operators acting on a separable Hilbert space $\mathcal{H}$, equipped with the usual operator norm; the involution on $\mathfrak{B}(\mathcal{H})$ is given by taking the adjoint operator. Equality (2) is a consequence of the following calculation: since the operator A^*A is selfadjoint and positive, its norm $\|A^*A\|$ is equal to

$$\begin{aligned} \|A^*A\| &= \sup_{x\in\mathcal{H},\|x\|=1} (A^*Ax, x) = \sup_{x\in\mathcal{H},\|x\|=1} (Ax, Ax) \\ &= \sup_{x\in\mathcal{H},\|x\|=1} \|Ax\|^2 = \|A\|^2. \end{aligned} \tag{3}$$

Note that any subalgebra $\mathfrak{A}' \subset \mathfrak{A}$ of a C^*-algebra $\mathfrak{A}$ that is closed under the norm $\|\cdot\|$ and invariant under the involution is also a C^*-algebra.

A *morphism* (or $*$-*homomorphism*) of a C^*-algebra $\mathfrak{A}_1$ into a C^*-algebra $\mathfrak{A}_2$ is any homomorphism π that preserves the involution:

$$(\pi(A))^* = \pi(A^*). \tag{4}$$

Assertion 1. *Every morphism $\pi\colon \mathfrak{A}_1 \to \mathfrak{A}_2$ of a C^*-algebra $\mathfrak{A}_1$ into a C^*-algebra $\mathfrak{A}_2$ is norm-preserving*:

$$\|\pi(A)\|_2 = \|A\|_1, \qquad A \in \mathfrak{A}_1. \tag{5}$$

We refer to [7] for a proof.

Assertion 2. *Let $\mathcal{J} \subset \mathfrak{A}$ be a two-sided closed ideal of the C^*-algebra $\mathfrak{A}$. Then $\mathcal{J}$ is invariant under the involution, and the quotient algebra $\mathfrak{A}/\mathcal{J}$ (with the naturally induced involution) is a C^*-algebra.*

For a proof, see [13].

ASSERTION 3. *Let* $\pi\colon \mathfrak{A}_1 \to \mathfrak{A}_2$ *be a morphism of* C^**-algebras. Then the kernel* $\operatorname{Ker}\pi = \{A \in \mathfrak{A}_1\colon \pi(A) = 0\}$ *is a closed two-sided ideal of* $\mathfrak{A}_1$*, and the image* $\Im\pi = \{B \in \mathfrak{A}_2\colon B = \pi(A)$ *for some* $A \in \mathfrak{A}_1\}$ *is a closed* C^**-subalgebra of* $\mathfrak{A}_2$ *isomorphic to the algebra* $\mathfrak{A}_1/\operatorname{Ker}\pi$.

For a proof, see [13].

Since all separable Hilbert spaces $\mathcal{H}$ are mutually isomorphic, the same is true of the algebras $\mathfrak{B}(\mathcal{H})$. In a sense, $\mathfrak{B}(\mathcal{H})$ is a maximal C^*-algebra; more precisely, we have the next result.

ASSERTION 4. *Every* C^**-algebra is isomorphic to a* C^**-subalgebra of* $\mathfrak{B}(\mathcal{H})$.

For a proof, see [13].

An element $A \in \mathfrak{A}$, where $\mathfrak{A}$ is an algebra with involution, is said to be hermitian (or selfadjoint, or real) if $A^* = A$; A is called a projection if A is hermitian and $A^2 = A$; finally, when $\mathfrak{A}$ contains a unit 1, A is unitary if $A^*A = AA^* = 1$.

ASSERTION 5. *A commutative* C^**-algebra with* 1 *is isomorphic to the* C^**-algebra of all continuous complex-valued functions defined on a suitable compact space* X *(with involution given by* $f^* = \overline{f}$*, where the bar denotes complex conjugate). Every commutative* C^**-algebra is isomorphic to the* C^**-algebra of all continuous functions defined on a suitable locally compact space* X *and tending to zero at infinity.*

For a proof, see [13].

ASSERTION 6. *If a* C^**-algebra* $\mathfrak{A} \subset \mathfrak{B}$ *is a* C^**-subalgebra of a* C^**-algebra* $\mathfrak{B}$ *and* $A \in \mathfrak{A}$ *is invertible in* $\mathfrak{B}$*, then* $A^{-1} \in \mathfrak{A}$.

PROOF. Clearly, $A^* \in \mathfrak{A}$ and $AA^* \in \mathfrak{A}$ are also invertible elements of $\mathfrak{B}$, i.e., there exists an element $T \in \mathfrak{B}$ such that $AA^*T = TAA^* = 1$. Consider the smallest closed subalgebra $\mathfrak{A}_0$ generated by the elements AA^*, T, and 1 (i.e., the closure $\mathfrak{B}$ in of all polynomials in these elements). This is a commutative C^*-algebra, and when it is represented as the algebra of continuous functions on a compact space X, the element AA^* corresponds to a function $f(x)$, $x \in X$, which is never zero, and the element T is represented by the function $1/f(x)$. Since $1/f(x)$ lies in the subalgebra of functions generated by $f(x)$ and 1 (by the Stone-Weierstrass theorem), $\mathfrak{A}_0$ coincides with the algebra generated by the element AA^* and 1, i.e., $T \in \mathfrak{A}$. Hence $A^{-1} = A^*T \in \mathfrak{A}$.

The *spectrum* $\operatorname{Sp} A$ of an element $A \in \mathfrak{A}$, where $\mathfrak{A}$ is an algebra with 1, is the set $\operatorname{Sp} A = \{\lambda \in \mathbb{C}\colon$ the element $A - \lambda 1$ is not invertible in $\mathfrak{A}\}$.

ASSERTION 7. *The spectrum of any hermitian element* $A \in \mathfrak{A}$ *is real.*

The proof follows easily from Assertions 4 and 5.

ASSERTION/DEFINITION 8. *A hermitian element* A *of a* C^**-algebra* $\mathfrak{A}$ *is said to be positive if one of the following equivalent conditions holds*:

1. $\operatorname{Sp} A \subset [0, \infty)$.
2. $A = B^*B$ *for some* $B \subset \mathfrak{A}$.
3. $A = B^2$ *for some hermitian element* $B \in \mathfrak{A}$.
4. *There exists a faithful morphism* $\pi\colon \mathfrak{A} \to \mathfrak{B}(\mathcal{H})$ *(i.e.,* $\operatorname{Ker}\pi = 0$*) such that* $(\pi(A)x, x) \geq 0$ *for all* $x \in \mathcal{H}$.
5. *The preceding property holds for every faithful morphism* $\pi\colon \mathfrak{A} \to \mathfrak{B}(\mathcal{H})$.

A proof is given in [13].

DEFINITION. A state $\langle\cdot\rangle$ on a C^*-algebra $\mathfrak{A}$ with 1 is a linear functional on $\mathfrak{A}$ that takes nonnegative values on positive elements of $\mathfrak{A}$ and satisfies $\langle 1\rangle = 1$.

For every state $\langle\cdot\rangle$ on a C^*-algebra $\mathfrak{A}$ we have the equality

$$\langle A^*\rangle = \langle\overline{A}\rangle, \qquad A \in \mathfrak{A}. \tag{6}$$

Indeed, by condition 1) of Assertion 8, Assertion 4, and the spectral theorem, every hermitian element $A \in \mathfrak{A}$ can be written as a difference of positive elements: $A = C - B$, $C \geq 0$, $B \geq 0$, and hence the state $\langle\cdot\rangle$ takes real values on hermitian elements. But any element $A \in \mathfrak{A}$ is expressible in the form $A = B_1 + iB_2$, where B_1, B_2 are hermitian, hence (6) follows.

ASSERTION 9. *A state is necessarily a continuous linear functional on $\mathfrak{A}$ with norm equal to* 1. *More generally, every positive linear functional $\langle\cdot\rangle$ on a C^*-algebra $\mathfrak{A}$ (i.e., taking nonnegative values on positive elements) is continuous and has norm equal to $\langle 1\rangle$. If $\langle\cdot\rangle$ is a continuous linear functional on a C^*-algebra with* 1 *and satisfies $\langle 1\rangle = 1$, then $\langle\cdot\rangle$ is a state if and only if its norm is equal to* 1.

For a proof, see [13].

The weak $*$-topology on the space $\mathfrak{A}^*$ dual to $\mathfrak{A}$ is generated by the basis of open sets

$$G_A^{a,b} = \{f \in \mathfrak{A}^* \colon b < f(A) < a\}.$$

Evidently, convergence in $\mathfrak{A}^*$ with respect to the weak $*$-topology coincides with pointwise convergence of functionals.

ASSERTION 10. *The set of all states $\mathcal{P} = \mathcal{P}(\mathfrak{A})$ on a C^*-algebra $\mathfrak{A}$ is a convex weakly-$*$-compact subset of $\mathfrak{A}^*$.*

For a proof, see [13].

The extreme points of $\mathcal{P}$ are called *pure* states. Recall that a point $x \in \mathcal{P}$ of a convex set $\mathcal{P}$ is called an extreme point if and only if it cannot be expressed in the form $x = \lambda_1 x_1 + \lambda_2 x_2$, where $x_1, x_2 \in \mathcal{P}$, $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$.

ASSERTION/DEFINITION 11. A morphism $\pi\colon \mathfrak{A} \to \mathfrak{B}(\mathcal{H})$ of a C^*-algebra $\mathfrak{A}$ into $\mathfrak{B}(\mathcal{H})$ is said to be irreducible if one of the following equivalent conditions is satisfied:

1. Every element $C \in \mathfrak{B}(\mathcal{H})$ commuting all $\pi(A)$, $A \in \mathfrak{A}$, is of the form $C = \lambda 1$ for some scalar λ.
2. The space $\mathcal{H}$ contains no nontrivial subspace invariant under all the operators $\pi(A)$, $A \in \mathfrak{A}$.
3. The closure of $\pi(\mathfrak{A})$ in the weak topology of the algebra $\mathfrak{B}(\mathcal{H})$ coincides with $\mathfrak{B}(\mathcal{H})$.
4. Every nonzero vector $x \in \mathcal{H}$ is cyclic for $\pi(\mathfrak{A})$, i.e., the set $\pi(\mathfrak{A})x$ is dense in $\mathcal{H}$ (see [13]).

A morphism $\pi\colon \mathfrak{A} \to \mathfrak{B}(\mathcal{H})$ for which there exists at least one cyclic vector $x \in \mathcal{H}$ is called *cyclic*.

The GNS (Gelfand-Naĭmark-Segal) construction. To each state $\langle\cdot\rangle = \rho(\cdot)$ on a C^*-algebra with 1, the GNS construction canonically associates a Hilbert space $\mathcal{H}_\rho$ together with a morphism $\pi_\rho\colon \mathfrak{A} \to \mathfrak{B}(\mathcal{H}_\rho)$ which is very convenient for studying the properties of ρ. The space $\mathcal{H}_\rho$ is constructed as follows. Define a nonnegative hermitian form on $\mathfrak{A}$ by

$$(A, B) = \langle B^* A\rangle. \tag{7}$$

The set $\mathcal{N} = \{A\colon (A, A) = 0\}$ is a closed left ideal in $\mathfrak{A}$, as is easily seen from the two inequalities

$$|(A, B)|^2 \leq (A, A)(B, B) \tag{8}$$

(Schwarz inequality) and

$$|\langle B^*AB\rangle| < \|A\|\langle B^*B\rangle, \tag{9}$$

which follows from Assertion 8 and the fact that the linear functional $f(A) = \langle B^*AB\rangle$ is positive. Thus, $\mathfrak{A}/\mathcal{N}$ is a prehilbert space with the inner product

$$([A],[B]) = (A,B), \tag{10}$$

where $[A]$ denotes the coset in $\mathfrak{A}/\mathcal{N}$ containing A.

Let $\mathcal{H}_{\mathrm{GNS}} \equiv \mathcal{H}_\rho$ denote the completion of $\mathfrak{A}/\mathcal{N}$ with respect to the inner product (10). The GNS morphism π_ρ of the algebra $\mathfrak{A}$ into the algebra $\mathfrak{B}(\mathcal{H}_\rho)$ is defined as follows.

On the dense linear subspace $\mathfrak{A}/\mathcal{N} \subset \mathcal{H}_{\mathrm{GNS}}$ define the operator

$$\pi_\rho(A)[B] = [AB] \in \mathfrak{A}/\mathcal{N}. \tag{11}$$

This is well defined because $\mathcal{N}$ is a left ideal of $\mathfrak{A}$. It follows from (9) that $\pi_\rho(A)$ is bounded on $\mathfrak{A}/\mathcal{N}$, and its norm in $\mathfrak{B}(\mathcal{H}_\rho)$ satisfies

$$\|\pi_\rho(A)\| \le \|A\|. \tag{12}$$

Thus, $\pi_\rho(A)$ extends by continuity to all of $\mathcal{H}_{\mathrm{GNS}}$ so that inequality (12) is preserved.

ASSERTION 12. *A state ρ is pure if and only if the morphism π_ρ is irreducible.*

For a proof, see [13].

We consider the following example of the GNS construction, which we will frequently have occasion to use. Let $\mathfrak{A} = \mathfrak{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, and let the state on $\mathfrak{A}$ be defined by the formula

$$\langle A\rangle = (A\Phi_0, \Phi_0),$$

where Φ_0 is a unit vector in $\mathcal{H}$. Then $\mathcal{H}_{\mathrm{GNS}}$ is unitarily equivalent to the space $\mathcal{H}$, with $[E]$ going into Φ_0. Indeed, the ideal $\mathcal{N}$ consists of the elements $A \in \mathfrak{B}(\mathcal{H})$ for which

$$\Phi_0 \in \operatorname{Ker} A.$$

Moreover, for every $A \in \mathfrak{B}(\mathcal{H})$ the class $[A] \in \mathfrak{B}(\mathcal{H})/\mathcal{N}$ consists of the operators $A' \in \mathfrak{B}(\mathcal{H})$ for which

$$A'\Phi_0 = A\Phi_0.$$

By associating to each coset $[A]$ the vector $A\Phi_0$, we thus get the required mapping of $\mathcal{H}_{\mathrm{GNS}}$ into the space $\mathcal{H}$. It is easy to verify that $\pi(A) = A$, $A \in \mathfrak{B}(\mathcal{H})$.

We mention here the von Neumann algebras, which are a special class of C^*-algebras. These are the subalgebras $\mathfrak{A} \subseteq \mathfrak{B}(\mathcal{H})$ that are invariant under the adjoint operation and are closed in the weak topology of $\mathfrak{B}(\mathcal{H})$. The topological complications in the structure of C^*-algebras disappear in a sense when we consider von Neumann algebras. Thus, for example, if X is a compact space with a finite measure μ, then the weak closure of the C^*-algebra of operators in $\mathfrak{B}(L_2(X,\mu))$, acting as multiplication by continuous functions $f(x)$ on X, forms a von Neumann algebra consisting of multiplication operators by essentially bounded functions $f \in L_\infty(X,\mu)$ with norm $\|f\| = \operatorname{ess\,sup}_{x\in X} |f(x)|$.

§3. Fock spaces and second-quantization operators

A Fock space is a Hilbert space with an additional tensor product structure. Let $\mathcal{H}$ be a separable Hilbert space. Unless the contrary is indicated, $\mathcal{H}$ will always be taken to be complex and equipped with an inner product that is antilinear in the second argument.

We will define the *symmetric* (boson) Fock space $\mathcal{F}_s = \mathcal{F}_s(\mathcal{H})$ on $\mathcal{H}$ and the *antisymmetric* (fermionic) Fock space $\mathcal{F}_a = \mathcal{F}_a(\mathcal{H})$. It is convenient first to introduce the more general space

$$\mathcal{F} = \mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)}, \tag{1}$$

where $\mathcal{F}^{(0)} = \mathbb{C}$ (the space of constants), and

$$\mathcal{F}^{(n)} = \mathcal{F}^{(n)}(\mathcal{H}) = \mathcal{H}^{\otimes n} = \underbrace{\mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{n \text{ times}}$$

(where $\otimes$ denotes tensor product of Hilbert spaces [36]). The space $\mathcal{F}$ in (1) is a direct sum of Hilbert spaces, i.e., the space of sequences

$$F = (f_0, f_1, \dots, f_n, \dots), \qquad f_n \in \mathcal{F}^{(n)}, \tag{2}$$

with finite norm

$$(F, F) = \|F\|^2 = \sum_{n=0}^{\infty} \|f_n\|^2. \tag{3}$$

Note that if $\mathcal{H} = L_2(\Omega, \Sigma, \mu)$, then $\mathcal{F}^{(n)}$ is the space of square-integrable functions $f(x_1, \dots, x_n)$, $x_i \in \Omega$, $i = 1, \dots, n$, with respect to the measure μ^n on Ω^n.

The symmetric group S_n acts naturally on $\mathcal{F}^{(n)}$ by permuting the factors in the tensor products $\varphi_1 \otimes \cdots \otimes \varphi_n$, $\varphi_i \in \mathcal{H}$.

We write $\mathcal{F}_s^{(n)}$ and $\mathcal{F}_a^{(n)}$ for the subspaces of symmetric (resp., antisymmetric) elements of $\mathcal{F}$ (i.e., which are either invariant under permutations or else are multiplied by the sign of the permutation, respectively).

Definition 1.

$$\mathcal{F}_s = \bigoplus_{n=0}^{\infty} \mathcal{F}_s^{(n)}, \qquad \mathcal{F}_a = \bigoplus_{n=0}^{\infty} \mathcal{F}_a^{(n)}.$$

In the case noted above, when $\mathcal{H}$ is a function space, $\mathcal{F}_s^{(n)}(\mathcal{F}_a^{(n)})$ consists of the symmetric (antisymmetric) functions. We will call $\mathcal{F}_s^{(n)}$ and $\mathcal{F}_a^{(n)}$ the *n-particle spaces.*

The second-quantization functor. If an operator U with $\|U\| \le 1$ is given on $\mathcal{H}$, we write $\Gamma(U)$ for the operator on $\mathcal{F}$ which on each subspace $\mathcal{F}^{(n)}$ acts on an element $\varphi_1 \otimes \cdots \otimes \varphi_n$ by the formula

$$\Gamma(U)(\varphi_1 \otimes \cdots \otimes \varphi_n) = (U\varphi_1) \otimes \cdots \otimes (U\varphi_n) \tag{4}$$

and which is extended by linearity and continuity to all of $\mathcal{F}$. Note that $\mathcal{F}_s$ and $\mathcal{F}_a$ are invariant under $\Gamma(U)$, and the restriction of $\Gamma(U)$ to these subspaces will again be denoted by $\Gamma(U)$. If U is unitary, then so is $\Gamma(U)$ on $\mathcal{F}_s$ and $\mathcal{F}_a$.

Let H be a selfadjoint operator with domain $D \subseteq \mathcal{H}$. Let $d\Gamma(U)$ be the operator on $\mathcal{F}$ acting on vectors of the form $\varphi_1 \otimes \cdots \otimes \varphi_n$, $\varphi_i \in D$, by

$$d\Gamma(H)(\varphi_1 \otimes \cdots \otimes \varphi_n) = H\varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_n + \varphi_1 \otimes H\varphi_2 \otimes \varphi_3 \otimes \cdots \otimes \varphi_n + \cdots + \varphi_1 \otimes \cdots \otimes H\varphi_n \tag{5}$$

and extended by linearity to the linear span of the $\varphi_1 \otimes \cdots \otimes \varphi_n$. Formula (5) is obtained by formally differentiating $\Gamma(e^{itH})$ at the point $t = 0$. We can also regard $d\Gamma(U)$ as acting on $\mathcal{F}_s$ and $\mathcal{F}_a$, on which its closure is selfadjoint.

Creation and annihilation operators. Given $\varphi \in \mathcal{H}$, we define the creation $a^*(\varphi)$ and annihilation operators $a(\varphi)$ on $\mathcal{F}(\mathcal{H})$ by

$$a(\varphi)\Omega = 0, \qquad a^*(\varphi)\Omega = (0, \varphi, 0, \dots),$$

where $\Omega = (1, 0, 0, \dots)$ is the "vacuum" vector in Fock space,

$$\begin{aligned} a(\varphi)(\varphi_1 \otimes \cdots \otimes \varphi_n) &= \sqrt{n}(\varphi_1, \varphi)(\varphi_2 \otimes \cdots \otimes \varphi_n), \\ a^*(\varphi)(\varphi_1 \otimes \cdots \otimes \varphi_n) &= \sqrt{n+1}(\varphi \otimes \varphi_1 \otimes \cdots \otimes \varphi_n). \end{aligned} \tag{6}$$

It is readily checked that

$$(a(\varphi))^* = a^*(\varphi).$$

These operators extend by linearity to operators defined on the dense domain $\mathcal{F}_0(\mathcal{H}) \subset \mathcal{F}$ consisting of the finite sequences in (2), i.e., those for which all $f_n = 0$ when $n \geq n_0$ for some n_0. We write $\mathcal{F}_0(\mathcal{H}) = \mathcal{F}_0$.

Define linear operators on $\mathcal{F}$ by the formulas

$$P_\pm(\varphi_1 \otimes \cdots \otimes \varphi_n) = \frac{1}{n!} \sum_{\pi \in S_n} (\pm)^\pi \varphi_{\pi(1)} \otimes \cdots \otimes \varphi_{\pi(n)}.$$

They are the orthogonal projections onto $\mathcal{F}_s^{(n)}$ and $\mathcal{F}_a^{(n)}$, respectively.

Definition 2. Set

$$a_\pm(\varphi) = P_\pm a(\varphi), \qquad a_\pm^*(\varphi) = P_\pm a^*(\varphi) \tag{7}$$

on $\mathcal{F}_{s,0}$ and $\mathcal{F}_{a,0}$, respectively. We will often omit the subscripts $\pm$ here when the space $\mathcal{F}_s$ or $\mathcal{F}_a$ is clear from the context. In the case when the Fock space consists of sequences of symmetric (antisymmetric) functions

$$\Phi = (f_0, f_1(x_1), \dots, f_n(x_1, \dots, x_n), \dots)$$

the creation and annihilation operators act by the formulas

$$\begin{aligned} (a(\varphi)\Phi)_n(x_1, \dots, x_n) &= \sqrt{n+1} \int_\Omega f_{n+1}(\xi, x_1, \dots, x_n)\overline{\varphi}(\xi) d\mu(\xi), \\ (a^*(\varphi)\Phi)_n(x_1, \dots, x_n) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n (\pm)^{i-1} \varphi(x_i) f_{n-1}(x_1, \dots, \check{x}_i, \dots, x_n), \end{aligned} \tag{7^a}$$

where variables marked with $\check{a}$ are omitted.

REMARK 1. If the norm in (3) is replaced by the norm

$$\|F\|^2 = \sum_{n=0}^{\infty} \frac{1}{n!}\|f_n\|^2,$$

then a^* and a again have the form (7^a), but without the factors $1/\sqrt{n}$ and $\sqrt{n+1}$, which are difficult to remember.

In the boson case we have the canonical commutation relations (CCR)

$$\begin{gathered}[a(f), a(g)] = 0 = [a^*(f), a^*(g)], \\ [a(g), a^*(f)] = (f, g)1,\end{gathered} \tag{8}$$

while for the fermion case we have the canonical anticommutation relations (CAR)

$$\begin{gathered}\{a(f), a(g)\} = 0 = \{a^*(f), a^*(g)\} \stackrel{\text{def}}{=} a^*(f)a^*(g) + a^*(g)a^*(f), \\ \{a^*(f), a(g)\} = (f, g)1.\end{gathered} \tag{9}$$

In fact, an even simpler approach is to *define* the creation and annihilation operators by specifying their action on the vacuum vector Ω and then imposing the CCR or CAR. Specifically, by virtue of the commutation (or anticommutation) relations, the action of the operator $a(\varphi)$ on the vector

$$\frac{1}{\sqrt{n!}} a_{\pm}^*(\varphi_1) \dots a_{\pm}^*(\varphi_n)\Omega = (0, \dots, 0, P_{\pm}(\varphi_1 \otimes \dots \otimes \varphi_n), 0, \dots) \tag{9^a}$$

must give the vector

$$\frac{1}{\sqrt{n}} \sum (\pm)^k (\varphi_k, \varphi) \frac{1}{((n-1)!)^{1/2}} a_{\pm}^*(\varphi_1) \dots \check{a}_{\pm}^*(\varphi_k) \dots a_{\pm}^*(\varphi_n)\Omega,$$

which coincides with definition (6) and (7^a).

It is not difficult to see that on $\mathcal{F}_a$

$$\|a(f)\| = \|a^*(f)\| = \|f\|. \tag{10}$$

Also, the mappings

$$a(f)\colon \mathcal{F}_s^{(n)} \to \mathcal{F}_s^{(n-1)}, \qquad a^*(f)\colon \mathcal{F}_s^{(n-1)} \to \mathcal{F}_s^{(n)}$$

have norm

$$\|a(f)\|_{n,n-1} = \|a^*(f)\|_{n-1,n} = \sqrt{n}\|f\|. \tag{11}$$

This implies that the set of finite sequences is a dense set of analytic vectors for $a^{\#}(f)$, where $a^{\#} = a$ or a^*, i.e., for all $F \in \mathcal{F}_{s,0}$ and all $z \in \mathbb{C}$

$$\sum_{n=0}^{\infty} \frac{|z|^n}{n!}\|a^{\#}(f)F\| < \infty. \tag{12}$$

This is also true for the operators

$$\begin{aligned}\Phi(f) &= \frac{1}{\sqrt{2}}[a^*(f) + a(f)],\\ \Pi(f) &= \frac{i}{\sqrt{2}}[a^*(f) - a(f)].\end{aligned} \tag{13}$$

The operators $\Phi(f)$ and $\Pi(f)$ defined on $\mathcal{F}_{s,0}$ are essentially selfadjoint (by the well-known Nelson criterion [36]) and satisfy the following commutation relations:

$$\begin{aligned}[\Phi(f), \Phi(g)] = 0 = [\Pi(f), \Pi(g)],\\ [\Phi(f), \Pi(g)] = i(g, f).\end{aligned} \tag{14}$$

Their closures are selfadjoint and define unitary Weyl operators on the space $\mathcal{F}_s$

$$U(f) = \exp\{i\Phi(f)\}, \quad V(f) = \exp\{i\Pi(f)\}, \tag{15}$$

satisfying the relations

$$\begin{aligned}U(f)U(g) &= U(f+g) = U(g)U(f),\\ V(f)V(g) &= V(f+g) = V(g)V(f),\end{aligned} \tag{16}$$

$$\exp\{i(\Phi(f) + \Pi(g))\} = U(f)V(g)\exp\left\{\frac{i}{2}(f,g)\right\}$$

$$= V(g)U(f)\exp\left\{-\frac{i}{2}(f,g)\right\}. \tag{16^a}$$

Indeed, consider the family of operators

$$\Pi_t(g) \equiv U(tf)\Pi(g)U^{-1}(tf) = \exp\{it\Phi(f)\}\Pi(g)\exp\{-t\Phi(f)\}.$$

Then

$$\frac{d\Pi}{dt} = -(f,g)1,$$

i.e.,

$$\Pi_t(g) = \Pi(g) - (f,g)1t. \tag{16^b}$$

Therefore,

$$U(tf)V(g)U^{-1}(tf) = \exp\{i\Pi_t(g)\} = \exp\{-it(f,g)\}V(g).$$

Taking $t = 1$, we obtain the second equality in (16^a). The first equality in (16^a) follows from

$$\exp\{it(\Phi(f) + \Pi(g))\} = U(tf)V(tg)\exp\left\{\frac{it^2}{2}(f,g)\right\},$$

by taking $t = 1$, and this in turn follows from the fact that the derivatives of the right- and left-hand sides with respect to t are equal (using (16^b)) for $t = 0$; (16) is obvious.

The norm-closed subalgebra $W \subset \mathfrak{B}(\mathcal{F}_s(\mathcal{H}))$ generated by the operators $\{\Pi(g), \Phi(g), g \in \mathcal{H}\}$ is called the Weyl algebra.

Operators that conserve particle number. These are the operators acting on $\mathcal{F}_s$ (or $\mathcal{F}_a$) for which $\mathcal{F}_s^{(n)}$ (or $\mathcal{F}_a^{(n)}$) are invariant.

The simplest example is the "particle number operator"

$$d\Gamma(\mathbf{1})(F) = nF, \quad \text{if } F \in \mathcal{F}^{(n)}. \tag{17}$$

It is often necessary to express certain of these operators in terms of the creation and annihilation operators. We first note that monomials of the form

$$a^*(\varphi_1)\dots a^*(\varphi_n)a(g_1)\dots a(g_m) \tag{18}$$

preserve the number of particles if $n = m$.

Let us consider three important operators on $\mathcal{F}_s$ (or $\mathcal{F}_a$) in the case when $\mathcal{H} = L_2(R^\nu, dx)$. For such an $\mathcal{H}$ we introduce the "operator-valued distributions" $a^*(x)$, $a(x)$, $x \in R^\nu$, defined by

$$a(f) = \int \overline{f}(x)a(x)\,dx, \qquad a^*(f) = \int f(x)a^*(x)\,dx \tag{19}$$

and satisfying the commutation relations (in the case of $\mathcal{F}_s$)

$$\begin{aligned} [a(x), a^*(y)] &= \delta(x-y)1, \\ [a(x), a(y)] &= [a^*(x), a^*(y)] = 0 \end{aligned} \tag{20}$$

(with similar anticommutation relations holding for $\mathcal{F}_a$).

Using these symbols, it is convenient to express vectors and operators in $\mathcal{F}_s$ and $\mathcal{F}_a$ in the following form:

1) if $f(x_1, \dots, x_n)$ is a smooth symmetric (or antisymmetric) function, then we set

$$\begin{aligned} F &= \{0, \dots, 0, f(x_1, \dots, x_n), \dots\} \\ &= \frac{1}{\sqrt{n!}} \int f(z_1, \dots, z_n)a^*(z_1)\dots a^*(z_n)dz_1\dots dz_n\Omega \in \mathcal{F}_{s(a)}^{(n)} \end{aligned} \tag{21}$$

(cf. (9^a)).

2) To certain distributions K in $n+m$ variables one can associate the operator

$$\begin{aligned} A = \int_{R^{(n+m)\nu}} K(x_1, \dots, x_n, y_1, \dots, y_m)a^*(x_1)\dots a^*(x_n) \\ \times a(y_1)\dots a(y_m)\,dx_1\dots dx_n\,dy_1\dots dy_m, \end{aligned} \tag{22}$$

which acts on the vectors (21) by the following rule. Use the commutation rules (20) to move each factor $a(y_i)$ in (22) in the expression AF to the right of all the $a^*(z_1)\dots a^*(z_n)$; then use the rule $a(y_i)\Omega = 0$. This yields a linear combination of expressions of the form (21), which for a suitable class of kernels K will contain only smooth functions f.

Examples. 1) Let h be the operator on $\mathcal{H} = L_2(R^\nu, dx)$ given by multiplication by the smooth function $h(x)$. Then

$$d\Gamma(h) = \int_{R^\nu} h(x)a^*(x)a(x)\,dx \stackrel{\text{def}}{=} \int_{R^\nu} h(x)\delta(x-y)a^*(x)a(y)\,dx\,dy. \tag{23}$$

2) Let $h\varphi = -\Delta\varphi$, $\varphi \in \mathcal{H}$, Δ the Laplace operator. Then

$$d\Gamma(h) = \int (-\Delta a^*)(x)a(x)\,dx \stackrel{\text{def}}{=} \int (-\Delta\delta)(x-y)a^*(x)a(y)\,dx\,dy. \tag{24}$$

3) Consider the expression

$$\begin{aligned} &\int V(x,y)a^*(x)a^*(y)a(y)a(x)\,dx\,dy \\ &\qquad \stackrel{\text{def}}{=} \int V(x,y)\delta(x-x_1)\delta(y-y_1)a^*(x)a^*(y)a(y_1)a(x_1)\,dx\,dy\,dx_1\,dy_1, \end{aligned} \tag{25}$$

where $V(x,y)$ is a symmetric function (a pair interaction potential). According to our rule, on each subspace $\mathcal{F}^{(n)}$, $n \geq 2$, this operator acts as multiplication by the function

$$2 \sum_{1<i<j<n} V(x_i, x_j).$$

Gaussian representation of $\mathcal{F}_s$. Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $f_1, f_2, \ldots$, and let $\xi_i = \xi(f_i)$ be independent Gaussian random variables with zero mean $\langle \xi_i \rangle = 0$ and unit variance, defined on a probability space (Ω, Σ, μ). We can take Σ to be the smallest σ-algebra with respect to which all the ξ_i are measurable. If

$$f = \sum_i c_i f_i \in \mathcal{H},$$

then we set

$$\xi(f) \stackrel{\text{def}}{=} \sum_i c_i \xi(f_i), \tag{26}$$

where the series converges in the mean square sense.

Define an isomorphism

$$L_2(\Omega, \Sigma, \mu) \to \mathcal{F}_s(\mathcal{H}) \tag{27}$$

by the assignment

$$:\xi^{l_1}(f_1)\ldots\xi^{l_k}(f_k): \leftrightarrow (a^*(f_1))^{l_1}\ldots(a^*(f_k))^{l_k}\Omega, \tag{28}$$

if $f_1, \ldots, f_k$ are distinct basis elements (the Wick ordering $::$ is defined in [26]).

One can check that

$$\begin{aligned} a^*(f)(:\xi(f_1)\ldots\xi(f_k):) &=: \xi(f)\xi(f_1)\ldots\xi(f_k): \\ a(f)(:\xi^l(f):) &= l:\xi^{l-1}(f): \end{aligned} \tag{29}$$

Oscillator representation for $\mathcal{F}_s$. If we identify the spaces $L_2(R^1, dx)$ and $L_2(R^1, \frac{e^{-y^2/2}}{\sqrt{2\pi}}dy)$ via the mapping

$$f(x) \leftrightarrow f(x)(2\pi)^{-1/4}e^{-x^2/4} \tag{30}$$

and use the identification (28)

$$L_2\left(R^1, \frac{1}{2\pi}e^{-y^2/2}dy\right) \leftrightarrow \mathcal{F}_s(\mathbb{C}) \tag{31}$$

we obtain an identification

$$L_2(R^1, dx) \leftrightarrow \mathcal{F}_s(\mathbb{C}). \tag{32}$$

The vacuum vector in $L_2(R^1, dx)$ is $(2\pi)^{-1/4}e^{-x^2/4}$, which is the eigenfunction with smallest eigenvector of the energy operator (Hamiltonian) for the harmonic oscillator

$$H_0 = \frac{1}{2}a^*a = \frac{1}{2}\left(-2\frac{d^2}{dx^2} + \frac{x^2}{2} - 1\right), \tag{33}$$

where

$$a = \frac{1}{2}\left(\frac{x}{\sqrt{2}} + \sqrt{2}\frac{d}{dx}\right), \qquad a^* = \frac{1}{2}\left(\frac{x}{\sqrt{2}} - \sqrt{2}\frac{d}{dx}\right). \tag{34}$$

A similar identification of $L_2(R^N, d^Nx)$ with the space $\mathcal{F}(\mathbb{C}^N)$ can be obtained by taking tensor powers of the isomorphism (32).

Gauss (Grassmann) representation for $\mathcal{F}_a$. This representation is analogous to the Gaussian representation for $\mathcal{F}_s$. Let $G(\mathcal{H})$ be the Grassmann algebra with 1 over the Hilbert space $\mathcal{H}$, i.e., the Grassmann algebra with generators $\{\psi(e_i),\ i = 1, \dots\}$, where $\{e_i\}$ is an orthonormal basis in $\mathcal{H}$ and

$$\{\psi(e_i), \psi(e_j)\} = \psi(e_i)\psi(e_j) + \psi(e_j)\psi(e_i) = 0. \tag{35}$$

To each monomial

$$\psi(e_{i_1}) \dots \psi(e_{i_k}), \qquad i_1 < i_2 < \dots < i_k, \tag{35^a}$$

we associate the element

$$a^*(e_{i_1}) \dots a^*(e_{i_k})\Omega. \tag{36}$$

This map extends to an isomorphism

$$\overline{G}(\mathcal{H}) \leftrightarrow \mathcal{F}_a(\mathcal{H}),$$

where $\overline{G}(\mathcal{H})$ is the completion of $G(\mathcal{H})$ in the norm induced by the inclusion $G(\mathcal{H}) \subset \mathcal{F}_a(\mathcal{H})$ via (36). For any $f = \sum c_i e_i \in \mathcal{H}$ we write

$$\psi(f) = \sum c_i \psi(e_i) \in \overline{G}(\mathcal{H}).$$

It is easy to verify that $a^*(f)$ acts on $\overline{G}(\mathcal{H})$ as left multiplication by $\psi(f)$, and $a(f)$ acts as left differentiation $\partial/\partial\psi(f)$ (see [5]). It is defined as follows. For $f = e_{i_0}$ we have

$$\frac{\partial}{\partial\psi(e_{i_0})}\psi(e_{i_1}) \dots \psi(e_{i_k}) = \begin{cases} 0 & \text{if } i_0 \neq i_s \text{ for all } s = 1, \dots, k, \\ (-1)^{s-1}\psi(e_{i_1}) \dots \check{\psi}(e_{i_s}) \dots \psi(e_{i_k}) & \text{if } i_0 = i_s. \end{cases} \tag{37}$$

For an arbitrary $f = \sum c_i e_i$ we define

$$\frac{\partial}{\partial\psi(f)} = \sum_i c_i \frac{\partial}{\partial\psi(e_i)}. \tag{38}$$

Holomorphic representation for $\mathcal{F}_s$. Let $\mathcal{H} = l_2(\mathcal{N}) = \mathbb{C}^N$, $\mathcal{N} = \{1, \dots, N\}$ and let R_N be the Hilbert space of holomorphic functions $f(z_1, \dots, z_N)$ on $\mathbb{C}^N$ with the inner product

$$(f, g) = \frac{1}{\pi^N} \int f\overline{g} \exp\Big\{-\sum_{k=1}^{N} |z_k|^2\Big\} \prod dx_k\, dy_k, \qquad z_k = x_k + iy_k. \tag{39}$$

We identify

$$R_N \leftrightarrow \mathcal{F}_s(l_2(\mathcal{N})) \tag{40}$$

using the formula

$$z_1^{k_1} \dots z_N^{k_N} \leftrightarrow (a_1^*)^{k_1} \dots (a_N^*)^{k_N} \Omega. \tag{41}$$

Here the operators a_k^* are given by multiplication by z_k, and $a_k f = \partial f / \partial z_h$.

Holomorphic representation for $\mathcal{F}_a$. In contrast to the algebra G considered above, here we introduce the algebra $G_2(l_2(\mathcal{N}))$ over $l_2(\mathcal{N})$ with twice the number of anticommuting generators $\psi_1, \dots, \psi_n, \overline{\psi}_1, \dots, \overline{\psi}_n$. The algebra $G_1(l_2(\mathcal{N})) \subset G_2(l_2(\mathcal{N}))$ of polynomials in the generators $\{\psi_i,\ i = 1, \dots, N\}$ will be called the algebra of holomorphic functions, and we give it an inner product by the formula

$$(f, g) = \int f\overline{g} \prod_{i=1}^{N} \exp\{-\overline{\psi}_i \psi_i\} \prod_{i=1}^{N} d\overline{\psi_i}\, d\psi_i, \tag{42}$$

where $\int \prod_{i=1}^{N} d\overline{\psi}_i\, d\psi_i$ is the Berezin integral with respect to the algebra $G_2(l_2(\mathcal{N}))$ (see [5] and also §5 below), and $\overline{g}$ is obtained from g by applying the antilinear involution θ in $G_2(l_2(\mathcal{N}))$ acting on the generators by the formula

$$\theta\psi_i = \overline{\psi}_i, \qquad \theta\overline{\psi}_i = \psi_i \tag{43}$$

and which is extended to the whole algebra $G_2(l_2(\mathcal{N}))$ by the rule

$$\theta(AB) = \theta(B)\theta(A).$$

It is not difficult to show that the basis (35^a) is an orthonormal basis in $G_2(l_2(\mathcal{N}))$. The mapping of $G_1(l_2(\mathcal{N}))$ into $\mathcal{F}_a(l_2(\mathcal{N}))$ is again of the form (36), and the operators $a(f)$ and $a^*(f)$ act on $G_1(l_2(\mathcal{N}))$ as before.

Free dynamics in Fock spaces. Let a selfadjoint operator h be defined on the one-particle space $\mathcal{H}$. Then we can define the operators $\Gamma(e^{ith})$ on both $\mathcal{F}_s(\mathcal{H})$ and $\mathcal{F}_a(\mathcal{H})$. The operators

$$\Gamma(e^{ith}) = \exp\{it d\Gamma(h)\}$$

form a unitary group of transformations of the space $\mathcal{F}_s(\mathcal{H})$ or $\mathcal{F}_a(\mathcal{H})$. That is, we have a Schrödinger dynamics in these spaces, which is usually called the *free dynamics* (generated by h). The corresponding Heisenberg dynamics

$$\tau_t(A) = \exp\{it d\Gamma(h)\} A \exp\{-it d\Gamma(h)\}$$

is also called the *free dynamics in the algebras* $\mathfrak{B}(\mathcal{F}_a(\mathcal{H}))$ or $\mathfrak{B}(\mathcal{F}_s(\mathcal{H}))$.

§4. The CAR algebra and its free dynamics. Dynamics of a system of interacting fermions. Analog of Robinson's theorem

1. The algebra of canonical anticommutation relations (CAR).

DEFINITION 1. The C^*-algebra $\mathfrak{A}(\mathcal{H}) \subset \mathfrak{B}(\mathcal{F}_a(\mathcal{H}))$ generated by all the operators $a(f)$, $f \in \mathcal{H}$, acting on the antisymmetric Fock space $\mathcal{F}_a(\mathcal{H})$ is called the *algebra of canonical anticommutation relations* (CAR) *over* $\mathcal{H}$.

Let $\mathcal{H}_\Lambda \subset \mathcal{H}$ be any ordered family of subspaces indexed by the elements Λ of an ordered set, so that $\mathcal{H}_{\Lambda_1} \subset \mathcal{H}_{\Lambda_2}$ if $\Lambda_1 < \Lambda_2$, and suppose that these subspaces generate all of $\mathcal{H}$, i.e., $\overline{\cup \mathcal{H}_\Lambda} = \mathcal{H}$. Then we get a representation of the algebra $\mathfrak{A}(\mathcal{H})$ of the form

$$\mathfrak{A}(\mathcal{H}) = \bigcup_\Lambda \mathfrak{A}(\mathcal{H}_\Lambda), \tag{1}$$

where $\mathfrak{A}(\mathcal{H}_\Lambda) \subset \mathfrak{A}(\mathcal{H})$ is the subalgebra of $\mathfrak{A}(\mathcal{H})$ generated by the operators $\{a(f), a^*(f), f \in \mathcal{H}_\Lambda\}$. Since $\mathfrak{A}(\mathcal{H}_{\Lambda_1})$ imbeds homeomorphically and isometrically in $\mathfrak{A}(\mathcal{H}_{\Lambda_2})$ for $\Lambda_1 < \Lambda_2$, expression (1) defines a quasilocal structure on $\mathfrak{A}(\mathcal{H})$ (see 3.0). The algebras $\mathfrak{A}_\Lambda \equiv \mathfrak{A}(\mathcal{H}_\Lambda)$ are called the algebras of local elements. In particular, if $\mathcal{H} = L_2(R^\nu, dx)$ or $\mathcal{H} = L_2(Z^\nu)$, one takes the index set Λ to be the collection of bounded subsets of R^ν (or Z^ν) ordered by inclusion; then $\mathcal{H}_\Lambda = L_2(\Lambda, dx)$ (or $\mathcal{H}_\Lambda = l_2(\Lambda)$). Note that $\mathfrak{A}_\Lambda$ is a superalgebra: the closure of the polynomials in $a^\#(f)$ consisting of linear combinations of monomials $\prod a^\#(f)$ with an even number of factors is the even subspace, while the closure of the span of the odd monomials forms the odd subspace. Further, if $\Lambda \cap \Lambda' \neq \varphi(\Lambda \subset R^\nu$ or $\Lambda \subset Z^\nu)$, then the algebra $\mathfrak{A}_{\Lambda \cup \Lambda'} = \mathfrak{A}_\Lambda \otimes \mathfrak{A}_{\Lambda'}$, where the tensor product $\otimes$ is taken in the graded sense (see [23]). In particular, the above imbedding $\mathfrak{A}_{\Lambda_1} \to \mathfrak{A}_{\Lambda_2}$ takes $A \in \mathfrak{A}_{\Lambda_1}$ into $A \otimes 1_{\Lambda_2 \setminus \Lambda_1} \in \mathfrak{A}_{\Lambda_2}$.

If $\dim \mathcal{H} = n < \infty$, we can express $\mathfrak{A}(\mathcal{H})$ as

$$\mathfrak{A}(\mathcal{H}) = \mathfrak{A}_1 \otimes \cdots \otimes \mathfrak{A}_n. \tag{2}$$

Here $\mathfrak{A}_i$ is the algebra generated by the operators $a(e_i)$, $a^*(e_i)$, where $\{e_1, \ldots, e_n\}$ is an orthonormal basis in $\mathcal{H}$. Since each $\mathfrak{A}_i$ is isomorphic to the algebra of 2×2 matrices, we see from (2) that $\mathfrak{A}(\mathcal{H})$ is isomorphic to $\mathfrak{M}_{2^n}$, the matrices of order 2^n, i.e., it coincides with the algebra $\mathfrak{B}(\mathcal{F}_a(\mathcal{H}))$. This is no longer true when $\mathcal{H}$ is infinite dimensional; in this case the algebra $\mathfrak{A}(\mathcal{H}) \subset \mathfrak{B}(\mathcal{F}_a(\mathcal{H}))$ does not contain any nonzero compact operator (see [7]).

2. Free dynamics on the CAR algebra. Let h be a selfadjoint operator acting on the one-particle subspace $\mathcal{H}$ and let τ_t be the free dynamics induced by h on the algebra $\mathfrak{B}(\mathcal{F}_a(\mathcal{H}))$ (see §3).

LEMMA 1. *The* CAR *algebra* $\mathfrak{A}(\mathcal{H}) \subset \mathfrak{B}(\mathcal{F}_a(\mathcal{H}))$ *is invariant under the dynamics* τ_t.

PROOF. We claim that

$$\tau_t a^\#(f) = a^\#(e^{-ith} f), \tag{3}$$

from which the statement of the lemma follows at once. To prove (3), choose an orthonormal basis $\{f_i\}$ in $\mathcal{H}$ (belonging to the domain D_h of the operator h) and denote by $c_{ij} = (hf_i, f_j)$ the matrix elements of h in this basis. Then the operator $d\Gamma(h)$ is equal to

$$H = d\Gamma(h) = \sum c_{ij} a^*(f_i) a(f_j), \tag{4}$$

as can be verified by direct calculation (the series (4) converges in the strong topology of $\mathfrak{B}(\mathcal{F}_a(\mathcal{H}))$. Writing $a_t(f) = \tau_t a(f)$, we then obtain that

$$\frac{da_t(f_k)}{dt} = \tau_t(i[H, a_t(f)]) = \tau_t\left(i\sum_j c_{kj}a_t(f_j)\right) = \tau_t(a_t(-ihf_k)). \tag{5}$$

¿From the initial condition $a_0(f_k) = a(f_k)$, we get that

$$a_t(f_k) = a(e^{-ith}f_k). \tag{6}$$

Formula (3) now follows from (6) by linearity.

When $\mathcal{H} = L_2(R^\nu, dx)$ or $\mathcal{H} = l_2(Z^\nu)$, we will consider translationally invariant one-particle Hamiltonians, i.e., Hamiltonians that commute with the group U_x of translations $(U_x f)(y) = f(y-x)$, $x \in R^\nu$ (or $x \in Z^\nu$). Such an operator h on $L_2(R^\nu, dx)$ is given on the set $S(R^\nu) \subset L_2(R^\nu, dx)$ by the formula

$$(hf)(x) = \int_{R^\nu} \widehat{h}(x-y)f(y)\,dy, \qquad x \in R^\nu, \tag{7}$$

where $\widehat{h}$ is a distribution such that the operator (7) is essentially selfadjoint on $S(R^\nu)$. In the case when $\mathcal{H} = l_2(Z^\nu)$, a translationally invariant one-particle Hamiltonian has the form

$$(hf)(x) = \sum_{Z^\nu} \widehat{h}(x-y)f(y). \tag{8}$$

We now take the Fourier transform

$$f(x) \to \widetilde{f}(k) = \int_{R^\nu} \exp\{itk\}f(x)\,dx, \qquad k \in R^\nu,$$

or

$$\widetilde{f}(k) = \sum_{Z^\nu} \exp\{itk\}f(x), \qquad k \in T^\nu,$$

where T^ν is the ν-dimensional torus; (7) and (8) then become multiplication operators

$$(h\widetilde{f})(k) = \widetilde{h}(k)f(k), \qquad k \in R^\nu \quad (\text{or } k \in T^\nu), \tag{9}$$

where $\widetilde{h}$ is the Fourier transform of the function $\widehat{h}$. The function $\widetilde{h}(k)$ is generally assumed to be a sufficiently smooth function of k.

In the case of a continuous space R^ν, the most interesting examples involve Hamiltonians given by the functions $\widetilde{h}(k) = k^2$ or $\widetilde{h}(k) = \sqrt{k^2 + m^2}$, $m \neq 0$.

3. Construction of a dynamics for a system of interacting fermions in R^ν (analog of Robinson's theorem). Consider the one-particle Hamiltonian

$$hf = \frac{1}{2}\Delta f, \tag{10}$$

where is the Laplace operator on $L_2(R^\nu, dx)$. The corresponding group $\exp\{ith\}$ is given by the formula

$$(e^{ith}f)(x) = \frac{1}{(-2\pi it)^{1/2}} \int_{R^\nu} \exp\left\{i\frac{(x-y)^2}{2t}\right\} f(y)\,dy. \tag{11}$$

We write

$$H_0 = d\Gamma(h) = \frac{1}{2} \int\limits_{R^\nu \times R^\nu} (\Delta\delta)(x-y) a^*(x) a(y)\, dx\, dy. \tag{12}$$

We consider now bounded operators on $\mathcal{F}_a(L_2(R^\nu, dx))$ of the form

$$V = \sum_{s=1}^{M} V_s, \tag{13}$$

where V_s is the monomial

$$V_s = a^*(f_1^{(s)}) \dots a^*(f_{m_s}^{(s)}) a(g_1^{(s)}) \dots a(g_{l_s}^{(s)}), \tag{14}$$

and $f_i^{(s)}$, $g_j^{(s)}$, $i = 1, \dots, m_s$, $j = 1, \dots, l_s$ are functions in $S(R^\nu)$.

Our constructions also apply to the more general case of operators of the form

$$\begin{aligned} V_s = \int \int K_s(x_1, \dots, x_{m_s}, y_1, \dots, y_{l_s}) a^*(x_1) \dots a^*(x_{m_s}) \\ \times a(y_1) \dots a(y_{l_s})\, dx_1 \dots dx_{m_s}\, dy_1 \dots dy_{l_s}, \end{aligned} \tag{15}$$

where $K_s(x_1, \dots, x_{m_s}, y_1, \dots, y_{l_s}) \in S(R^{\nu(m_s+l_s)})$ are antisymmetric in the variables $x_1, \dots, x_{m_s}$ and $y_1, \dots, y_{l_s}$ taken separately.

We assume that whenever a sum V_s appears in (13), so does its conjugate V_s^*, so that the operator V is selfadjoint

$$V^* = V. \tag{16}$$

Thus for every real λ, the operator

$$H = H_0 + \lambda V,$$

defined on the domain $D_{H_0} \subset \mathcal{F}_a$ of the operator H_0 is essentially selfadjoint by Kato's criterion (see [36]).

Operators of the type (13), (14) (or (15)) will be called completely smooth operators. However, they are no longer translationally invariant, i.e., they do not commute with the group of translations

$$U_x = \Gamma(U_x), \qquad x \in R^\nu, \tag{17}$$

acting on $\mathcal{F}_a$.

Nevertheless, we can use V to construct a translationally invariant operator W on $\mathcal{F}_a$ by "averaging" V over all its "translations". More precisely, consider for each $x \in R^\nu$ the operator

$$V_x = U_x V U_x^{-1} = \sum_s V_s(x), \qquad V_s(x) = U_x V_s U_x^{-1} \tag{18}$$

and define W by

$$W = \int_{R^\nu} V_x\, dx = \sum_{s=1}^{M} W_s \tag{19}$$

where each operator W_s is of type (15), with kernels of the form

$$\begin{aligned}\mathcal{K}_s(x_1,\dots,x_{m_s},y_1,\dots,y_{l_s}) \\ = \int K_s(x_1+x,\dots,x_{m_s}+x,y_1+x,\dots,y_{l_s}+x)\,dx\end{aligned} \tag{20}$$

and such that

$$\begin{aligned}\mathcal{K}_s(x_1+x_0,\dots,x_{m_s}+x_0,y_1+x_0,\dots,y_{l_s}+x_0) \\ = \mathcal{K}_s(x_1,\dots,x_{m_s},y_1,\dots,y_{l_s}).\end{aligned} \tag{21}$$

The operator W is now translationally invariant, but in general it is unbounded. We will assume that for all s

$$l_s > 0 \tag{22}$$

(and consequently, in view of the selfadjointness condition (16), $m_s > 0$). In this case $V\Omega = 0$, and so $W\Omega = 0$ also.

It follows that W is defined and symmetric on the set $\mathcal{F}_{a,0}(S(R^\nu))$ of all finite sequences

$$\{f_0, f_1, \dots, f_n, 0, \dots\} \in \mathcal{F}_a, \qquad n = 0, 1, 2, \dots \tag{23}$$

with components $f_k \in S(R^{\nu k})$, $k \le n$.

THEOREM 2. *The sequences* (23) *in the set* $\mathcal{F}_{a,0}(S(R^\nu))$ *are analytic vectors for the operator*

$$H = H_0 + \lambda W \tag{24}$$

and consequently, for every real λ, H *is essentially selfadjoint on* $\mathcal{F}_{a,0}(S(R^\nu))$ *by the Nelson criterion* (*see* [36]).

The proof is completely analogous to the proof of Theorem 3 below.

Thus if condition (22) holds, a dynamics $U_t = \exp\{itH\}$ is defined on the space $\mathcal{F}_a$, and with it the associated Heisenberg dynamics

$$\tau_t(A) = U_t A U_t^{-1}, \qquad A \in \mathfrak{B}(\mathcal{F}_a). \tag{25}$$

When condition (22) is violated, the operator W, and hence also H, may fail to be densely defined. This will be true, for instance, when

$$V = \int_{R^\nu} f(y)a^*(y)\,dy + \int_{R^\nu} \overline{f}(y)a(y)\,dy, \qquad f \in S(R^\nu)$$

in which case the operator

$$W = c\int_{R^\nu} a^*(y)\,dy + \overline{c}\int_{R^\nu} a(y)\,dy \tag{26}$$

cannot be defined on any vector in $\mathcal{F}_a$.

Thus in this case we cannot associate any (Schrödinger) dynamics on $\mathcal{F}_a$ to the formal expression (26). However, we will now show that (26) is associated with a suitably constructed Heisenberg dynamics on the CAR algebra $\mathfrak{A} \subset \mathfrak{B}(\mathcal{F}_a)$. For every cube $\Lambda \subset R^\nu$ we set

$$H_\Lambda = H_0 + \lambda W_\Lambda, \tag{27}$$

where

$$W_\Lambda = \int_\Lambda V_x\, dx. \tag{28}$$

Then W_Λ is a bounded operator and H_Λ, defined on the domain $D_H \subset \mathcal{F}_a$, is selfadjoint.

Define the group of automorphisms $\alpha_t^\Lambda(\cdot)$ (Heisenberg dynamics) of the algebra $\mathfrak{A}$ by the formula

$$\alpha_t^\Lambda(A) = \exp\{itH_\Lambda\} A \exp\{-itH_\Lambda\}, \qquad A \in \mathfrak{A}. \tag{29}$$

Using equation (5.4.0), we can rewrite $\alpha_t^\Lambda(A)$ as the series

$$\begin{aligned}\alpha_t^\Lambda(A) = \tau_t(A) + \sum (i\lambda)^n \iint\limits_{0<t_1<\cdots<t_n<t} [\tau_{t_1}(W_\Lambda),\\ [\tau_{t_2}(W_\Lambda), [\dots [\tau_{t_n}(W_\Lambda), \tau_t(A)]\dots]]]\, dt_1 \dots dt_n,\end{aligned} \tag{30}$$

where τ_t denotes the free Heisenberg dynamics (3)

$$\tau_t(A) = \exp\{itH_0\} A \exp\{-itH_0\} \tag{31}$$

on the algebra $\mathfrak{B}(\mathcal{F}_a)$. Since the norm of each term in the series (30) is bounded by

$$(2\lambda)^n \|W_\Lambda\|^n \|A\| t^n/n! \le (2\lambda)^n \|V\|^n |\Lambda|^n \|A\| t^n/n! \tag{32}$$

the series is norm-convergent.

THEOREM 3. *Let the operator H_0 be given by* (12) *and V be defined the sum* (13), *where in each term $m_s + l_s$ is even, and assume that the kernels $K_s \in S(R^{\nu(m_s+l_s)})$ have compact support. Then the limit*

$$\lim_{\Lambda \uparrow R^\nu} \alpha_t^\Lambda(A) = \alpha_t(A) \tag{33}$$

exists for all t and all local elements $A \in \mathfrak{A}^0$ of the CAR *algebra ($\mathfrak{A}^0$ being the set of local elements), and we have norm-convergence. The map $A \to \alpha_t(A)$, $A \in \mathfrak{A}^0$ extends by continuity to a group of $*$-automorphisms of the* CAR *algebra $\mathfrak{A}$ (it will again be denoted by α_t).*

Moreover, for sufficiently small $|t|$ the element $\alpha_t(A)$, $A \in \mathfrak{A}^0$ is expressible as a norm-convergent series

$$\begin{aligned}\alpha_t(A) = \tau_t(A) + \sum_{n=1}^\infty (i\lambda)^n \iint\limits_{0<t_1<\cdots<t_n<t} dt_1 \dots dt_n\\ \times \iint\limits_{R^{\nu n}} dx_1 \dots dx_n [V_{x_1,t_1}[\dots [V_{x_n,t_n}, \tau_{t_n}(A)]\dots]],\end{aligned} \tag{34}$$

where we have written

$$V_{x,t} = \tau_t(V_x). \tag{35}$$

PROOF. We begin by proving the last statement of the theorem, i.e., that the series (35) converges in the operator norm. Partition the space R^ν into unit cubes $\{B_y, y \in Z^\nu\}$, where y is the center of the cube B_y. In each space $L_2(B_y)$ choose an orthonormal basis $\{\varphi_i^y, i = 1, 2, \dots\}$; extending each function φ_i^y by zero outside the cube B_y, we get an orthonormal basis $\{\varphi_i^y, y \in Z^\nu, i = 1, 2, \dots\}$ in the whole space $L_2(R^\nu)$. Let $\Gamma = \{\gamma_y, y \in Z^\nu\}$ be a finite multi-index, where for each $y \in Z^\nu$ γ_y denotes a finite subset of the natural numbers, where only finitely many of them are nonempty. Write $\operatorname{supp}\Gamma = \{y\colon \gamma_y \neq \varphi\}$ and for each pair Γ, $\widetilde{\Gamma}$ let $\mathcal{A}_{\Gamma,\widetilde{\Gamma}}$ be the monomial

$$\mathcal{A}_{\Gamma,\widetilde{\Gamma}} = \prod_{y\in\operatorname{supp}\Gamma} \prod_{i\in\gamma_y} a^*(\varphi_i^y) \prod_{\widetilde{y}\in\operatorname{supp}\widetilde{\Gamma}} \prod_{\widetilde{i}\in\widetilde{\gamma}_{\widetilde{y}}} a(\varphi_{\widetilde{i}}^{\widetilde{y}}), \tag{36}$$

where in the first product $\prod_{i\in\gamma_y} a^*(\varphi_i^y)$ the factors are arranged in order of increasing index $i \in \gamma_y$, while in the second product $\prod_{y\in\operatorname{supp}\Gamma}$ the factors are arranged in increasing lexicographic order as determined by the points in $\operatorname{supp}\Gamma$; the same ordering is used for the produce of the $a(\varphi_{\widetilde{i}}^{\widetilde{y}})$.

We will need the next lemma.

LEMMA 4. *Let the local element*

$$\begin{aligned}\mathcal{A} = \int K(x_1, \dots, x_l, y_1, \dots, y_{\widetilde{l}}) a^*(x_1) \dots a^*(x_l) \\ \times a(y_1) \dots a(y_{\widetilde{l}})\, dx_1 \dots\, dx_l\, dy_1 \dots dy_l,\end{aligned} \tag{37}$$

be given, where the function K is C^∞ with compact support contained in the set $G^{l+\widetilde{l}} \subset R^{\nu(l+\widetilde{l})}$, $G \subset R^\nu$ a bounded domain.

Then we have the expansion

$$\begin{aligned}\tau_t(A) &= \sum_{\Gamma,\widetilde{\Gamma}} C_t^{(K)}(\Gamma, \widetilde{\Gamma}) \mathcal{A}_{\Gamma,\widetilde{\Gamma}}, \\ |\Gamma| &= \sum_y |\gamma_y| = l, \\ |\widetilde{\Gamma}| &= \sum_{\widetilde{y}} |\widetilde{\gamma}_{\widetilde{y}}| = \widetilde{l},\end{aligned} \tag{38}$$

where the coefficients $C_t^{(k)}(\Gamma, \widetilde{\Gamma})$ satisfy the estimate

$$\begin{aligned}|C_t^{(K)}(\Gamma, \widetilde{\Gamma})| < \prod_{y\in\operatorname{supp}\Gamma} \frac{1}{[|\gamma_y|(d(y,G)+1)^{\nu+1}]^{|\gamma_y|}} \\ \times \prod_{\widetilde{y}\in\operatorname{supp}\widetilde{\Gamma}} \frac{1}{[|\widetilde{\gamma}_{\widetilde{y}}|(d(\widetilde{y},G)+1)^{\nu+1}]^{|\widetilde{\gamma}_{\widetilde{\gamma}}|}} (t^{\nu/2+1})^{l+\widetilde{l}} R^{l+\widetilde{l}}.\end{aligned} \tag{39}$$

Here $R = R(K)$ is a constant depending on the kernel K, and $d(y, G)$ is the distance of the point y from the set G.

This lemma will be proved below; for the time being, we use it to prove the theorem. It will be helpful to introduce the operators

$$\widetilde{V}_{z,s} = \int_{B_z} V_s(x)\, dx, \tag{40}$$

where $V_s(x) = U_x V_s U_x^{-1}$ is one of the terms in the sum (13). Since $V_{z,s}$ is again of the form (37), $\tau_t(\widetilde{V}_{z,s})$ admits an expansion (38) with coefficients $C_t^{z,s}(\Gamma, \widetilde{\Gamma})$ bounded as in (39), except that $d(y, G)$ can be replaced by the distance $|z - y|$.

In the above terminology, we can write the nth term of the series (34) as

$$\int_{0<t_1<\cdots<t_n<t} \cdots \int \sum_{(z_1,s_1,\Gamma_1,\widetilde{\Gamma_1}),\ldots,(z_n,s_n,\Gamma_n,\widetilde{\Gamma_n})} \prod C_{t_1}^{z_1,s_1}(\Gamma_1, \widetilde{\Gamma_1}) \ldots C_{t_n}^{z_n,s_n}(\Gamma_n, \widetilde{\Gamma_n}) C_t^{(K)}(\Gamma, \widetilde{\Gamma})[\mathcal{A}_{\Gamma_1,\widetilde{\Gamma}_1}, [\ldots [\mathcal{A}_{\Gamma_n,\widetilde{\Gamma_n}}, \mathcal{A}_{\Gamma,\widetilde{\Gamma}}]]]. \tag{41}$$

Since each monomial $\mathcal{A}_{\Gamma_i,\widetilde{\Gamma}_i}$ contains an even number of operators $a^{\#}$, the n-fold commutator in (41) is nonzero only if the following condition holds: for each i,

$$\big(\operatorname{supp}\Gamma_i \bigcup \operatorname{supp}\widetilde{\Gamma}_i\big) \bigcap \Big[\bigcup_{j=i+1}^{n} \big(\operatorname{supp}\Gamma_j \bigcup \operatorname{supp}\widetilde{\Gamma}_j\big) \bigcup \big(\operatorname{supp}\Gamma \bigcup \operatorname{supp}\widetilde{\Gamma}\big) \Big] \neq \emptyset. \tag{42}$$

Moreover, the norm of the commutator is at most 2^n. For fixed Γ and $\widetilde{\Gamma}$, summing first over $(z_1, s_1, \Gamma_1, \widetilde{\Gamma}_1)$, then over $(z_2, s_2, \Gamma_2, \widetilde{\Gamma}_2)$, and so on, and then finally over Γ and $\widetilde{\Gamma}$, we find from (39) and (41), using also that $|\Gamma_i| \leq \max_{s=1,\ldots,M} m_s$, $|\widetilde{\Gamma}_i| \leq \max_{s=1,\ldots,M} \widetilde{m}_s$, that the sum in (41) is bounded by

$$t^{(\nu/2+1)(L+\widetilde{L})n} \frac{(n + l + \widetilde{l})!}{(l + \widetilde{l})!} R_1^n R(K), \tag{43}$$

where $L = \min_{s=1,\ldots,M} m_s$, $\widetilde{L} = \min_{1,\ldots,M} l_s$, and $R_1 = R_1(V)$.

Integrating next over $t_1, \ldots, t_n$, we obtain finally that the norm of the nth term of the series is bounded by the expression

$$t^{(\nu/2+1)(L+\widetilde{L})n+n} \frac{(n + l + \widetilde{l})!}{n!(l + \widetilde{l})!} R_1^n R(K). \tag{44}$$

It follows that (34) converges for $|t| < t_0 = t_0(V)$. Our estimates also show that the series for $\alpha_t^{\Lambda}(A)$, which is analogous to (34) but in which the integration over $R^{\nu n}$ in the nth term is replaced by an integration over Λ^n, converges termwise (in norm) to the series (34).

Thus, we have shown that (33) holds for small t and every local element $A \in \mathfrak{A}$. Since $\|\alpha_t^{(\Lambda)}(A)\| = \|A\|$, the same relation holds for the limit $\alpha_t(A)$. Since the local algebra $\mathfrak{A}^0$ is dense in $\mathfrak{A}$, the map $\alpha_t(A)$ extends to a $*$-automorphism of the algebra $\mathfrak{A}$. Moreover, for all $A \in \mathfrak{A}$ and $|t| < t_0$, we have

$$\alpha_t^{(\Lambda)}(A) \to \alpha_t(A), \Lambda \uparrow R^{\nu} \tag{45}$$

(convergence in norm).

Furthermore, since

$$\alpha_{t_1}^{\Lambda}(\alpha_{t_2}^{\Lambda}(A)) = \alpha_{t_1+t_2}^{\Lambda}(A) \tag{46}$$

when $|t_1|, |t_2| < t_0$, the left-hand side has a limit as $\Lambda \uparrow R^{\nu}$, which can be used to define the automorphisms α_t for $|t| < 2t_0$. Continuing in this way, we define a group of automorphisms $\alpha_{t_1}\alpha_{t_2} = \alpha_{t_1+t_2}$. Relation (33) is proved similarly. This completes the proof of the theorem.

PROOF OF LEMMA 4. We note that for $\mathcal{A}$ of the form (37), we have

$$\tau_t(\mathcal{A}) = \int_{(R^\nu)^{l+\tilde{l}}} \widetilde{K}(x_1,\dots,x_l,y_1,\dots,y_{\tilde{l}})a^*(x_1)\dots a^*(x_l) \times a(y_1)\dots a(y_{\tilde{l}})\,dx_1\dots dx_l\,dy_1\dots dy_{\tilde{l}},$$

where

(47)
$$\widetilde{K}(x_1,\dots,x_l,y_1,\dots,y_{\tilde{l}}) = \int \prod_{i=1}^{l} g_t(x_i-\xi_i)\prod_{j=1}^{\tilde{l}}\overline{g}_t(y_j-\eta_j)K(\xi_1,\dots,\xi_l,\eta_1,\dots,\eta_{\tilde{l}})\,d\xi_1\dots d\xi_l\,d\eta_1\dots d\eta_{\tilde{l}},$$

and $g_t(x-y)$ is the kernel of the operator $\exp\{ith\}$ on $L_2(R^\nu)$ (see [36]). The coefficients $C_t^{(k)}(\Gamma,\widetilde{\Gamma})$ in the expansion(38) are given by

(48)
$$C_t^{(K)}(\Gamma,\widetilde{\Gamma}) = \int \widetilde{K}(x_1,\dots,x_l,y_1,\dots,y_{\tilde{l}})\varphi_\Gamma(x_1,\dots,x_l) \times \varphi_{\widetilde{\Gamma}}(y_1,\dots,y_{\tilde{l}})\,dx_1\dots dx_l\,dy_1\dots dy_{\tilde{l}},$$

where the function $\varphi_\Gamma(x_1,\dots,x_l)$ is obtained by the antisymmetrization of the function $\varphi^{y_1}_{i_1,1}(x_1)\dots\varphi^{y_1}_{i_1,k_1}(x_{k_1})\varphi^{y_2}_{i_2,1}(x_{k_1+1})$ where $\{y_1<y_2<\dots<y_n\}=\operatorname{supp}\Gamma$, and $\{i_{s,1}<i_{s,2}<\dots<i_{s,k_s}\}=\gamma_s$. The estimate (39) is thus a consequence of the following bound for the function $\widetilde{K}$:

(49)
$$|\widetilde{K}(x_1,\dots,x_l,y_1,\dots,y_{\tilde{l}})| < C(K)\prod_i \frac{1}{(d(x_i,G)+1)^{\nu+1}}\prod_j \frac{1}{(d(y_j,G)+1)^{\nu+1}}t^{(\nu/2+1)(l+\tilde{l})}$$

if we recall the familiar inequality $a_1\dots a_n < \left(\frac{a_1+\dots+a_n}{n}\right)^n$. It suffices to prove (49) for a function of a single variable; let

$$\overline{f}(x) = \frac{1}{(2\pi t)^{\nu/2}}\int \exp\left\{\frac{i(x-\xi)^2}{2t}\right\}f(\xi)\,d\xi,$$

where $f(\xi)$ is a C^∞ function with compact support contained in G. We will show that

(50)
$$|\overline{f}(x)| < \frac{ct^{\nu/2+1}}{(d(x,G)+1)^{\nu+1}},$$

where the constant $c=c(f)$ depends only on the function f. Suppose that $d(x,G)>1$, and choose a coordinate system with origin at a point $y_0\in\partial G$ closest to x, and such that the negative x_1 axis passes through x. Denoting the coordinate of the point x by $-\xi, \xi = d(x,y_0)$,

we find that

$$\begin{aligned}\overline{f}(-\xi) =&\frac{1}{(2\pi t)^{\nu/2}}\int_0^\infty dy_1\int_{-\infty}^\infty\cdots\int_{-\infty}^\infty dy_2\dots dy_\nu\exp\left\{\frac{i(y_1+\xi)^2}{2t}\right\}\\ &\times\prod_{j=2}^{\nu}\exp\left\{\frac{iy_j^2}{2t}\right\}f(y_1,y_2,\dots,y_\nu)\\ =&\frac{1}{(2\pi it)^{\nu/2}}t\int_0^\infty dy_1\int_{-\infty}^\infty\cdots\int_{-\infty}^\infty\exp\left\{\frac{i(y_1+\xi)^2}{2t}\right\}\frac{i(y_1+\xi)}{t}\\ &\times\prod_{j=2}^{\nu}\exp\left\{\frac{iy_j^2}{2t}\right\}\frac{f(y_1,y_2,\dots,y_\nu)}{\imath(y_1+\xi)}\,dy_2\dots dy_\nu\\ =&\frac{-t}{(2\pi t)^{\nu/2}}\int_0^\infty\int_{-\infty}^\infty\cdots\int_{-\infty}^\infty\exp\left\{\frac{i(y+s)^2}{2t}\right\}\prod_{j=2}^{\nu}\exp\left\{\frac{iy_j^2}{2t}\right\}\\ &\times\frac{\partial}{\partial y_1}\frac{f(y_1,y_2,\dots,y_\nu)}{i(y_1+\xi)}\,dy_1\dots dy_\nu,\end{aligned}$$

where in the last equality we integrate by parts with respect to the variable y_1. Continuing this procedure n times, we get that

$$\begin{aligned}\overline{f}(-\xi)=\frac{(-t)^n}{(2\pi t)^{\nu/2}}\int_0^\infty\int_{-\infty}^\infty\cdots\int_{-\infty}^\infty\exp\left\{\frac{i(y_1+\xi)^2}{2t}\right\}\prod_{j=2}^{\nu}\exp\left\{\frac{iy_j^2}{2t}\right\}\\ \times\left[\sum_{k=0}^{n}C_{k,n}\frac{f_{y_1}^{(n-k)}(y_1,\dots,y_\nu)}{(y_1+\xi)^{n+k}}\right]dy_1\dots dy_\nu,\end{aligned}$$

where the $C_{k,n}$ are certain universal coefficients. From the last expression, we find that for every n,

$$|\widetilde{f}(\xi)|<\frac{t^{n-\nu/2}}{\xi^n}c_n(f).$$

The result (50) now follows upon taking $n=\nu+1$.

REMARK. An existence theorem for a Heisenberg dynamics for lattice quantum systems was first established by Robinson (see [37]). Our constructions are in many respects similar to his.

§5. Linear dynamics for fermion and boson systems

1. Linear dynamics on the CAR algebra. Let X be a countable set and consider the Fock space $\mathcal{F}_a(\mathcal{H})$, where $\mathcal{H}=l_2(X)$, and the quadratic form

$$\begin{aligned}H=&\frac{1}{2}\sum_{x_1,x_2}b_{x_1,x_2}a^*(f_{x_1})a^*(f_{x_2})\\ &+\frac{1}{2}\sum_{x_1,x_2}b^*_{x_1,x_2}a(f_{x_1})a(f_{x_2})+\sum_{x_1,x_2}c_{x_1,x_2}a^*(f_{x_1})a(f_{x_2}),\end{aligned}\tag{1}$$

in the creation and annihilation operators; here $\{f_x, x\in X\}$ is an orthonormal basis in the space $\mathcal{H}$,

$$b_{x_1,x_2}=-b_{x_2,x_1},\qquad c_{x_1,x_2}=\overline{c}_{x_2,x_1},\qquad b^*_{x_1,x_2}=\overline{b}_{x_2,x_1},$$

and the matrices $B = \{b_{x_1,x_2}\}$, $B^* = \{b_{x_1,x_2}\}$, and $C = \{c_{x_1,x_2}\}$ define bounded operators on $l_2(X)$ ($C^* = C$).

When $B = 0$, (1) defines a selfadjoint operator on $\mathcal{F}_a(\mathcal{H})$ and the associated dynamics

$$\tau_t(A) = \exp\{itH\}A\exp\{-itH\}$$

on the CAR algebra $\mathfrak{A}(\mathcal{H})$ is a free dynamics for which the one-particle operator h has the matrix C with respect to the basis $\{f_x,\, x \in X\}$.

We have already observed that when $B \neq 0$, expression (1) may in general fail to define any operator on $\mathcal{F}_a(\mathcal{H})$ with a dense domain. However, we can associate to H a Heisenberg dynamics on the CAR algebra just as we did in the previous section to define the dynamics α_t. Let $\Lambda \subset X$ be a finite subset of X and let

$$\begin{aligned} H_\Lambda =& \frac{1}{2}\sum_{x_1,x_2\in\Lambda} b_{x_1,x_2}a^*(x_1)a^*(x_2) \\ &+ \frac{1}{2}\sum_{x_1,x_2\in\Lambda} b^*_{x_1,x_2}a(x_1)a(x_2) + \sum_{x_1,x_2\in\Lambda} c_{x_1,x_2}a^*(x_1)a(x_2), \end{aligned} \tag{2}$$

where $a^\#_x = a^\#(f_x)$. Clearly, H_Λ is a bounded selfadjoint operator on $\mathcal{F}(\mathcal{H})$, so that we have the dynamics

$$\tau^\Lambda_t(A) = \exp\{itH_\Lambda\}A\exp\{-itH_\Lambda\}. \tag{3}$$

We will see below that the CAR algebra $\mathfrak{A}(\mathcal{H}) \subset \mathfrak{B}(\mathcal{F}_a(\mathcal{H}))$ is invariant under this dynamics.

LEMMA 1. *For every element $A \in \mathfrak{A}(\mathcal{H})$ the limit*

$$\tau_t(A) = \lim_{\Lambda\uparrow X} \tau^\Lambda_t(A), \qquad t \in R^1, \tag{4}$$

exists (operator norm convergence), and the limit maps τ_t take $\mathfrak{A}(\mathcal{H})$ into itself for $t \in R^1$.

PROOF. We consider the action of the dynamics τ^Λ_t on the generators $\{a^\#_x,\, x \in X\}$ of the CAR algebra. Evidently,

$$a^\#_{\Lambda,x}(t) = a^\#_x \quad \text{for } x \notin \Lambda, \tag{5}$$

where we have written $a^\#_{\Lambda,x}(t) = \tau^\Lambda_t(a^\#_x)$. In addition, for all $t \in R^1$ the operators $a^\#_{\Lambda,x}(t)$ satisfy the anticommutation relations, and in the expressions for H_Λ the generators $a^\#_x$ can be replaced by $a^\#_{\Lambda,x}(t)$ for any fixed t. Hence we get

$$\begin{aligned} \frac{da_{\Lambda,x}(t)}{dt} &= i[H_\Lambda, a_{\Lambda,x}(t)] = -i\sum_{x'\in\Lambda} c_{x,x'}a_{\Lambda,x'}(t) - i\sum_{x'\in\Lambda} b_{x,x'}a^\#_{\Lambda,x}(t), \\ \frac{da^*_{\Lambda,x}(t)}{dt} &= i[H_\Lambda, a^*_{\Lambda,x}(t)] = i\sum_{x'\in\Lambda} c_{x,x'}a^*_{\Lambda,x'}(t) - i\sum_{x'\in\Lambda} b^*_{x,x'}a_{\Lambda,x'}(t). \end{aligned} \tag{6}$$

We have thus obtained a linear system of differential equations for the operators $\{a^\#_{\Lambda,x}(t),\, x \in \Lambda\}$, whose solution can be written in the form

$$\begin{aligned} a_{\Lambda,x}(t) &= \sum_{x'\in\Lambda} g^\Lambda_{x,x'}(t)a_{x'} + \sum_{x'\in\Lambda} h^\Lambda_{x,x'}(t)a^*_x, \qquad x \in \Lambda, \\ a^*_{\Lambda,x}(t) &= \sum_{x'\in\Lambda} \overline{g}^\Lambda_{x,x'}(t)a^*_{x'} + \sum_{x'\in\Lambda} \overline{h}^\Lambda_{x,x'}(t)a_{x'}, \end{aligned}$$

where the block matrix

$$\begin{pmatrix} G_\Lambda(t) & H_\Lambda(t) \\ \overline{H}_\Lambda(t) & \overline{G}_\Lambda(t) \end{pmatrix} = \exp\left\{ -it \begin{pmatrix} C_\Lambda & B_\Lambda \\ B_\Lambda^* & -C_\Lambda \end{pmatrix} \right\}, \tag{7}$$

$$G_\Lambda(t) = \{g^\Lambda_{x,x'}(t), x, x' \in \Lambda\}, \qquad H_\Lambda(t) = \{h^\Lambda_{x,x'}(t), x, x' \in \Lambda\}.$$

Here $C_\Lambda = \{c_{x,x'},\ x, x' \in \Lambda\}$, and similarly for B_Λ and B^*_Λ. When $\Lambda \uparrow X$, the matrix (7) converges strongly to the operator

$$\begin{pmatrix} G(t) & H(t) \\ \overline{H}(t) & \overline{G}(t) \end{pmatrix} = \exp\left\{ -it \begin{pmatrix} C & B \\ B^* & -C \end{pmatrix} \right\} \tag{8}$$

acting on the space $l_2(X) \oplus l_2(X)$, hence we see from (6) and (8) that the limit of $\tau_t^\Lambda(a_x^\#)$ exists (in the norm of $\mathfrak{A}(\mathcal{H})$) and is equal to

$$\begin{aligned} \tau_t(a_x) &= \lim_{\Lambda\uparrow X} \tau_t^\Lambda(a_x) = \sum_{x'\in X} g_{x,x'}(t) a_{x'} + \sum_{x'\in X} h_{x,x'}(t) a^*_{x'}, \\ \tau_t(a^*_x) &= \lim_{\Lambda\uparrow X} \tau_t^\Lambda(a^*_x) = \sum_{x'\in X} \overline{g}_{x,x'}(t) a^*_{x'} + \sum_{x'\in X} \overline{h}_{x,x'}(t) a_{x'}, \end{aligned} \tag{9}$$

where $\{g_{x,x'}(t)\}$ and $\{h_{x,x'}(t)\}$ are the elements of the matrices $G(t)$ and $H(t)$, respectively.

From what we have proved, it follows that the limit (4) exists for every element $A \in \mathfrak{A}(\mathcal{H})$, and also that τ_t is a dynamics on $\mathfrak{A}(\mathcal{H})$. The lemma is proved.

2. Linear dynamics on the Weyl algebra. We recall that a Weyl algebra is a C^*-algebra $\eta(\mathcal{H}) \subset \mathfrak{B}(\mathcal{F}_s(\mathcal{H}))$ of bounded operators acting on $\mathcal{F}_s$ generated by the Weyl operators $\{U(f), V(f), f \in \mathcal{H}\}$ (see §3). We consider once again the quadratic form (see (13.3))

$$H = \frac{1}{2}\sum_{x_1,x_2} a_{x_1,x_2}\pi_{x_1}\pi_{x_2} + \frac{1}{2}\sum_{x_1,x_2} b_{x_1,x_2}\Phi_{x_1}\Phi_{x_2} + \frac{1}{2}\sum_{x_1,x_2} c_{x_1,x_2}(\Phi_{x_1}\pi_{x_2} + \pi_{x_1}\Phi_{x_2}) \tag{10}$$

in the creation and annihilation operators $a^\#(f)$, $f \in \mathcal{H}$, where $\mathcal{H} = l_2(X)$. Here for convenience we have expressed H in terms of the operators

$$\begin{aligned} \pi(f) &= \frac{i}{\sqrt{2}}(a^*(f) - a(f)), \\ \Phi(f) &= \frac{i}{\sqrt{2}}(a^*(f) + a(f)) \end{aligned}$$

and set $\pi_x = \pi(f_x)$, $\Phi_x = \Phi(f_x)$; $\{f_x, x \in X\}$ is an orthonormal basis of $\mathcal{H}$ indexed by the elements of the countable set X; $A = \{a_{x_1,x_2}\}$, $B = \{b_{x_1,x_2}\}$, and $C = \{c_{x_1,x_2}\}$ are symmetric real matrices defining bounded selfadjoint operators on $l_2(X)$, and they satisfy the conditions

$$A > 0, \qquad B > 0, \qquad C^2 < A^{1/2}BA^{1/2}. \tag{11}$$

In general, expression (10) need not define a selfadjoint operator on $\mathcal{F}_s(\mathcal{H})$. In order to use it to construct a dynamics in the Weyl algebra, we consider (as in the case of fermion systems) the operator

$$\begin{aligned} H_\Lambda =& \frac{1}{2}\sum_{x_1,x_2\in\Lambda} a_{x_1,x_2}\pi_{x_1}\pi_{x_2} \\ &+ \frac{1}{2}\sum_{x_1,x_2\in\Lambda} b_{x_1,x_2}\Phi_{x_1}\Phi_{x_2} + \frac{1}{2}\sum_{x_1,x_2\in\Lambda} c_{x_1,x_2}(\Phi_{x_1}\pi_{x_2} + \pi_{x_1}\Phi_{x_2}), \end{aligned} \tag{12}$$

where $\Lambda \subset X$ is a finite subset.

LEMMA 2. *When* (11) *holds, the operator* H_Λ *is selfadjoint on* $\mathcal{F}_S(\mathcal{H})$.

PROOF. It is easy to check that when (11) is satisfied, the expression for H_Λ in terms of the canonical linear transformations

$$\widehat{\pi}_X^\Lambda = \sum_{x' \in \Lambda} U_{x,x'}^\Lambda \pi_{x'}, \qquad \widehat{\Phi}_X^\Lambda = \sum_{x' \in \Lambda} V_{x,x'}^\Lambda \Phi_{x'}, \qquad x \in \Lambda, \tag{13}$$

where the matrices $U^\Lambda = \{U_{x_1,x_2}^\Lambda,\ x_1, x_2 \in \Lambda\}$, $V^\Lambda = \{V_{x_1,x_2}^\Lambda,\ x_1, x_2 \in \Lambda\}$ satisfy the condition

$$V^\Lambda = ((U^\Lambda)^{\mathrm{tr}})^{-1},$$

can be reduced to the following form:

$$H_\Lambda = \omega \sum_{x \in \Lambda} [(\widehat{\pi}_x^\Lambda)^2 + (\widehat{\Phi}_x^\Lambda)^2 + \mu_x (\widehat{\pi}_x^\Lambda \widehat{\Phi}_x^\Lambda + \widehat{\Phi}_x^\Lambda \widehat{\pi}_x^\Lambda)], \tag{14}$$

where $\omega > 0$, and $|\mu_x| < 1$ for all $x \in \Lambda$. As before, the operators $\{\widehat{\pi}_x^\Lambda, \widehat{\Phi}_x^\Lambda\}$ satisfy the commutation relations (14.3),

$$[\widehat{\pi}_x^\Lambda, \widehat{\pi}_{x'}^\Lambda] = [\widehat{\Phi}_x^\Lambda, \widehat{\Phi}_{x'}^\Lambda] = 0, \qquad [\widehat{\pi}_x^\Lambda, \widehat{\Phi}_{x'}^\Lambda] = -i\delta_{x,x'}. \tag{15}$$

It is known (see [46]) that any two representations of a finite system of operators $\{\widehat{\pi}_x^\Lambda, \widehat{\Phi}_x^\Lambda,\ x \in \Lambda\}$ that satisfy the commutation relations (15) must be unitarily equivalent. One such representation can be defined in the space $L_2(R^{|\Lambda|}, d^{|\Lambda|}x)$ by setting

$$\pi_x f = i \frac{df}{dq_x}, \qquad \Phi_x f = q_x f,$$
$$f = f\{q_x, x \in \Lambda\} \in L_2(R^{|\Lambda|}, d^{|\Lambda|}x)$$

so that the operator H_Λ takes the form

$$H_\Lambda f = \omega \sum_{x \in \Lambda} \left[-\frac{\partial^2 f}{\partial q_x^2} + q_x^2 f + \mu_x \left[q_x \frac{\partial f}{\partial q_x} + \frac{\partial}{\partial q_x} q_x f \right] \right] = \omega \sum_{x \in \Lambda} H_x.$$

When $|\mu_x| < 1$, the selfadjointness and semiboundedness of each term H_x follow easily from the KLMN theorem (a modification of the familiar Kato-Rellich theorem, see [36]). This completes the proof of the lemma.

We have thus defined a dynamics

$$\tau_t^\Lambda(A) = \exp\{itH_\Lambda\} A \exp\{-itH_\Lambda\}$$

on the subalgebra $\mathfrak{A}_\Lambda = \mathfrak{B}(\mathcal{F}_S(\mathcal{H}_\Lambda)) \subset \mathfrak{B}(\mathcal{F}_s(\mathcal{H}))$, where $\mathcal{H}_\Lambda \subset \mathcal{H}$ is the subspace spanned by the vectors $\{f_x, x \in \Lambda\}$.

LEMMA 3. *For each local element* A *of the Weyl algebra (i.e.,* $A \in \mathfrak{A}_{\Lambda_0}$ *for some finite* $\Lambda_0 \subset X$*), the strong limit*

$$\lim_{\Lambda \uparrow X} \tau_t^\Lambda(A) = \tau_t(A)$$

exists for every $t \in R^1$. The transformations $A \to \tau_t(A)$, $t \in K_1$, extend to the whole Weyl algebra and define a dynamics on it.

PROOF. Consider the operators

$$\pi_x^\Lambda(t) = \tau_t^\Lambda(\pi_x), \qquad \Phi_x^\Lambda(t) = \tau_t^\Lambda(\Phi_x), \qquad x \in \Lambda.$$

Arguing as in the case of fermion systems and using the commutation relations (15), we obtain the differential equations

$$\frac{d\pi_x^\Lambda(t)}{dt} = -\sum_{x' \in \Lambda} b_{x,x'} \Phi_{x'}^\Lambda(t) - \sum_{x' \in \Lambda} c_{x,x'} \pi_{x'}^\Lambda(t),$$

$$\frac{d\Phi_x^\Lambda(t)}{dt} = \sum_{x' \in \Lambda} a_{x,x'} \pi_{x'}^\Lambda(t) + \sum_{x' \in \Lambda} c_{x,x'} \Phi_{x'}^\Lambda(t), \qquad x \in \Lambda$$

for the operators $\pi_x^\Lambda(t)$ and $\Phi_x^\Lambda(t)$. These equations have the solution

$$\pi_x^\Lambda(t) = \sum_{x' \in \Lambda} g_{x,x'}^\Lambda(t) \pi_{x'}^\Lambda + \sum_{x' \in \Lambda} h_{x,x'}^\Lambda(t) \Phi_{x'}^\Lambda,$$

$$\Phi_x^\Lambda(t) = \sum_{x' \in \Lambda} d_{x,x'}^\Lambda(t) \pi_{x'}^\Lambda + \sum_{x' \in \Lambda} k_{x,x'}^\Lambda(t) \Phi_{x'}^\Lambda,$$

where

$$\begin{pmatrix} G_\Lambda & H_\Lambda \\ D_\Lambda & K_\Lambda \end{pmatrix} = \exp\left\{ it \begin{pmatrix} -B_\Lambda & -C_\Lambda \\ C_\Lambda & A_\Lambda \end{pmatrix} \right\},$$

$G_\Lambda(t) = \{g_{x,x'}^\Lambda(t)\}$, $H_\Lambda(t) = \{h_{x,x'}^\Lambda(t)\}$, and so on, and A_Λ, B_Λ, C_Λ are defined in the same way as for fermion systems. As $\Lambda \uparrow X$, the matrix $\left(\begin{smallmatrix} G_\Lambda & H_\Lambda \\ D_\Lambda & K_\Lambda \end{smallmatrix}\right)$ converges strongly to the matrix $\exp\left\{ it \left(\begin{smallmatrix} -B & -C \\ C & A \end{smallmatrix}\right) \right\} \equiv \left(\begin{smallmatrix} G & H \\ D & K \end{smallmatrix}\right)$ and thus the operators $\Phi_x^\Lambda(t)$ and $\pi_x^\Lambda(t)$ converge strongly on elements of $\mathcal{F}_{s,0}$ to the operators $\Phi_x(t)$ and $\pi_x(t)$

$$\Phi_x(t) = \sum_{x'} g_{x,x'}(t) \pi_{x'} + \sum_{x'} h_{x,x'}(t) \Phi_{x'} = \pi(f_x(t)) + \Phi(\widetilde{f}_x(t)),$$

where $f_x(t) = \sum_{x'} g_{x,x'}(t) f_{x'}$, $\widetilde{f}(t) = \sum_{x'} h_{x,x'}(t) f_{x'}$ and $\pi_x(t)$ is given by a similar expression. This implies (see [36]) that $\exp\{i\Phi_x^\Lambda(t)\}$ and $\exp\{i\pi_x^\Lambda(t)\}$ converge strongly to the operators

$$\exp\{i\Phi_x(t)\} = \exp\{\pi(f(t)) + \Phi(\widetilde{f}(t))\} V(f(t)) U(\widetilde{f}(t)) \exp\{i(f, \widetilde{f})\},$$

which lie in the Weyl algebra. That τ_t defines a dynamics on the Weyl algebra follows from the fact that the matrices $\left(\begin{smallmatrix} G(t) & H(t) \\ D(t) & K(t) \end{smallmatrix}\right)$ define a continuous transformation group on $l_2(X) \oplus l_2(X)$ (see [36]). This proves the lemma.

We note that the free dynamics on the algebra $\mathfrak{B}(\mathcal{F}_s(\mathcal{H}))$ defined by the Hamiltonian

$$H = \sum h_{x_1,x_2} a^*(f_{x_1}) a(f_{x_2}) = d\Gamma(h),$$

where $h_{x_1,x_2} = (hf_{x_1}, f_{x_2})$, coincides with the linear dynamics generated by the expression

$$\overline{H} = \frac{1}{2}\Big(\sum_{x_1,x_2} h_{x_1,x_2} \pi_{x_1} \pi_{x_2} + \sum_{x_1,x_2} h_{x_1,x_2} \Phi_{x_1} \Phi_{x_2} \Big).$$

(This can be seen from the fact that for any finite $\Lambda \subset X$, H_Λ and $\overline{H}_\Lambda$ differ only by a constant.) ¿From this and the lemma just proved, we conclude that the Weyl algebra $\mathfrak{A} \subset \mathfrak{B}(\mathcal{F}_s(\mathcal{H}))$ is invariant under the free dynamics on $\mathfrak{B}(\mathcal{F}_s(\mathcal{H}))$.

§6. Random dynamics (stochastic Langevin equations)

The Markov processes with local interaction give rise to a large class of systems with a random dynamics, to be considered in the next section. Here we will analyze a closely related random dynamics that is described by an infinite system of stochastic differential equations. The example examined here is also of further interest for its relation with the physical concept of "stochastic quantization" (see the review in [29]). From the viewpoint of constructing Gibbs fields on the infinite lattice Z^ν, the meaning of the constructions to be given here is as follows. There is a class of Gibbs modifications μ of the Gaussian field on the lattice Z^ν for which one can explicitly produce an infinite system of stochastic differential equations which are such that μ is an invariant distribution for the Markov process defined by the solution of the equations, and such that their time evolution ν_t converges to the Gibbs distribution μ as $t \to \infty$ for a large class of initial distributions ν_0.

1. System of stochastic differential equations (the Langevin equations). The main object of our study will be an infinite system of equations of the form

$$d\xi_x(t) = F_x(\{\xi_y(t), y \in Z^\nu\})dt + dW_x(t), \quad x \in Z^\nu, \tag{1}$$

with initial conditions $\xi_x(0) = \xi_x^0$, $x \in Z^\nu$. Here $\{\xi_x(t),\ x \in Z^\nu\}$ is a system of random processes to be determined below; they are indexed by the points x in Z^ν and are defined for all times $t \geq 0$; $\{\xi_x^0,\ x \in Z^\nu\} \subset R^{Z^\nu}$ is some fixed configuration on Z^ν; and $\{W_x(t),\ x \in Z^\nu\}$ is a set of independent Wiener stochastic processes starting at zero, $W_x(0) = 0$, $x \in Z^\nu$, and defined on the probability space (Ω, Σ, μ). Finally, $\{F_x(\{\xi_y,\ y \in Z^\nu\}),\ x \in Z^\nu\}$ is a system of functions defined on the set R^{Z^ν} of all configurations $\{\xi_y,\ y \in Z^\nu\} \in R^{Z^\nu}$. In fact, in the present case we will see that each function F_x depends on the configurations $\{\xi_y\}$ in some finite fixed neighborhood of the point x,

$$F_x = F_x(\{\xi_y, y : |x - y| < d\}).$$

Intuitively, equations (1) say that the increment $d\xi_x$ of the process ξ_x in the time interval $(t, t+dt)$ is the sum of a "deterministic" part $F_x dt$ (depending on the values of all the processes $\{\xi_y,\ y \in Z^\nu\}$ at the time t), plus the random increment dW_x of the Wiener process. Although this interpretation of the system of equations (1) is very convenient, to define it formally we first rewrite (1) in integral form,

$$\xi_x(t) = \xi_x^0 + \int_0^t F_x(\{\xi_y(t), y \in Z^\nu\})d\tau + W_x(t). \tag{2}$$

In the example considered here, the functions F_x have the following form. Let U^0 be the quadratic form

$$U^0 = a_0 \sum_{x \in Z^\nu} \xi_x^2 + \sum_{\substack{x \neq y \\ x,y \in Z^\nu}} a_{x-y}\xi_x\xi_y, \tag{3}$$

where the function $a = \{a_u,\ u \in Z^\nu\}$ satisfy the following conditions:

1) a has compact support, i.e., $a_u = 0$ for $|u| > d$;
2) a is even, $a_u = a_{-u}$;
3) $a_0 > \sum_{u \neq 0} |a_u|$.

The last condition ensures that the form U^0 is positive definite. Let $P(\cdot)$ be an even polynomial of degree at least 4^x, with leading coefficient 1 (and such that $P(0) = 0$). Consider the formal Hamiltonian

$$U = U^0 + \varepsilon \sum_{x \in Z^\nu} P(\xi_x), \tag{4}$$

where $\varepsilon > 0$ and set

$$F_x = -\frac{1}{2}\frac{\partial U}{\partial \xi_x} = -a_0 \xi_x - \sum_{y \neq x} a_{y-x}\xi_y - \frac{\varepsilon}{2}P'(\xi_x). \tag{5}$$

Before studying the infinite system (2) (or (1)), let us consider its "spatial truncation", i.e., the finite system of equations

$$\xi_x^\Lambda(t) = \xi_x^0 + \int_0^t F_x^\Lambda(\{\xi_y^\Lambda(\tau), y \in \Lambda\})d\tau + W_x(t), \qquad x \in \Lambda, \tag{6}$$

or in differential form,

$$\begin{gathered} d\xi_x^\Lambda = F_x^\Lambda(\{\xi_y^\Lambda, y \in \Lambda\})dt + dW_x(t), \qquad x \in \Lambda, \\ \xi_x^\Lambda(0) = \xi_x^0, \qquad x \in \Lambda, \end{gathered} \tag{7}$$

where

$$F_x^\Lambda = -\frac{1}{2}\frac{\partial U_\Lambda}{\partial \xi_x} = -a_0\xi_x - \sum_{\substack{y \in \Lambda \\ y \neq x}} a_{y-x}\xi_y - \frac{\varepsilon}{2}P'(\xi_x), \qquad x \in \Lambda,$$

and

$$U_\Lambda = \sum_{x,y \in \Lambda} a_{x-y}\xi_x\xi_y + \varepsilon \sum_{x \in \Lambda} P(\xi_x). \tag{8}$$

We will use the following theorem on existence and uniqueness of solutions for the system (6). Let $\Sigma_t^\Lambda \subset \Sigma$ denote the σ-subalgebra of Σ generated by the values $\{W_x(\tau),\ 0 < \tau < t\}$ of the Wiener processes $\{W_x,\ x \in \Lambda\}$.

THEOREM 1. *There exists a unique multivariate random process $\xi^\Lambda(t) = \{\xi_x^\Lambda(t),\ x \in \Lambda\}$, $t \in [0, \infty)$, which satisfies* (6) *and is defined on the probability space (Ω, Σ, μ) for almost all $\omega \in \Omega$. It is a Markov process, and for every $t \in [0, \infty)$ its values $\{\xi_x^\Lambda(t)\}$ are measurable with respect to the σ-algebra Σ_t^Λ. Its sample paths $\xi_x^\Lambda(t; \omega)$ are continuous in t for almost all $\omega \in \Omega$.*

The proof of this theorem can be found in various textbooks on the theory of stochastic processes (see, e.g., [8]) and will not be given here.

We note that the form of the functions F_x, and in particular the condition that $\varepsilon > 0$, is essential for the existence of a solution for all t. This is easily understood by considering the "deterministic" differential equation

$$\frac{d\xi}{dt} = -a\xi + \xi^3,$$

which has the property that for large enough initial conditions ξ^0, all solutions $\xi(t)$ become infinite in a finite positive time t. The additional diffusion associated with the Brownian motion

can no longer correct this effect. We note further that the fact that the process $\{\xi_t^\Lambda\}$ is Markov (and also measurable with respect to Σ_Λ^t) is clear from our heuristic discussion of the equations (1) if we recall that the multivariate Wiener process $W^\Lambda(t) = \{W_x(t),\ x \in \Lambda\}$ has the Markov property.

We will solve the system of stochastic equations (2) by taking the limit as $\Lambda \uparrow Z^\nu$ of the solutions for the finite systems (6) subject, however, to the requirement that the initial conditions $\{\xi_x^0,\ x \in Z^\nu\}$ do not grow very rapidly at infinity. Specifically, let $\mathcal{H} = l_2(Z^\nu,\ e^{-|x|}) \subset R^{Z^\nu}$ be the set of configurations $\xi = \{\xi_x,\ x \in Z^\nu\}$ such that

$$\|\xi\|^2 \equiv \sum_{x \in Z^\nu} |\xi_x|^2 e^{-|x|} < \infty,$$

where

$$|x| = \sum_{i=1}^{\nu} |x_i|, \qquad x = (x_1, \dots, x_\nu) \in Z^\nu.$$

THEOREM 2. *Let the initial conditions $\xi^0 = \{\xi_x^0,\ x \in Z^\nu\} \in \mathcal{H}$ be given. Then there exists an infinite-dimensional stochastic random process $\xi(t) = \{\xi_x(t),\ x \in Z^\nu\}$, $t \in [0, \infty)$ which satisfies equations* (2) *and is such that for every $x \in Z^\nu$ the sample paths $\xi_x(t) = \xi_x(t; \omega)$ are continuous for almost all ω, and the configuration $\{\xi_x(t; \omega)\} \in \mathcal{H}$ for every t and almost all ω. The random process $\xi(t)$ is a Markov process, and for every t its values $\{\xi_x(t; \omega)\}$ are measurable with respect to the σ-algebra Σ^t generated by the values of the Wiener processes $\{W_x(\tau),\ x \in Z^\nu,\ 0 < \tau \le t\}$.*

PROOF. Let U be a smooth function on the space R^Λ and let $\xi^\Lambda(t) = \{\xi_x^\Lambda(t),\ x \in \Lambda\}$ be a solution of system (6). Consider the process $\eta(t) = U(\xi^\Lambda(t))$. It turns out that the differential $d\eta$ of this process has the form (Ito's formula)

$$\begin{aligned} d\eta &= \sum_x U'_{\xi_x}(\xi^\Lambda(t)) d\xi_x^\Lambda + \frac{1}{2} \sum_x U''_{\xi_x \xi_x}(\xi^\Lambda(t))\, dt \\ &= \sum_x \left(U'_{\xi_x} F_x + \frac{1}{2} U''_{\xi_x \xi_x} \right) dt + \sum_x U'_{\xi_x}\, dW_x. \end{aligned} \tag{9}$$

The additional summand $1/2 \sum U''_{\xi_x, \xi_x}(\xi^\Lambda(t))\, dt$ is present because $(d\xi\, d\xi_x^\Lambda)^2 = F_x^2 dt^2 + 2F_x\, dt\, dW_x + (dW_x)^2$ has the same order of smallness as $(dW_x)^2 \sim dt$ (we refer to [11] for a detailed derivation of equations (9) and (9′) below). Thus, the increment for the process $\eta(t)$ during a finite time is equal to

$$U(\xi^\Lambda(t_1)) - U(\xi^\Lambda(t_2)) = \int_{t_1}^{t_2} \left(\sum_x \left(U'_{\xi_x} F_x + \frac{1}{2} U''_{\xi_x \xi_x} \right) \right) dt + \sum_x \int_0^t U'_{\xi_x}\, dW_x. \tag{9′}$$

We must now explain how the integrals $\int U'_{\xi_x}(\xi^\Lambda)\, dW_x$, which are called *Ito stochastic integrals,* are to be interpreted. Let $\eta(t) = \eta(t; \omega)$ be a random process defined on the space (Ω, Σ, μ) such that for each t its value $\eta(t, \omega)$ is measurable with respect to the σ-algebra Σ^t, and for almost all $\omega \in \Omega$ the integral $\int_0^t \eta^2(\tau, \omega)\, d\tau < \infty$. Then the stochastic integral $\int_0^t \eta(\tau)\, dW_x(\tau)$ is defined as the limit in probability of the following integral sums:

$$\sum_{k=0}^{n-1} \eta(t_k)(W_x(t_{k+1}) - W_x(t_k)), \tag{10}$$

where $0 = t_0 < t_1 < \cdots < t_n = t$ is an arbitrary partition of the interval $[0, t]$ (a proof that the limit $\int_0^t \eta(\tau)\, dW_x(\tau)$ exists can be found, e.g., in [11]). It follows immediately from the definition of stochastic integral that its mean value is zero:

$$\left\langle \int_0^t \eta(\tau) dW_x(\tau) \right\rangle = 0. \tag{11}$$

Indeed, this is true for every integral sum in (10), since in each term of the sum the random variables $\eta(t_k)$ and $W_x(t_{k+1}) - W_x(t_k)$ are independent and $\langle W_x(t_{k+1}) - W_x(t_k)\rangle = 0$. Now consider the function

$$U(\xi^\Lambda(t)) = \|\xi^\Lambda(t)\|^2_{\mathcal{H}} = \sum_{x\in\Lambda} (\xi^\Lambda_x(t))^2 e^{-|x|}. \tag{12}$$

LEMMA 3. *For every finite $\Lambda \subset Z^\nu$, $U(\xi^\Lambda(t))$ satisfies the estimate*

$$\left\langle \sum_{x\in\Lambda} (\xi^\Lambda_x(t))^2 e^{-|x|} \right\rangle \le \left(\sum_{x\in\Lambda} (\xi^0_x)^2 e^{-|x|} + D_1 \right) e^{Rt} + D_2 \tag{13}$$

for all $t > 0$, where the constants $R > 0$, $D_1 > 0$, $D_2 > 0$ do not depend on Λ.

PROOF. Applying Ito's formula to the function (12) and averaging the resulting equality with respect to the measure μ, we get by (11) that

$$\begin{aligned}\left\langle \sum_{x\in\Lambda} (\xi^\Lambda_x(t))^2 e^{-|x|} \right\rangle = &\sum_{x\in\Lambda} (\xi^0_x)^2 e^{-|x|} - 2\int_0^t \left[\sum_{x,y\in\Lambda} a_{x-y} \langle \xi^\Lambda_x(\tau)\xi^\Lambda_y(\tau)\rangle e^{-|x|} \right] d\tau \\ &- \varepsilon \int_0^t \sum_{x\in\Lambda} e^{-|x|} \langle \xi^\Lambda_x(\tau) P'(\xi^\Lambda_x(\tau))\rangle + t \sum_{x\in\Lambda} e^{-|x|}.\end{aligned}$$

Note that for all $\xi \in R^1$ the polynomial $\xi P'(\xi)$ is bounded from below by

$$\xi P'(\xi) > -c, \qquad c > 0.$$

Using the inequality

$$|\xi^\Lambda_x \xi^\Lambda_y| < \frac{1}{2}((\xi^\Lambda_x)^2 + (\xi^\Lambda_y)^2),$$

we find that

$$\left\langle \sum_{x\in\Lambda} (\xi^\Lambda_x(t))^2 e^{-|x|} \right\rangle < \sum_{x\in\Lambda} (\xi^0_x)^2 e^{-|x|} + R\int_0^t \left\langle \sum_{x\in\Lambda} (\xi^\Lambda_x(\tau))^2 e^{-|x|} \right\rangle d\tau + Bt, \tag{14}$$

where $R = \sum_{u\in Z^\nu} |a_\nu| e^{-|u|}$ and $B = (c\varepsilon + 1) \sum_{x\in Z^\nu} e^{-|x|}$. We write $s(t) = \left\langle \sum_{x\in\Lambda} (\xi^\Lambda_x(t))^2 e^{-|x|} \right\rangle$ and consider the solution of the equation

$$\widetilde{s}(t) = R\int_0^t \widetilde{s}(\tau) d\tau + Bt + s(0).$$

It is clear from (14) that $s(t) \le \widetilde{s}(t)$. On the other hand, $\widetilde{s}(t)$ is a solution of the equation $\widetilde{s}' = R\widetilde{s}(t) + B$, $\widetilde{s}(0) = s(0)$. Upon solving this equation we obtain the bound (13).

LEMMA 4. *Let Λ_1 and Λ_2 be finite subsets of Z^ν (with $0 \in \Lambda_1 \subset \Lambda_2$) and let $\xi^{\Lambda_1}(t)$ and $\xi^{\Lambda_2}(t)$ be the corresponding solutions of the equations. Then we have the following estimates*:

1. *The estimate*

$$\sum_{x\in\Lambda_1} \langle(\xi_x^{\Lambda_1}(t) - \xi_x^{\Lambda_2}(t))^2\rangle e^{-2|x|} < e^{-\rho(0,\partial\Lambda_1)} M(\|\xi^0\|_{\mathcal{H}}^2 + \widetilde{D})e^{R_2 t}, \tag{15}$$

where M, $\widetilde{D}$, and R_2 are constants independent of Λ_1 and Λ_2, and $\rho(0,\partial\Lambda_1)$ is the distance from the point $0 \in Z^n$ to the set $\partial\Lambda_1$.

2. *The estimate*

$$\left\langle \max_{0\le\tau\le t} \sum_{x\in\Lambda_1} (\xi_x^{\Lambda_1}(\tau) - \xi_x^{\Lambda_2}(\tau))e^{-2|x|} \right\rangle < e^{-\rho(0,\partial\Lambda_1)} \widetilde{M}(\|\xi^0\|_{\mathcal{H}}^2 + \overline{D})e^{\overline{R}t}, \tag{15$'$}$$

where the constants $\widetilde{M}$, $\overline{D}$, and $\overline{R}$ do not depend on Λ_1 and Λ_2.

PROOF. We observe that

$$\begin{aligned}
&d\Big(\sum_{x\in\Lambda_1} (\xi_x^{\Lambda_1}(t) - \xi_x^{\Lambda_2}(t))\Big)^2 e^{-2|x|} \\
&\quad= 2\sum_{x\in\Lambda_1} e^{-2|x|}(\xi_x^{\Lambda_1}(t) - \xi_x^{\Lambda_2}(t))(d\xi_x^{\Lambda_1}(t) - d\xi_x^{\Lambda_2}(t)) \\
&\quad= 2\sum_{x\in\Lambda_1} e^{-2|x|}(\xi_x^{\Lambda_1}(t) - \xi_x^{\Lambda_2}(t))(F_x^{\Lambda_1}(\xi^{\Lambda_1}) - F_x^{\Lambda_2}(\xi^{\Lambda_2}))\,dt \\
&\quad= -2\sum_{x\in\Lambda_1} e^{-2|x|}(\xi_x^{\Lambda_1}(t) - \xi_x^{\Lambda_2}(t)) \\
&\qquad\qquad\times\Big[\sum_{y\in\Lambda_1} a_{x-y}(\xi_y^{\Lambda_1}(t) - \xi_y^{\Lambda_2}(t)) - \sum_{y\in\Lambda_2\setminus\Lambda_1} a_{x-y}\xi_y^{\Lambda_2}\Big] \\
&\qquad- \varepsilon\sum_{x\in\Lambda_1} e^{-2|x|}(\xi_x^{\Lambda_1}(t) - \xi_x^{\Lambda_2}(t))(P'(\xi_x^{\Lambda_1}) - P'(\xi_x^{\Lambda_2})).
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{x\in\Lambda_1} (\xi_x^{\Lambda_1}(t) - \xi_x^{\Lambda_2}(t))^2 e^{-2|x|} &< R_1\int_0^t \Big(\sum_{x\in\Lambda_1} (\xi_x^{\Lambda_1}(\tau) - \xi_x^{\Lambda_2}(\tau))^2 e^{-2|x|}\Big)d\tau \\
&\quad+ D_1 e^{-\rho(0,\partial\Lambda_1)}\int_0^t \Big(\sum_{y\in\Lambda_2}\Big(\xi_y^{\Lambda_2}(\tau)e^{-2|y|}\Big)\Big)\,d\tau,
\end{aligned} \tag{16}$$

where we have used the estimate

$$(\xi-\eta)(P'(\xi) - P'(\eta)) > -K(\xi-\eta)^2$$

for a suitable constant $K \ge 0$.

Averaging the inequality (16) with respect to ω and applying estimate (13) of the previous lemma, and also the arguments used in its proof, we obtain (15).

We note that inequality (16) remains valid if the left-hand side is replaced by the expression

$$\max_{0<s<t} \sum_{x\in\Lambda_1} (\xi_x^{\Lambda_1}(s) - \xi_x^{\Lambda_2}(s))e^{-2|x|}$$

After averaging and using (15) and (13), this gives (15′).

We now choose a sequence of cubes $\Lambda_N \subset Z^\nu$ with sides of length $L_N = N$ and center at 0. We will show that for any fixed point $x_0 \in Z^\nu$, for almost all the trajectories $\xi_{x_0}^{\Lambda_N}(t;\omega) = \xi_{x_0}^N(t;\omega)$ of the processes $\xi_{x_0}^{\Lambda_N}(t)$ converge uniformly on any finite time interval $[0,T]$. For simplicity, we take $x_0 = 0$ and for each N consider the set

$$E_N = \left\{ \max_{0<s<T} |\xi_0^N(s;\omega) - \xi_0^{N+1}(s;\omega)| > e^{-N/3} \right\}.$$

In view of the estimate (15′) and Chebyshev's inequality, we find that

$$\mu(E_N) < K(T)\frac{e^{-N}}{\varepsilon_N^2} < K(T)e^{-N/3}.$$

Since the series $\sum_N \mu(E_N) < \infty$, by the Borel-Cantelli theorem [1], there exists a set $\widetilde{\Omega} \subset \Omega$ of full measure such that for any $\omega \in \widetilde{\Omega}$ we have

$$\max_{0<s<T} |\xi_0^N(s;\omega) - \xi_0^{N+1}(s;\omega)| < e^{-N/3}$$

for all N greater than some $N_0 = N_0(\omega)$. Thus, the sequence of functions $\xi_0^N(t;\omega)$, $\omega \in \widetilde{\Omega}$, converges uniformly on every finite time interval. The limit functions $\xi_0^\infty(t;\omega)$ are the sample paths of the process $\xi_0(t)$. Similarly, one proves the existence of the limit process $\xi_x^\infty(t)$ for every point $x \in Z^\nu$. The remaining assertions in the theorem now follow easily. In particular, Lemma 3 implies that $\{\xi_x^\infty(t;\omega)\} \subset \mathcal{H}$ for every t for almost all ω.

2. Reduced dynamics on the space of measures. For any configuration $\xi^0 = \{\xi_x^0\} \in \mathcal{H}$ and any $t \geq 0$, let $P_t(\cdot \mid \xi^0)$ denote the distribution in R^{Z^ν} of the values of the infinite-dimensional random vector $\xi(t) = \{\xi_x(t;\omega),\ x \in Z^\nu$, where $\xi(t) = \xi^\infty(t)$ is the solution (constructed in the previous subsection) of the system of equations (2) with the initial conditions $\xi^0 = \{\xi_x^0\}$. By the results proved above, the distribution $P_t(\cdot \mid \xi^0)$ is concentrated on the set $\mathcal{H} \subset R^{Z^\nu}$, i.e.,

$$P_t(\mathcal{H}|\xi^0) = 1. \tag{17}$$

Let $\mathfrak{M}$ denote the class of probability distributions ν in R^{Z^ν} concentrated on $\mathcal{H}$, and define a family of transformations on $\mathfrak{A}$ by

$$(V_t\nu)(A) = \int_{\mathcal{H}} P_t(A|\xi^0)\, d\nu(\xi^0), \qquad A \subseteq \mathcal{H}. \tag{18}$$

Since the process $\{\xi^\infty(t)\}$ is Markov, the transformations V_t form a semigroup,

$$V_{t_1+t_2} = V_{t_1}V_{t_2}. \tag{19}$$

In addition, for each finite $\Lambda \subset Z^\nu$ we consider the semigroup V_t^Λ of transformations of probability measures on R^Λ given by

$$(V_t^\Lambda \nu)(\cdot) = \int_{R^\Lambda} P_t^\Lambda(\cdot|\xi_\Lambda^0)\, d\nu(\xi_\Lambda^0), \tag{20}$$

where $P_t^\Lambda(\cdot \mid \xi_\Lambda^0)$ is the distribution of the values of the process $\xi^\Lambda(t)$, where $\xi^\Lambda(t)$ is the solution of (6) starting at the point $\xi_\Lambda^0 \in R^\Lambda$ at time $t = 0$. It turns out that if the initial measure $\nu = \nu_0$ on R^Λ has a density $p^0(\xi^\Lambda)$ with respect to the Lebesgue measure $d^\Lambda\xi$ on R^Λ, then so do all the measures $\nu_t = V_t\nu_0$, and their densities $p_t(\xi^\Lambda)$ satisfy the Chapman-Kolmogorov differential equation

$$\frac{\partial p_t}{\partial t} = \frac{1}{2}\sum_{x\in\Lambda} \frac{\partial^2 p_t}{\partial \xi_x} - \sum_{x\in\Lambda} \frac{\partial}{\partial \xi_x}(F_x^\Lambda p_t) \tag{21}$$

with initial condition $p|_{t=0} = p_0$, where the F_x^Λ are the coefficients in equation (6). We note that the distribution $\nu_{\text{equil}}^\Lambda$ with density

$$p_{\text{equil}}(\xi^\Lambda) = \frac{1}{Z_\Lambda}\exp\{-U_\Lambda(\xi^\Lambda)\}, \tag{22}$$

where U_Λ is the truncated Hamiltonian (8) and the normalization factor Z_Λ is given by

$$Z_\Lambda = \int_{R^\Lambda} \exp\{-U_\Lambda(\xi^\Lambda)\} d^\Lambda\xi$$

is a stationary point of the semigroup V_t:

$$V_t^\Lambda \nu_{\text{equil}}^\Lambda = \nu_{\text{equil}}^\Lambda \tag{22$'$}$$

for all t, as follows immediately by substituting (22) in the right-hand side of equation (21).

When $\varepsilon = 0$, the measure $\nu_{\text{equil}}^\Lambda$ coincides with the Gaussian measure $\nu_{\text{gauss}}^\Lambda$ with density

$$p_{\text{gauss}}^\Lambda(\xi) = \frac{1}{Z_\Lambda^{\text{gauss}}}\exp\Big\{-\sum_{x,y\in\Lambda} a_{x-y}\xi_x\xi_y\Big\}. \tag{23}$$

¿From (8), (22), and (23) we see that for $\varepsilon > 0$, $\nu_{\text{equil}}^\Lambda$ is the Gibbs modification of the measure $\nu_{\text{gauss}}^\Lambda$ by means of the interaction $\varepsilon\sum_{x\in\Lambda} P(\xi_x)$:

$$\frac{d\nu_{\text{equil}}^\Lambda}{d\nu_{\text{gauss}}^\Lambda} = \frac{1}{\widetilde{Z}_\Lambda}\exp\Big\{-\varepsilon\sum_{x\in\Lambda} P(\xi_x)\Big\}, \tag{24}$$

where

$$\widetilde{Z}_\Lambda = \Big\langle \exp\Big\{-\varepsilon\sum_{x\in\Lambda} P(\xi_x)\Big\}\Big\rangle_{\nu_{\text{gauss}}^\Lambda}.$$

As is shown in [26], under our assumptions on the function $\{a_u,\ u \in Z^\nu\}$, for small enough $\varepsilon > 0$ the thermodynamic limit

$$\nu_{\text{equil}} = \lim_{\Lambda\uparrow Z^\nu} \nu_{\text{equil}}^\Lambda \tag{24$'$}$$

of the measures $\nu_{\text{equil}}^\Lambda$ exists.

THEOREM 5. *The measure ν_{equil} is concentrated on $\mathcal{H}$ and invariant under the semigroup V_t.*

This theorem will be proved simultaneously with the next theorem. Let Q be an arbitrary polynomial of even degree with leading coefficient 1. Let ν_Q^Λ be the Gibbs modification of the Gaussian measure $\nu_{\text{gauss}}^\Lambda$ associated with the interaction $\varepsilon \sum_{x\in\Lambda} Q(\xi_x)$,

$$\frac{d\nu_Q^\Lambda}{d\nu_{\text{gauss}}^\Lambda} = \frac{1}{Z_{\Lambda,Q}} \exp\left\{ -\varepsilon \sum_{x\in\Lambda} Q(\xi_x)\right\} \tag{25}$$

where $Z_{\Lambda,Q}$ is a normalizing factor. As already indicated, the limit measure

$$\nu_Q = \lim_{\Lambda\uparrow Z^\nu} \nu_Q^\Lambda$$

exists for sufficiently small $\varepsilon > 0$.

THEOREM 6. *The measure ν_Q is concentrated on $\mathcal{H}$ and for small enough ε, $\nu_Q^t = V_t\nu_Q$ converges to ν_{equil} as $t \to \infty$. More precisely, for every fixed local function F_A we have*

$$\langle F_A\rangle_{\nu_Q^t} \to \langle F_A\rangle_{\nu_{\text{equil}}} \quad \text{as } t \to \infty \tag{26}$$

and the convergence is exponentially fast.

We recall that for a finite set $A \subset Z^\nu$, a local function F_A of the field configurations $\xi = \{\xi_x, x \in Z^\nu\}$ is one that depends only on the values of the configurations of ξ on the set A.

A proof of this theorem will now be briefly sketched.

First consider the case $\varepsilon = 0$. Then one checks easily that for every Λ, the distribution $\mu_\Lambda^0(\cdot \mid \xi_\Lambda^0)$ of the values of the field $\{\xi_x^\Lambda(t)\}$ on $\Lambda \times R^1$, generated by the solution of equations (6) with $\varepsilon = 0$, is Gaussian for all initial conditions ξ_0^Λ. Now consider the field $\{\widetilde{\xi}_x(t)\}$ on $\Lambda \times R^1$ with random initial conditions ξ_0^Λ distributed according to the law $\nu_{\text{gauss}}^\Lambda$. The distribution of the field $\{\widetilde{\xi}_x(t)\}$ is equal to

$$\int \mu_\Lambda^0(\cdot|\xi_0) d\nu_{\text{gauss}}^\Lambda(\xi_0) = \mu_0^\Lambda \tag{27}$$

and is Gaussian. Its mean value is $\langle\widetilde{\xi}_x(t)\rangle = 0$ for all $x \in \Lambda$ and $t > 0$, and its covariance

$$\langle\widetilde{\xi}_x(t)\widetilde{\xi}_y(t)\rangle = C_{t,s}^\Lambda(x,y)$$

is readily seen by explicit computation to satisfy the estimate

$$|C_{t,s}^\Lambda(x,y)| < B\exp\{-\gamma|t-s| - \gamma|x-y|\}, \tag{28}$$

where $B > 0$ and $\gamma > 0$ are absolute constants (independent of Λ).

For $\varepsilon > 0$ we consider the analogous measure μ

$$\mu_{\varepsilon,Q}^\Lambda(\cdot) = \int_{R^\Lambda} \mu_\varepsilon^\Lambda(\cdot|\xi_0^\Lambda) d\nu_Q^\Lambda(\xi_0),$$

which is the distribution of the field $\{\overline{\xi}_x^\Lambda(t)\}$ on $\Lambda \times R^1$ obtained by solving equations (6) with $\varepsilon > 0$ and initial conditions $\{\xi_0^\Lambda\}$ distributed according to the law ν_Q^Λ. For any $T > 0$ we denote

by $\mu_0^{\Lambda,T}$ and $\mu_\varepsilon^{\Lambda,T}$ the distributions of the restrictions of the fields $\{\widetilde{\xi}_x^\Lambda(t)\}$ and $\{\overline{\xi}_x^\Lambda(t)\}$ to the set $\Lambda \times [0,T] \subset \Lambda \times R^1$ (generated by the distributions μ_0^Λ and μ_ε^Λ, respectively). One can show that $\mu_\varepsilon^{\Lambda,T}$ is absolutely continuous with respect to $\mu_0^{\Lambda,T}$, and that for almost all configurations $\xi^{\Lambda,T} = \{\xi_x^\Lambda(t)\}$, $x \in \Lambda$, $t \in [0,T]$ the density $d\mu_\varepsilon^{\Lambda,T}/d\mu_0^{\Lambda,T}$ is given by

$$\begin{aligned}\frac{d\mu_\varepsilon^{\Lambda,T}}{d\mu_0^{\Lambda,T}}(\xi^{\Lambda,T}) = \exp\Big\{ &-\frac{\varepsilon}{2}\sum_x \int_0^T P'(\xi_x^{\Lambda,T}(\tau))dW_x(\tau) \\ &-\frac{\varepsilon^2}{8}\sum_{x\in\Lambda}\int_0^T [P'(\xi_x^{\Lambda,T}(\tau))]^2 d\tau\Big\}\frac{d\nu_Q^\Lambda}{d\nu_{\text{gauss}}^\Lambda}(\xi^{\Lambda,T}(0))\end{aligned} \tag{29}$$

(this is the Girsanov formula, see [11]).

If we now use Ito's formula for the increment

$$\begin{aligned}\frac{\varepsilon}{2}\sum_{x\in\Lambda}P(\xi_x^{\Lambda,T}(T)) - \frac{\varepsilon}{2}\sum_{x\in\Lambda}P(\xi_x^{\Lambda,T}(0)) &= \frac{\varepsilon}{2}\sum_{x\in\Lambda}\int_0^T P'(\xi_x^{\Lambda,T}(\tau))F_x(\xi^{\Lambda,T}(\tau))\,d\tau \\ &+ \frac{\varepsilon}{4}\sum_{x\in\Lambda}\int_0^T P''(\xi_x^{\Lambda,T}(\tau))\,d\tau + \frac{\varepsilon}{2}\int_0^T\sum_{x\in\Lambda}P'(\xi_x^{\Lambda,T}(\tau))\,dW_x(\tau),\end{aligned}$$

and also equation (28) for $d\nu_Q^\Lambda/d\nu_{\text{gauss}}^\Lambda$, we can rewrite (29) in the form

$$\begin{aligned}\frac{d\mu_\varepsilon^{\Lambda,T}}{d\mu_0^{\Lambda,T}} = &\frac{1}{\widetilde{Z}_\Lambda}\exp\Big\{-\frac{\varepsilon}{2}\sum_{x\in\Lambda}[P(\xi_x^{\Lambda,T}(T)) - P(\xi_x^{\Lambda,T}(0))] \\ &+\frac{\varepsilon}{4}\sum_{x\in\Lambda}\int_0^T P''(\xi_x^{\Lambda,T}(\tau))\,d\tau - \frac{\varepsilon}{2}\sum_{x,y}\int_0^T P'(\xi_x^{\Lambda,T}(\tau))a_{x-y}\xi_y^{\Lambda,T}(\tau)\,d\tau \\ &-\frac{3}{8}\varepsilon^2\sum_{x\in\Lambda}\int_0^T[P'(\xi_x^{\Lambda,T}(\tau))]^2 d\tau - \varepsilon\sum_{x\in\Lambda}Q(\xi_x^{\Lambda,T}(0))\Big\}.\end{aligned}$$

Thus, the measure $\mu_\varepsilon^{\Lambda,T}$ is a Gibbs modification of the $\mu_0^{\Lambda,T}$. For small ε, we can use the bound (28) on the covariance of the Gaussian measure $\mu_0^{\Lambda,T}$ to derive a cluster expansion for $\mu_\varepsilon^{\Lambda,T}$ which is similar to the above-noted cluster expansion for the measure ν_Q^Λ given in [26]. Using this cluster expansion one can show that the limit

$$\mu_\varepsilon = \mu_\varepsilon^{Z^\nu,R} = \lim_{\substack{\Lambda\uparrow Z^\nu \\ T\to\infty}} \mu_\varepsilon^{\Lambda,T} \tag{30}$$

exists, and the limit measure μ_ε again admits a cluster expansion. The limit μ_ε is the distribution for the field $\{\xi_x^\infty(t)\}$ given by the solutions of the infinite system of equations (1) with random initial conditions distributed according to the law ν_Q. It follows from this that for every t_0 the distribution $\mu_\varepsilon^{t_0}$ of the values of the field $\{\xi_x^\infty(t)\}$ at a fixed time $t = t_0$ coincides with $V_{t_0}\nu_0$:

$$\mu_\varepsilon^{t_0} = V_{t_0}\nu_Q.$$

Next, using the cluster expansion for the distribution μ_ε one can show that the field $\{\overline{\xi}_x(t)\}$ decays exponentially. This implies in particular that for any local function F_A, the mean $\langle F_A(\xi^\infty(t_0))\rangle_{\mu_\varepsilon} = \langle F_A(\xi)\rangle_{\mu_\varepsilon^{t_0}}$ converges to the limit $\langle F_A(\xi)\rangle_{\overline{\nu}}$ exponentially fast, and the limit measure $\overline{\nu}$ (which is independent of the distribution ν_0 of the initial conditions) coincides with the stationary measure for the semigroup V_t. Taking $Q = P$, we see from (22′) that all the marginal distributions $\mu_\varepsilon^{\Lambda,T,t_0}$ for $\mu_\varepsilon^{\Lambda,T}$ coincide with $\nu_{\text{equil}}^\Lambda$. From this, we obtain upon taking the limits (24) and (30) that $\overline{\nu} = \nu_{\text{equil}}$, which is the assertion of Theorem 6.

§7. Marginally closed Markov chains with local interaction

The Markov processes (or chains in the discrete-time case) with local interaction comprise an extensive class of random fields which are often encountered in applications yet also comparatively simple to study.

1. Basic definitions. We consider the random field $\xi = \{\xi(x,t),\ (x,t) \in Z^\nu \times Z_+^1\}$ on the semilattice $Z_+^{\nu+1} = Z^\nu \times Z_+^1$ (where Z_+^1 is the positive "time" lattice) with values in some finite or countable set S. A *configuration* $\{\xi(x,t)\}$ of this field will be represented as a sequence of field configurations $\xi_t = \{\xi_t(x) = \xi(x,t),\ x \in Z^\nu\}$ on the lattice Z^ν at fixed times. We say that the field forms a Markov chain if for every $t > 0$ the conditional distribution of the configuration ξ_{t+1}, subject to the condition that the configurations $\xi_0 = \overline{\xi}_0, \dots, \xi_t = \overline{\xi}_t$ are specified at all the previous times, depends only on the last configuration $\overline{\xi}_t$:

$$\Pr(\xi_{t+1} \in A | \overline{\xi}_t, \overline{\xi}_{t-1}, \dots, \overline{\xi}_0) = \Pr(\xi_{t+1} \in A | \xi_t = \overline{\xi}). \tag{1}$$

Here $A \subset \Omega = S^{Z^\nu}$ is an arbitrary set of field configurations on Z^ν.

A Markov chain will be called a chain with local interaction if for every finite subset $X \subset Z^\nu$ and every collection of values $S_X = \{s_x,\ x \in X\}$, the conditional probability

$$\Pr(\xi_{t+1}|_X = S_X | \xi_t = \overline{\xi}) = A_X^t(S_X, \overline{\xi}|_{Q+X}) \tag{2}$$

depends only on the values of the configuration $\overline{\xi}$ in a Q-neighborhood $Q + X = \bigcup_{x \in X}(Q + x)$ of the set X. Here $Q \subset Z^\nu$ is some fixed finite set (a neighborhood of the point $0 \in Z^\nu$), and $Q + x$ is the translate of Q by the vector $x \in Z^\nu$; $\xi_{t+1}|_X$ denotes the restriction of the configuration to the set X (similarly for $\overline{\xi}|_{Q+X}$), and $A_X^t(S_X, S'_{X+Q})$ is some function of the pair of configurations (S_X, S'_{X+Q}). We will henceforth assume that $A_X(\cdot,\cdot)$ does not depend on t (i.e., the transition probabilities are uniform in time). Finally, we shall say that a Markov chain is conditionally independent if the values of the configuration ξ_{t+1}, subject to the condition that the configuration $\xi_t = \overline{\xi}$ is fixed, are independent:

$$\Pr(\xi_{t+1}|_X = S_X | \xi_t = \overline{\xi}) = \prod_{x \in X} p_x(s_x | \overline{\xi}|_{Q+x}), \tag{3}$$

where

$$p_x(s | S_{Q+x}) \equiv A_{\{x\}}(s; S_{Q+x}), \quad s \in S \tag{4}$$

is a family of probability distributions on the space S depending on the configurations S_{Q+x} on the set $Q + x$. We will also assume that this family is "spatially uniform" in Z^ν. This means that

$$A_{\{x\}}(s; S_{Q+x}) = A_0(s; S_Q^x) \equiv p(s | S_Q^x),$$

where S_Q^x is the configuration on Q obtained from S_{Q+x} by translation by $-x$. A chain is conditionally linear if the conditional probabilities satisfy

$$p(s|S_Q) = p(\xi_{t+1}(0) = s | \xi_t|_Q = S_Q) = \sum_{y \in Q} a_y(s, S_Q(y)), \tag{5}$$

where $\{a_y(s,s')\}$ is a set of functions of the pairs (s,s').

2. The BBKY equations and marginal closedness. Definition (2) can be rewritten as

$$\Pr(\xi_{t+1}|_X = S_X|\xi_t = \overline{\xi}) = A_X(S_X, S'_{X+Q}) \prod_{z\in X+Q} \delta_{\overline{\xi}(z),s'_{X+Q}(z)} \tag{6}$$

where $\delta_{s,s'}$ is the Kronecker symbol. Averaging (6) over the values of ξ_t, we find from this that the probabilities

$$P_X(S_X, t+1) = Pr(\xi_{t+1}|_X = S_X) \tag{6^a}$$

(sometimes called the correlation functions) are given by

$$\begin{aligned} P_X(S_X, t+1) &= \sum_{S'_{Q+X}} A_X(S_X, S'_{X+Q}) \left\langle \prod \delta_{\overline{\xi}(z),s'(z)} \right\rangle \\ &= \sum A_X(S_X, S'_{X+Q}) P_{X+Q}(S'_{X+Q}, t) \end{aligned}$$

where $\langle\cdot\rangle$ denotes an average over the values of ξ_t. The hierarchy of equations

$$P_X(S_X, t+1) = \sum_{S'_{Q+X}} A_X(S_X, S'_{X+Q}) P_{X+Q}(S'_{X+Q}, t), \tag{7}$$

expressing the finite-dimensional distributions of the chain at time $t+1$ in terms of the finite-dimensional distributions at time t, is similar in structure to the well-known BBGKY (Bogolyubov, Born, Green, Kirkwood, Yvon [35]) hierarchy for the evolution of a system of correlation functions in classical statistical physics. We note that the hierarchy (7) is difficult to analyze, because it expresses the probabilities $P_X(\cdot, t+1)$ only in terms of the probabilities $P_Y(\cdot, t)$, where $|Y| > |X|$. However, in some cases one can use the consistency condition

$$P_X(S_X, t) = \sum_{S'_Y} P_{X \bigcup Y}(S_{X \bigcup Y}, t),$$

for the correlation functions to transform the right-hand side of (7) so that only probabilities $P_Y(\cdot, t)$ with $|Y| \le |X|$ appear. Here $Y \subset Z^\nu$ is an arbitrary set disjoint from X, and $S'_Y = S_{X\cup Y}$ is a configuration on $X \cup Y$ such that $S_{X\cup Y}|_X = S_X$, $S_{X\cup Y}|_Y = S'_Y$.

Definition 1. A Markov chain with local interaction is said to be *marginally closed* if for every initial distribution P_0 (i.e., probability distribution for the values of the configuration ξ_0 at time zero), for all $t \ge 0$, $X \subset Z^\nu$, and S_X, the probabilities $P_X(S_X, t+1)$ can be expressed in the form

$$P_X(S_X, t+1) = \sum_Y \sum_{S_Y} A_{X,Y}(S_X, S_Y) P_Y(S_Y, t), \tag{8}$$

where the sum Σ_Y is over all nonempty $Y \subset X \cup Q$ such that $|Y| \le |X|$, and $\{A_{X,Y}(S_X, S_Y)$, $X \subset Z^\nu$, $Y \subset X \cup Q\}$ is a system of functions of pairs of configurations (S_X, S_Y) depending only on the conditional probabilities (2) (i.e., independent of the initial distribution P_0).

Remark. A Markov chain with marginally closed conditional distributions is also sometimes said to be marginally closed.

LEMMA 1. *If a Markov chain is conditionally independent, then it is marginally closed if and only if it is conditionally linear* (*see equation* (5)).

PROOF 1. Sufficiency. Assume that the conditional probabilities are conditionally linear. Then it follows from (5) and (6) that

$$P_X(S_X|\xi_t=\overline{\xi}) = \prod_{x\in X}\Big(\sum_{y\in Q}\sum_{s'} a_y(s_x,s')\delta_{s',\overline{\xi}(x+y)}\Big). \tag{9}$$

Multiplying out the expressions in parentheses and noting that

$$\delta^2_{s,\overline{\xi}(z)} = \delta_{s,\overline{\xi}(x)},$$

we reduce the right-hand side of (9) to

$$\sum_Y \sum_{S'_Y} \widehat{A}(S_X,S'_Y)\prod_{z\in Y}\delta_{s'(z),\overline{\xi}(z)},$$

where the sum is over all $Y\subset X+Q$ and $|Y|\le|X|$. Upon averaging over the values of the configuration $\overline{\xi}$, we obtain (8).

2. Necessity. Let $X=\{0\}$ be the one-point set coinciding with $0\in Z^\nu$ and $t=0$. Then

$$\Pr(\xi_1(0)=s)\equiv P_{\{0\}}(s,1)=\sum_{S_Q} A_{\{0\}}(s,S_Q)P_Q(S_Q,0). \tag{10}$$

Since the chain is marginally closed, the right-hand side of (10) can be rewritten as

$$\sum_{y\in Q}\sum_{s'}\widehat{A}_{\{0\},\{y\}}(s,s')p_{\{y\}}(s',0).$$

For any fixed configuration $\overline{S}_Q$ on Q we consider the initial distribution p_0 such that

$$\Pr(\xi_0|_Q=\overline{S}_Q)=1.$$

In this case, $P_Q(\overline{S}_Q,0)=1$ and $p_{\{y\}}(s',0)=\delta_{s',\overline{s}_Q(y)}$. Hence we obtain

$$\Pr(\xi_1(0)=s|\xi_0|_Q=\overline{S}_Q)\equiv A_{\{0\}}(s,\overline{S}_Q)=\sum_{y\in Q}\widehat{A}_{\{0\},\{y\}}(s,\overline{s}_Q(y)).$$

Upon setting

$$a_y(s,s')=\widehat{A}_{\{0\},\{y\}}(s,s')$$

we arrive at the assertion of the lemma.

It is clear that for a conditionally linear chain, the functions $\{a_y(s,s'),\ y\in Q\}$ satisfy the following conditions. For any configuration $S'_Q=\{s'_Q(y),\ y\in Q\}$ the following conditions are satisfied:

a) the inequality

$$\sum_{y\in Q} a_y(s,s'_Q(y))\ge 0.$$

b) the equality

$$\sum_s \sum_y a_y(s, s'_Q(y)) = 1. \tag{11}$$

For every $y \in Q$ and $s \in S$, define the quantities

$$\sum_{s\in S} a_y(s, s') = q_y(s'). \tag{11'}$$

As follows from (11), for every set of values $S'_Q = \{s'(y),\, y \in Q\}$,

$$\sum_{y\in Q} q_y(s'(y)) = 1.$$

Thus, $q_y(s')$ does not depend on s': $q_y(s') \equiv q_y$,

$$\sum_{y\in Q} q_y = 1.$$

Notice that if instead of the functions $a_y(s, s')$ we consider the functions

$$\widetilde{a}_y(s, s') = a_y(s, s') + c_y(s), \tag{12}$$

where $\{c_y(s),\, y \in Q\}$ is a set of functions on S such that for all $s \in S$

$$\sum_{y\in Q} c_y(s) = 0. \tag{13}$$

then the conditional probabilities (9) remain unchanged.

LEMMA 2. *One can choose a set of functions* $\{c_y(s)\}$ *satisfying* (13), *such that the transformed system of functions* $\widetilde{a}_y(s, s')$ *in* (12) *is positive*

$$\widetilde{a}_y(s, s') \geq 0.$$

PROOF. Fix $s = s_0$ and for every $y \in Q$ set

$$d_y(s_0) = d_y = \min_{s'\in S} a_y(s, s'). \tag{14}$$

Note that by condition a) above,

$$\sum_{y\in Q} d_y = \sum_y{}^{+} d_y + \sum_y{}^{-} d_y \geq 0,$$

where $\sum^{\pm}$ denotes a sum over those y for which $d_y > 0$ ($d_y < 0$, respectively).

We now set

$$c_y(s^0) = c_y = \begin{cases} |d_y| & \text{if } d_y \leq 0, \\ \frac{\sum^- d_y}{\sum^+ d_y} d_y & \text{if } d_y > 0. \end{cases}$$

It follows from this definition that condition (13) is satisfied. Moreover, (14) implies that for every $y \in Q$ we have $c_y + d_y \geq 0$. Thus, $a_y(s_0, s) + c_y(s_0) > 0$ for all y, s_0, s. The lemma is proved.

We will henceforth always assume that

$$a_y(s, s') \geq 0.$$

3. Ergodic behavior of conditionally linear chains. In what follows, we will study invariant measures and the asymptotic behavior of conditionally independent and marginally closed chains for large times t.

In particular, we will show that this behavior is completely determined by the behavior of the one-point correlation function, i.e., by the set of probabilities $\{P_{\{x\}}(s_{\{x\}}, t),\ x \in Z^\nu\}$. The probabilities $P_X(S_X, t)$ introduced above (see (6^a)) are usually called n-point correlation functions, if $|X| = n$.

It follows from (3) that

$$P_{\{x\}}(s, t+1) = \sum_{s', y} a_y(s, s') P_{\{x+y\}}(s', t). \tag{15}$$

If the initial distribution P_0 is assumed invariant under translations of the lattice Z^ν, then the same will be true of the distribution P_t for the configurations at any time t and, in particular, the one-point correlation function $P_{\{x\}}(s, t)$ is independent of x:

$$P_{\{x\}}(s, t) = P(s, t).$$

¿From this and (15), we find that

$$P(s, t+1) = \sum_{s'} \left(\sum_{y} a_y(s, s') \right) P(s', t) = \sum_{s'} b(s, s') P(s', t), \tag{16}$$

where $b(s, s') = \sum_y a_y(s, s')$. Since $b(s, s') \geq 0$ and $\sum_s b(s, s') = 1$, the matrix $B = \{b(s, s')\}$ can be regarded as the transition probability matrix of a homogeneous Markov chain with set of states S here ($b(s, s')$ is the probability for a particle to go from s' to s). We will denote this chain by L;[2] it turns out that the asymptotic behavior of our original chain $\{\xi_{x,t}\}$ with values in S is uniquely determined by the properties of the Markov chain L.

We now recall some notions relating to (homogeneous) Markov chains with finitely or countably many states. For any two states $s_1, s_2 \in S$ let $B^n = \{b^{(n)}_{s_1,s_2}\}$ denote the matrix of transition probabilities after n steps: $b^{(n)}_{s_1,s_2}$ is the probability that the chain, originally in state s_1, will be in state s_2 after n steps; let $f^{(n)}_{s_1,s_2}$ denote the probability that a transitions from s_1 to s_2 occurs for the first time in the nth step. The probability of ever reaching s_2 from s_1 is given by the sum $f^{\infty}_{s_1,s_2} = \sum_n f^{(n)}_{s_1,s_2}$. In particular, $f^{\infty}_s = f^{\infty}_{s,s}$ is the probability of returning to the state s. A state s is said to be recurrent if $f^{\infty}_s = 1$ and nonrecurrent if $f^{\infty}_s < 1$. It is easy to verify that s is nonrecurrent if and only if

$$\sum_n b^{(n)}_{s,s} < \infty \tag{17}$$

(in this case the series $\sum_n b^{(n)}_{s_1,s_2} < \infty$ automatically converges for every pair s_1, s_2). The sum $\mu_s = \sum_{n=1}^{\infty} n f^{(n)}_{s,s}$ is called the average time of return. A recurrent state s for which $\mu_s = \infty$ is called a null-recurrent state.

In order for s to be a null-recurrent state, it is necessary and sufficient that $\sum_n b^{(n)}_{s,s} = \infty$, but

$$b^{(n)}_{s,s} \to 0, \quad n \to \infty. \tag{18}$$

[2]Here by L we mean a family of Markov chains having the same transition probability matrix; the members of the family differ only in their initial distributions.

The state s is said to be *periodic* with period T if $b_{s,s}^{(n)} = 0$ for all n not divisible by T, and $b_{s,s}^{(Tk)} > 0$ for all $k = 1, 2, \dots$.

Recurrent nonnull aperiodic states (i.e., which are not periodic) are called ergodic states. For ergodic states we have the asymptotic relation $b_{s_1,s_2}^{(n)} = \mu_{s_1}^{-1} f_{s_2,s_1}^{\infty} + o(1)$ as $n \to \infty$, and in particular, $b_{s,s}^{(n)} = \mu_s^{-1} + o(1)$ as $n \to \infty$. For a non-null-recurrent periodic state with period T, we have $b_{s,s}^{(kT)} \to T\mu_s^{-1}$ as $k \to \infty$. We say that a state s_2 is *reachable* from s_1 if $f_{s_1,s_2}^{\infty} = 1$. A Markov chain is said to be *irreducible* if every state is reachable from every other state. It is a fact that all states of an irreducible Markov chain are of the same type: they are either all nonrecurrent, null-recurrent, periodic (with the same period), or, finally, ergodic.

In the case of an arbitrary Markov chain, all the recurrent states can be divided uniquely into classes, each forming an irreducible Markov chain. For an irreducible Markov chain with ergodic states (such a chain is called ergodic) there exists a unique stationary (i.e., time-independent) probability distribution $\pi(s) = \mu_s^{-1}$ for the states s.

For more information on all the above, see, e.g., [45].

After this general review, we return to the study of the correlation functions.

THEOREM 3. *Let all the states of the Markov chain L be nonrecurrent or null-recurrent. Then for any translationally invariant initial distribution P_0 of the original chain $\{\xi_{x,t}\}$ all the correlation functions satisfy*

$$\lim_{t\to\infty} P_X(S_X, t) = 0.$$

PROOF. It follows from (16) that

$$P(s,t) = \sum_{s'} b_{s,s'}^{(t)} P(s', 0),$$

where $P(s', 0)$ is the one-point correlation function for the initial distribution. Relations (17) and (18) imply that $P(s,t) \to 0$ as $t \to \infty$, for every s. We then get the assertion of the theorem by using the estimate $P_X(S_X, t) \le P_{\{x\}}(S_X(x), t)$, where $x \in X$ is an arbitrary point of X and $S_X(x)$ is the value of the configuration S_X at x.

To analyze the other cases we will need the following representation for the correlation functions $P_X(S_X, t)$. Upon repeatedly iterating the equality

$$P_X(S_X, t) = \sum_{S_{X+Q}} P(S_X | S_{X+Q}) P_{X+Q}(S_{X+Q}, t-1),$$

we obtain

$$\begin{aligned} P_X(S_X, t) = \sum_{S_{X+Q}^{(1)}, S_{X+Q+Q}^{(2)} \dots S_{X+Q+Q+\dots+Q}} & P(S_X | S_{X+Q}^{(1)}) \\ & \times P(S_{X+Q}^{(1)} | S_{X+Q+Q}^{(2)}) \dots P(S_{X+(t-1)Q}^{(t-1)} | S_{X+tQ}^{(t)}) P_{X+tQ}(S_{X+tQ}^{t}, 0), \end{aligned} \tag{19}$$

where

$$\begin{aligned} X + kQ &= X + \underbrace{Q + \dots + Q}_{k \text{ times}} \\ &= \{x + y_1 + y_2 + \dots + y_k, \quad x \in X, \quad y_i \in Q, \quad i = 1, \dots, k\}. \end{aligned}$$

Next, inserting the expression

$$P(S^k_{X+kQ}|S^{(k+1)}_{x+(k+1)Q}) = \prod_{z\subseteq X+Q}\sum_{y\in Q} a_y(s^{(k)}(z), s^{(k+1)}(z+y)),$$

into (19), we get

$$\begin{aligned} P_X(S_X,t) = \sum_{y_1,\dots,y_t}\ \sum_{S^{(1)}_{X+Q},S^{(2)}_{X+2Q},\dots,S^{(t)}_{X+tQ}} & \\ \times a_{y_1}(S_X(x), S^{(1)}_{X+Q}(x+y_1)a_{y_2}(S^{(1)}_{X+Q}(x+y_1), & \\ S^{(2)}_{X+2Q}(x+y_1+y_2)\dots a_{y_t}(S^{(t-1)}_{X+(t-1)Q}(x+y_1+\dots+y_{t-1}), & \\ S^{(t)}_{X+tQ}(x+y_1+\dots+y_t)P_0(S^{(t)}_{X+Qt},0). & \end{aligned}$$

Each term in this sum can be described with the aid of the following graph G, constructed on the points of the semilattice $Z^\nu \times Z^1_+$.

1) On the slice $Y_t = Z^\nu \times \{t\}$ the set of vertices of the graph G coincides with the set X.

2) Exactly one edge "descends" from each vertex (x,t'), where $0 < t' \le t$, i.e., there is exactly one edge of the form $\{(x,t'),(x',t'-1)\}$, where $x'-x \in Q$.

3) There is at least one "upward" directed edge entering each vertex (x,t'), $0 \le t' < t$, i.e., an edge of the form $\{(x',t'+1),(x,t')\}$, with $x-x' \in Q$.

The set of all such graphs will be denoted by $R(X,t)$. By a marked graph, we mean a pair (G,S_G), where G is a graph as described above, and S_G is a configuration: $S_G = \{S_G(v), v \in G\}$, $S_G(v) \in S$, defined on the vertices of G.

We define the contribution of any marked graph (G,S_G) to be the quantity

$$P(G,S_G) = \prod_{(v,v')\in G} a_{x'-x}(S_G(v), S'_G(v')), \tag{20}$$

where the product is taken over all edges (v,v') of G ($v=(x,t)$, $v'=(x',t')$, $t=t'+1$, $x'-x\in Q$).

From equation (19) we obtain that

$$P_X(S_X,t) = \sum_{\substack{G\in R(X,t)\\ S_G:S_G|_X=S_X}} P(G,S_G)\Big\langle \prod_v \delta_{S(v),\xi_0(v)}\Big\rangle, \tag{21}$$

where the product Π_v is over the vertices in the zeroth slice Y_0, $v=(x,0)$, and the average $\langle\cdot\rangle$ is over the configurations in the zeroth slice.

Let $R_c(X,t) \subseteq R(X,t)$ denote the set of connected graphs. It is evident that a graph $G \in R(X,t)$ is connected if and only if it contains exactly one vertex in the zeroth slice.

4. Dual random walk and estimates obtained using it. To a correlation function $P_X(S_X,t)$ we associate the following random walk involving several particles on the lattice Z^ν, the walk occurring in "reversed" time $\tau = 0,1,\dots,t$. At time zero there are n particles located at the points of the set $X \subset Y_t$ ($|X|=n$). The particle at $x\in X$ may, independently of the other particles, jump during one-time step to another point $x+y$ with probability q_y, independently of how the other particles behave. If two (or more) particles jump to the same point, then they "coalesce" to form a new particle. During the next time step, the same rule

governs the motion of the old and newly formed particles. Clearly, any trajectory of the motion of this system of particles during the time $\tau = t$ gives rise to a graph $G \in R(X,t)$. Let the graph G be fixed. Then it follows easily from the definition (11′) of the probabilities q_y (and their independence of the state s' in (11′)) that

$$\sum_{S_G:\ S_G|_{Y_0}-\text{fixed}} P(G,S_G) = \prod_{\text{edges of } G} q_y, \tag{22}$$

where the sum on the left is over all configurations on the graph G with arbitrary but fixed values at the vertices of the zeroth slice, and the product on the right is taken over the edges $\{(x,t),\ (x',t+1)\}$ of the graph G, and $y = x' - x$. Formula (22) implies, in particular, the estimate

$$\sum_{\substack{S_G:\ S_G|_{Y_0}-\text{fixed}\\ S_G|_{Y_1}-\text{fixed}}} P(G,S_G) \le \prod_{\text{edges of } G} q_y, \tag{23}$$

where the sum on the left is over all configurations S_G with fixed values on the slices Y_0 and Y_1.

The case of dimensions $\nu = 1$ and 2.

THEOREM 4. *Let $\nu = 1$ or $\nu = 2$, and assume that the Markov chain L is irreducible, with all its states ergodic. Then for any initial distribution, every correlation function $P_X(S_X,t)$ converges to a limit $P_X(S_X,\infty)$ as $t \to \infty$, and the limit is given by*

$$P_X(S_X,\infty) = \sum_{\tau=1}^{\infty} \sum_{G\in\overline{\mathcal{R}}_c(X,\tau)} \sum_{s} \pi(s) \sum_{S_G:S_G(x,0)=s;S_G|_{Y_\tau}=S_X} P(G,S_G). \tag{24}$$

Here $\pi(s)$ is the stationary distribution of the chain L, and $\overline{R}_c(X,\tau)$ is the set of connected graphs having more than one vertex lying in the slices $Y_{\tau'}$, $\tau' > 0$ (i.e., they become connected only on the last slice Y_0). The sum $\sum_{S_G}$ is over all configurations S_G of the graph G with the property that they take the value s at the last point in the slice Y_0, and their restriction to the slice Y_τ is equal to S_X. The quantities $P_X(S_X,\infty)$ are the correlation functions of the (unique) stationary distribution of the original chain.

PROOF. We consider only the part of the sum (21) corresponding to disconnected graphs G. Note that for a fixed set X and time t, the quantity

$$P_{\text{discon}}(X,t) = \sum_{G\in\mathcal{R}_{\text{discon}}(X,t)} \prod_{y \text{ an edge of } G} q_y$$

is just the probability that at least two particles, say those at the points $x_1, x_2 \in X$, do not coalesce during the entire motion extending over t time steps. Considering their difference $z(t) = x_1(t) - x_2(t)$, we may regard the point $z(t) \in Z^\nu$ as the coordinate of a particle walking along the lattice Z^ν with transition probabilities

$$P_{z\to z'} = \sum_{y_1,y_2:y_1-y_2=z'-z} q_{y_1} q_{y_2}. \tag{25}$$

It is clear from this formula that $z(t)$ describes a symmetric random walk on the lattice (i.e., $\Pr(z - z') = \xi) = \Pr(z - z' = -\xi)$).

It is known that for a symmetric random walk in dimensions $\nu = 1, 2$, the probability that the particle will eventually reach the coordinate origin is equal to 1 (see [40]). Since the number of initial pairs (x_1, x_2) is finite (equal to $\binom{n}{2}$), we see that the probability

$$P_{\text{discon}}(x,t) \to 0 \quad \text{as } t \to \infty.$$

The disconnected part of the sum in (21) admits the estimate

$$\begin{aligned} &\sum_{G\in\mathcal{R}_{\text{discon}}(X,t)} \sum_{S_G} P(G,S_G)\left\langle \prod_v \delta_{s(v),\xi_0(v)}\right\rangle \\ &\qquad < P_{\text{discon}}(X,t) \sup_{v_1,\dots,v_n} \sum_{s(v_1),\dots,s(v_n)} P(\xi_0(x(v_1)) \\ &\qquad = s(v_1),\dots,\xi_0(x(v_n)) = s(v_n)), \end{aligned} \tag{26}$$

where the supremum is taken over all the vertices in the zeroth slice.

If the initial distribution P_0 is such that its one-particle correlation function is equal to $\overline{P}(\xi_0(x) = s) = \pi(s)$, then the series (21) is dominated by $\sum_{\tau=1}^{\infty}\sum_s \pi(s)P_c(\tau)$, where $P_c(\tau)$ is the probability that all the particles will coalesce together after exactly τ time steps. It follows that the series in (24) converges. Moreover, the connected part of the sum (24) in this case converges to $P_X(S_X,\infty)$. In the general case, for fixed t the difference $\overline{P}_X(S_X,t) - P_X(S_X,\infty)$, where $\overline{P}_X(S_X,t)$ is the correlation function for the initial distribution $\overline{P}_0$, is dominated by

$$\sum_\tau P_c(\tau)\sum_s |\pi(s) - p_{t-\tau}(s)|, \tag{27}$$

where $p_{t-\tau}(s)$ is the probability of the state s for the Markov chain L at time $t-\tau$ for some fixed initial distribution $p_0(s)$. But the ergodicity of the chain L implies that (27) tends to zero, proving the theorem.

A similar result is also valid for the case when the Markov chain L splits into a finite number of irreducible ergodic classes.

When the Markov chain L is periodic with period $T_0 > 1$, the correlation functions $P_X(S_X, T_0t+T)$, where $0 \le T < T_0$, converge to a limit $P_X(S_X,\infty,T)$ given by equation (24), with $\pi(s)$ replaced by $\pi_T(s)$, where $\pi_T(s) = \lim_{t\to\infty} p_{T_0t+T}(s)$, and $p_{T_0t+T}(s)$ is the distribution of the Markov chain L at the time T_0t+T (it is determined by the initial distribution of L). If this initial distribution coincides with the stationary distribution for each of the periodic classes, then the process behaves periodically in the strict sense.

Case of higher dimension, $\nu \ge 3$. In this case the disconnected graphs G in equation (24) give a nonzero contribution to the limit correlation functions. Note that the average

$$\left\langle \prod_v \delta_{s(v),\xi_0(v)}\right\rangle$$

over the initial distribution occurring in (21) can as usual be expressed in terms of semi-invariants,

$$\left\langle \prod_v \delta_{s(v),\xi_0(v)}\right\rangle = \sum_{\alpha=(V_1,\dots,V_k)} \prod_{i=1}^{k} {\left\langle \prod_{v\in V_i}{}' \delta_{s(v),\xi_0(v)}\right\rangle}, \tag{28}$$

the sum being taken over all partitions $\alpha = (V_1, \dots, V_k)$ of the set of vertices of the graph G lying in the zeroth slice, and

$$\left\langle \prod_{v \in V}{}' \delta_{s(v),\xi_0(v)} \right\rangle \equiv \widetilde{P}_0(V, S_V)$$

is the semi-invariant of the set of variables $\{\delta_{s(v),\xi_0(v)}\}$, computed with respect to the initial distribution. Using the decomposition (28), we obtain that

$$\begin{aligned} P_X(S_X, t) &= \sum_{G, S_G} \sum_{\alpha = (V_1, \dots, V_k)} P(G, S_G) \prod \widetilde{P}_0(V_i, S_{V_i}) \\ &= \sum_{G, S_G} P(G, S_G) \Big[\prod_{v_i} P_0(v_i, s_{v_i}) + \sum_{\alpha \neq 0} \prod_i \widetilde{P}_0(V_i, S_{V_i}) \Big]. \end{aligned} \tag{29}$$

The first term in square brackets corresponds to the partition $\alpha = 0$ of the set of initial vertices of G into one-point blocks, while the second term corresponds to the sum over all the other partitions.

We now make an important hypothesis concerning the initial distribution P_0, namely, that it is translationally invariant and such that

$$\sum_{V : 0 \in V} \sum_{S_V} \widetilde{P}_0(V, S_V) < \infty \tag{30}$$

for every n.

LEMMA 5. *Under condition* (30),

$$\lim_{t \to \infty} \sum_{G, S_G} P(G, S_G) \sum_{\alpha \neq 0} \prod_{i=1}^{k} \widetilde{P}_0(V_i, S_i) = 0. \tag{31}$$

PROOF. For each set $V \subset Z^\nu$ pick the smallest (with respect to the lexicographic ordering) point $y \in V$ and let $V^0 = V - y$ be the translation of V by y; then the smallest point in V^0 is the origin. For any graph G and any partition $\alpha = (V_1, \dots, V_k)$ we denote by $\{V_1^0, \dots, V_k^0\}$ the corresponding collection of translated sets. Further, because the distribution P_0 is translationally invariant, we can rewrite the sum (31) as

$$\sum_{k=1}^{n-1} \sum_{V_1^0, \dots, V_k^0} \prod \widetilde{P}(V_i^0, S_i) \sum_{y_1, \dots, y_k} \sum_{G, S_G} P(G, S_G), \tag{32}$$

where the sum $\sum_{V_1^0, \dots, V_k^0}$ is over the k-tuples of sets $(V_1^0, \dots, V_k^0)$ of which at least one set consists of at least two points; the sum $\sum_{y_1, \dots, y_k}$ is over the k-tuples $(y_1, \dots, y_k)$ such that the sets $(V_1^0 + y_1, \dots, V_k^0 + y_k)$ are pairwise disjoint; and $\sum_{G, S_G}$ is over the graphs (G, S_G) whose set V of initial vertices coincides with $\cup (V_i^0 + y_i)$, and the values $S_G|_{V_i^0 + y_i}$ with the values S_i.

Let us now consider $A = A(x_1, \dots, x_n, V_1^0, \dots, V_k^0, t)$, namely, the event that some pair of particles initially present at the points x_i, x_j will, upon executing a random walk in reverse

time, turn up at time t at the points $\overline{x}_j$, $\overline{x}_i$ belonging to the set $V_l + y_l$ for some $l = 1, \dots, k$, with $y_l \in Z^\nu$. Evidently,

$$\sum_{y_1,\dots,y_k} \sum_{G, S_G} P(G, S_G) < \Pr(A(x_1, \dots, x_n, V_1^0, \dots, V_k^0, t)). \tag{33}$$

On the other hand, $\Pr(A(x_1, \dots, x_n, V_1^0, \dots, V_k^0, t)) \to 0$ as $t \to \infty$. Indeed, for a fixed pair x_i, x_j consider the difference $z_{ij} = x_i - x_j$. Then the event A means that $z_{ij}(t)$ belongs to the set of points of the form $\{x - x',\, x, x' \in V_l^0,\, l = 1, \dots, k\}$. However, the probability for $z_{ij}(t)$ to be in a fixed bounded set is well known to tend to zero. The lemma now follows from the bound (33), our hypothesis (30), and the representation (32).

Thus, we need to investigate

$$\sum_{G, S_G} P(G, S_G) \prod_v \widetilde{P}_0(v, s(v)), \tag{34}$$

where the product is over all initial vertices of the graph G.

The collection of graphs $R = R(X, t)$ is the union of the subsets R^k of graphs G having exactly k connected components. We begin by considering the class R^n. Every graph $G \in R^n$ is a union of n nonintersecting paths $\Gamma_i = \{(x_{i,0}, 0), (x_{i,1}, 1), \dots, (x_{i,t}, t)\}$, $i = 1, \dots, n$. If $n = 1$ we get that

$$\sum_{G, S_G} P(G, S_G) \langle \delta_{v, \xi(v)} \rangle = \sum_{s'} b^{(t)}(s(x_1), s') P(s', 0) = P_{x_1}(s(x_1), t), \tag{35}$$

where $b^{(t)}$ is the matrix of transition probabilities after t steps in the Markov chain L.

For $n > 1$ we include the class $R^n(X, S_X, t)$ in the larger class $\widetilde{R}^n(X, S_X, t)$ whose elements are n-tuples $(\Gamma_1, \dots, \Gamma_n)$ of marked paths that may intersect (i.e., several paths Γ_i may pass through the same point $x \in Z^\nu$, and may have different markings at x). The contribution from each such n-tuples is given by

$$P(\Gamma_1, \dots, \Gamma_n) = \prod_{i,l} a_{x_{i,l} - x_{i,l+1}}(s(x_{i,l+1}), s(x_{i,l})) \prod_i P(s(x_{i0}, 0), 0).$$

Let $\widetilde{R}^{n, t-r} \subset \widetilde{R}^n$ be the collection of all sets $\{\Gamma_1, \dots, \Gamma_n\} \subset \widetilde{\mathcal{R}}^n$ whose paths Γ_i have a last intersection at time $t-\tau-1$, no two paths Γ_i and Γ_j intersecting at the times $t-\tau, t-\tau+1, \dots, t$. Thus,

$$\begin{aligned}
&\sum_{G \in \mathcal{R}^{(n)}} P(G, S_G) \prod \widetilde{P}_0(s(v), 0) \\
&\qquad = \prod P_{x_i}(s(x_i), t) - \sum_{\tau=0}^{t-2} \sum_{\{\Gamma_1, \dots, \Gamma_n\} \in \widetilde{\mathcal{R}}^{(n), t-\tau}} P(\Gamma_1, \dots, \Gamma_n) \\
&\qquad = \prod P_{x_i}(s(x_i), t) \\
&\qquad\quad - \sum_{\tau=0}^{t-2} \sum_{G_\tau, S_{G_\tau}} \sum_{\{y_i, s_i'\}} \prod a_{y_i}(s(x_{i,t-\tau}), t-\tau) p(s_i', t-\tau-1),
\end{aligned}$$

where G_τ is any graph in $\widetilde{R}(X, S_X, t-\tau)$, and the sum $\sum_{\{y_i, s_i'\}}$ is over all sets $\{y_i, s_i'\}$, with s_i' running over S, and the sets $(y_1, \dots, y_n)$, $y_n \in Q$ are such that at least two of the points $x_{i,t-\tau} + y_i$ coincide.

We claim that the last sum converges as $t \to \infty$ to the expression

$$\prod_i \pi(s(x_i)) - \sum_{\tau=0}^{\infty} \sum_{(G,S_G)\in\mathcal{R}^n(S_X,X,\tau)} P(G, S_G) \sum_{\{y_i,s_i'\}} \prod_i a_{y_i}(s_i(x_{i0}), s_i')\pi(s_i'), \tag{36}$$

where the sum over $\{y_i, s_i'\}$ is as above. Note that since we are summing over the sets $\{y_1, \dots, y_n\}$, the sum $\sum_{(G,S_G)}$ includes only graphs G with the property that at least two initial points are within a distance $\le \operatorname{diam} Q$ of each other. Thus the sum $\sum_{\tau=0}^{\infty} \sum_{(G,S_G)\in\mathcal{R}^n(S_X,X,\tau)}$ is dominated by $n(n+1)/2$ multiplied by the average number of times the random walk, beginning at the point $x_i - x_j$, enters the set Q prior to reaching the coordinate origin. Let $g_0(x, x')$ denote the average number of times a random walk starting at x' reaches the point x before reaching the origin; we recall that $g_0(x, x')$ is bounded by $g(x, x)$, and since for $\nu \ge 3$ a random walk is nonrecurrent, we have $g_0(x, x) < \sum p^t(0, 0) < \infty$ (see [40]). It follows easily from this that (35) converges to (36) as $t \to \infty$.

The contribution to $P_X(S_X, t)$ from graphs in the class $\mathcal{R}^k(X, t)$, $k < n$, converges as $t \to \infty$ to the sum

$$\begin{aligned} &\sum_{\tau=1}^{\infty} \sum_{(G,S_G)\in\widetilde{\mathcal{R}}^k(X,\tau)} P(G, S_G) \prod_i \pi(s_i(0)) \\ &\quad - \sum_{\tau_1=1}^{\infty} \sum_{\tau_2=0}^{\infty} \sum_{(G,S_G)\in\mathcal{R}^k(X,\tau_1,\tau_2)} P(G, S_G) \sum_{\{y_i,s_i'\}} \prod_i a_{y_i}(s_i(0), s_i')\pi(s_i'). \end{aligned} \tag{37}$$

Here $\widetilde{\mathcal{R}}^{(k)}(X, \tau) \subset \mathcal{R}^k(X, t)$ denotes the class of k-component graphs, which during the time interval $[1, \tau)$ have more than k components, and $\mathcal{R}^k(X, \tau_1, \tau_2) \subset \mathcal{R}^k(X, \tau_1 + \tau_2)$ is the class of k-component graphs having exactly k paths at each time τ_2, $\tau_2 + 1, \dots, \tau_2 + \tau_1$, but more than k paths prior to $t = \tau_2$.

Based on the above results, we arrive at the following:

THEOREM 6. *Suppose that an initial translationally invariant distribution P_0 satisfies condition* (30) *and that the chain L on S consists of finitely many classes of essentially aperiodic ergodic states. Then the limit of the correlation functions $P_X(S_X, t)$ as $t \to \infty$ exists and is given by equations* (36) *and* (37), *in which $\pi(s)$ is the invariant measure of the chain L to which the distributions $P(\overline{s}, t)$ on S, with initial distribution $P(\overline{s}, 0) = P_0(S(x) = \overline{s})$, converge. These limits*

$$P_X(S_X) = \lim_{t\to\infty} P_X(S_X, t)$$

define a translationally invariant stationary distribution of the chain on S^{Z^ν}. Conversely, for any such distribution the correlation functions are given by equations (36), (37) *for a suitably chosen invariant measure $\pi(s)$ of the chain L. Among all such stationary distributions there exists only one that satisfies condition* (30).

CHAPTER II

CONSTRUCTION OF AN EQUILIBRIUM DYNAMICS

An equilibrium dynamics is one in which small fluctuations occur about an "equilibrium" position. In the case of classical systems, this is primarily the dynamics of measures that are absolutely continuous with respect to some Gibbs measure, or the dynamics in a Hilbert space of functions on an infinite phase space which are square-integrable with respect to the Gibbs measure. For quantum systems, one has a dynamics in the GNS representation, constructed using limit Gibbs states (KMS states or ground states) on a suitable algebra of observables. In this chapter we shall, among other things, discuss in detail the Euclidean approach to the study of such a dynamics.

§1. Ground and temperature states

A C^*-dynamical system is a pair $(\mathfrak{A}, \alpha_t)$, where $\mathfrak{A}$ is a C^*-algebra and α_t a strongly continuous one-parameter group of automorphisms of $\mathfrak{A}$. The strong continuity means that for each fixed $A \in \mathfrak{A}$, the curve $\alpha_t(A)$ is continuous with respect to the norm on $\mathfrak{A}$.

A state $\langle\cdot\rangle$ is said to be invariant (relative to α_t) if for all $A \in \mathfrak{A}$ we have $\langle A\rangle = \langle\alpha_t(A)\rangle$. We will consider only invariant states with the following property.

DEFINITION 1. A state $\langle\cdot\rangle$ is said to be β-invariant (with respect to α_t), $0 \le \beta \le \infty$ if for all $A_1, A_2 \in \mathfrak{A}$ there exists a bounded analytic function $F_{A_1,A_2}(z)$ in the strip $0 < \operatorname{Im} z < \beta$ which is continuous on the closure of the strip and satisfies for all real $z = t$

1) $F_{A_1,A_2}(t) = \langle A_1\alpha_t(A_2)\rangle$,

2) $F_{A_1,A_2}(t + i\beta) = \langle\alpha_t(A_2)A_1\rangle$, for $0 \le \beta < \infty$.

When $\beta = \infty$ such a state is called a *ground* state, while for $\beta < \infty$ it is called a KMS (Kubo-Martin-Schwinger) state or a *temperature state* (with temperature β^{-1}).

REMARK 1. The requirement that F be bounded is superfluous for $\beta < \infty$ (see [7]).

Note that an equilibrium state with $\beta \neq 0$ is invariant under the dynamics α_t. Indeed, for $\beta < \infty$ the function $F_{E,A}(t) = \langle\alpha_t(A)\rangle$ can be continued periodically to the entire plane and hence must be constant (since it is bounded). If $\beta = \infty$, then for $A = A^*$ the function $F_{E,A}(t) = \langle\alpha_t(A)\rangle$ is real on the real axis, and by the reflection principle can be analytically continued into the lower halfplane; consequently it is constant, again by virtue of its boundedness. For arbitrary A, we use the decomposition $A = A_1 + iA_2$, where A_1 and A_2 are hermitian.

For $\beta = 0$, Definition 1 says that the state is a trace (i.e., $\langle A_1A_2\rangle = \langle A_2A_1\rangle$ for all $A_1, A_2 \in \mathfrak{A}$). It is invariant under any dynamics defined by inner automorphisms

$$\alpha_t(A) = U_tAU_t^{-1}, \qquad U_t \in \mathcal{A} \tag{1}$$

where $\{U_t\}$ is a group of elements in $\mathfrak{A}$, and also under any dynamics that can be "approximated" by inner automorphisms.

The basic objects of study in quantum statistical physics are the time correlation functions

$$\Gamma_{A_1,A_2,\dots,A_n}(t_1,\dots,t_n) = \langle \alpha_{t_1}(A_1)\alpha_{t_2}(A_2)\dots\alpha_{t_n}(A_n)\rangle, \tag{2}$$

where $\langle\cdot\rangle$ is a β-equilibrium state. Techniques from the spectral theory of operators are well suited for analyzing their large-time asymptotic behavior. To this end one considers the GNS representation π of the C^*-algebra $\mathfrak{A}$ corresponding to an invariant state $\langle\cdot\rangle$. The Hilbert space $\mathcal{H}_{\mathrm{GNS}}$ on which this representation acts is called the *physical* Hilbert space $\mathcal{H}_{\mathrm{phys}}$ (for the given state $\langle\cdot\rangle$ and dynamics α_t), and the cyclic vector $\Omega = [E]$ is called the *physical vacuum.* Because the state $\langle\cdot\rangle$ is invariant, the transformation $[A] \to [\alpha_t(A)]$ preserves the norm $\|\cdot\|_{\mathrm{phys}}$ on $\mathcal{H}_{\mathrm{phys}}$ (recall that $[A]$ is the image of the element $A \in \mathfrak{A}$ under the quotient map $\mathfrak{A} \to \mathfrak{A}/\mathcal{N}$, where $\mathcal{N}$ is the ideal of "null" elements (see 2.1)). Thus this map is well defined on $\mathcal{H}_{\mathrm{phys}}$ and extends to a strongly continuous unitary group U_t acting on $\mathcal{H}_{\mathrm{phys}}$. The strong continuity of U_t is a consequence of its weak continuity on the dense set $\pi(A)\Omega$, $A \in \mathfrak{A}$, the latter property following from the KMS condition. By Stone's theorem [36] the group U_t can be represented in the form $U_t = \exp\{itH\}$, where $H = H_{\mathrm{phys}}$ is a selfadjoint operator called the (physical) *Hamiltonian.* Recall that the representation $\pi\colon \mathfrak{A} \to \mathfrak{B}(\mathcal{H}_{\mathrm{phys}})$ is defined by the formula $\pi(A)[B] = [AB]$ (see 2.1).

LEMMA 1. *For all $A \in \mathfrak{A}$ and $t \in R^1$*

$$\begin{aligned} \pi(\alpha_t(A)) &= \exp\{itH_{\mathrm{phys}}\}\pi(A)\exp\{-itH_{\mathrm{phys}}\}, \\ \exp\{itH_{\mathrm{phys}}\}\Omega &= \Omega. \end{aligned} \tag{3}$$

PROOF. By definition,

$$U_t\pi(A)\Omega = \pi(\alpha_t(A))\Omega. \tag{4}$$

This gives the second equality in (3). For any $B \in \mathfrak{A}$ we have

$$\begin{aligned} U_t\pi(A)[B] &= U_t\pi(A)\pi(B)\Omega = U_t\pi(AB)\Omega \\ &= \pi(\alpha_t(AB))\Omega = \pi(\alpha_t(A)\alpha_t(B))\Omega \\ &= \pi(\alpha_t(A))\pi(\alpha_t(B))\Omega = \pi(\alpha_t(A))U_t\pi(B)\Omega \\ &= \pi(\alpha_t(A))U_t[B]. \end{aligned}$$

Thus, for any $\xi \in \mathcal{H}_{\mathrm{GNS}}$

$$U_t\pi(A)\xi = \pi(\alpha_t(A))U_t\xi,$$

which is the first equality in (3).

In terms of the operator H_{phys}, the correlation functions are of the form

$$\begin{aligned} &\Gamma_{A_1,A_2,\dots,A_n}(t_1,\dots,t_n) \\ &\quad = (\Omega, \pi(A_1)\exp\{i(t_2-t_1)H_{\mathrm{phys}}\}\pi(A_2)\dots\exp\{i(t_n-t_{n-1})H_{\mathrm{phys}}\}\pi(A_n)\Omega), \end{aligned} \tag{5}$$

if $t_n > t_{n-1} > \dots > t_1$.

We next give some convenient criteria for a state to be equilibrium, which will be used below to construct equilibrium states.

LEMMA 2. *An invariant state is a ground state if and only if* $H_{\text{phys}} \geq 0$.

PROOF. If $H_{\text{phys}} \geq 0$ then the representation (5) shows that $\langle A_1 \alpha_t(A_2)\rangle = \Gamma_{A_1 A_2}(0,t)$ is analytic and bounded in the upper halfplane. Conversely, if $E_{(-\infty,-\varepsilon)}\mathcal{H}_{\text{phys}} \neq 0$ for some $\varepsilon > 0$, where E_A is a spectral resolution of the identity for H_{phys}, then there exists an $A \in \mathfrak{A}$ such that $E_{(-\infty,-\varepsilon)}[A] \neq 0$. Then the function $F_{A,E}(t) = (\Omega, \exp\{itH_{\text{phys}}\}A\Omega)$ cannot be continued to a bounded function in the upper halfplane.

REMARK 2. It follows from (3) that $H_{\text{phys}}\Omega = 0$. Thus, if $\langle\cdot\rangle$ is a ground state, then its eigenvalue coincides with the lower bound of the spectrum of H_{phys}.

REMARK 3. We will occasionally consider a representation of the algebra $\mathfrak{A}$ and dynamics α_t on some Hilbert space $\mathcal{H}$ which is unitarily equivalent to the GNS representation, generated by a state $\langle\cdot\rangle$ invariant under the dynamics α_t. Such a representation will again be called a GNS representation, and the operator H (or H_{phys}) generating the group U_t will be called the Hamiltonian.

Equilibrium states on the algebra $\mathfrak{B}(\mathcal{H})$. Let $B = \mathfrak{B}(\mathcal{H})$ be the algebra of all bounded operators acting on a separable Hilbert space.

LEMMA 3. *Every dynamics* α_t *in* $\mathfrak{B}(\mathcal{H})$ *is of the form*

$$\alpha_t(A) = \exp\{itQ\}A\exp\{-itQ\}, \tag{6^a}$$

where Q *is a selfadjoint (in general, unbounded) operator on* $\mathcal{H}$.

We refer to [7] for the proof.

LEMMA 4. *For any trace-class operator* $\rho \geq 0$ *with* $\operatorname{Tr}\rho = 1$, *the formula*

$$\langle A\rangle = \operatorname{Tr}\rho A, \qquad A \in \mathfrak{B}(\mathcal{H}), \tag{6^b}$$

defines a state on $\mathfrak{B}(\mathcal{H})$, *and every state on* $\mathfrak{B}(\mathcal{H})$ *has a unique representation of the form* (6^b).

For the proof we refer to [46].

The operator ρ is generally called the *density matrix* for the state (6^b).

Let $\mathfrak{M}_n$ denote the algebra of matrices of order n.

LEMMA 5. *Every linear functional* $\rho(A)$ *on a matrix algebra* $\mathfrak{M}_n$ *with the properties*

$$\begin{gathered}\rho(A_1A_2) = \rho(A_2A_1), \qquad A_1A_2 \in \mathfrak{M}_n,\\ \rho(1) = 1,\end{gathered} \tag{7}$$

is of the form

$$\langle A\rangle = \frac{1}{n}\operatorname{Tr}A. \tag{8}$$

PROOF. It follows from (7) that $\langle A\rangle$ is the same for all similar matrices:

$$\langle SAS^{-1}\rangle = \langle A\rangle$$

and is consequently a function of the coefficients of the characteristic polynomial of A. By linearity, $\langle A\rangle = c\operatorname{Tr}A$; (8) now follows from the normalization condition (7).

LEMMA 6. *In order for the dynamics* (6^a) *in* $\mathfrak{B}(\mathcal{H})$ *to have a* β*-equilibrium state* $(0 < \beta < \infty)$*, it is necessary and sufficient that the operator* $e^{-\beta Q}$ *be of trace class. In this case the* β*-equilibrium state is unique and its density matrix is given by*

$$\rho = (\operatorname{Tr} e^{-\beta Q})^{-1} \exp\{-\beta Q\}. \tag{8'}$$

PROOF. We first prove that the condition is sufficient. A direct calculation shows that a state (6) with density matrix (8′) is a β-equilibrium state for the dynamics (6^a). We first prove uniqueness for the finite-dimensional case $\mathcal{H} = \mathbb{C}^n$, i.e., $\mathfrak{B}(\mathcal{H}) = \mathfrak{M}_n$. In this case the operators $\exp\{izQ\}$ are defined for all complex z; the function $F_{A_1,A_2}(z)$ is also defined for all z by condition 1) above, and condition 2) can be rewritten as

$$\langle A_1 \alpha_{z+i\beta}(A_2)\rangle = \langle \alpha_z(A_2) A_1\rangle$$

for all complex z. Setting $z = 0$, we obtain that

$$\langle A_1 e^{-\beta Q} A_2 e^{\beta Q}\rangle = \langle A_2 A_1\rangle.$$

Upon setting $A_3 = e^{-\beta Q} A_2$, we get that

$$\langle A_1 A_3 e^{-\beta Q}\rangle = \langle e^{\beta Q} A_3 A_1\rangle. \tag{9}$$

We now introduce the linear functional

$$\rho(A) = \frac{1}{\langle e^{\beta Q}\rangle} \langle e^{\beta Q} A\rangle,$$

which by (9) satisfies

$$\rho(e^{-\beta Q} A_1 A_3 e^{\beta Q}) = \rho(A_3 A_1). \tag{10}$$

Since ρ is invariant under the dynamics $(\alpha_z, z \in \mathbb{C})$, we get from (10) that

$$\rho(A_1 A_2) = \rho(A_2 A_1) \quad \text{and } \rho(\mathbf{1}) = 1,$$

i.e.,

$$\langle e^{\beta Q} A\rangle = c \operatorname{Tr} A, \tag{11}$$

where c is a constant. Finally, (8′) follows from (11).

Passing now to the case of an arbitrary separable infinite-dimensional Hilbert space $\mathcal{H}$, we note that since $e^{-\beta Q}$ is a trace-class operator, the spectrum of Q consists of isolated eigenvalues of finite multiplicity that tend to $+\infty$ and are such that for every $\lambda < \infty$, the space $\mathcal{H}_\lambda = E_\lambda \mathcal{H}$ is finite-dimensional, where E_λ is the spectral projection of Q. Since the spaces $\mathcal{H}_\lambda$ are invariant under Q, it follows easily from the preceding arguments that the operator $\rho_\lambda = E_\lambda \rho$ on $\mathcal{H}_\lambda$ is of the form $\rho_\lambda = \operatorname{const} \exp\{-\beta Q_\lambda\}$, where $Q_\lambda = E_\lambda Q$. This shows that ρ is of the form (8).

To prove the necessity of the condition of the lemma, we show that $\operatorname{Ker} \rho = \{0\}$. Assuming the contrary, let $\mathcal{H}_0 = \operatorname{Ker} \rho \neq \{0\}$. The decomposition

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1, \qquad \mathcal{H}_1 = \mathcal{H}_0^\perp$$

corresponds to a decomposition of any operator $A \in \mathfrak{B}(\mathcal{H})$ as a block matrix

$$A \sim \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix},$$

where $A_{00}\colon \mathcal{H}_0 \to \mathcal{H}_0$, $A_{01}\colon \mathcal{H}_1 \to \mathcal{H}_0$, and so on. The matrix for ρ is of the form

$$\rho = \begin{pmatrix} 0 & 0 \\ 0 & \rho_1 \end{pmatrix},$$

where ρ_1 is the part of ρ acting on $\mathcal{H}_1$, and the matrix for e^{itQ} is

$$e^{itQ} = \begin{pmatrix} e^{iQ_0 t} & 0 \\ 0 & e^{iQ_1 t} \end{pmatrix},$$

where Q_i is the part of Q acting on the invariant subspace $\mathcal{H}_i$, $i = 0, 1$ (recall that Q commutes with ρ). Computing $\langle B\alpha_t(A)\rangle$ and $\langle \alpha_t(A)B\rangle$, we further find that

$$\langle \alpha_t(A)B\rangle = \operatorname{Tr} \rho_1(e^{iQ_1 t}A_{10}e^{-iQ_0 t}B_{01} + e^{itQ_1}A_{11}e^{-itQ_1}B_{11}),$$
$$\langle B\alpha_t(A)\rangle = \operatorname{Tr} \rho_1(B_{10}e^{iQ_0 t}A_{01}e^{-iQ_1 t} + B_{11}e^{itQ_1}A_{11}e^{-itQ_1}).$$

We now get a contradiction by choosing $A_{01} = 0$ and $B_{11} = 0$, and A_{10} and B_{10} such that the function

$$\operatorname{Tr} \rho_1(B_{10}e^{iQ_0 t}A_{01}e^{-iQ_1 t}) \neq 0$$

of t is not identically zero. Furthermore, arguing as above, we obtain that $\rho = \operatorname{const} e^{-\beta Q}$, and hence $\operatorname{Tr} e^{-\beta Q} < \infty$. This proves the lemma.

The next lemma describes all the ground states for the dynamics (6^a).

LEMMA 7. *A necessary and sufficient condition for the dynamics* (6^a) *to have a ground state is that* Q *be bounded from below and that the infimum of its spectrum be an eigenvalue. Under these conditions the ground state is given by a density matrix* ρ *for which*

$$\operatorname{Ker} \rho \supseteq E_0^{\perp}, \quad \operatorname{Im} \rho \subset E_0,$$

where E_0 *is the eigensubspace corresponding to the eigenvalue* $\lambda_0 = \inf \sigma(Q)$*, and* $\sigma(Q)$ *denotes the spectrum of* Q*.*

The proof is similar to the proof of Lemma 6.

REMARK 4. If the dynamics (6^a) has a β-equilibrium state for some $\beta = \beta_0$ then it has a β-equilibrium state for all $\beta > \beta_0$, and this state satisfies the assumptions of Lemma 7. In this situation we have $\dim E_0 < \infty$ and the states $\langle\cdot\rangle_\beta$ converge as $\beta \to \infty$ to a ground state with density matrix

$$\rho = \frac{1}{\dim E_0} P_{E_0},$$

where P_{E_0} is the projection of $\mathcal{H}$ onto the subspace E_0.

Let H be a selfadjoint semibounded operator acting on a Hilbert space $\mathcal{H}$ and having a (normalized) eigenvector Φ_0 with eigenvalue λ_0 (i.e., $\lambda_0 = \inf \sigma(H)$). We consider on $\mathfrak{B}(\mathcal{H})$ the state

$$\langle A\rangle = (A\Phi_0, \Phi_0), \tag{11^a}$$

which by Lemma 7 is a ground state for the dynamics

$$\tau_t(A) = \exp\{itH\}A\exp\{-itH\} \tag{11^b}$$

on the algebra $\mathfrak{B}(\mathcal{H})$. As we have already seen (2.1), the GNS space $\mathcal{H}_{\mathrm{GNS}}$ constructed using the state (11^a) is canonically isomorphic to the Hilbert space $\mathcal{H}$, the isomorphism taking the element $[E] \in \mathcal{H}_{\mathrm{GNS}}$ into Φ_0, and $\pi(A) = A$ for every $A \in \mathfrak{B}(\mathcal{H})$.

LEMMA 8. *Under the above isomorphism between* $\mathcal{H}_{\mathrm{GNS}}$ *and* $\mathcal{H}$*, the operator* H_{phys} *on* $\mathcal{H}$ *goes into the operator* $H - \lambda_0$.

PROOF. According to equations (3), (11^b) the operator $\exp\{itH_{\mathrm{GNS}}\}\exp\{-itH\}$ commutes with every $A \in \mathfrak{B}(\mathcal{H})$, hence it must be a multiple of the identity operator, i.e., $\exp\{itH_{\mathrm{GNS}}\} = \alpha(t)\exp\{itH\}$, where $\alpha(t)$ is some function of t. Applying this equality to the vector Φ_0, we obtain from the equality $\exp\{itH_{\mathrm{GNS}}\}\Phi_0 = \Phi_0$ that $\alpha(t) = \exp\{i\lambda_0 t\}$, from which the statement of the lemma follows.

COROLLARY. *For the free dynamics* τ_t *on the algebras* $\mathfrak{B}(\mathcal{F}_{a,s}(\mathcal{H}))$ *constructed from the one-particle Hamiltonian* h *acting on the space* $\mathcal{H}$*, the operators* $H^{a,s}_{\mathrm{GNS}}$ *generated by the state*

$$\langle A\rangle = \langle A\Omega, \Omega\rangle, \qquad A \in \mathfrak{B}(\mathcal{F}_{a,s}(\mathcal{H})),$$

and acting on the spaces $\mathcal{F}_{a,s}(\mathcal{H})$ *are given by*

$$H^{a,s}_{\mathrm{GNS}} = d\Gamma^{a,s}(h).$$

Here $\Omega \in \mathcal{F}_{a,s}(\mathcal{H})$ *denotes the vacuum vector.*

Thermodynamic limit of equilibrium states. Let a quasilocal structure be given on the C^*-algebra $\mathfrak{A}$, i.e., $\mathfrak{A}$ is the closure of the inductive limit $(\mathfrak{A}_0 = \cup\mathfrak{A}_\Lambda)$ of "local subalgebras" $\mathfrak{A}_\Lambda$ labeled by the elements of an ordered index set $\{\Lambda\}$, such that for $\Lambda_1 < \Lambda_2$ we are given a homomorphic and isometric imbedding $\mathfrak{A}_{\Lambda_1} \to \mathfrak{A}_{\Lambda_2}$. In such a case the dynamics α_t and its equilibrium states on the algebra $\mathfrak{A}$ are often constructed by taking the thermodynamic limit of the dynamics $\alpha_\Lambda(t)$ and their equilibrium states $\langle\cdot\rangle_\Lambda$ defined on the local algebras $\mathfrak{A}_\Lambda$. More precisely, we have the next lemma.

LEMMA 9. *Assume that for every local algebra* $\mathfrak{A}_\Lambda$ *there are defined a dynamics* $\alpha^\Lambda_t\colon \mathfrak{A}_\Lambda \to \mathfrak{A}_\Lambda$ *and a* β*-equilibrium state* $\langle\cdot\rangle_\beta$ $(0 < \beta \le \infty)$*. Suppose, moreover, that the limits*

$$\lim_{\Lambda\uparrow}\langle A\rangle_\Lambda = \langle A\rangle \tag{12}$$

and

$$\lim_{\Lambda\uparrow}\alpha^\Lambda_t(A) = \alpha_t(A), \tag{13}$$

exist for every local element A*, where the convergence in* (13) *is with respect to the norm and is uniform in* t *in an interval* $|t| < t_0$*, where* t_0 *does not depend on* A*. Then the transformations* $\alpha_t(A)$ *can be extended (by continuity) to the whole algebra* $\mathfrak{A}$*, and can also be defined for all values* $t \in R^1$ *(using the group law* $\alpha_{t_1+t_2} = \alpha_{t_1}\alpha_{t_2}$*) in such a way that the transformations determine a dynamics on the algebra* $\mathfrak{A}$*. Moreover, the state* (12) *(extended to all of* $\mathfrak{A}$*) is a* β*-equilibrium state for the dynamics* α_t.

The straightforward proof will be omitted.

REMARK 5. If the quasilocal algebra $\mathfrak{A}$ is contained in $\mathfrak{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, and convergence in (13) is taken in the strong operator topology on $\mathfrak{B}(\mathcal{H})$ (i.e., $\alpha^\Lambda_t(A)X \to \alpha_t(A)X$ with respect to the norm on $\mathcal{H}$ for every $X \in \mathcal{H}$), then the limit transformation $\alpha_t(A)$, $A \in \mathfrak{A}$, is known to be a dynamics on $\mathfrak{A}$, and the limit state $\langle\cdot\rangle$ on $\mathfrak{A}$ is also a β-equilibrium state for α_t.

LEMMA 10. *Assume as in Lemma* 9 *that for each local algebra* $\mathfrak{A}_\Lambda \subset \mathfrak{A}$ *we have a dynamics* α_t^Λ *with a* β*-equilibrium state* $\langle\cdot\rangle_\Lambda$ *such that conditions* (12) *and* (13) *are satisfied. Suppose, moreover, that GNS representations* π_Λ *of the algebras* $\mathfrak{A}_\Lambda$ *and* U_t^Λ *of the dynamics* α_t^Λ *are given which all act on the same Hilbert space* $\mathcal{H}$*. Suppose that the limits*

$$\lim_{\Lambda\uparrow} \pi_\Lambda(A) = \pi(A), \quad A \in \mathfrak{A}, \tag{14}$$

(*with respect to the operator norm on* $\mathfrak{B}(\mathcal{H})$) *exist for every local element* $A \in \mathfrak{A}$ *and that*

$$\lim_{\Lambda\uparrow} U_t^\Lambda = U_t \quad \text{for all } t. \tag{15}$$

Then $\pi(A)$ *defines a GNS representation of* $\mathfrak{A}$ *and is generated by the limit equilibrium state* (12), *and the group* U_t *is a GNS representation of the dynamics* (13) (*generated by the state* (12)).

We will omit the proof of this lemma.

REMARK. As in the previous lemma, the norm convergence in (14) and (15) can be replaced by strong convergence. In that case, if π is known to be a representation of $\mathfrak{A}$ in $\mathfrak{B}(\mathcal{H})$ and the U_t form a unitary group of operators on $\mathcal{H}$, then it follows that π and U_t give a GNS representation of the algebra $\mathfrak{A}$ and dynamics α_t relative to the state (12).

§2. Ground state for an infinite system of harmonic oscillators

Here we will analyze a simple example of a system whose ground state can be explicitly computed.

Consider the quadratic form

$$H = \sum_{x\in X} \pi_x^2 + \frac{1}{4} \sum_{x_1,x_2\in X} a_{x_1,x_2}\Phi_{x_1}\Phi_{x_2}, \tag{1}$$

where $\pi_x = \pi(f_x)$, $\Phi_x = \Phi(f_x)$ are selfadjoint operators on the Fock space $\mathcal{F}_s(\mathcal{H})$ obtained from the creation and annihilation operators $a_x^* = a^*(f_x)$ and $a_x = a(f_x)$ by formulas (13.3.1); $\{f_x,\ x \in X\}$ is an orthonormal basis in $\mathcal{H}$ (X is a countable set), and $A = \{a_{x_1,x_2},\ x_1, x_2 \in X\}$ is a symmetric positive matrix. As we saw in 5.1, this form defines a dynamics τ_t on the Weyl algebra $\mathfrak{A}(\mathcal{H}) \subset \mathfrak{B}(\mathcal{F}_s(\mathcal{H}))$ which is a limit of finite-dimensional dynamics τ_t^Λ as $\Lambda \uparrow X$, $\Lambda \subset X$ a finite subset of X:

$$\tau_t^\Lambda(A) = \exp\{itH_\Lambda\} A \exp\{-itH_\Lambda\}, \tag{2}$$

where

$$H_\Lambda = \sum_{x\in\Lambda} \pi_x^2 + \frac{1}{4} \sum_{x_1,x_2\in\Lambda} a_{x_1,x_2}\Phi_{x_1}\Phi_{x_2}. \tag{3}$$

As we shall see below, the operator H_Λ has a ground-state vector Φ_0^Λ which defines a ground state

$$\langle A\rangle_\Lambda = (A\Phi_0^\Lambda, \Phi_0^\Lambda) \tag{4}$$

relative to the dynamics (2) on the algebra $\mathfrak{B}(\mathcal{F}_s(\mathcal{H}_\Lambda))$.

LEMMA 1. *For every local element D of the Weyl algebra $\mathfrak{A}(\mathcal{H})$ the limit*

$$\lim_{\Lambda\uparrow X}\langle D\rangle_\Lambda = \langle D\rangle, \tag{5}$$

exists and defines a state on $\mathfrak{A}(\mathcal{H})$. It is a ground state for the dynamics τ_t.

PROOF. Define a representation of the operators $\{\pi_x,\ \Phi_x, x\in\Lambda\}$ on the space $L_2(R^\Lambda,\ d^{|\Lambda|}x)$ by

$$\pi_x f = i\frac{df}{dq_x}, \qquad \Phi_x f = q_x f, \qquad f\{q_x, x\in\Lambda\}\in L_2(R^\Lambda, d^{|\Lambda|}x).$$

Then the operator H takes the form

$$H_\Lambda f = -\sum_{x\in\Lambda}\frac{d^2 f}{dq_x^2} + \frac{1}{4}\Big(\sum_{x_1,x_2\in\Lambda} a_{x_1,x_2}q_{x_1}q_{x_2}\Big)f.$$

Note that the orthogonal change of variables

$$q_x = \sum_{x'\in\Lambda} U^\Lambda_{x,x'}q'_{x'}, \qquad x\in\Lambda, \tag{6}$$

takes H_Λ into a sum of operators:

$$H_\Lambda f = -\sum_x \frac{d^2 f}{dq'^2_x} + \frac{1}{4}\Big(\sum_x \lambda^\Lambda_x (q'_x)^2\Big)f, \tag{6^a}$$

where the $\lambda^\Lambda_x > 0$ are the eigenvalues of the matrix $A^\Lambda = \{a_{x,x'},\ x,x'\in\Lambda\}$.

The ground-state vector for such an operator is given by

$$\Phi_0 = \text{const}\exp\Big\{-\frac{1}{4}\sum \lambda_x^{1/2}(q'_x)^2\Big\}. \tag{7}$$

Returning to the original variables q_x, we obtain that

$$\Phi_0 = \text{const}\exp\Big\{-\frac{1}{4}\sum_{x,x'\in\Lambda} b^\Lambda_{x,x'}q_x q_{x'}\Big\}, \tag{7$'$}$$

where $B^\Lambda = \{b^\Lambda_{x,x'}\} = (A^\Lambda)^{1/2}$.

Next, by the commutation relations (14.3.1) we can write any monomial in the Weyl algebra in the form

$$D = U_{x_1}\dots U_{x_k}V_{x'_1}\dots V_{x'_{k'}} = \exp\Big\{i\sum_{y\in Y} n_y\Phi_y\Big\}\exp\Big\{i\sum_{y'\in Y'} s_{y'}\pi_{y'}\Big\}, \tag{8}$$

where $\{y_i\}$ is the set of distinct points in the collection $\{x_1,\dots,x_k\}$, and n_y is the multiplicity of y in $\{x_1,\dots,x_k\}$; and similarly for Y' and $\{s_y\}$. We will henceforth assume that the set Λ contains Y and Y', and to facilitate the computations we express D as a product

$$D = \exp\Big\{i\sum_{y\in\Lambda} n_y\Phi_y\Big\}\exp\Big\{i\sum_{y'\in\Lambda} s_{y'}\pi_{y'}\Big\},$$

where we set $n_y = 0$ for those $y \in \Lambda$ not contained in Y (and similarly for $s_{y'}$). For the above representations of Φ_y and π_y on $L_2(R^\Lambda, d^{|\Lambda|}x)$, the action of the operators $\exp\{i\Sigma n_y \Phi_y\}$ and $\exp\{i\Sigma s_{y'}\pi_{y'}\}$ is given by the formulas

$$\exp\left\{i\sum n_y\Phi_y\right\}f = \prod_{y\in\Lambda}\exp\{iq_y n_y\}f,$$

$$\exp\left\{i\sum s_{y'}\pi_{y'}\right\}f = f\{q_y - s_y, y\in\Lambda\}.$$

Thus,

$$\begin{aligned}(D\Phi_0^\Lambda, \Phi_0^\Lambda) = \text{const}\int_{R^\Lambda}\exp\left\{i\sum_{x\in\Lambda} iq_x n_x\right\}\\ \times\exp\left\{-\frac{1}{4}\sum_{x_1,x_2} b^\Lambda_{x_1,x_2}(q_{x_1}-s_{x_1})(q_{x_2}-s_{x_2})\right\}\\ \times\exp\left\{-\frac{1}{4}\sum_{x_1,x_2} b^\Lambda_{x_1,x_2}q_{x_1}q_{x_2}\right\}\prod_{x\in\Lambda}dq_x.\end{aligned}$$

The evaluation of the integral is straightforward and leads to the final formula

$$\begin{aligned}\langle D\rangle_\Lambda = \exp\Big\{-\frac{1}{2}\sum_{y_1,y_2\in Y} c^\Lambda_{y_1,y_2}n_{y_1}n_{y_2}\\ -\frac{1}{8}\sum_{y_1,y_2\in Y'} b^\Lambda_{y_1,y_2}s_{y_1}s_{y_2} + \frac{i}{2}\sum_{y\in Y\cap Y'} n_y s_y\Big\},\end{aligned}\tag{9}$$

where $C^\Lambda = \{c^\Lambda_{y_1,y_2}\} = (B^\Lambda)^{-1} = (A^\Lambda)^{-1/2}$.

We see from (9) that the limit $\lim_{\Lambda\uparrow X}\langle\cdot\rangle_\Lambda$ exists and is given by equation (9), with the matrices C^Λ and B^Λ replaced respectively by $C = A^{-1/2}$ and $B = A^{1/2}$. The lemma is proved.

The change of variable $(\lambda_x^\Lambda)^{1/4}q'_x = q''_x$ takes the operator H_Λ in (6^a) into

$$H_\Lambda = \sum \lambda_x^{1/2}(a_x^\Lambda)^* a_x^\Lambda + \text{const},$$

where

$$\alpha_x^\Lambda f = \frac{1}{\sqrt{2}}q''_x f + \frac{df}{dq''_x}, \qquad (\alpha_x)^* f = \frac{1}{\sqrt{2}}q''_x f - \frac{df}{dq'_x}.$$

Passing now to the operators $\overline{a}_x^\Lambda = \sum_{x'}\overline{U}^\Lambda_{x,x'}a^\Lambda_{x'}$ and $(\overline{a}_x^\Lambda)^* = \sum_{x'}\overline{U}^\Lambda_{x,x'}(a^\Lambda_{x'})^*$, where $\overline{U}^\Lambda = \{\overline{U}^\Lambda_{x,x'}\}$ is the orthogonal matrix inverse to U_Λ appearing in equation (6), we get finally that

$$H_\Lambda = \sum_{x',x} b^\Lambda_{x,x'}(\overline{a}_x^\Lambda)^*\overline{a}^\Lambda_{x'} + \text{const}\,.$$

From this and the corollary to Lemma 8 in §1, it follows that the operator H^Λ_{GNS} constructed from the ground state for the dynamics (2) coincides with the operator

$$\overline{H}_\Lambda = \sum_{x',x\in\Lambda} b^\Lambda_{x,x'}a_x a_{x'},$$

acting on the space $\mathcal{F}_s(\mathcal{H})$.

Using further Lemma 8 of §1, we arrive at the next result.

LEMMA 2. *Let τ_t be the limit dynamics on the Weyl algebra $W(\mathcal{H}) \subset \mathfrak{B}(\mathcal{F}_s(\mathcal{H}))$ constructed by equation* (1), *and let the limit state $\langle\cdot\rangle$ on $W(\mathcal{H})$ (see* (5)) *be a ground state for τ_t. Then τ_t has a GNS representation relative to $\langle\cdot\rangle$ which acts on the space $\mathcal{F}_s(\mathcal{H})$ and coincides with the free dynamics whose Hamiltonian is*

$$H = \sum_{x',x\in X} b_{x,x'} a_x^* a_x,$$

and the matrix $B = \{b_{x,x}\} = A^{1/2}$.

§3. A free quasistate

We recall that a quasistate on an algebra $\mathfrak{A}$ with unit 1 is a linear functional $\langle\cdot\rangle$ on $\mathfrak{A}$ such that $\langle 1\rangle = 1$. An even quasistate on the CAR superalgebra $\mathfrak{A}(\mathcal{H})$ (i.e., one for which $\langle A\rangle = 0$ for all odd elements $A \in \mathfrak{A}(\mathcal{H})$) is said to be *Gaussian* if

$$\begin{gathered}\langle a^{\#}(f_1)\dots d^{\#}(f_{2k})\rangle = \sum(-1)^{|\pi|}\prod\langle a^{\#}(f_{i_s})a^{\#}(f_{j_s})\rangle,\\ a^{\#}(f) = a^*(f) \quad \text{or } a(f),\end{gathered} \tag{1}$$

where the sum is over all partitions of the index set $\{1,\dots,2k\}$ into k pairs (i_1,j_1), $(i_2,j_2),\dots,(i_k,j_k)$, where $i_s < j_s$, and $|\pi|$ is the sign of the permutation $\pi = (i_1,j_1,i_2,j_2,\dots,i_k,j_k)$ (see [26]).

A quasistate on a CAR algebra is said to be gauge invariant if it is nonzero only on monomials that contain the same number of creation and annihilation operators. In particular, a Gaussian quasistate is gauge invariant if and only if

$$\langle a(f)a(g)\rangle = \langle a^*(f)a^*(g)\rangle = 0 \tag{2}$$

for all pairs $f, g \in \mathcal{H}$.

A Gaussian gauge-invariant quasistate on a CAR algebra is called a *free quasistate.* Obviously, such a state is determined by its values on the monomials $a^*(f)a(g)$,

$$\langle a^*(f)a(g)\rangle = (Bf,g), \tag{3}$$

where B is a bounded operator on $\mathcal{H}$. It follows from (3) that

$$\langle a(g)a^*(f)\rangle = ((E-B)f,g). \tag{4}$$

LEMMA 1. *The free quasistate defined by the operator B is a state if and only if B is selfadjoint and*

$$0 \le B \le E. \tag{5}$$

PROOF. The necessity follows from the equality

$$\langle a^*(f)a(g)\rangle = \overline{\langle a^*(g)a(f)\rangle}$$

and positivity:

$$\langle a^*(f)a(f)\rangle \ge 0, \quad \langle a(f)a^*(f)\rangle \ge 0.$$

In the case of a one-dimensional space $\mathcal{H} = \mathbb{C}^1$, every nonconstant positive even element in the algebra $\mathfrak{A}(\mathcal{H})$ has the form

$$caa^* + ba^*a, \qquad c \geq 0, \quad b \geq 0,$$

and condition (5) means that quasistates are positive on such elements. Now let the dimension of $\mathcal{H}$ be arbitrary, and take B to be a finite-rank selfadjoint operator satisfying the condition (5). We can split $\mathcal{H}$ into an orthogonal direct sum

$$\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_s \oplus \mathcal{H}_0,$$

where $\mathcal{H}_0 = \operatorname{Ker} B$ and the $\mathcal{H}_i$ are one-dimensional subspaces spanned by the eigenvectors of B, with eigenvalues λ_i, $0 < \lambda_i \leq 1$. Then the CAR algebra is expressible as a tensor product of superalgebras (see [23]):

$$\mathfrak{A}(\mathcal{H}) = \mathfrak{A}(\mathcal{H}_1) \otimes \cdots \otimes \mathfrak{A}(\mathcal{H}_s) \otimes \mathfrak{A}(\mathcal{H}_0),$$

and the quasistate $\langle\cdot\rangle$ is a tensor product of quasistates on the $\mathfrak{A}(\mathcal{H}_i)$, $i = 0, 1, \ldots, s$; the quasistate $\langle\cdot\rangle_0$ on $\mathfrak{A}(\mathcal{H}_0)$ is nonzero only on $1 \in \mathfrak{A}(\mathcal{H}_0)$ and is therefore a state, while the quasistates on $\mathfrak{A}(\mathcal{H}_i)$, $i = 1, \ldots, s$, are states by what we have proved above. Since a tensor product of states is again a state (this holds for even states under tensor product of superalgebras), we see that $\langle\cdot\rangle$ is a state. Any selfadjoint operator B satisfying (5) can be approximated in the weak topology by finite-rank operators B_n that satisfy (5). Since the states $\langle\cdot\rangle_B$ approximate the quasistate $\langle\cdot\rangle$, it follows that the latter is a state.

LEMMA 2. *Let the algebra $\mathfrak{A}(\mathcal{H})$ be equipped with a free dynamics τ_t which is generated by a one-particle Hamiltonian h acting on $\mathcal{H}$. Then the free quasistate $\langle\cdot\rangle_B$ defined by the operator B is invariant under τ_t if and only if H commutes with B.*

The proof follows from equations (6.4.1) and (3).

Wick monomials on a CAR algebra. Let a Gaussian quasistate (not necessarily gauge invariant) be defined on the CAR algebra $\mathfrak{A}(\mathcal{H})$. Then, just as in the case of functions of a Gaussian system of random variables (see [26]), we can introduce the Wick ordering operation $:\cdot:$ on the algebra $\mathfrak{A}(\mathcal{H})$. Namely, given monomials $a^\#(f_1)\ldots a^\#(f_n) \equiv a^\#_T$, where $T = \{f_1, \ldots, f_n\}$ is an ordered set of elements $f_i \in \mathcal{H}$, we define $:\cdot:$ inductively by the formula

$$: a^\#_T := \sum : a^\#_{T'} : \langle a^\#_{T\setminus T'}\rangle (-1)^{\pi(T,T')}, \tag{6}$$

where the sum is over all subsets $T' \subseteq T$ (ordered as before by inclusion), the subset $T \setminus T'$ is the complement of T' in T, and $\pi(T, T')$ is the sign of the permutation $T \to (T \setminus T', T')$. The operation $:\cdot:$ extends by linearity to all elements in $\mathfrak{A}(\mathcal{H})$.

Properties of the Wick operation.

I. We have the following formula

$$: a^\# := \sum a^\#_{T'} \langle a^\#_{T\setminus T'}\rangle (-1)^{\pi(T,T') + \frac{|T - T'|}{2}} \tag{7}$$

and its inversion

$$a^\# = \sum_{\substack{T' \subseteq T \\ |T\setminus T'| \text{ even}}} : a^\#_{T'} : \langle a^\#_{T\setminus T'}\rangle (-1)^{\pi(T,T')}.$$

(Note that $\langle a^{\#}_{T\setminus T}\rangle \neq 0$ only if $|T-T'|$ is even.)

II. For any two elements A_1, A_2 in $\mathfrak{A}(\mathcal{H})$,

$$(:A_1A_2:)^* =: A_2^*A_1^*: . \tag{8}$$

III. For any two-ordered sets

$$T=(f_1,\dots,f_n), \qquad S=(g_1,\dots,g_m), \tag{9}$$

$$\langle: a^{\#}_T :: a^{\#}_S :\rangle = \begin{cases} 0, & |T|\neq|S|, \\ \sum_\pi (-1)^{|\pi|}\langle a^{\#}(f_1)a^{\#}_{g_{\pi(1)}}\rangle\dots\langle a^{\#}(f_n)a^{\#}_{g_{\pi(n)}}\rangle, \end{cases}$$

where the sum is over all permutations $\pi = \begin{pmatrix} 1 & \dots & n \\ \pi(1) & \dots & \pi(n)\end{pmatrix}$ and $|\pi|$ is the sign of π.

Many other properties of the Wick ordering for a Gaussian system of random variables (see [26]) also carry over to the case of the algebra $\mathfrak{A}(\mathcal{H})$ by using the rule of signs.

Wick ordering can also be defined for Gaussian quasistates on a Grassmann algebra (see [26]) by means of equations (6)–(9) (the Grassmann algebra can also be viewed as a subalgebra of the CAR generated solely by creation operators).

We note that if a free quasistate $\langle\cdot\rangle_B$ is invariant under a free dynamics τ_t on the algebra $\mathfrak{A}(\mathcal{H})$ generated by a one-particle Hamiltonian h, then τ_t acts on the Wick monomials $:a^{\#}(f_1)\dots a^{\#}(f_n):$ by the formula

$$\tau_t(:a^{\#}(f_1)\dots a^{\#}(f_n):) =: a^{\#}(e^{ith}f_1)\dots a^{\#}(e^{ith}f_n): . \tag{9^a}$$

This follows immediately from equation (6.4.1) for the action of the free dynamics on the monomials $a^{\#}_T$, and equation (7).

We now compute the GNS representation of the algebra $\mathfrak{A}(\mathcal{H})$ generated by a free state $\langle\cdot\rangle_B$ which is invariant under the free dynamics; we will confine ourselves to the case when the eigenvectors $\{e_1,\dots,e_n,\dots\}$ of the operator B form a basis in $\mathcal{H}$.

If

$$\begin{aligned} T&=(e_{i_1},e_{i_2},\dots,e_{i_k}), \quad i_1<i_2<\dots<i_k, \\ S&=(e_{j_1},e_{j_2},\dots,e_{j_m}), \quad j_1<j_2<\dots<j_m \end{aligned}$$

are any two finite subsets of vectors in the basis, we introduce the Wick monomials

$$:a^{\#}_T a_S: \tag{10}$$

where

$$a^*_T = a^*(e_{i_1})\dots a^*(e_{i_k}), \qquad a_S = a(e_{j_1})\dots a(e_{j_m}).$$

Lemma 3. *Relative to the inner product*

$$(A,C)=\langle C^*A\rangle_B, \quad A,C\in\mathfrak{A}(\mathcal{H})$$

the monomials (10) *with distinct pairs* (T,S) *and* (T',S') *are orthogonal, and the squared norm of a monomial* (10) *is equal to*

$$(:a^{\#}_T a_S:,:a^{\#}_T a_S:) = \prod_{s=1}^{k}(1-\lambda_{i_s})\prod_{p=1}^{m}\lambda_{j_p}, \tag{11}$$

where λ_i is the eigenvalue of B corresponding to the eigenvector e_i.

The proof follows from a direct application of properties II and III above.

We see from (11) that the monomials $:a_T^{\#}a_S:$ have nonzero norm if and only if the set T contains no eigenvectors of B with eigenvalue 1, and S contains no eigenvectors of B with eigenvalue 0. Thus, the space $\mathcal{H}_{\mathrm{GNS}}$ is spanned by the monomials $:a_T^{\#}a_S:$, where T consists of eigenvectors of B belonging to the space

$$\mathcal{H}_1 = \mathcal{H} \ominus \mathrm{Ker}(E - B), \tag{12}$$

and S consists of eigenvectors of B contained in the space

$$\mathcal{H}_2 = \mathcal{H} \ominus \mathrm{Ker}\, B. \tag{13}$$

This shows that the GNS representation can be realized on a tensor product

$$\mathcal{F}_a(\mathcal{H}_1) \otimes \mathcal{F}_a(\mathcal{H}_2^*) = \mathcal{F}_a(\mathcal{H}_1 \oplus \mathcal{H}_2^*), \tag{13^a}$$

of Fock spaces (the tensor product being considered in the sense of superspaces, see [23]), if to each normalized monomial

$$:a_T^{\#}a_S: \subset \mathcal{H}_{\mathrm{GNS}}, \qquad T = (e_{i_1}, \dots, e_{i_k}), \quad S = (e_{j_1}, \dots, e_{j_m})$$

we associate the vector

$$\begin{aligned} &b_1^*(e_{i_1}^{(1)}) \dots b_1^*(e_{i_k}^{(1)})\Omega_1 \otimes b_2^*(e_{j_1}^{(2)}) \dots b_2^*(e_{j_m}^{(2)})\Omega_2 \\ &\qquad = \widehat{b}^*(e_{i_1}^{(1)}) \dots \widehat{b}^*(e_{i_k}^{(1)})\widehat{b}(e_{j_1}^{(2)}) \dots \widehat{b}(e_{j_m}^{(2)})\widehat{\Omega}. \end{aligned}$$

Here $\mathcal{H}_2^*$ is the Hilbert space obtained from $\mathcal{H}_2$ by changing the inner product:

$$(f, g)_{\mathcal{H}_2^*} = \overline{(f, g)}_{\mathcal{H}_2} = (g, f)_{\mathcal{H}_2}, \tag{13^b}$$

and $\Omega_1, \Omega_2, \widehat{\Omega} = \Omega_1 \otimes \Omega_2$ are the vacuum vectors in the Fock spaces $\mathcal{F}_a(\mathcal{H}_1)$, $\mathcal{F}_a(\mathcal{H}_2^*)$, and $\mathcal{F}_a(\mathcal{H}_1 \oplus \mathcal{H}_2^*)$, respectively; $\{b_1^*(f), f \in \mathcal{H}_1\}$, $\{b_2^*(f), f \in \mathcal{H}_2\}$, and $\{\widehat{b}^*(f), f \in \mathcal{H}_1 \oplus \mathcal{H}_2^*\}$ are the ordinary creation operators acting on the respective spaces; and $e_i^{(1)}$ and $e_j^{(2)}$ denote vectors in the basis $\{e_i\}$ in $\mathcal{H}$, regarded as elements of the spaces $\mathcal{H}_1$ and $\mathcal{H}_2^*$, respectively (and hence as elements of $\mathcal{H}_1 \oplus \mathcal{H}_2^*$).

Assume further that a free dynamics τ_t is defined on the algebra $\mathfrak{A}(\mathcal{H})$ which is generated by a one-particle Hamiltonian h commuting with the operator B (and thus the free state $\langle\cdot\rangle_B$ is invariant under τ_t). We may assume that the eigenvectors $\{e_i\}$ of B introduced above are also eigenvectors for h.

Then by (9^a), the monomials $:a_T^*a_S:$ are eigenvectors for the automorphisms τ_t

$$\tau_t(:a_T^*a_S:) = \prod_{p=1}^{k} \exp\{-it\lambda_p\} \prod_{s=1}^{m} \exp\{it\lambda_s\} :a_T^*a_S:, \tag{14}$$

where $T = (e_{i_1}, \dots, e_{i_k})$, $S = (e_{j_1}, \dots, e_{j_m})$, and λ_i is the eigenvalue of h corresponding to the vector e_i. It follows from (14) that the monomial $:a_T^*a_S:$ is an eigenvector for $\mathcal{H}_{\mathrm{GNS}}$ with eigenvalue $\sum_{p=1}^{k} \lambda_{i_p} - \sum_{s=1}^{m} \lambda_{j_s}$.

Regarded on the space $\mathcal{F}_a(\mathcal{H}_1 \oplus \mathcal{H}_2^*)$ described above, the operator H_{GNS} thus coincides with the operator

$$H_{\mathrm{GNS}} = \sum_{e_i^{(1)} \in \mathcal{H}_1} \lambda_i \widehat{b}^*(e_i^{(1)})\widehat{b}(e_i^{(1)}) + \sum_{e_j^{(2)} \in \mathcal{H}_2^*} (-\lambda_j)\widehat{b}^*(e_j^{(2)})\widehat{b}(e_j^{(2)}) = d\Gamma(\widehat{h}), \tag{15}$$

where $\widehat{h} = h_1 \oplus (-h_2)$, and h_1 and h_2 are the parts of the operator h acting on the invariant subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

All the results obtained above for the case of an operator B with a pure-point spectrum also remain valid in general, i.e., we have the following lemma.

LEMMA 4. *Let the algebra $\mathfrak{A}(\mathcal{H})$ admit a free state $\langle\cdot\rangle_B$ which is defined by the operator B and is invariant under the dynamics τ_t generated by a Hamiltonian h that commutes with B. Then the GNS representation of $\mathfrak{A}(\mathcal{H})$ generated by the state $\langle\cdot\rangle_B$ can be defined on the Fock space $\mathcal{F}_a(\mathcal{H}_1 \oplus \mathcal{H}_2^*)$, where the subspaces $\mathcal{H}_1$ and $\mathcal{H}_2^*$ are defined by equations* (12), (13), *and* (13^b), *in such a way that the operator H_{GNS} corresponding to the dynamics τ_t has the form* (15).

Equilibrium states for a free dynamics. Let a free dynamics τ_t on the CAR algebra $\mathfrak{A}(\mathcal{H})$ be defined by a one-particle Hamiltonian h.

LEMMA 5. *For each β, $0 < \beta < \infty$, there exists a unique β-equilibrium state for the free dynamics τ_t; it is free and defined by the operator*

$$B = (E + e^{\beta h})^{-1}. \tag{16}$$

PROOF. We first show that the free state $\langle\cdot\rangle_B$ with B given by (16) is a β-equilibrium state. Setting $A_1 = a^*(f)$ and $A_2 = a(g)$, we have

$$F_{A_1,A_2}(t) = \langle a^*(f)a(e^{ith}g)\rangle = (e^{-ith}(E + e^{\beta h})^{-1}f, g).$$

This function can be extended to the strip $0 < \operatorname{Im} z \leq \beta$, and for $z = t + i\beta$ it is equal to

$$F_{A_1,A_2}(t + i\beta) = (e^{-ith}e^{\beta h}(E + e^{\beta h})^{-1}f, g). \tag{17}$$

On the other hand, we get from (4.3) that

$$\langle\tau_t(A_2)A_1\rangle = (e^{-ith}(E - (E + e^{\beta h})^{-1})f, g),$$

which coincides with (17). In a similar way one verifies the KMS condition (2) for the case when $A_1 = a(g)$ and $A_2 = a^*(f)$. In the general case, we have $A_1 =: a_T^{\#} :$ and $A_2 =: a_S^{\#} :$, where $T = (f_1, \ldots, f_k)$ and $S = (g_1, \ldots, g_s)$, and the ordering $: \cdot :$ is with respect to the state $\langle\cdot\rangle_B$. The function $F_{A_1,A_2}(t) = \langle A_1\tau_t(A_2)\rangle_B$ is then expressible as a sum of products of the functions $\langle a^{\#}(f_k)a^{\#}(e^{ith}g_j)\rangle_B$, each of which extends into t in the strip $0 < \operatorname{Im} z \leq \beta$ and coincides for $z = t + i\beta$ with the function $\langle a^{\#}(e^{ith}g_j)a^{\#}(f_k)\rangle_B$. On the other hand, when $k+s$ is even, the function $\langle\tau_t(A_2)A_1\rangle$ is given by exactly the same sum of products of the functions $\langle a^{\#}(e^{ith}g_j)a^{\#}(f_k)\rangle_B$. Thus the free state defined by an operator B of the form (16) is β-equilibrium.

We next prove that this state is unique for the case when the eigenvectors $\{e_i\}$ of h span $\mathcal{H}$. We show first of all that a β-equilibrium state must satisfy $\langle a_T^* a_S\rangle = 0$ whenever the sets $T = (e_{i_1}, \ldots, e_{i_k})$ and $S = (e_{j_1}, \ldots, e_{j_m})$ are distinct. Indeed, in this case there exists a vector, say $e_{j_l} \in S$, different from all the vectors in T (or else, T contains a vector different from all the vectors in S). It suffices to consider the first case and set $A_1 = a_T^* a_{S\setminus\{e_{j_l}\}}$, $A_2 = a(e_{j_l})$. Then

$$\begin{aligned} F_{A_1,A_2}(t) &= \exp\{-it\lambda_{j_l}\}\langle a_T^* a_S\rangle, \\ \langle\tau_t(A_2)A_1\rangle &= -\exp\{-it\lambda_{j_l}\}\langle a_T^* a_S\rangle \end{aligned}$$

and hence equality (2) is possible only if $\langle a_T^* a_S\rangle = 0$. Since the other case is treated in the same way, we find that $T = S$. Now, again setting $A_1 = a_T^* a_{T\setminus\{e_{i_k}\}}$ and $A_2 = a(e_{i_k})$, we obtain that

$$F_{A_1,A_2}(t + i\beta) = \exp\{-it\lambda_{i_k} + \lambda_{i_k}\beta\}\langle a_T^* a_T\rangle.$$

On the other hand,

$$\langle \tau_t(A_1)A_2\rangle = -\exp\{-it\lambda_{i_k}\}\langle a_T^* a_T\rangle + \exp\{-it\lambda_{i_k}\}(-1)^{k-1}\langle a^*_{T\setminus\{e_{i_k}\}} a_{T\setminus e_{i_k}}\rangle.$$

Together with the KMS condition 2), this gives

$$\langle a_T^* a_T\rangle = \frac{(-1)^{k-1}}{(\exp\{\lambda_{i_k}\beta\}+1)}\langle a^*_{T\setminus\{e_{i_k}\}} a_{T\setminus\{e_{i_k}\}}\rangle.$$

Continuing in this way, we find that

$$\langle a_T^* a_T\rangle = (-1)^{|\pi|}\prod_{s=1}^{k}\frac{1}{(\exp\{\lambda_{i_s}\beta\}+1)},$$

where $|\pi|$ is the sign of the permutation $\pi = (1, k+1, 2, k+2, \dots, 2k-1, k, 2k)$, i.e., on all the monomials $a_T^* a_S$ the state $\langle\cdot\rangle$ coincides with the free state defined by the operator (16). A proof that the β-equilibrium state is unique for a dynamics τ_t with an arbitrary operator h can be found in [49]. Since

$$\operatorname{Ker} B = \{0\}, \qquad \operatorname{Ker}(E-B) = \{0\}$$

for the operator (16), we have $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, and thus the GNS space induced by the β-equilibrium state for the dynamics τ_t coincides with the space $\mathcal{F}_a(\mathcal{H}\oplus\mathcal{H}^*)$, where $\mathcal{H}^*$ is obtained from by $\mathcal{H}$ changing the inner product (13^b), and the operator H_{GNS} is given by

$$H_{\mathrm{GNS}} = d\Gamma(h\oplus(-h)).$$

LEMMA 6. *The free state $\langle\cdot\rangle_B$ is a ground state for the free dynamics τ_t generated by h if and only if*

$$B = E_{(-\infty,0)} + B_0 E_{\{0\}} \tag{18}$$

where B_0 is a selfadjoint operator on the space $\mathcal{H}_0 = E_{\{0\}}\mathcal{H}$ such that $0 < B_0 < 1$, and $\{E_\Delta,\ \Delta\subset R^1\}$ is the spectral family of projections for the operator h.

PROOF. Consider $A_1 = a^*(f)$ and $A_2 = A(g)$. Then the function $F_{A_1,A_2}(t) = \langle A_1\tau_t(A_2)\rangle$ has the form

$$F_{A_1,A_2}(t) = \langle e^{-it\lambda}Bf, g\rangle = \int e^{-ith}d\rho_{Bf,g}(\lambda), \tag{19}$$

where $\rho_{Bf,g}(\lambda) = (E_\lambda Bf, g)$ is the spectral measure. The function (19) can be extended to a bounded analytic function in the upper halfplane if and only if

$$\operatorname{supp}\rho_{Bf,g}(\lambda)\in(-\infty,0].$$

It follows that

$$B = E_{(-\infty,0]}B', \tag{20}$$

where B' is a selfadjoint operator acting on the space $\mathcal{H}_{(-\infty,0]} = E_{(-\infty,0]}\mathcal{H}$ and commuting with h. Choosing $A_1 = a(g)$ and $A_2 = a^*(f)$, we now obtain that

$$F_{A_1,A_2}(t) = \int e^{it\lambda} d\rho_{(E-B)f,g}(\lambda) \tag{21}$$

and the functions $F_{A_1,A_2}(t)$ can be continued into the upper halfplane for all f and g if and only if

$$E - B = E_{[0,\infty)}B'', \tag{22}$$

where B'' is a selfadjoint operator on $\mathcal{H}_{[0,\infty)}$ commuting with h. Relation (18) now follows easily from (20) and (22). For arbitrary Wick monomials $A_1 =: a_T^* a_S :$ and $A_2 =: a_{T'}^* a_{S'} :$ of the form (10), the function $F_{A_1,A_2}(t)$ is a sum of products of the form $\langle a^*(f)a(e^{ith}g)\rangle$ and $\langle a(g)a^*(e^{ith}f)\rangle$, and by what has already been proved, each factor extends analytically into the upper halfplane. The same is thus true of $F_{A_1,A_2}(t)$, and the lemma is proved.

We again obtain that the GNS representation for a free dynamics with ground state (18) is defined on the space

$$\mathcal{F}_a(\mathcal{H}_1 \oplus \mathcal{H}_2^*),$$

where $\mathcal{H}_1 = \mathcal{H}_+ \oplus \mathcal{H}_{\{0\}} \ominus \mathrm{Ker}(E - B_0)$, $\mathcal{H}_2 = \mathcal{H}_- \oplus \mathcal{H}_{\{0\}} \ominus \mathrm{Ker}\, B_0$, and $\mathcal{H}_+ = E_{(0,\infty)}\mathcal{H}$, $\mathcal{H}_- = E_{(-\infty,0)}\mathcal{H}$, $\mathcal{H}_{\{0\}} = \mathrm{Ker}\, h$. The GNS operator is of the form

$$d\Gamma(h_+ \oplus (-h_-)),$$

where $h_\pm = h|_{\mathcal{H}_\pm}$.

Let a one-particle operator be defined on $\mathcal{H}$, i.e., a positive Hamiltonian h with discrete spectrum

$$0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \ldots,$$

where the eigenvalues are listed in increasing order and $\lambda_n \to \infty$ as $h \to \infty$. Let $\{\varphi_n\}$ be an orthonormal set of eigenfunctions corresponding to the λ_n. Let

$$h_\mu = h + \mu E,$$

where μ is real, be a one-parameter family of Hamiltonians. The spectrum of the operator $d\Gamma(h_\mu)$ on $\mathcal{F}_a(\mathcal{H})$ consists of eigenvalues of the form

$$(\lambda_{i_1} + \mu) + \cdots + (\lambda_{i_k} + \mu),$$

where $i_1 < i_2 < \cdots < i_k$ are integers, and the associated eigenvectors are given by

$$\Phi_{\{i_1,\ldots,i_k\}} = a^*(\varphi_{i_1}) \ldots a^*(\varphi_{i_k})\Omega.$$

Evidently, if $\mu > 0$ the smallest eigenvalue of $d\Gamma(h_\mu)$ is zero, corresponding to the eigenvector $\Omega = \Phi_{\min}$. If $\mu \le 0$ then there exists an integer r such that

$$\lambda_r \le -\mu < \lambda_{r+1}.$$

Then it is obvious that the smallest eigenvalue of $d\Gamma(h_\mu)$ is equal to $(\lambda_0 + \mu) + \cdots + (\lambda_r + \mu)$, corresponding to the eigenvector

$$\Phi_{\min} = a^*(\varphi_0) \ldots a^*(\varphi_r)\Omega. \tag{23}$$

LEMMA 7. *The state* $\langle A\rangle_{\min} = \langle A\Phi_{\min}, \Phi_{\min}\rangle$ *is a free ground state for the dynamics* τ_t^μ *generated by* h_μ.

PROOF. We first consider the case $\mu \geq 0$. The gauge invariance of the state $\langle\cdot\rangle_{\min}$ follows from the equality

$$(a(f_1)\dots a(f_n)a^*(g_1)\dots a^*(g_m)\Omega,\Omega) \\ = (a^*(g_1)\dots a^*(g_m)\Omega, a^*(f_n)\dots a^*(f_1)\Omega) = 0 \quad \text{for } n \neq m.$$

The Gaussian property is a consequence of

$$(a(f_1)\dots a(f_n)a^*(g_1)\dots a^*(g_n)\Omega,\Omega) \\ = \sum(-1)^{k-1}(f_ng_k)(a(f_1)\dots a(f_{n-1})a^*(g_1)\dots a^*(\check{g}_k)\dots a^*(g_m)\Omega,\Omega),$$

which is obtained by repeatedly moving $a(f_n)$ to the right of all the $a^*(g_k)$. Continuing this procedure, we obtain an expansion of the form (1). When $\mu < 0$ we consider the following $*$-automorphism γ of the CAR algebra (a canonical transformation), which acts on the generators $a^\#(f)$ by the formulas

$$\begin{aligned} \gamma(a(f)) &\equiv \widetilde{a}(f) = a^*(f^*), \\ \gamma(a^*(f)) &\equiv \widetilde{a}^*(f) = a(f^*), \end{aligned} \tag{24^a}$$

where $f^* = \sum_{k=0}^r \overline{c}_k\varphi_k$, if $f = \sum_{k=0}^r c_k\varphi_k \in E^\mu_{(-\infty,0]}\mathcal{H} = \mathcal{H}_\mu$ and $\{E^\mu_\Delta\}$ is a spectral family for the operator h_μ, and

$$\begin{aligned} \gamma(a(f)) &\equiv \widetilde{a}(f) = a(f), \\ \gamma(a^*(f)) &\equiv \widetilde{a}^*(f) = a^*(f), \quad f \in \mathcal{H}_\mu^\perp = \overline{\mathcal{H}}_\mu. \end{aligned} \tag{24^b}$$

Note that for the new system of generators $(\widetilde{a}^*(f), \widetilde{a}(f))$, which satisfy the anticommutation condition, $\Phi_{\min}$ given by (23) is a vacuum vector:

$$\widetilde{a}(f)\Phi_{\min} = 0, \qquad f \in \mathcal{H},$$

and the representation of this system in $\mathcal{F}_a(\mathcal{H})$ is unitarily equivalent to the standard Fock representation of the creation and annihilation operators $\{b^*(f), b(f)\}$ in the space $\mathcal{F}_a(\widetilde{\mathcal{H}})$, where $\widetilde{\mathcal{H}} = \mathcal{H}_\mu^* \oplus \overline{\mathcal{H}}_\mu$, and $\mathcal{H}_\mu^*$ is $\mathcal{H}_\mu$ with the altered inner product (13^b). Thus the proof that $(A\Phi_{\min}, \Phi_{\min})$ is free follows from the previous arguments. The corresponding operator B is of the form $B = E^\mu_{(-\infty,0]}$, and hence by Lemma 6 $\langle\cdot\rangle_{\min}$ is a ground state for the dynamics τ_t^μ.

The space $\mathcal{H}_{\mathrm{GNS}}$ associated to this state coincides with the Fock space $\mathcal{F}_a(\widetilde{\mathcal{H}})$, and the operator H^μ for the dynamics τ_t^μ has the form

$$\sum_k |\lambda_k + \mu| b_k^* b_k = d\Gamma(|h + \mu|).$$

We note that when written in terms of the operators $\widetilde{a}_k^*$ and $\widetilde{a}_k$, the operators $d\Gamma(h+\mu) = \sum_k(\lambda_k+\mu)a_k^*a_k$ have the form

$$\sum_k |\lambda_k + \mu|\widetilde{a}_k^* a_k + \left(\sum_{i=0}^r (\lambda_i + \mu)\right) E,$$

i.e., apart from a constant they are unitarily equivalent to the operator H^μ_{GNS}. In the case when the spectrum of h is arbitrary, the new system of operators $\widetilde{a}^*(f)$, $\widetilde{a}(f)$ obtained via the canonical transformation (24^a), (24^b) may fail in general to be unitarily equivalent to the standard system of Fock operators, and therefore the operator $d\Gamma(h+\mu) + sE$ may not be unitarily equivalent to H^μ_{GNS} for any constant s.

§4. Fock representation for the dynamics of free systems

Here we will consider some examples showing that for certain free systems the operator H_{GNS}, defined by a free dynamics and equilibrium state, is unitarily equivalent to a Hamiltonian of the form $d\Gamma(h)$ on a suitably chosen Fock space $\mathcal{F}_{a,s}(\mathcal{H})$, where h is a one-particle operator on $\mathcal{H}$.

1. The classical ideal gas. Recall that the state space for an infinite ideal gas is the collection Ω of all locally finite (unordered) sets $X = \{(q_i, v_i)\}$, $(q_i, v_i) \in R^\nu \times R^\nu$, i.e., sets such that only finitely many elements $(q_i, v_i) \in X$ are contained in any bounded region $G \subset R^\nu \times R^\nu$. The Gibbs distribution $\mu_0 = \mu_0^{\rho,\beta}$ for an ideal gas coincides with the distribution of the Poisson field in $R^\nu \times R^\nu$ given by the measure

$$d\lambda = \rho\left(\frac{\beta}{2\pi}\right)^{\nu/2} \exp\left\{-\frac{\beta v^2}{2}\right\} dq\, dv. \tag{1}$$

This field can be described as follows:

1) The number $n_B(X)$ of elements (q, v) in the configuration X contained in a bounded set $B \subset R^\nu \times R^\nu$ obeys a Poisson distribution

$$\Pr(n_B(X) = n) = \frac{(\lambda(B))^n}{n!} e^{-\lambda(B)}, \tag{2}$$

where the measure $\lambda(B)$ of B is given by equation (1).

2) For any two disjoint sets $B_1, B_2 \subset R^\nu$, the quantities $n_{B_1}(X)$ and $n_{B_2}(X)$ are independent.

This distribution is described more fully in [21]. If $\mathfrak{M}$ denotes the collection of all finite sets $X \in \Omega$, we have a partition

$$\mathfrak{M} = \bigcup \mathfrak{M}_n,$$

where $\mathfrak{M}_n$ is the collection of all n-element sets; $\mathfrak{M}_n$ can be represented as the quotient

$$\mathfrak{M}_n = (R^\nu \times R^\nu)^n / S_n$$

of the space $(R^\nu \times R^\nu)^n$ of ordered sequences $\{(q_1, v_1), \dots, (q_n, v_n)\}$ by the action of the group S_n permuting the elements. We can thus introduce on each space $\mathfrak{M}_n$ the measure

$$dX = \frac{dq_1 dv_1 \dots dq_n dv_n}{n!}, \tag{3}$$

which we can regard as a measure on the whole space $\mathfrak{M}_n$.

Remark. It is clear from the definition (3) that the set of collections $\{(q_1, v_1), \dots, (q_n, v_n)\}$ for which at least two elements coincide has measure zero, i.e., the collection $\mathfrak{M}' \subset \mathfrak{M}$ of all finite subsets $X \subset R^\nu \times R^\nu$ (i.e., collections all of whose elements are distinct) has full measure. For the same reason, the collection $\Omega' \subset \Omega$ of all locally finite subsets $X \subset R^\nu \times R^\nu$ has full measure μ_0 (see [21] for more details).

The correlation function $\rho_0(X) = \rho_{\mu_0}(X)$ on $\mathfrak{M}$ for the Poisson field given above is equal to

$$\rho_0(X) = \rho^{N(X)} \prod_i \left(\frac{\beta}{2\pi}\right)^{\nu/2} \exp\left\{-\frac{\beta v_i^2}{2}\right\}, \quad X = \{(q_i, v_i)\} \in \mathfrak{M}, \tag{4}$$

where $N(X)$ is the number of elements in X. A definition of the correlation function for any point field, i.e., for any distribution on Ω, is given in [14]. The following property will be important for us. Let $\varphi(X)$, $X \in \mathfrak{M}$, be a finite function on the space $\mathfrak{M}$, i.e., such that

$$\varphi(X) \neq 0, \quad X = \{(q_i, v_i)\}$$

only if $N(X) < N_0 = N_0(\varphi)$ and $X \subset G$, where $G = G(\varphi)$ is a bounded subset of $R^\nu \times R^\nu$. Then the functional

$$F_\varphi(\omega) = \sum_{X \subseteq \omega} \varphi(X) \tag{5}$$

is defined for all $\omega \in \Omega$. A functional of the time (5) is usually called a *summing* functional. The formula

$$\langle F_\varphi(\omega) \rangle_\mu = \int_{\mathfrak{M}} \varphi(X) \rho_\mu(X)\, dX \tag{6}$$

can be shown to hold for any point field in $R^\nu \times R^\nu$ with probability distribution μ on Ω for which the correlation function $\rho_\mu(X)$ exists.

If the function φ in (5) does not have compact support but is integrable on $\mathfrak{M}$ with respect to σ-finite measure $\rho_\mu dX$ then the summing functional (5) is defined for almost all ω with respect to μ, and equation (6) continues to hold as before.

Now consider the space $\mathcal{H} = L_2(R^\nu \times R^\nu, d\lambda)$ of functions $f(q, v)$ on $R^\nu \times R^\nu$. The symmetrized nth tensor power

$$(\mathcal{H}^{\otimes n})^{\mathrm{sym}} = \mathcal{H}^{(n)}_{\mathrm{sym}}$$

(with inner product equal to $1/n!$ times the usual inner product on $\mathcal{H}^{(n)}_{\mathrm{sym}}$) is easily seen to coincide with the Hilbert space $L_2(\mathfrak{M}_n, \rho_0(X)dX)$, where the correlation function $\rho_0(X)$, $X \in \mathfrak{M}_n$, is given by (4). Thus, the Fock space $\mathcal{F}_s(\mathcal{H})$ coincides with $L_2(\mathfrak{M}, \rho_0 dX)$.

We now define a unitary transformation of $L_2(\mathfrak{M}, \rho_0 dX)$ onto the space $L_2(\Omega, \mu_0)$, as follows. For each finite function $f \in L_2(\mathfrak{M}, \rho_0 dX)$ we define another finite function

$$\varphi_f(X) = \int_{\mathfrak{M}} f(X \cup \widetilde{X})(-1)^{N(\widetilde{X})} \rho_0(\widetilde{X})\, d\widetilde{X} \tag{7}$$

and the summing functional

$$F^f(\omega) \equiv F_{\varphi_f}(\omega) = \sum_{X \subseteq \omega} \varphi_f(X). \tag{8}$$

Lemma 1. *The transformation $f \to F^f$ is isometric*:

$$\int_{\mathfrak{M}} f(X) \overline{g}(X) \rho_0(X) dX = \langle F^f \overline{F^g} \rangle_{\mu_0} \tag{9}$$

and after extension to all of $L_2(\mathfrak{M}, \rho_0 dX)$ gives a unitary equivalence with all of $L_2(\Omega, \mu_0)$.

PROOF. Let $f \in L_2(\mathfrak{M}_n, \rho_0 dX)$. We will show that for every finite element $g \in L_2(\mathfrak{M}_m, \rho_0 dX)$, $m \le n$, the inner product

$$(F^f, F_g) = \langle F^f \overline{F_g} \rangle_{\mu_0},$$

where F_g is defined by (5), is equal to

$$(F^f, F_g) = \begin{cases} \int_{\mathfrak{M}_n} f(X)\overline{g}(X)\rho_0(X)dX, & m = n, \\ 0, & m < n. \end{cases} \tag{10}$$

Indeed,

(11)

$$\begin{aligned} \langle F^f \overline{F_g} \rangle_{\mu_0} &= \int_\Omega \Big(\sum_{X \subset \omega} \varphi_f(X) \Big) \Big(\sum_{X' \subset \omega} \overline{g}(X') d\mu_0(\omega) \Big) \\ &= \int_\Omega \sum_{X \subset \omega} \Bigg(\sum_{\substack{X_1, X_2, X_3 : X_1 \cap X_2 = X_2 \cap X_3 = X_1 \cap X_3 = \emptyset \\ X = X_1 \cup X_2 \cup X_3}} \varphi_f(X_1 \cup X_2)\overline{g}(X_2 \cup X_3) \Bigg) d\mu_0(\omega) \\ &= \int_{\mathfrak{M}} \Bigg(\sum_{\substack{X_1, X_2, X_3 : X_1 \cap X_2 = X_2 \cap X_3 = X_1 \cap X_3 = \emptyset \\ X = X_1 \cup X_2 \cup X_3}} \varphi_f(X_1 \cup X_2)\overline{g}(X_2 \cup X_3) \Bigg) \rho_0(X)\, dX \\ &= \int_{\mathfrak{M}} \varphi_f(X_1 \cup X_2)\overline{g}(X_2 \cup X_3)\rho_0(X_1)\rho_0(X_2)\rho_0(X_3)\, dX_1\, dX_2\, dX_3. \end{aligned}$$

The last equality in (11) is based on some straightforward combinatorics and also uses the multiplicative form of $\rho_0(X)$. Furthermore, in all of these formulas X_1, X_2, X_3 may be taken to be finite subsets of $R^\nu \times R^\nu$ (see the above Remark). Inserting φ_f from (7) into the last equality and using the readily verified identity

$$\int_{\mathfrak{M}} \Big(\sum_{X' \subseteq X} \varphi(X') \Big) h(X)\, dX = \int_{\mathfrak{M}^2} \varphi(X_1) h(X_2 \cup X_1)\, dX_1\, dX_2,$$

we find that

$$\begin{aligned} \langle F^f \overline{F_g} \rangle_{\mu_0} = \int_{\mathfrak{M}^3} &\Big(\sum_{\tilde{X} \subseteq X} (-1)^{N(\tilde{X})} \Big) \\ &\times f(X_2 \cup X_2)\overline{g}(X_2 \cup X_3)\rho_0(X_1)\rho_0(X_2)\rho_0(X_3) dX_1 dX_2 dX_3. \end{aligned}$$

Since $\sum_{\tilde{X} \subseteq X} (-1)^{N(\tilde{X})} = 0$, if $X_1 \neq \emptyset$, we obtain finally that

$$\langle F^f \overline{F_g} \rangle_{\mu_0} = \int_{\mathfrak{M}^2} f(X_2) g(X_2 \cup X_3)\rho_0(X_2)\rho_0(X_3)\, dX_2\, dX_3,$$

from which (10) follows. Since for $f \in L_2(\mathfrak{M}_n, \rho_0\, dX)$ we have $\varphi_f = f + \widetilde{f}$, where $\widetilde{f} \in \bigoplus_{k<n} L_2(\mathfrak{M}_k, \rho_0\, dX)$, it follows from (10) that for $f \in L_2(\mathfrak{M}_n, \rho_0(X)\, dX)$ and $g \in L_2(\mathfrak{M}_m, \rho_0\, dX)$, we have

$$(F^f, F^g) = \delta_{m,n} \int_{\mathfrak{M}_n} f(X)\overline{g}(X)\rho_0(X)\, dX.$$

This already shows that the mapping (7) is isometric. Now let $F_\Lambda(\omega) = F_\Lambda(X)$, where $X = \omega \cap \Lambda$ for a bounded set $\Lambda \subset R^\nu \times R^\nu$, be a local function, depending only on the part of the configuration ω contained in Λ. Then any such F_Λ can be expressed as a summing function (5) by setting

$$\varphi(X) = \sum_{X' \subseteq X} (-1)^{N(X \setminus X')} F_\Lambda(X'),$$

where $X \subset \Lambda$ is a finite configuration in Λ. The summing functionals (5) are thus dense in $L_2(\Omega, \mu_0)$. On the other hand, the map $f \to \varphi_f$ (7) is a bijection of the set of finite bounded functions with itself. In fact, its inverse is given by

$$f(X) = \int_{\mathfrak{M}} \varphi_f(X \cup \widetilde{X}) \rho_0(\widetilde{X})\, d\widetilde{X}.$$

Thus, the map $f \to F^f$ extends to a unitary transformation from $L_2(\mathfrak{M}, \rho_0 dX)$ onto $L_2(\Omega, \mu_0)$.

REMARKS. 1. Let $\widetilde{\mathcal{H}}_n \subset L_2(\Omega, \mu_0)$ be the image of the space $L_2(\mathfrak{M}_n, \rho_0 dX)$ under the map (8). To get a better idea of the structure of these spaces, we consider the form of the functionals $F^f \in \mathcal{H}_n$ for $n = 0, 1, 2$. Clearly, $\mathcal{H}_0$ is the space of constants, while $\mathcal{H}_1$ is the closure of the functionals of the form

$$F(\omega) = \sum_{(q,v) \in \omega} f(q, v) + f_0, \tag{11^a}$$

where

$$f_0 = -\rho \left(\frac{\beta}{2\pi} \right)^{\nu/2} \int_{R^\nu \times R^\nu} f(q, v) \exp \left\{ -\frac{\beta v^2}{2} \right\} dq\, dv,$$

and the function f is bounded with compact support in $R^\nu \times R^\nu$. The term f_0 in (11) ensures that the functional (11) is orthogonal to all constants. The space $\mathcal{H}_2$ is the closure of the summing functionals of the form

$$F = \sum_{\{(q_1,v_1),(q_2,v_2)\} \subset \omega} f((q_1, v_1)(q_2, v_2)) + \sum_{(q,v) \in \omega} f_1(q, v), \tag{12}$$

where

$$f_1(q, v) = -\int_{R^\nu \times R^\nu} f((q, v)(\widetilde{q}, \widetilde{v})) \rho_0(\widetilde{q}, \widetilde{v})\, d\widetilde{q}\, d\widetilde{v},$$

where $f((q_1, v_1), (q_2, v_2))$ is a symmetric bounded function with compact support in $(R^\nu \times R^\nu)^2$. The second term in (12) ensures that F is orthogonal to the constants and to functionals of the form $\sum_{(q,v) \in \omega} g(q, v)$.

2. Let $B \subset R^\nu$ be a bounded set and let the function $f \in L_2(\mathfrak{M}_n, \rho_0 dX)$ have the form

$$f(X) = \prod_{i=1}^{n} \chi_B(q_i), \qquad X = \{(q_i, v_i)\} \in \mathfrak{M}_n,$$

where $\chi_B(\cdot)$ is the characteristic function of the set B. Then a simple calculation shows that

$$F^f(\omega) = P_n^{\rho|B|}(\kappa_B(\omega)),$$

where $\kappa_B(\omega)$ is the number of particles in the configuration ω that lie in the set B, $\kappa \geq 0$ is an integer, and $P^\mu(\kappa)$ is the Charlier polynomial

$$P_n^\mu(s) = \sum_{k=0}^{n} (-1)^{n-k} \frac{s^{(k)}}{k!(n-k)!} \mu^{n-k},$$

where

$$s^{(k)} = \begin{cases} s(s-1)\dots(s-k+1) = \frac{s!}{k!}, & k > 0, \\ 1, & k = 0. \end{cases}$$

These polynomials are orthogonal with respect to the Poisson distribution on Z_+^1 with rate μ:

$$\sum_{s=0}^{\infty} P_n^\mu(s) P_m^\mu(s) \frac{\mu^s e^{-\mu}}{s!} = \frac{\delta_{m,n}}{n!} \mu^n.$$

We now consider the dynamics τ_t on the space Ω given by

$$\begin{aligned} \tau_t \omega = \omega_t = \{(q_i^t, v_i^t)\}, \qquad \omega = \{(q_i \dots v_i)\}, \\ q_i^t = q_i + v_i t, \qquad v_i^t = v_i. \end{aligned} \tag{13}$$

Since certain locally finite configurations whose velocities increase very rapidly at infinity may cease to be locally finite (i.e., may collapse, see 1.1) after a finite time t for the dynamics (13), we will need the following lemma to ensure that our definition is correct.

LEMMA 2. *The space Ω contains a set Ω' of full μ_0 measure such that for every $\omega \in \Omega'$, we have $\tau_t \omega \in \Omega'$, for all $t \geq 0$.*

The proof of this lemma is omitted.

REMARK. Recall that we have already encountered in 1.1 the analogous assertion for $\nu = 1$, but in the more complicated case of a nonideal gas.

We may thus consider the dynamics in Ω'.

LEMMA 3. *For all ρ and β, the Poisson measure $\mu_0 = \mu_0^{\rho,\beta}$ on Ω' is invariant under the dynamics* (13).

PROOF. Define a transformation of the function $F(\omega)$ on Ω' by the formula

$$(U_t F)(\omega) = F(\tau_t^{-1} \omega).$$

It is easy to see that for summing functionals F_f of the form (5), we have

$$U_t F_f = F_{\widetilde{U}_t f}, \tag{14}$$

where

$$(\widetilde{U}_t f)(X) = f(\tau_t^{-1} X), \qquad X \in \mathfrak{M}. \tag{15}$$

Under the transformation $\tau_t \colon \mathfrak{M} \to \mathfrak{M}$, the measure dX and correlation function $\rho_0(X)$ are easily seen to be invariant; from equations (6) and (14) we conclude that $\langle U_t F_f \rangle_{\mu_0} = \langle F_f \rangle_{\mu_0}$. This remains true for any function $F \in L_2(\mathfrak{M}, \rho_0 dX)$, which

proves the assertion of the lemma and thereby establishes that the group $\{U_t, t \in R^1\}$ is unitary on $L_2(\Omega, \mu_0)$.

Under the correspondence between the Fock space $\mathcal{F}_s(\mathcal{H}) = L_2(\mathfrak{M}, \rho_0 dX)$ and $L_2(\Omega, \mu_0)$ given in Lemma 1 (here $\mathcal{H} = L_2(R^\nu \times R^\nu, d\lambda)$), the dynamics U_t on $L_2(\Omega, \mu_0)$ is unitarily equivalent to the dynamics $\widetilde{U}_t$ on $L_2(\mathfrak{M}, \rho_0 dX)$. The latter dynamics is readily seen to be of the form $\widetilde{U}_t = \Gamma(u_t)$, where u_t is the one-particle dynamics in given by

$$(u_t f)(q, v) = f(q - vt, v).$$

Its infinitesimal generator h is

$$(hf)(q, v) = iv\frac{\partial f}{\partial q}$$

whence the infinitesimal generator H of the dynamics $\widetilde{U}_t$ is equal to

$$Hf = d\Gamma(h)f = \sum_k iv_k \frac{\partial f}{\partial v_k}, \qquad f \in L_2(\mathfrak{M}, \rho_0 dX).$$

We note that the spectrum of h is Lebesgue and fills the entire line $(-\infty, \infty)$. Thus H has a Lebesgue spectrum with infinite multiplicity which also fills all of $(-\infty, \infty)$.

2. Stochastic gas. We now consider another stochastic dynamics defined on the space Ω of all locally finite collections $\omega = \{q_i\}$, $q_i \in R^\nu$.

For each fixed configuration $\omega \in \Omega$ we consider the collection of independent Wiener processes $Y = \{y_{q_i}\}$ starting at points in ω:

$$y_{q_i}(0) = q_i, \qquad i = 1, 2, \ldots.$$

The distribution of the process $Y(t)$ in the space $S_\omega = \bigotimes_{q_i \in \omega} S_{q_i}$ of all collections of trajectories $w_t^\omega = \{w_{q_i}(t)\}$, where S_{q_i} is the space of trajectories of an individual Brownian motion starting at q_i, is the product $dW = \prod dW_{q_i}$ of the Wiener measures defined on each space S_{q_i}.

LEMMA 4. *For any locally finite configuration ω, almost all (with respect to the measure dW) sets of trajectories $w_t^\omega = \{w_{q_i}(t)\}$ are such that for every $0 \le t_0 < \infty$ the configuration*

$$\omega_{t_0} = \{w_{q_i}(t_0), q_i \in \omega\}$$

is locally finite.

A stochastic dynamics $\{\omega_t\}$ is thus defined on the space Ω; we denote by $P^t_{\omega_0}(\cdot)$ the associated transition measure,

$$P^t_{\omega_0}(A) = Pr(\omega_t \in A|_{\omega_{t=0}} = \omega_0).$$

Then for any probability measure $\mu = \mu_0$ on Ω we can define the evolution of the measures $\{\mu_t,\ 0 \le t < \infty\}$ in the usual way by setting

$$\mu_t(A) = \int_\Omega P^t_\omega(A)\, d\mu_0(\omega). \tag{16}$$

LEMMA 5. *The distribution μ_0^ρ of the Poisson field in R^ν generated by the measure ρdx on R^ν is invariant under the evolution* (16) *for any* $0 < \rho < \infty$.

The proof is similar to that of Lemma 3, and can also be found in [16].

Thus, the unitary group U_t on $L_2(\Omega, \mu_0^\rho)$ generated by our dynamics can be carried over via the isomorphism (8) to the Fock space $\mathcal{F}_s(\mathcal{H})$, where $\mathcal{H} = L_2(R^\nu, \rho dx)$, and it coincides there with the group $\Gamma(u_t)$; $u_t = e^{it\Delta}$ is the group acting on $L_2(R^\nu, \rho dx)$, where Δ is the Laplace operator. The infinitesimal generator of the group U_t thus coincides with $d\Gamma(\Delta)$.

REMARK. A similar construction is valid for any dynamics on Ω generated by the independent dynamics of the individual particles in a configuration $\omega \in \Omega$. The Poisson measure μ_0^ρ is invariant under every such dynamics, and the corresponding group U_t on $\mathcal{F}_s(L_2(R^\nu, \rho dx))$ is of the form $\Gamma(u_t)$, where u_t is the one-particle dynamics in $L_2(R^\nu, \rho dx)$.

§5. The Euclidean approach

1. Brief sketch of the method. We recall that in the introductory chapter (5.0) we discussed the two basic ideas that form the basis of the general method in mathematical physics called the "Euclidean approach":

1) Introduction of an "imaginary time", i.e., the operator semigroups $\exp\{-tH\}$, $t > 0$, are studied in place of the unitary group $\exp\{itH\}$, $-\infty < t < \infty$, where H is the Hamiltonian of the system. By studying the semigroup $\exp\{-tH\}$ one can extract a certain amount of information concerning the spectral properties of the operator H which can then be brought to bear on the analysis of $\exp\{itH\}$.

2) The renormalized Feynman-Kac-Nelson formula: let the operator H acting on the Hilbert space $L_2(X, d\nu_0)$ be of the form

$$H = H_0 + V,$$

where H_0 is selfadjoint on $L_2(X, d\nu_0)$ and coincides with the infinitesimal generator of the stochastic semigroup $\mathcal{J}_t^0$ for some stationary Markov process $\xi = \{\xi_t, t \in R^1\}$ with values in X and invariant measure $d\nu_0$, and V is a multiplication operator given by multiplication by a real-valued function on X; moreover, H has a ground-state eigenvector Φ_0 with eigenvalue λ_0. One can then construct a new stationary Markov process $\eta = \{\eta_t, t \in R^1\}$ with values in X and invariant measure $d\nu = |\Phi(x_0)|^2 d\nu_0$ such that the infinitesimal generator H^{renorm} of the associated stochastic semigroup $\mathcal{J}_t$ acting on $L_2(X, d)$ is unitarily equivalent to the operator $H - \lambda_0 E$. The probability distribution μ in the space $\Omega = X^{R^1}$ of trajectories $\{x_t,\ t \in R^1\}$ of the process η is obtained as the Gibbs limit of the distribution μ_0 for the process ξ in Ω:

$$\mu = \lim_{T \uparrow R^1} \mu_T, \tag{2}$$

where the distribution μ_T (T is a finite closed interval on the R^1 axis) is determined by the density

$$\frac{d\mu_T}{d\mu_0} = \frac{1}{Z_T} \exp\left\{ - \int_T V(x(\tau)) d\tau \right\}, \tag{3}$$

and Z_T is a normalization factor,

$$Z_T = \int_\Omega \exp\Big\{ - \int_T V(x(\tau))d\tau \Big\} d\mu_0.$$

The ground state of the operator H^{renorm} is the function $\Phi_0 \equiv 1 \in L_2(X, d\nu)$ with zero eigenvalue, and the rest of the spectrum of H^{renorm} describes the energy of the "excited states" of the system.

The same method can also be used for infinite-dimensional quantum systems (fields). The only difference is that now there is no initial Hamiltonian H, and the Hamiltonians $H_\Lambda(1)$ are defined for finite subsystems of the infinite system (the latter are indexed by the elements Λ of an increasing ordered set); the subsystems are described by the Hilbert spaces $\mathcal{H}_\Lambda = L_2(X_\Lambda, d\nu_\Lambda^0)$, where the $X_\Lambda \subset X$ are subsets of some set X. For every Λ we construct as described above a stationary Markov process η_Λ with values in X_Λ and distributions μ_Λ in the space $\Omega_\Lambda = X_\Lambda^{R^1}$, and then take the limit $\Lambda \uparrow$. This gives a limit stationary Markov process $\eta = \{\eta_t,\ t \in R^1\}$ with state space X and distribution μ,

$$\mu = \lim_{\Lambda\uparrow} \mu_\Lambda$$

on $\Omega = X^{R^1}$. The infinitesimal generator H^{renorm} for the stochastic semigroup $\mathcal{J}_t$ of this process is declared to be the Hamiltonian of the infinite system (in its "ground state").

There are a few infinite systems for which the dynamics τ_t can be constructed by taking a limit $\Lambda \uparrow$ of local dynamics

$$\tau_t^\Lambda = \exp\{itH_\Lambda\} A \exp\{-itH_\Lambda\}, \qquad A \in \mathfrak{B}(\mathcal{H}_\Lambda)$$

in a suitable quasilocal algebra $\mathfrak{A} = \bigcup_\Lambda \mathfrak{A}_\Lambda$, where $\mathfrak{A}_\Lambda \subseteq \mathfrak{B}(\mathcal{H})$ and the ground state $\langle A\rangle$ for this dynamics is obtained as the limit of ground states of the form

$$\langle A\rangle_\Lambda = (A\Phi_0^\Lambda, \Phi_0^\Lambda)_{\mathcal{H}_\Lambda}, \qquad A \in \mathfrak{A}_\Lambda, \tag{4}$$

as $\Lambda \uparrow$ (here Φ_0^Λ is the (normalized) ground-state eigenvector for the Hamiltonian H_Λ). For such systems, H^{renorm} turns out to be unitarily equivalent to the operator H_{GNS} for the dynamics τ_t on the space $\mathcal{H}_{\text{GNS}}$ constructed from the ground state $\langle\cdot\rangle$.

We will now illustrate this general scheme in the case of the following very simple example. Let an infinite system of interacting oscillators be given on the lattice Z^ν (this system was analyzed in §2). The formal Hamiltonian is given by (see 1.2)

$$H = \sum_{x\in Z^\nu} p_x^2 + \frac{1}{4}\sum a_{x_1,x_2} q_{x_1} q_{x_2}, \tag{5}$$

where $\{p_x, q_x, x \in Z^\nu\}$ is the system of "momentum" and position operators, and $A = \{a_{x_1,x_2},\ x_1, x_2 \in Z^\nu\}$ is a infinite matrix defining a strictly positive operator on $l_2(Z^\nu)$. The introduction of the formal Hamiltonian (5) means that for every finite set $\Lambda \subset Z^\nu$ one must choose a Hamiltonian

$$\begin{aligned} H_\Lambda f &= \sum_{x\in\Lambda} p_x^2 f + \frac{1}{4}\Big(\sum_{x_1,x_2\in\Lambda} a_{x_1,x_2} q_{x_1} q_{x_2} \Big) f \\ &= -\sum_{x\in\Lambda} \frac{\partial^2 f}{\partial q_x^2} + \frac{1}{4}\Big(\sum_{x_1,x_2\in\Lambda} a_{x_1,x_2} q_{x_1} q_{x_2} \Big) f, \end{aligned} \tag{6}$$

acting on the space $L_2(R^\Lambda, d^{|\Lambda|}q)$. As we have already shown in §2, the operator $H_\Lambda - \lambda_0 E$ (where λ_0 is the smallest eigenvalue of H_Λ) is unitarily equivalent to the operator

$$H_\Lambda^{\text{renorm}} = -\sum_{x\in\Lambda} \frac{\partial^2 f}{\partial q_x^2} + \sum_{x_1,x_2\in\Lambda} b^\Lambda_{x_1,x_2} q_{x_1} \frac{\partial f}{\partial q_{x_2}},$$

acting on the space $L_2(R^\Lambda, p(q)d^{|\Lambda|}q)$, where

$$p(q) = \text{const} \exp\left\{ -\frac{1}{2} \sum_{x_1,x_2\in\Lambda} b^\Lambda_{x_1,x_2} q_{x_1} q_{x_2} \right\}, \tag{8}$$

and

$$B_\Lambda = \{b^\Lambda_{x_1,x_2}\} = A_\Lambda^{1/2}, \qquad A_\Lambda = \{a_{x_1,x_2}, x_1, x_2 \in \Lambda\}.$$

Here $H_\Lambda^{\text{renorm}}$ is the infinitesimal generator of a stochastic semigroup for the stationary Gaussian Markov process $\xi_t^\Lambda = \{\xi_t^\Lambda(x),\, x \in \Lambda\}$ with mean $\langle \xi_t^\Lambda(x)\rangle = 0$ and covariance matrix

$$\langle \xi^\Lambda_{t_1}(x_1), \xi^\Lambda_{t_2}(x_2)\rangle = (A_\Lambda^{1/2} \exp\{-A_\Lambda^{1/2}|t_1 - t_2|\})_{x_1,x_2}. \tag{9}$$

Letting $\Lambda \uparrow Z^\nu$, we now obtain an infinite-dimensional Gaussian stationary process $\xi_t = (\xi_t,\, t \in Z^\nu\}$ with zero mean and covariance matrix of the form (9), where A_Λ is replaced by the matrix A. The stochastic semigroup $\mathcal{J}_t$ for this process acts on the space $\mathcal{H}_{\text{phys}} = L_2(R^{Z^\nu}, d\nu_0)$ of functionals of ξ_t that depend on the values $\{\xi_{t=0}(x)\}$ at the time $t = 0$. Here ν_0 is a Gaussian measure with zero mean and covariance matrix $A^{-1/2}$.

Passing to the new Markov process

$$\eta_t(x) = \sum_{x'\in Z^\nu} C_{x,x'} \xi_t(x'),$$

where the matrix $C = \{C_{x_1,x_2}\} = A^{1/4}$, we obtain that for every fixed $t = t_0$ the random variables $\{\eta_{t_0}(x)\}$ are independent and have a normal distribution $d\mu_0(x)$ with variance 1. The covariance of this process is equal to

$$\langle \eta_t(x)\eta_{t'}(x')\rangle = (\exp\{-A^{1/2}|t - t'|\})_{x,x'}. \tag{10}$$

The space $\mathcal{H}_{\text{phys}} = L_2(r^{Z^\nu}, d\nu_0)$ coincides with the space $L_2(R^{Z^\nu}, d\widehat{\nu}_0)$ of functionals of the values $\{\eta_{t=0}(x)\}$ of the process at time $t = 0$, and $d\nu_0$ is the product $\prod_{x\in 2^\nu} d\mu_0(x)$ of the normal distributions.

Choosing an orthonormal basis

$$:\eta^{k_1}_{t=0}(x_1) :: \eta^{k_2}_{t=0}(x_2) : \cdots : \eta^{k_s}_{t=0}(x_s) := \Phi_{(x_1,k_1),\ldots,(x_s,k_s)}, \tag{11}$$

of $L_2(R^{Z^\nu}, d\widehat{\nu}_0)$, we can identify $L_2(R^{Z^\nu}, d\widehat{\nu}_0)$ with the Fock space $\mathcal{F}_s(l_2(Z^\nu))$, as was explained in 3.1. Next, a straightforward calculation of the matrix elements for the operator $\mathcal{J}_t$ in the basis (11):

$$(\mathcal{J}_t \Phi_{(x_1,k_1),\ldots,(x_s,k_s)} \Phi_{(x'_1,k'_1),\ldots,(x'_s,k'_s)}) \tag{12}$$

together with equation (10) and the standard diagram technique for computing means of the type $\langle : \xi_{X_1} : \cdots : \xi_{X_m} : \rangle$, where $\xi_{X_i} = \prod_{x \in X_i} \xi_x^{(i)}$ and the $\{\xi_x^{(i)}, x \in X_i\}$ are sets of random variables (see [26] for more details), shows that under the above identification of $L_2(R^{Z^\nu}, d\widehat{\nu}_0)$ with $\mathcal{F}_s(l_2(Z^\nu))$, $\mathcal{J}_t$ goes into the operator $\Gamma(\exp(-tA^{1/2}))$ on $\mathcal{F}_s(l_2(Z^\nu))$, and hence H^{renorm} is unitarily equivalent to the operator $d\Gamma(A^{1/2})$. On the other hand, we showed in §2 that for the dynamics τ_t on the Weyl algebra constructed from the formal Hamiltonian (5), the operator H_{GNS} found for the ground state $\langle \cdot \rangle$

$$\langle A \rangle = \lim_{\Lambda \uparrow Z^\nu} (A\Phi_0^\Lambda, \Phi_0^\Lambda),$$

of the Weyl algebra is also unitarily equivalent to $d\Gamma(A^{1/2})$, i.e., H_{GNS} is unitarily equivalent to H^{renorm} (here Φ_0^Λ is a normalized ground-state eigenvector for the Hamiltonian (6)).

The setup described above also applies to the more complicated anharmonic-oscillator Hamiltonian given formally by

$$\begin{aligned} H_{\text{anharm}} &= \sum_{x \in Z^\nu} p_x^2 + \sum_{x_1, x_2 \in Z^\nu} a_{x_1, x_2} q_{x_1} q_{x_2} + \lambda \sum_{x \in Z^\nu} Q(q_x) \\ &= H_{\text{harm}} + V, \end{aligned} \tag{13}$$

where H_{harm} is the formal Hamiltonian (5) for harmonic oscillators with the potential $V = \lambda \sum_{x \in Z^\nu} Q(q_x)$ and $Q(\cdot)$ is a polynomial bounded from below. However, to construct the distribution μ for the limit Markov process corresponding to the Hamiltonian (13), it is technically easier to use a Gibbs modification of the limit Gaussian measure $\mu_0 = \mu_{\text{harm}}$ already constructed above for harmonic oscillators, i.e., to set

$$\mu = \lim_{\substack{\Lambda \uparrow Z^\nu \\ T \uparrow R^1}} \mu_{\Lambda, T}, \tag{14}$$

where

$$\frac{d\mu_{\Lambda,\Gamma}}{d\mu_{\text{harm}}} = \frac{1}{Z_{\Lambda,T}} \exp\left\{ -\lambda \sum_{x \in \Lambda} \int_T Q(q_x(\tau))\, d\tau \right\}, \tag{15}$$

and $Z_{\Lambda,T}$ is a normalization factor.

We note that the proof that the limit (14) exists requires subtle methods in statistical physics and quantum field theory (e.g., cluster expansions, or correlation inequalities, etc.).

Finally, we can use the remark at the end of 5.0 and take as the starting point for the construction of the limit measure not the formal Hamiltonian H, but rather the formal Euclidean classical action $S(\{q_x(\tau)\})$, which for a system of anharmonic oscillators has the form

$$\begin{aligned} S_{\text{Eucl}}(\{q_x(\tau)\}) &= -i \int \left(\frac{1}{2} \sum_x \dot{q}_x^2 + \frac{1}{2} \sum_{x_1, x_2} a_{x_1, x_2} q_{x_1} q_{x_2} + \sum_x Q(q_x) \right) d\tau \\ &= i(S_{\text{Eucl}}^{\text{harm}} + S_{\text{Eucl}}^{\text{anharm}}). \end{aligned} \tag{16}$$

The term in (16) quadratic in $\{q_x(\tau)\}$ generates a Gaussian measure μ_0 for harmonic oscillators, and its Gibbs modification by means of the perturbation $iS_{\text{Eucl}}^{\text{anharm}}$ coincides

with the modification (14) and (15). This last approach is convenient because it makes it unnecessary to introduce the hamiltonians H_Λ explicitly, and it is the one most often used in the Euclidean approach to the models of Euclidean quantum field theory to be described below.

We conclude this section by stressing that, as we have seen from the examples given above, one very convenient way to study infinite physical systems in the ground state, which avoids the need to explicitly construct the Heisenberg dynamics (in some suitable C^*-algebra), is to construct the Markov field and introduce the associated Hilbert space $\mathcal{H}_{\text{phys}}$ and a stochastic semigroup acting on it (i.e., what is usually referred to as a "Euclidean object"). Then the dynamics itself (more precisely, the associated GNS representation for the ground state) is defined on the space $\mathcal{H}_{\text{phys}}$ in terms of H^{renorm}, the infinitesimal generator of the semigroup $\mathcal{J}_t$.

2. Euclidean quantum fields (general axioms). For Euclidean quantum field models, the appropriate "Euclidean object" (when fermion fields are present) turns out to be less familiar than the random probabilistic processes encountered in the above consideration of boson models. We will therefore begin our treatment of the Euclidean approach to the quantum field models with a discussion on a widely adopted system of axioms.

Field algebra. A *boson field* algebra $\mathcal{E}_B^n = \mathcal{E}_B^n(R^{\nu+1})$ is a commutative topological algebra with unit 1 (and a metrizable topology) and a distinguished set of generators $\{\xi_\varphi,\ \varphi \in S^n(R^{\nu+1})\}$ indexed by the elements of the Schwartz space $S^n(R^{\nu+1})$ of infinitely differentiable functions that decay rapidly at infinity (see [38]) and are defined on $R^{\nu+1}$ with values in $\mathbb{C}^n$ (or R^n). The map

$$S^n(R^{\nu+1}) \to \mathcal{E}_B\colon \varphi \to \xi_\varphi$$

is required to be linear and continuous. Thus, $\mathcal{E}_B$ is the completion of the commutative algebra of polynomials in the elements $\{\xi_\varphi\}$ of $\mathcal{E}_B$. A *fermion* field algebra $\mathcal{E}_F^n = \mathcal{E}_F^n(R^{\nu+1})$ is a (topological) Grassmann (super) algebra with unit 1, and having a distinguished set of odd (anticommuting) generators $\{\psi_\varphi, \overline{\psi}_\varphi,\ \varphi \in S^n(R^{\nu+1})\}$, indexed by the elements $\varphi \in S^n(R^{\nu+1})$. The maps $S^n(R^{\nu+1}) \to \mathcal{E}_F^n\colon \varphi \to \psi_\varphi$ and $\varphi \to \overline{\psi}_\varphi$ are required to be linear and continuous. The *full* field algebra $\mathcal{E} = \mathcal{E}^{n,m}(R^{\nu+1}) = \mathcal{E}_B^n(R^{\nu+1}) \otimes \mathcal{E}_F^n(R^{\nu+1})$ is the tensor product of the boson and fermion algebras.

A *Euclidean quantum field* is a quasistate $\langle\cdot\rangle$, defined either on a boson algebra $\mathcal{E}_B^n$ (boson field) or fermion algebra $\mathcal{E}_F^n$ (pure fermion field), or on the full algebra $\mathcal{E}^{n,m}$.

Positivity in the sense of Nelson and Symanzik. We assume that the boson part of the field $\langle\cdot\rangle$ on $\mathcal{E}_B^n$, i.e., the restriction of the quasistate $\langle\cdot\rangle$ to the boson subalgebra $\mathcal{E}_B^n$, is realized as some generalized random field. More precisely, this means that there exists a probability space $(\Omega, \Sigma, \mu\ )$ (Σ a σ-algebra, μ a probability measure defined on Σ) such that the algebra $\mathcal{E}_B$ is (topologically) isomorphic to the algebra $\widehat{L}_\infty(\Omega, \Sigma, \mu) = \bigcap_{1\le p<\infty} L_p(\Omega, \Sigma, \mu)$ and for every element $F \in \mathcal{E}_B$

$$\langle F\rangle = \int_\Omega \widetilde{F}\, d\mu,$$

where $\widetilde{F} \in \widehat{L}_\infty(\Omega, \Sigma, \mu)$ is the image of F under the imbedding $\mathcal{E}_B \to \widehat{L}_\infty(\Omega, \Sigma, \mu)$; and the topology on the algebra $\widehat{L}_\infty$ is defined by a countable family of norms $\|\cdot\|_{L_p}$. It

follows from these hypotheses that the Schwinger functions (the moment functions $\langle \xi_{\varphi_1}, \dots, \xi_{\varphi_n} \rangle$ for the boson field) are separately continuous in each variable $\varphi_i \in S^n(R^{\nu+1})$. In the case of a general field $\langle \cdot \rangle$ on the full field algebra, this property is postulated for the complete Schwinger functions

$$\langle \xi_{\varphi_1} \dots \xi_{\varphi_k} \psi_{\varphi_{k+1}} \dots \psi_{\varphi_{k+s}} \overline{\psi}_{\varphi_{k+s+1}} \dots \overline{\psi}_{\varphi_{k+s+m}} \rangle.$$

Translation invariance. We will always assume that the field $\langle \cdot \rangle$ is translationally invariant. This means that the quasistate $\langle \cdot \rangle$ is preserved by every homomorphism $U_s \colon \mathcal{E} \to \mathcal{E}$ of the algebra $\mathcal{E}$ into itself, $s \in R^{\nu+1}$ acting on the generators by the formula

$$U_s \xi_\varphi = \xi_{\varphi(\cdot - s)}$$

(and similarly for ψ_φ and $\overline{\psi}_\varphi$):

$$\langle U_s A \rangle = \langle A \rangle. \tag{17}$$

Physical positivity (Osterwalder-Schroeder positivity). We first introduce the following involutions:

i) In the space R^ν,

$$\mathcal{V}x = (-x^0, x^1, \dots, x^\nu), \qquad x = (x^0, x^1, \dots, x^\nu).$$

(One usually calls x^0 the *time* coordinate, and ν is the time involution.)

ii) In the space of functions

$$\begin{aligned} (\mathcal{V}_{\text{bos}}\varphi)(x) &= \overline{\varphi}(\mathcal{V}x), \qquad \varphi \in S^n(R^{\nu+1}), \\ (\mathcal{V}_{\text{ferm}}\varphi)(x) &= \varepsilon\overline{\varphi}(\mathcal{V}x), \qquad \varphi \in S^m(R^{\nu+1}), \end{aligned} \tag{17'}$$

where $\overline{\varphi}$ is the vector whose coordinates are the complex conjugates of those of φ, and ε is an $m \times m$ matrix satisfying $\varepsilon^2 = E$.

iii) In the algebra $\mathcal{E}$, with the action on the generators given by

$$\Theta\xi_\varphi = \xi\nu_{\text{bos}\,\varphi}, \qquad \Theta\psi_\varphi = \overline{\psi}\nu_{\text{ferm}\,\varphi}, \qquad \Theta\overline{\psi}_\varphi = \psi\nu_{\text{ferm}\,\varphi}.$$

The involution Θ defined on the generators is then extended to a continuous antilinear involution of the field algebra $\mathcal{E}$ by means of the relations

$$\Theta(A_1A_2) = \Theta(A_2)\Theta(A_1), \qquad A_1, A_2 \in \mathcal{E}. \tag{17''}$$

The field $\langle \cdot \rangle$ is said to be Θ-invariant if for every $A \in \mathcal{E}$

$$\langle \Theta A \rangle = \overline{\langle A \rangle}.$$

Evidently, for any Θ-invariant field $\langle \cdot \rangle$ the bilinear form

$$(A_1, A_2) = \langle \Theta A_1, A_2 \rangle \tag{18}$$

is Hermitian.

We denote by $\mathcal{E}^+ \subset \mathcal{E}$ the subalgebra of $\mathcal{E}$ with unit 1 generated by the elements $\{\xi_\varphi, \psi_\varphi, \overline{\psi}_\varphi\}$ such that $\varphi \subset \operatorname{supp} R^\nu_+ = \{x \colon x^0 > 0\}$ (the "future" algebra). A Θ-invariant field is said to be OS-positive if the Hermitian form (1) is nonnegative on the subalgebra $\mathcal{E}^+$, i.e.,

$$(A, A) = \langle \Theta A A \rangle \geq 0$$

for all $A \in \mathcal{E}^+$.

The physical Hilbert space. Let $\langle\cdot\rangle$ be a scalar OS-positive field. We use the nonnegative Hermitian form (18) to define an inner product on $\mathcal{E}^+$. Let $I_0 \subset \mathcal{E}^+$ be the subspace of all $A \in \mathcal{E}^+$ for which

$$(A, A) = 0.$$

We define an inner product in the quotient space $\mathcal{E}^+/I_0$ by

$$([A_1], [A_2]) = (A_1, A_2),$$

where $[A] \in \mathcal{E}^+/I_0$ denotes the coset class of the element $A \in \mathcal{E}^+$; the completion of $\mathcal{E}^+/I_0$ with respect to this inner product is called the *physical* Hilbert space $\mathcal{H}_{\text{phys}}$.

The transfer matrix. Consider the map

$$U_\tau : \mathcal{E}^+ \to \mathcal{E}^+ : A \to U_\tau A, \qquad \tau > 0,$$

where we write $U_\tau = U_{\{\tau,0,\dots,0\}}$ and the vector $(\tau, 0, 0, \dots, 0) \in R^{\nu+1}$ is directed along the positive time axis.

LEMMA 1. *For every $A \in \mathcal{E}^+$ and $\tau > 0$ we have the inequality*

$$(U_\tau A, U_\tau A) \le (A, A). \tag{19}$$

For the proof, see [38].

It follows from (19), in particular, that $U_\tau I_0 \subset I_0$, so that we have a well-defined operator on $\mathcal{H}_{\text{phys}}$:

$$\mathcal{J}_\tau : \mathcal{H}_{\text{phys}} \to \mathcal{H}_{\text{phys}},$$

acting on the elements $[A] \in \mathcal{E}_+/I_0$ by the formula

$$\mathcal{J}_\tau [A] = [U_\tau A]. \tag{19^a}$$

LEMMA 2. *The family of operators $\{\mathcal{J}_\tau, \tau \ge 0\}$ forms a strongly continuous semigroup of selfadjoint contractions on $\mathcal{H}_{\text{phys}}$.*

PROOF. That $\mathcal{J}_\tau$ is a semigroup is obvious; its strong continuity and the contraction property follow from (19), while the selfadjointness of $\mathcal{J}_\tau$ is a consequence of the evident equality

$$\Theta U_\tau = U_{-\tau}\Theta. \tag{20}$$

The semigroup $\mathcal{J}_\tau$ is usually called the *transfer matrix* of the field (or more precisely, the *transfer matrix semigroup*).

Lemma 2 and Stone's theorem [36] imply that

$$\mathcal{J}_\tau = \exp\{-\tau H\},$$

where H is a nonnegative selfadjoint operator acting on $\mathcal{H}_{\text{phys}}$. Plainly, $[1] \equiv \Omega$, where $1 \in \mathcal{E}^+$ is the identity of the algebra $\mathcal{E}$, is a ground-state vector for H with zero eigenvalue,

$$H\Omega = 0.$$

The operator $H = H_{\text{phys}}$ is called the (physical) *Hamiltonian* of the field, and Ω is its *vacuum* vector.

Momentum operators. It is clear that every homomorphism $U_{0,\overline{s}} \equiv U_{\overline{s}}$, where $\overline{s} \in R^\nu$, takes the algebra $\mathcal{E}^+$ into itself, and also, by (17) and (20), $\mathcal{H}_{\text{phys}}$ into itself. Thus it generates a strongly continuous unitary group $U_{\overline{s}}$, $\overline{s} \in R^\nu$ (the group of spatial translations) in $\mathcal{H}_{\text{phys}}$ which commutes with the semigroup $\mathcal{J}_\tau$. By Stone's theorem we have

$$U_{\overline{s}} = \exp\{i(s^1 P_1 + \cdots + s^\nu P_\nu)\}, \quad \overline{s} = (s^1, \ldots, s^\nu),$$

where $P = (P_1, \ldots, P_\nu)$ is a set of mutually commuting selfadjoint operators, called the *momentum operators.* They all commute with the Hamiltonian H, and

$$P_k \Omega = 0, \quad k = 1, \ldots, \nu.$$

Other symmetries. If a group of homomorphisms G acting on the algebra $\mathcal{E}$ corresponds to symmetries of the field (such as spatial rotations, "isotopic rotations", and so on) that take $\mathcal{E}$ into itself, preserve the quasistate $\langle\cdot\rangle$, and commute with the time translations U_τ, then we have a unitary representation $g \to T_g$, $g \in G$ of G on $\mathcal{H}_{\text{phys}}$ which commutes with the Hamiltonian H_{phys} of the field.

3. Markov field (boson case). In practice, one often uses another (equivalent) construction of the space $\mathcal{H}_{\text{phys}}$ and semigroup $\mathcal{J}_\tau$ which is based on the reversible Markov property and the stability of the boson field under reflection. We assume here that the Nelson-Symanzik axiom is satisfied, that the Euclidean boson translationally invariant field is realized as a probability distribution μ on a measure space (Ω, Σ), and that the field algebra $\mathcal{E}$ coincides with the algebra $\widehat{L}_\infty(\Omega, \Sigma, \mu) = \bigcap_{1 \le p < \infty} L_p(\Omega, \Sigma, \mu)$. For each t and $\varepsilon > 0$, we define $\Sigma_t^\varepsilon \subset \Sigma$ to be the σ-algebra generated by the random variables $\{\xi_\varphi, \varphi \in S^n(R^{\nu+1})\}$, for which the function φ is supported in the strip $\{x : |x^0 - t| < \varepsilon\}$.

We write $\Sigma_t = \bigcap_{\varepsilon > 0} \Sigma_t^\varepsilon$. The boson field $\langle\cdot\rangle$ will be said to have the *reversible Markov* property if it is Θ-invariant and for all $A_+ \in \mathcal{E}^+$ and $A_- \in \mathcal{E}^-$ (the "past" algebra $\mathcal{E}^-$ is defined analogously to the "future" algebra $\mathcal{E}^+$), we have

$$M(A_+ A_- / \Sigma_0) = M(A_+ / \Sigma_0) M(A_- / \Sigma_0), \tag{20^a}$$

where $M(\cdot/\Sigma_0) \equiv \langle\cdot/\Sigma_0\rangle$ is the conditional expectation relative to the σ-algebra Σ_0 (see [11]).

A reversible Markov field is said to be *stable under reflection* if every element $F \in \widehat{L}_\infty(\Omega, \Sigma, \mu)$ transforms under the involution Θ by the formula

$$\Theta F = \overline{F}.$$

LEMMA 3. *Let the boson field $\langle\cdot\rangle$ be stable under reflection and have the reversible Markov property. Then it is OS-positive, and the space $\mathcal{H}_{\text{phys}}$ constructed in the previous section is canonically isomorphic to the space $\mathcal{H}_{\text{phys}} = \widehat{L}_\infty(\Omega, \Sigma, \mu)$. This isomorphism is defined on the elements $[A] \in \mathcal{E}^+/I_0$, $A \in \mathcal{E}^+ = \widehat{L}_\infty(\Omega, \Sigma, \mu)$ by the formula*

$$[A] \to \langle A/\Sigma_0\rangle \in \widehat{L}_\infty(\Omega, \Sigma_0, \mu)$$

(here the σ-algebra Σ_+ is generated by the variables $\{\xi_\varphi, \operatorname{supp}\varphi \subseteq R_+^{\nu+1}\}$). Under this isomorphism, the semigroup $\mathcal{J}_\tau$ on $\mathcal{H}_{\text{phys}}$ goes into the unitarily equivalent semigroup $\widetilde{\mathcal{J}}_\tau$ acting on $L_2(\Omega, \Sigma_0, \mu)$ by the formula

$$\overline{\mathcal{J}}_\tau f = \langle U_\tau f/\Sigma_0\rangle = P_0 U_\tau f, \qquad f \in L_2(\Omega, \Sigma_0, \mu),$$

where U_τ is the unitary operator on $L_2(\Omega, \Sigma, \mu)$ corresponding to "time translation" by τ, and P_0 is the projection of $L_2(\Omega, \Sigma, \mu)$ onto the subspace $L_2(\Omega, \Sigma_0, \mu)$.

PROOF. For any $A \in \mathcal{E}^+ = \widehat{L}(\Omega, \Sigma_+, \mu)$ we have the decomposition

$$A = A_0 + A', \tag{21}$$

where $A_0 = \langle A/\Sigma_0 \rangle \in \widehat{L}_\infty(\Omega, \Sigma_0, \mu)$ and $\langle A'/\Sigma_0 \rangle = 0$. Further,

$$(A, A) = \langle\langle \Theta A A/\Sigma_0 \rangle\rangle = \langle\langle \Theta A/\Sigma_0 \rangle \langle A/\Sigma_0 \rangle\rangle,$$

where we have used the Markov property. From the decomposition (21) and the stability under reflection, we see finally that

$$(A, A) = (A_0, A_0) = \int_\Omega |A_0|^2 d\mu.$$

The remaining assertions in the lemma are proved similarly.

We note that the spatial translations $x \to x + (0, \overline{s})$, $\overline{s} \in R^\nu$, in $R^{\nu+1}$ generate a group of unitary operators $\{U_{\overline{s}}, \overline{s} \in R^\nu\}$ on $\mathcal{H}_{\text{phys}}$ that commute with the transfer matrix $\mathcal{J}_\tau$.

Example. Free scalar neutral field in $R^{\nu+1}$. In this case the generators ξ_φ are indexed by real functions $\varphi \in S(R^{\nu+1})$. The field is Gaussian with zero mean and covariance

$$\langle \xi_{\varphi_1} \xi_{\varphi_2} \rangle = c((-\Delta^{(\nu+1)} + m^2)^{-1} \varphi_1, \varphi_2) = c \int_{R^{\nu+1}} \frac{\widetilde{\varphi}_1(p) \overline{\widetilde{\varphi}_2(p)}}{p^2 + m^2}\, dp, \tag{22}$$

where $c > 0$, $m \geq 0$ ($m > 0$ for dimensions $\nu = 0, 1$), $\Delta^{(\nu+1)}$ is the Laplace operator on $R^{\nu+1}$, $(\cdot, \cdot)$ is the inner product in $L_2(R^{\nu+1}, d\overline{x})$, and $\widetilde{\varphi}(p)$ is the Fourier transform of $\varphi \in S^{R^{\nu+1}}$. The above field is called a *Nelson field* (see [58]), or (for $\nu = 1$) an Ornstein-Uhlenbeck field (process), see 5.0 above.

The formal Euclidean action for the field (22) is

$$S_{\text{Eucl}} = ic \int_{R^{\nu+1}} L(\varphi)\, d^{\nu+1} x,$$

where the Lagrangian $L(\varphi)$ is given by

$$L(\varphi) = (\nabla \varphi(x))^2 + m\varphi^2(x). \tag{23}$$

LEMMA 4. *The Gaussian field* (22) *is stable under reflection and has the reversible Markov property*; *it is invariant under the motions of the space $R^{\nu+1}$. Every scalar neutral* (*i.e., real*) *Gaussian field with these properties is of the form* (22).

We will not prove this lemma here (see [32, 33]). We observe further that the Markov property of (22) reflects the local nature of the Lagrangian (23) for this field.

One finds without difficulty that the space $\mathcal{H}_{\text{phys}}$ for this field is isomorphic to the Fock space $\mathcal{F}_s(\mathcal{H}_{-1/2})$, where $\mathcal{H}_{-1/2} = \mathcal{H}_{-1/2}(R^\nu)$ is the Sobolev space of functions of $x \in R^\nu$ with the inner product

$$((-\Delta^{(\nu)} + m^2)^{-1/2} \varphi_1, \varphi_2) = \int_{R^\nu} \frac{\widetilde{\varphi}_1(p) \overline{\widetilde{\varphi}_2(p)}}{(p^2 + m^2)^{1/2}}\, dp,$$

and $-\Delta^{(\nu)}$ is the Laplace operator in R^ν. The transfer matrix $\mathcal{J}_\tau$ coincides with the operator

$$\Gamma(\exp\{-(-\Delta^{(\nu)}+m^2)^{1/2}\tau\}), \qquad \tau>0$$

on $\mathcal{F}_s(\mathcal{H}_{-1/2})$.

More complicated (non-Gaussian) boson scalar Markov fields can be obtained by modifying the Nelson field by a self-interaction of the form

$$V=\int :P(\xi(x)):\, dx,$$

where P is an even polynomial of degree, at least four, which is bounded from below, and $:\cdot:$ denotes the Wick ordering corresponding to the Nelson Gaussian field (22) (see [26]). Such modifications in the case when $\nu=1$ have been widely studied using a rather difficult and refined technique based in part on cluster expansions (see the books [12] and [38]).

4. The Markov fermion field.

A. Digression on probability theory on a Grassmann algebra. We will need some results from the theory of quasistates on a Grassmann algebra; this theory has some formal similarities with ordinary probability theory.

Let $\mathfrak{A}=\mathfrak{A}_N$, where $N=\{1,\dots,n\}$, be the Grassmann algebra with 1 on finitely many generators $\alpha_1,\dots,\alpha_n$. Every element $\mathfrak{A}$ of can be expressed uniquely in the form

$$f=\sum_{T\subseteq N} C_T\alpha_T, \tag{24}$$

where the sum is over all subsets of N, and

$$\alpha_T=\alpha_{i_1}\alpha_{i_2}\dots\alpha_{i_k}, \qquad T=\{i_1<i_2<\dots<i_k\}.$$

We observe that an element f has a two-sided (left and right) inverse if and only if its free coefficient C_φ is nonzero. Every invertible $f\in\mathfrak{A}$ can be written uniquely as

$$f=C_\emptyset\exp\{\widehat{f}\}, \tag{25}$$

where $\widehat{f}\in\mathfrak{A}$ is an element with $C_\varphi=0$. Conversely, every element of the form (25) is invertible, with inverse given by

$$f^{-1}=C_\emptyset^{-1}\exp\{-\widehat{f}\}.$$

We note that the algebra $\mathfrak{A}$ is the tensor product

$$\mathfrak{A}=\mathfrak{A}_1\otimes\dots\otimes\mathfrak{A}_n,$$

(in the sense of superalgebras, see [23]), where each $\mathfrak{A}_k$, $k=1,\dots,n$, is the two-dimensional Grassmann algebra with 1 on the single generator α_k.

The Berezin integral. There exists a standard quasistate $\langle\cdot\rangle_0$, called the Berezin integral, on the algebra $\mathfrak{A}$. It is defined as the product of the quasistates $\langle\cdot\rangle_0^k$ on the algebras $\mathfrak{A}_k$, $k = 1, \dots, n$, given by

$$\langle 1\rangle_0^k = 0, \qquad \langle\alpha_k\rangle_0^k = 1.$$

Thus, for an element f expressed in the form (24) we have

$$\langle f\rangle_0 = C_N, \tag{25'}$$

where C_N is the coefficient of the highest monomial $\alpha_1\alpha_2\dots\alpha_n$. The standard notation for the quasistate $\langle\cdot\rangle_0$ is

$$\langle f\rangle_0 \equiv \int_{\mathfrak{A}} f\, d\alpha_1 \dots d\alpha_n \equiv \int_{\mathfrak{A}} f\, d\alpha_N. \tag{26}$$

The density of a quasistate.

LEMMA 5. *Every quasistate $\langle\cdot\rangle$ on the algebra $\mathfrak{A}$ is uniquely representable in the form*

$$\langle f\rangle = \int_{\mathfrak{A}} fg\, d\alpha_N, \tag{27}$$

where $g \in \mathfrak{A}$ is called the density of $\langle\cdot\rangle$.

PROOF. Let

$$M_T = \langle\alpha_T\rangle$$

be the moment of the quasistate $\langle\cdot\rangle$. Then if we set

$$g = \sum_{T\subseteq N} (-1)^{\pi(T)} M_{T'}\alpha_T, \tag{28}$$

where $T' = N \setminus T = (i'_1, \dots, i'_{n-k})$, $T = (i_1, \dots, i_k)$, and $\pi(T)$ is the sign of the permutation

$$(i'_1, \dots, i'_{n-k}, i_1, \dots, i_k),$$

we see that (27) is satisfied for all $f \in \mathfrak{A}$. The uniqueness of the representation (27) is obvious.

Note that (28) shows that for an even quasistate (i.e., one which vanishes on all odd elements $f \in \mathfrak{A}$) and even n, the density g is an even element of $\mathfrak{A}$. We will always assume henceforth without further mention that $\langle\cdot\rangle$ is an even quasistate and n is even.

A quasistate $\langle\cdot\rangle$ on $\mathfrak{A}$ will be called *regular* if its density g is invertible, i.e., according to (28), if the total moment is nonzero:

$$M_N \neq 0. \tag{28'}$$

Partial Berezin integral. Let $T \subset N$ be a subset with an even number of elements and let $\mathfrak{A}_T \subset \mathfrak{A}_N$ be a Grassmann subalgebra of $\mathfrak{A}_N$. We define the partial Berezin integral

$$\int_{\mathfrak{A}_{N-T}} f \, d\alpha_{N\setminus T}$$

to be the linear map $\mathfrak{A} \to \mathfrak{A}_T$ given on monomials α_S, $S \subseteq N$, by

$$\int \alpha_S d\alpha_{N\setminus T} = \begin{cases} 0 & \text{if } N \setminus T \not\subseteq S, \\ (-1)^{\pi(S,T)}\alpha_{T\cap S} & \text{if } N \setminus T \subseteq S, \end{cases}$$

where $\pi(S,T)$ is the sign of the permutation taking all the elements $i_j \in T \cap S$ in the set $S = \{i_1, \dots, i_k\}$ to the left of all the elements in $N \setminus T \subseteq S$ (and preserving the ordering of the elements in $T \cap S$ and $N \setminus T$ considered separately). It is easily verified that the repeated Berezin integral satisfies

$$\int_{\mathfrak{A}_T} \left(\int_{\mathfrak{A}_{N-T}} f d\alpha_{N\setminus T} \right) d\alpha_T = \int_{\mathfrak{A}} f \, d\alpha_N. \tag{29}$$

For any subset $T \subseteq N$ of even cardinality, we write $\langle\cdot\rangle_T$ for the restriction of the quasistate to the subalgebra $\mathfrak{A}_T$, and g_T is the density of $\langle\cdot\rangle_T$ on $\mathfrak{A}_T$. It follows easily from (29) that

$$g_T = \int_{\mathfrak{A}_{N\setminus T}} g \, d\alpha_{N\setminus T}.$$

According to (28′), the quasistate $\langle\cdot\rangle_T$ is regular if

$$M_T \neq 0.$$

The characteristic functional. Let $\langle\cdot\rangle$ be a quasistate on an algebra $\mathfrak{A}$ with density g_T and let $F = \{f_1, \dots, f_s\}$ be a set of odd elements on $\mathfrak{A}$. We enlarge $\mathfrak{A}$ by adding new Grassmann generators $\eta_1, \dots, \eta_s$. The partial integral

$$\int_{\mathfrak{A}} \exp\{f_1\eta_1 + f_2\eta_2 + \cdots + f_s\eta_s\} g \, d\alpha_N = \psi_F \in \widehat{\mathfrak{A}}, \tag{28″}$$

is called the *characteristic functional* of F (here $\widetilde{\mathfrak{A}}$ is the Grassmann algebra with the unit on the generators $\eta_1, \dots, \eta_s$).

Moment formulas. The linear operations $f\partial/\partial\eta_i$ and $\partial f/\partial\eta_i$ (left and right differentiation) for $f \in \widehat{\mathfrak{A}}$ are defined on the algebra $\mathfrak{A}$ as follows. On monomials η_T they are given by the formula

$$\frac{\partial\eta_T}{\partial\eta_i} = \begin{cases} 0, & i \notin T, \\ (-1)^{\sigma_T^{\text{left}}(i)}\eta_{T\setminus\{i\}}, & i \in T, \end{cases}$$

where $\sigma_T^{\text{left}}(i)$ is the number of factors in η_T preceding η_i. The right derivative $\eta_T\partial/\partial\eta_i$ is given by the same formula, except that $\sigma_T^{\text{left}}(i)$ is replaced by $\sigma_T^{\text{right}}(i)$ the number of factors in η_T that follow η_i

LEMMA 6. *The formula*

$$\langle f_{i_1} \dots f_{i_k} \rangle = \left(\frac{\partial}{\partial \eta_{i_1}} \frac{\partial}{\partial \eta_{i_2}} \dots \frac{\partial}{\partial \eta_{i_k}} \psi_F \right)_{\eta=0} = \left(\frac{\partial}{\partial \eta_T} \psi_F \right)_{\eta=0} \tag{30}$$

holds, where $(h)_{\eta=0}$ *denotes the free coefficient on expansion* (24) *for* $h \in \widehat{\mathfrak{A}}$.

PROOF. The result (30) follows from the identity

$$\exp\left\{ \sum_i f_i \eta_i \right\} = \prod_i (1 + f_i \eta_i) = 1 + \sum_{\{i_1,\dots,i_k\}} f_{i_1} f_{i_2} \dots f_{i_k} \eta_{i_k} \dots \eta_{i_1}$$

for the exponential.

Conditional expectation. Let $\langle \cdot \rangle$ be a quasistate on $\mathfrak{A}$ and $T \subseteq N$ a subset of even cardinality such that the state $\langle \cdot \rangle_T$ on $\mathfrak{A}_T$ is regular. The *conditional expectation* $\langle f \mid \mathfrak{A}_T \rangle$ is defined as the linear map $\mathfrak{A} \to \mathfrak{A}_T$ given by

$$\langle f | \mathfrak{A}_T \rangle \equiv \langle f | T \rangle \stackrel{\text{def}}{=} \int_{\mathfrak{A}_{N \setminus T}} f g \, d\alpha_{N \setminus T} (g_T)^{-1} = g_T^{-1} \int_{\mathfrak{A}_{N \setminus T}} f g \, d\alpha_{N-T}, \tag{31}$$

where we recall that g_T is the density of the quasistate $\langle \cdot \rangle_T$ (the last equality in (31) follows from the evenness of the element g_T^{-1}). The element

$$g g_T^{-1} \in \mathfrak{A} \tag{32}$$

is called the *conditional density* of $\langle \cdot \rangle$ relative to the algebra $\mathfrak{A}_T$. We note that the definitions (31) and (32) are formally the same as corresponding concepts in the classical theory of probability.

It is not difficult to check that the conditional expectation $\langle \cdot \mid T \rangle$ has the following properties, which again are reminiscent of the properties of the classical conditional expectation.

(i) Let $T_1 \subseteq T_2 \subseteq N$ be subsets of $\mathcal{N}$ of even cardinality and such that the states $\langle \cdot \rangle_{T_1}$ and $\langle \cdot \rangle_{T_2}$ are regular. Then

$$\langle \langle f | T_2 \rangle | T_1 \rangle = \langle f | T_1 \rangle. \tag{33}$$

This is verified by a straightforward calculation, see [52].

(ii) If $h \in \mathfrak{A}_T$, then

$$\langle hf | T \rangle = h \langle f | T \rangle, \qquad \langle fh | T \rangle = \langle f | T \rangle h. \tag{33^a}$$

Moreover,

$$\langle 1 | T \rangle = 1.$$

(iii) For every $f \in \mathfrak{A}$, we have

$$\langle \langle f | T \rangle \rangle = \langle f \rangle.$$

The last two properties (ii) and (iii) uniquely characterize the conditional expectation in the classical case. In the Grassmann case, we have the following result.

LEMMA 7. *Let a quasistate $\langle\cdot\rangle$ be given on $\mathfrak{A}$ and let $T \subseteq N$ be a subset of even cardinality such that the quasistate $\langle\cdot\rangle_T$ is regular. Then there is just one map $\pi\colon \mathfrak{A} \to \mathfrak{A}_T$ having the properties* (ii) *and* (iii)*, i.e.,*

$$\pi(hf) = h\pi(f), \qquad \pi(fh) = \pi(f)h, \qquad h \in \mathfrak{A}_T, \quad \pi 1 = 1 \tag{34}$$

and

$$\langle\pi(f)\rangle = \langle f\rangle. \tag{35}$$

It coincides with the conditional expectation $\langle\cdot \mid T\rangle$.

PROOF. By (34) we see that

$$\pi(a_R) = (-1)^{\pi(S,P)} a_P \pi(a_S) = (-1)^{\pi(S,P)} a_P \sum_L K_L^S a_L,$$

where $P = R \cap T$, $S = R \cap (T \setminus N) \subset R$, $\pi(S,P)$ is the permutation taking the set P to the left of the set $S \subset R$, and K_L^S are the coefficients in the expansion of the element $\pi(a_S) \in \mathfrak{A}_T$. Further, by (35) we have

$$\langle\pi(a_R)\rangle = (-1)^{\pi(R,P)} \sum Q_{P,L} K_L^S = M_{P\cup S}, \tag{36}$$

where $Q_{P,L} = \langle a_P a_L\rangle$, $L, P \subseteq T$.

For fixed S and P running over the entire subset T, we can regard the right-hand side of (36) as a system of linear equations with matrix $\{Q_{P,L}\}$ for the coefficients $\{K_L^S, L \subseteq T\}$. One computes without difficulty that

$$\operatorname{Det}\{Q_{P,L}\} = \pm(\langle a_T\rangle)^{2^{|T|}} \neq 0$$

(in view of the regularity of $\langle\cdot\rangle_T$). Thus the coefficients K_L^S are uniquely determined, and hence π coincides with $\langle\cdot \mid T\rangle$.

Conditional characteristic functional. Let $\langle\cdot \mid T\rangle$ be a conditional expectation with density $g(g_T)^{-1}$, and let $F\{f_1, \dots, f_s\}$ be a set of odd elements of $\mathfrak{A}$. We again enlarge the algebra $\mathfrak{A}$ by adding new generators $\eta_1, \dots, \eta_s$ and consider the partial integral

$$\psi_F^{\mathrm{cond}} = \int_{\mathfrak{A}_{N-T}} \exp\{f_1\eta_1 + \dots + f_s\eta_s\} d\alpha_{N\setminus T} \in \mathfrak{A}_T \otimes \widehat{\mathfrak{A}}_N,$$

where $\widetilde{\mathfrak{A}}$ is the Grassmann algebra with 1 on the generators $\eta_1, \dots, \eta_s$; this integral is called the *conditional characteristic functional.* The conditional moments satisfy the formula

$$\langle f_{i_1} \dots f_{i_k} | T\rangle = \left(\frac{\partial}{\partial\eta_{i_k}} \dots \frac{\partial}{\partial\eta_{i_1}} \psi_F^{\mathrm{cond}}\right)_{\eta=0}, \tag{36′}$$

where $(h)_{\eta=0}$ for $h \in \mathfrak{A}_T \otimes \widehat{\mathfrak{A}}$ denotes the part of the expansion (24) for h that does not contain the generators η_i, $i = 1, \dots, s$.

Independence. The two sets $F = \{f_1, \dots, f_s\}$ and $H = \{h_1, \dots, h_j\}$ are said to be independent relative to the quasistate $\langle\cdot\rangle$ if for every $f \in \mathfrak{A}_F$ and $h \in \mathfrak{A}_H$ we have

$$\langle fh\rangle = \langle f\rangle\langle h\rangle,$$

where $\mathfrak{A}_F \subseteq \mathfrak{A}$ and $\mathfrak{A}_H \subseteq \mathfrak{A}$ are the subalgebras with 1 generated by the sets F and H, respectively.

Let $\mathfrak{A}_T$ be a subalgebra of $\mathfrak{A}$, where T is a set of even cardinality, for which the conditional expectation $\langle\cdot \mid T\rangle$ is defined. The system $F = \{f_1, \dots, f_s\}$ is independent of the algebra $\mathfrak{A}_T$ if and only if the conditional expectation for every element $f \in \mathfrak{A}_F$ is given by

$$\langle f|T\rangle = \langle f\rangle 1. \tag{36''}$$

¿From this it also follows easily that the system F is independent of $\mathfrak{A}_T$ if and only if the conditional characteristic functional satisfies

$$\psi_F^{\mathrm{cond}} \in \widehat{\mathfrak{A}}, \tag{36'''}$$

i.e., its expansion (24) does not involve any generators $\alpha_i \in \mathfrak{A}_T$.

Gaussian gauge-invariant quasistates. We consider the Grassmann algebra $\mathfrak{A}$ on $2n$ generators (n arbitrary), which are divided into two groups: $\alpha_1, \dots, \alpha_n, \overline{\alpha}_{\overline{1}}, \dots, \overline{\alpha}_{\overline{n}}$, where the indices are ordered as follows:

$$1 < \overline{1} < \dots < n < \overline{n}$$

(below we frequently omit the bar over the subscript k and write $\overline{\alpha}_k$ for $\overline{\alpha}_{\overline{k}}$).

In §3 above, we defined Gaussian gauge-invariant quasistates on a CAR algebra. This definition carries over *verbatim* to the case of a Grassmann algebra with generators $\alpha_1, \dots, \alpha_n, \overline{\alpha}_1, \dots, \overline{\alpha}_n$. Here we provide some information on Gaussian gauge-invariant quasistates and formulas associated with them. Each such state is defined by its covariance matrix $D = \{d_{ij}\}$, where

$$d_{ij} = \langle \alpha_i \overline{\alpha_j}\rangle.$$

We will need the formula (*Gaussian Berezin integral*)

$$\int \exp\left\{\sum c_{ij}\alpha_i\overline{\alpha}_j\right\} d\overline{\alpha}_n\, d\alpha_n \dots d\overline{\alpha}_1\, d\alpha_1 = \det\{c_{ij}\}, \tag{37}$$

where $C = \{c_{ij}\}$ is an arbitrary matrix. This formula follows easily by expanding out the exponential in expression (25′) for the Berezin integral.

LEMMA 8. *Let $\langle\cdot\rangle$ be a Gaussian gauge-invariant quasistate on the algebra $\mathfrak{A}$ with a nondegenerate covariance matrix D. Then the density of $\langle\cdot\rangle$ is equal to*

$$g = (\det\{-C\})^{-1} \exp\left\{-\sum c_{ij}\alpha_i\overline{\alpha}_j\right\}, \tag{38}$$

where the matrix $C = \{c_{ij}\}$ is equal to D^{-1}.

PROOF. We first observe that by (37), the element given by (38) is the density of some state $\langle\cdot\rangle$. We will show that this state is a Gaussian gauge-invariant quasistate with covariance matrix D. To this end we consider the characteristic functional of a system of generators $\alpha_1,\dots,\alpha_n,\overline{\alpha}_1,\dots,\overline{\alpha}_n$, which we write in a form somewhat different from (28″):

$$\begin{aligned}\psi &= \left\langle \exp\left\{\sum_i(\alpha_i\eta_i+\overline{\eta}_i\overline{\alpha}_i)\right\}\right\rangle\\ &= \det(-D)\int_{\mathfrak{A}}\exp\left\{\sum_{ij}c_{ij}\alpha_i\overline{\alpha}_j\right\}+\sum_i(\alpha_i\eta_i+\overline{\eta}_i\overline{\alpha}_i)\,d\alpha_N \qquad (39)\\ &= \exp\left\{\sum_{ij}d_{ij}\overline{\eta}_i\eta_j\right\}.\end{aligned}$$

Indeed, if we introduce the elements

$$\beta_i=\alpha_i-\sum_j d_{ij}\overline{\eta}_j,\qquad \overline{\beta}_i=\overline{\alpha}_i-\sum_j d_{ij}\eta_j$$

and use the change-of-variables formula for Berezin integrals (see [5]), we arrive at equation (39) after some straightforward algebra. For the characteristic functional as defined in (39), the moments $\langle\alpha_Q\rangle$, where $Q=T\cup\overline{T}$, $T=\{i_1<i_2<\dots<i_k\}$, $\overline{T}=\{\overline{j}_1<\dots<\overline{j}_s\}$, are given by the formula

$$\langle\alpha_Q\rangle=\frac{\partial}{\partial\overline{\eta}_{\overline{j}_s}}\dots\frac{\partial}{\partial\overline{\eta}_{\overline{j}_1}}\psi\frac{\partial}{\partial\eta_{i_1}}\dots\frac{\partial}{\partial\eta_{i_k}}. \qquad (40)$$

Applying this to ψ, we arrive at equation (1.3) for the moments of a Gaussian quasistate. It follows from (40) that

$$\langle\alpha_i\alpha_j\rangle=\langle\overline{\alpha}_i\overline{\alpha}_j\rangle=0,\qquad \langle\alpha_i\overline{\alpha}_j\rangle=d_{ij}.$$

The lemma is proved.

Wick ordering. In an algebra $\mathfrak{A}$ equipped with a Gaussian (gauge-invariant) quasistate, one can define the $:\cdot:$ (Wick ordering) operation in exactly the same way as for a free quasistate on the CAR algebra (see (6.3)). Moreover, the Wick monomials $:\alpha_Q:$ in $\mathfrak{A}$ can be defined using the Wick exponential:

$$\begin{aligned}:\exp\left\{\left(\sum_i\eta_i\alpha_i+\overline{\alpha}_i\overline{\eta}_i\right)\right\}:&=\exp\left\{\left(\sum_i\eta_i\alpha_i+\overline{\alpha}_i\overline{\eta}_i\right)\right\}\psi^{-1}\\ &=\exp\left\{\left(\sum_i\eta_i\alpha_i+\overline{\alpha}_i\overline{\eta}_i\right)-\sum_{ij}d_{ij}\overline{\eta}_i\eta_j\right\}.\end{aligned}$$

Namely, we have the formula

$$:\alpha_Q:=\left(\frac{\partial}{\partial\overline{\eta}_{i_k}}\dots\frac{\partial}{\partial\overline{\eta}_{i_1}}:\exp\left\{\left(\sum_i\overline{\eta}_i\alpha_i+\overline{\alpha}_i\eta_i\right)\right\}:\frac{\partial}{\partial\eta_{j_1}}\dots\frac{\partial}{\partial\eta_{j_s}}\right)_{\eta=0,\overline{\eta}=0},$$

where

$$Q = T \cup \overline{T}, \qquad T = \{j_1, \dots, j_s\}, \qquad \overline{T} = \{i_1, \dots, i_k\}$$

and $(\cdot)_{\eta=0,\overline{\eta}=0}$ is defined as in (36′).

In computing the moments and semi-invariants (see [26]) of the Wick monomials, one can use the diagram technique described in [26] (with proper allowance for the signs involved in the ordering of the factors). For example, the mean

$$\langle : \alpha_{Q_1} :: \alpha_{Q_2} : \rangle \tag{41}$$

is equal to the sum over all such "pairings" of the factors in α_{Q_1} and α_{Q_2} (as in the general formula in [26]), in which a factor in α_{Q_1} can be paired only with a factor in α_{Q_2}.

¿From this it follows in particular that the mean (41) is nonzero only when $Q_1 = T_1 \cup \overline{T}_1$ and $Q_2 = T_2 \cup \overline{T}_2$, where $|T_1| = |\overline{T}_2|$ and $|\overline{T}_1| = |T_2|$.

Conditional expectation. Let $\mathfrak{A}_T \subset \mathfrak{A}$ be a subalgebra with generators $\{\alpha_i, \overline{\alpha}_i, i \in T\}$. The restriction $\langle\cdot\rangle_T$ of a Gaussian gauge-invariant quasistate $\langle\cdot\rangle$ to $\mathfrak{A}_T$ is clearly again Gaussian gauge-invariant, and its covariance matrix is $D_T = \{d_{ij}, i, j \in T\}$. Recall that the matrix D_T is nonsingular. Then the density of $\langle\cdot\rangle_T$ on $\mathfrak{A}_T$ has the form (38), hence by (25) $\langle\cdot\rangle_T$ is regular and the conditional expectation $\langle\cdot \mid T\rangle$ is defined. The rule stated in the next lemma is useful for calculating $\langle\cdot \mid T\rangle$. We can write the matrix D in block form as

$$D = \begin{Bmatrix} D_{T,T} & D_{T,N\setminus T} \\ D_{N\setminus T,T} & D_{T\setminus N,T\setminus N} \end{Bmatrix},$$

where $D_{T,T} = D_T$, $D_{T,N\setminus T} = \{d_{ij}, i \in T, j \in N \setminus T\}$, and so on.

Lemma 9. A) *The following formulas are valid*:

$$\begin{aligned} \langle \alpha_i | T \rangle &= \sum_{j \in T} b_{ij} \alpha_j = \alpha_i^T, \qquad i \in N \setminus T, \\ \langle \overline{\alpha}_i | T \rangle &= \sum_{j \in T} c_{ij} \alpha_j = \overline{\alpha}_i^T, \qquad i \in N \setminus T, \end{aligned} \tag{41′}$$

where the matrices $B = \{b_{ij}\}$ *and* $C = \{c_{ij}\}$ *are given by*

$$B = D_{N\setminus T,T}(D_{T,T})^{-1}, \qquad C = (D_{T,T}^{-1} D_{T,N\setminus T})^{tr}. \tag{42}$$

B) *The elements*

$$\beta_i = \alpha_i - \langle \alpha_i | T \rangle, \qquad \overline{\beta}_i = \overline{\alpha}_i - \langle \overline{\alpha}_i | T \rangle, \qquad i \in N \setminus T$$

do not depend on the algebra $\mathfrak{A}_T$ (*with respect to the quasistate* $\langle\cdot\rangle$).

C) *The restriction* $\langle\cdot\rangle_F$ *of a quasistate* $\langle\cdot\rangle$ *to the Grassmann algebra* $\mathfrak{A}_F$ *with* 1 *generated by the set of elements* $F = \{\beta_i, \overline{\beta}_i, i \in N \setminus T\}$ *is a Gaussian gauge-invariant quasistate, and its covariance matrix* $D_F = \{\langle \beta_i \overline{\beta}_j \rangle_F\}$ *is equal to*

$$D_F = D_{N\setminus T,N\setminus T} - D_{N\setminus T,T} D_{T,T}^{-1} D_{T,N\setminus T}. \tag{43}$$

The proof of this lemma is based on an explicit calculation of the conditional characteristic functional for the set of generators $\{\alpha_i, \overline{\alpha}_i, i \in N \setminus T\}$,

$$\psi^{\mathrm{cond}} = \left\langle \exp\left\{ \sum_i (\alpha_i \eta_i + \overline{\eta}_i \overline{\alpha}_i) \right\} | T \right\rangle,$$

for which we find by Gaussian quadratures that

$$\psi^{\mathrm{cond}} = \exp\left\{ \sum_{i,j} d_{ij}^F \overline{\eta}_i \eta_j + \sum_{i,j} (b_{ij} \alpha_i \eta_j + c_{ij} \overline{\eta}_i \overline{\alpha}_j) \right\}. \tag{44}$$

Here the matrices $D_F = \{d_{ij}\}$, $B = \{b_{ij}\}$, and $C = \{c_{ij}\}$ are given by (43), (42). Formula (41′) now follows immediately by using a formula analogous to equation (40) for the conditional moments. Moreover, from (44) we find the expression

$$\exp\left\{ \sum_{i,j} d_{ij}^F \overline{\eta}_i \eta_j \right\} \tag{45}$$

for the conditional characteristic functional for the system $\{\beta_i, \overline{\beta}_i, i \in N \setminus T\}$, whence part B) follows upon recalling (36‴). Part C) also follows from (45), and the lemma is proved.

Using the explicit expression for the Wick monomials and the independence of the sets $\{\beta_i, \overline{\beta}_i, i \in N \setminus T\}$ and $\{\alpha_j, \overline{\alpha}_j, j \in T\}$, it is easy to show that the Wick monomial satisfies

$$: \beta_Q \alpha_S :=: \beta_Q : \cdot : \alpha_S : .$$

Furthermore, using (36′) and the preceding lemma, we see that

$$\langle : \beta_Q : | T \rangle = 0.$$

Thus for every monomial α_Q we have

$$\langle : \alpha_Q : | T \rangle =: \alpha_Q^T :,$$

where the monomial α_Q^T is obtained from α_Q by replacing all the factors α_i, $\overline{\alpha}_i$ by α_i^T and $\overline{\alpha}_i^T$, respectively.

B. Reversible Markov fermion fields. We return once more to the fermion Euclidean field, i.e., to a quasistate on the field algebra $\mathcal{E}_F$. The preceding constructions of "conditional probabilities" on a finite Grassmann algebra serve as a prototype for defining the conditional expectation on the infinite Grassmann algebra $\mathcal{E}_F$.

As in the case of the boson field, we consider ε-neighborhood

$$Y_t^\varepsilon = \{x : |x_0 - t| < \varepsilon\}$$

of the time slice $Y_t = \{x : x_0 = t\}$, and the Grassmann algebra $\mathcal{E}_t^\varepsilon$ generated by the elements ψ_φ, $\overline{\psi}_\varphi$, for which $\operatorname{supp} \varphi \subset Y_t^\varepsilon$. We write

$$\mathcal{E}_t = \bigcup_{\varepsilon > 0} \mathcal{E}_t^\varepsilon,$$

and evidently

$$\Theta \mathcal{E}_0 = \mathcal{E}_0.$$

Let a Θ-invariant quasistate be defined on $\mathcal{E}_F$. Define the Hermitian bilinear form on $\mathcal{E}_0$ by

$$(F_1, F_2) = \lim_{\varepsilon \to +0} \langle U_{-\varepsilon} \Theta F_1 U_\varepsilon F_2 \rangle, \qquad F_1, F_2 \in \mathcal{E}_0, \tag{46}$$

where U_ε is the homomorphism of $\mathcal{E}_F$ induced by translation by ε along the time axis.

DEFINITION. A quasistate $\langle\cdot\rangle$ on the fermion field algebra $\mathcal{E}_F$ is said to be reversibly Markov if:

1) it is Θ-invariant;

2) the Hermitian form (46) is nonnegative:

$$(F, F) \geq 0, \qquad F \in \mathcal{E}_0;$$

3) for every $t > 0$, the conditional expectation $\langle\cdot \mid \mathcal{E}_t\rangle$ is defined as a linear continuous map of $\mathcal{E}_F$ into the subalgebra $\mathcal{E}_t$ which satisfies properties ii) and iii) for the conditional expectation for a finite Grassmann algebra (see the preceding subsection). Here it is assumed that this mapping is unique;

4) for all t and any two elements $F_+ \in \mathcal{E}_t^+$ and $F^- \in \mathcal{E}_t^-$ (the algebras $\mathcal{E}_t^\pm$ are generated by the elements $\psi(\varphi)$, $\overline{\psi}(\varphi)$ with $\operatorname{supp}\varphi \subset R_t^\pm = \{x_0 > t, (+), \text{ or } x_0 < t, (-)\}$),

$$\langle F_+ F_- | \mathcal{E}_t\rangle = \langle F_+ | \mathcal{E}_t\rangle \langle F_- | \mathcal{E}_t\rangle.$$

We note that 1) and 3) imply that

$$\langle \Theta F | \mathcal{E}_0\rangle = \Theta \langle F | \varepsilon_0\rangle.$$

LEMMA 10. *Let a reversible Markov translationally invariant quasistate $\langle\cdot\rangle$ be given on the algebra $\mathcal{E}_F$. Then $\langle\cdot\rangle$ is OS-positive and the Hilbert space $\mathcal{H}_{\text{phys}} = \overline{\mathcal{E}_F^+/I_0}$ (see subsectition 2 above) is canonically isomorphic to the Hilbert space $\overline{\mathcal{H}}_{\text{phys}}$, where*

$$\overline{\mathcal{H}}_{\text{phys}} = \overline{\mathcal{E}_0/N_0}, \quad N_0 = \{F \in \mathcal{E}_0, (F, F) = 0\} \subset \mathcal{E}_0,$$

is the completion of the quotient space $\mathcal{E}_0/N_0$ with respect to the inner product

$$([F_1], [F_2]) \equiv (F_1, F_2), \qquad F_1, F_2 \in \mathcal{E}_0,$$

and $[F] = [F]_0 \in \mathcal{E}_0/N_0$ is the class of the element $F \in \mathcal{E}_0$. On the elements $[G] = [G]_+ \in \mathcal{E}_F^+/I_0$, $G \in \mathcal{E}_F^+$, this isomorphism is given by

$$[G]_+ \to [\langle G | \mathcal{E}_0\rangle]_0 \in \mathcal{E}_0/N_0.$$

Under the isomorphism the semigroup $\mathcal{J}_\tau$ goes into the semigroup $\overline{\mathcal{J}_\tau}$ acting on the elements $[F_0] \in \mathcal{E}_0/N_0$ by the formula

$$(46^a) \qquad \mathcal{J}_\tau [F]_0 = [\langle U_\tau F | \mathcal{E}_0\rangle]_0, \qquad \tau > 0,$$

where U_τ is the homomorphism of $\mathcal{E}_F$ induced by translation by $\tau > 0$ along the time axis.

The proof is similar to that of Lemma 3. As in the boson case, a unitary group $\{U_{\overline{s}}, \overline{s} \in R^\nu\}$ of spatial translations commuting with $\mathcal{J}_\tau$ acts on the space $\mathcal{H}_{\text{phys}}$.

Example: the free Dirac field in R^4. This is a Gaussian gauge-invariant field on the field $\langle\cdot\rangle$ algebra $\mathcal{E}_F$ with generators ψ_φ, $\overline{\psi}_\varphi$, where the functions $\varphi(x)$, $x \in R^4$, take values in the four-dimensional complex space $\mathbb{C}^4$. The covariance of this field is given by

$$\begin{aligned} \langle \psi_{\varphi_1} \overline{\psi}_{\varphi_2} \rangle &= \sum_{\alpha,\beta} \int_{R^4} \int_{R^4} \left(\sum_\mu \gamma_\mu \frac{\partial}{\partial x_\mu} + m \right)^{-1}_{\alpha\beta} (x, x'), \\ \varphi_1^\alpha(x) \overline{\varphi}_2^\beta(x') d^4x d^4x' &= \sum_{\alpha\beta} \int_{R^4} \left(\sum_\mu i\gamma_\mu p_\mu + m \right)^{-1}_{\alpha\beta} \widetilde{\varphi}_1^\alpha(p) \overline{\widetilde{\varphi}_2^\beta}(p)\, dp, \end{aligned} \tag{47}$$

where $\{\varphi^\alpha(x),\ \alpha = 0,1,2,3\}$ are the coordinates of the vector $\varphi(x) \in \mathbb{C}^4$ with respect to a basis of $\mathbb{C}^4$, and $\widetilde{\varphi}$ is the Fourier transform of the function $\varphi(x)$; the γ_μ, $\mu = 0,1,2,3$ are the four Dirac matrices, which are selfadjoint and satisfy the relations

$$\gamma_\mu \gamma_{\mu'} + \gamma_{\mu'} \gamma_\mu = 2\delta_{\mu\mu'}, \qquad \mu, \mu' = 0,1,2,3. \tag{47'}$$

We will assume here that the basis of $\mathbb{C}^4$ has been chosen so that the matrix γ_0 is diagonal,

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The formal Euclidean action corresponding to (47) is

$$\int \left[\sum_\alpha m \psi_\alpha(x) \overline{\psi}_\alpha(x) + \frac{1}{2} \sum_{\alpha,\beta} \psi_\alpha(x) \gamma_\mu^{\alpha\beta} \frac{\partial \overline{\psi}_\beta(x)}{\partial x_\mu} - \frac{\partial \psi_\alpha(x)}{\partial x_\mu} \gamma_\mu^{\alpha\beta} \overline{\psi}_\beta(x) \right] d^4x.$$

In the definition (17′) of the involution Θ, the matrix ε is taken equal to $\overline{\varepsilon} = \gamma_0$; the field is then Θ-invariant. The algebra $\mathcal{E}_0$ is generated by the elements

$$\psi_{\varphi(\overline{x})\delta(x_0)} = \psi_\varphi, \qquad \overline{\psi}_{\varphi(\overline{x})\delta(x_0)} = \overline{\psi}_\varphi,$$

where $\overline{x} = (x_1, x_2, x_3) \in R^3$, and $\varphi \in S^4(R^3)$ is a function on R^3 with values in $\mathbb{C}^4$. A simple calculation shows that on a generator $\overline{\psi}_\varphi$, $\varphi \in S^4(R^3)$ the Hermitian quadratic form (46) is equal to

$$\begin{gathered} (\overline{\psi}_\varphi, \overline{\psi}_\varphi) = \frac{\pi}{2} \int_{R^3} \left[\left(\frac{-i \sum_{\mu=1}^3 \gamma_\mu p_\mu + m}{\sqrt{p^2 + m^2}} \right) \gamma_0 + E \right]_{\alpha,\beta} \widetilde{\varphi}_\alpha(p) \overline{\widetilde{\varphi}_\beta(p)}\, d^3p, \\ p = (p_1, p_2, p_3), \end{gathered}$$

where $\widetilde{\varphi}(p) = \{\widetilde{\varphi}_\alpha(p),\ \alpha = 0,1,2,3\}$ is the Fourier transform of $\varphi \in S^4(R^3)$.

Similarly, we find that

$$(\psi_\varphi, \psi_\varphi) = \frac{\pi}{2} \int_{R^3} \left[E - \left(\frac{-i \sum_{\mu=1}^3 \gamma_\mu p_\mu + m}{\sqrt{p^2 + m^2}} \right) \gamma_0 \right]_{\alpha,\beta} \overline{\varphi}_\alpha(p) \varphi_\beta(p)\, d^3p.$$

It is not difficult to verify that for each $p \in R^3$ the matrices

$$A_\pm(p) = \left[E \pm \left(\frac{-i\sum_{\mu=1}^3 \gamma_\mu p_\mu + m}{\sqrt{p^2+m^2}}\right)\gamma_0\right]^{1/2}$$

are selfadjoint, $A_\pm(p) \geq 0$, and

$$\begin{aligned} &\operatorname{Ker} A_+(p) = \operatorname{Im}\ A_-(p) = C_-(p) \subset \mathbb{C}^4, \\ &\operatorname{Ker} A_-(p) = \operatorname{Im}\ A_+(p) = C_+(p) \subset \mathbb{C}^4 \end{aligned} \tag{48}$$

and

$$\dim C_-(p) = \dim C_+(p) = 2.$$

The restriction $A_\pm(p)|_{C_\pm(p)}$ coincides with the identity operator on the two-dimensional space $C_\pm(p)$, respectively. We can thus introduce the Hilbert spaces

$$\mathcal{H}_\pm = \bigoplus \int_{R^3} C_\pm(p)\, dp,$$

where $\bigoplus \int_{R^3} \dots d^3p$ denotes a direct integral of Hilbert spaces, i.e., just, space of functions $\{\varphi(p), p \in R^3\}$ such that for each p the value $\varphi(p) \in C_\pm(p)$, respectively.

The space $\mathcal{H}_+$ describes the state of a particle (electron), while $\mathcal{H}_-$ describes the state of an antiparticle (positron). The fact that for each fixed momentum $p \in R^3$ the spaces $C_\pm(p)$ are two-dimensional means that there are two possible values for the spin of a particle or antiparticle (see [4] for more details). We write

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-.$$

(According to (48), we have $\mathcal{H} = L_2(R^3, \mathbb{C}^4)$, the Hilbert space of functions with values in $\mathbb{C}^4$.)

LEMMA 11. *The Dirac field is a reversibly Markov field, and its state space $\mathcal{H}_{\text{phys}}$ coincides with the Fock space $\mathcal{F}_s(\mathcal{H})$. The transfer matrix $\mathcal{J}_\tau$ of the field coincides with the operator $\Gamma(\exp\{-th\})$ on $\mathcal{F}_s(\mathcal{H})$, where h acts on $\mathcal{H}$ by multiplication by the function $\sqrt{p^2+m^2}$. The Hamiltonian H of the Dirac field is equal to*

$$H = d\Gamma(h).$$

The simple proof will be omitted. Note that in this example, as in the case of the Nelson boson field, the Markov property of the field follows from the local nature of the Lagrangian.

5. Euclidean fields on a discrete space. In this book (Chapter 3) we will study Euclidean fields defined on a set $X \times R^1$ (or $X \times Z^1$), where the "space" X is a countable set, while R^1 (or Z^1) is the "time" (continuous or discrete). Such fields arise either by "discretizing" continuous Euclidean fields (see [38]), or as objects in their own right in certain models in statistical physics.

The concepts used here (Θ-invariance, OS-positivity, the physical Hilbert space, the transfer matrix, and the reversible Markov property) are analogous to those used in the discussion of Euclidean fields in a continuous space. However, in contrast to fields that take their values in a finite-dimensional linear space, for the case of a discrete space we will introduce a larger class of "chiral" fields, i.e., boson fields with values in an arbitrary (finite-dimensional) manifold (or even in a finite set). We will first describe these fields and then also consider fermion fields on a discrete set.

A. Boson Euclidean fields on a discrete set. We will limit ourselves to the discrete-time case Z^1; the case of continuous time R^1 is treated similarly.

Thus, assume that a random field

$$\xi = \{\xi_y, y = (x, x_0) \in X \times Z^1\},$$

is defined on the set $X \times Z^1 = Y$, i.e., we have a system of random variables ξ_y defined on a probability space (Ω, Σ, μ) indexed by the elements $y \in Y$ and taking values in some topological space S (the space of "spins"). In what follows we will assume without loss of generality that Ω coincides with the space S^Y of field configurations (i.e., of functions on Y with values in S), and that the σ-algebra Σ coincides with the Borel σ-algebra with respect to the Tychonoff topology on S^Y.

As above, we consider the field algebra $\mathcal{E} = \widehat{L}_p(\Omega, \Sigma, \mu) = \bigcap_{\infty > p \geq 1} L_p(\Omega, \Sigma, \mu)$. Suppose further that a finite set of scalar functions $\{\varphi_\alpha, \alpha \in M)$, M an index set, is defined on the space S so that the polynomials

$$\Phi_{y,\alpha}(\xi) = \varphi_\alpha(\xi_y), \qquad y \in Y, \quad \alpha \in M,$$

in the field variables for a dense set in $\mathcal{E}$. We assume also that if φ_α is in the set $\{\varphi_\alpha, \alpha \in M\}$ then so is its complex conjugate function $\overline{\varphi}_\alpha = \varphi_{\alpha^*}$. The map

$$* : M \to M : \alpha \to \alpha^* \tag{49}$$

is called *charge conjugation.*

For any $t \in Z^1$, let $\mathcal{E}_t^{\pm} \subset \mathcal{E}$ be the subalgebras generated by the field variables $\{\Phi_{\alpha,y}, \alpha \in M, y = (x, x^0), x^0 \geq t(+) \text{ or } x^0 \leq t(-)\}$, respectively.

We now make the following hypotheses regarding the field ξ, which are analogous to the ones made before for continuous fields.

Translational invariance. Let $\widetilde{Z}^\nu$ be a group of transformations of the space X which is isomorphic to the ν-dimensional lattice Z^ν and such that the quotient space $X/\widetilde{Z}^\nu$ (the orbit space for the action of the group $\widetilde{Z}^\nu$ on X) is finite.

Together with the transformations $s \in \widetilde{Z}^\nu$ on X, which will be called spatial translations, the time translations $(x, x^0) \to (x, x^0 - \tau)$ generate a group of transformations on Y isomorphic to the group $Z^{\nu+1}$, which we will call the group of *translations.* In addition, we introduce the time reflection ϑ: $(x, x^0) \to (x, -x^0)$. The translations and time reflection induce transformations in the space of field configurations, thereby giving rise to corresponding homomorphisms U_s, U_τ, and an antilinear involution Θ of the field algebra, together with corresponding transformations $U^*_{S(\tau)}$, Θ^* on the space of measures on Ω.

We assume that the random field ξ satisfies the following conditions:

1) it is translation invariant, i.e., its distribution μ does not change under the action of the translations $U^*_{S(\tau)}$;

2) it is reversible in time, i.e., μ is invariant under Θ^*;

3) ξ has the Markov property.

The definition of the Markov property for the field ξ on Y is identical to (20^a) given above for the case of Markov boson fields in the space $R^{\nu+1}$.

For any $\tau \in Z^1$ we denote by $X_\tau \subset Y$ the time slice $X_\tau = \{y = (x, \tau)\}$ and write Σ_τ for the σ-algebra generated by the values $\{\xi_y, y \in X_t\}$ for configurations in the slice X_τ.

Exactly as in the case of boson Markov fields on $R^{\nu+1}$, we define the physical Hilbert space $\mathcal{H}_{\text{phys}}$ by

$$\mathcal{H}_{\text{phys}} = L_2(\Omega, \Sigma_0, \mu) \subset L_2(\Omega, \Sigma, \mu).$$

It consists of all square-integrable functionals of the field ξ that depend only on the values of ξ on the zero time slice $X_0 \subset Y$.

The transfer matrix $\mathcal{J}_\tau$ on $\mathcal{H}_{\text{phys}}$ is given by the formula

$$\mathcal{J}_\tau f = \langle U_\tau f | \Sigma_0 \rangle = P_{\mathcal{H}_{\text{phys}}} U_\tau f, \qquad \tau > 0, \quad \tau \in Z^1,$$

where $\{U_\tau, \tau \in Z^\nu\}$ is the unitary group of operators on $L_2(\Omega, \Sigma, \mu)$ induced by the time-translation homomorphisms of the algebra $\mathcal{E}$, and $P_{\mathcal{H}_{\text{phys}}}$ is the projection of $L_2(\Omega, \Sigma, \mu)$ onto the subspace $\mathcal{H}_{\text{phys}}$. As before, we have

LEMMA 12. *The family of operators* $\{\mathcal{J}_\tau, \tau \in Z^1, \tau \geq 0\}$ *forms a semigroup of selfadjoint contraction operators on* $\mathcal{H}_{\text{phys}}$.

The selfadjointness of $\mathcal{J}_\tau$ follows from the time reversibility of the field, while the semigroup property follows from the translation invariance and the Markov property of ξ. The generator $\mathcal{J} = \mathcal{J}_1$ of the semigroup $\mathcal{J}_\tau$ is usually called the *transfer matrix* of the field. In analogy with the continuous time case, the Hamiltonian is the operator

$$H = \frac{1}{2} \ln \mathcal{J}^2.$$

The homomorphisms U_s of the algebra $\mathcal{E}$ induced by the spatial translations (and preserving the algebra $\mathcal{E}_0 = \mathcal{E}_0^+ \cup \mathcal{E}_0^-$) extend to a unitary group of operators $\{U_s, s \in \widetilde{Z}^\nu\}$ on $\mathcal{H}_{\text{phys}}$ that commute with the transfer matrix $\mathcal{J}$.

B. Gibbs modifications. As we have already noted several times, for many physical models the Euclidean field ξ that describes them is a Gibbs modification of some comparatively simple "free" field (independent or Gaussian). The general construction of such modifications, and also the principal method for studying them (the cluster expansion technique) are discussed in [26]. Here we briefly recall the main ideas and results relating to Gibbs fields, and we introduce some special assumptions regarding them.

Assume that a probability measure ν_0 is defined on the spin space S, and let the probability distribution $\mu_0 = \nu_0^Y$ on S^Y be the infinite product of identical copies of the measure ν_0 (i.e., the distribution of an independent field on Y).

1) Euclidean action. For each finite set $\Lambda \subset Y$ we define the Euclidean action $V_\Lambda(\xi)$ of the field to be the polynomial

$$V_\Lambda(\xi) = \sum_n R_n \prod_{y,\alpha} (\Phi_\alpha(\xi_y))^{n(y,\alpha)}, \tag{50}$$

in the field variables, where the sum is over the multi-indices $n = \{n(y, \alpha)\}$ (nonnegative integer-valued functions on $Y \times M$ that are nonzero at only finitely many points) such that

$$\operatorname{supp} n \subset \Lambda \times M.$$

The coefficients $\{R_n\}$ in (50) (which do not depend on Λ) are called the *potential of the*

interaction. The set $\mathfrak{A}$ of multi-indices n for which $R_n \neq 0$ is called the *support of the potential.* We make the following assumptions concerning the interaction potential $\{R_n\}$.

a) Translation invariance and time reversibility. Let g be a spatial translation, time translation, or time inversion acting on Y. Then

$$R_{gn} = R_n \tag{51}$$

(with the obvious action of g on the multi-indices n). It follows from (51) that

$$V_{g\Lambda}(\xi) = V_\Lambda(g^{-1}\xi), \qquad \Lambda \subset Y$$

(again, the action of g on the configuration ξ is defined in the obvious way).

b) Charge invariance. We postulate that

$$R_{n^*} = \overline{R}_n, \tag{52}$$

where $n^*(y,\alpha) = n(y,\alpha^*)$ and $*$ is the charge conjugation in M (see (49)); (52) implies that the action V_Λ is real.

We assume that an integer-valued metric ρ is defined on the set X and define a metric on Y by

$$\widehat{\rho}(y_1,y_2) = \rho(x_1,x_2) + |x_1^{(0)} - x_2^{(0)}|,$$
$$y_i = (x_i, x_i^0), \qquad i = 1,2.$$

Henceforth, if n is any multi-index, we denote by $\operatorname{ssupp} n$ the projection of the support $\operatorname{supp} n \subset Y \times M$ on Y.

c) Finiteness and Markov property. For any multi-index $n \in \mathfrak{A}$ (i.e., such that $R_n \neq 0$), the diameter of the set $\operatorname{ssupp} n$ is uniformly bounded:

$$\max_{n\in\mathfrak{A}} \operatorname{diam} \operatorname{ssupp} n < \infty \quad \text{(finiteness)}.$$

Moreover, for any two points $y_1 = (x_1, x_1^0)$, $y_2 = (x_2, x_2^0) \in \operatorname{ssupp} n$, we require

$$|x_1^{(0)} - x_2^{(0)}| \leq 1 \quad \text{(Markov property)}.$$

The degree of the multi-index $n = \{n(y,\alpha)\}$ is defined to be

$$|n| = \sum_{y,\alpha} n(y,\alpha).$$

d) Boundedness of the degree. The degrees of all the multi-indices in the support of the potential $\{R_n\}$ are assumed to be uniformly bounded:

$$\max_{n\in\mathfrak{A}} |n| < \infty.$$

REMARK. The various hypotheses on the interaction potential $\{R_n\}$ made above imply that the set of values taken by the coefficients R_n is finite, and therefore

$$\max_{n\in\mathfrak{A}} |R_n| < \infty.$$

2) Gibbs modification. For any finite set $\Lambda \subset Y$ we define the probability measure μ_Λ on (Ω, Σ) by means of the Gibbs modification of the measure μ_0,

$$\frac{d\mu_\Lambda}{d\mu_0}(\xi) = \frac{1}{Z_\Lambda} \exp\{-\beta V_\Lambda(\xi)\}, \quad \xi \in \Omega = S^Y, \tag{52^a}$$

where β is a parameter and the normalization factor Z_Λ (partition function) is given by

$$Z_\Lambda = \int_\Omega \exp\{-\beta V_\Lambda(\xi)\}\, d\mu_0.$$

LEMMA 13. *Let the set of field variables $\{\varphi_\alpha, \alpha \in M\}$ consist of bounded functions, and assume that the interaction potential $\{R_n\}$ satisfies all the hypotheses* a)–d). *Then there exists a number $\beta_0 > 0$ such that for all β with $|\beta| < \beta_0$:*

1) *The weak limit*

$$\mu = \lim_{\Lambda \uparrow Y} \mu_\Lambda \tag{53}$$

exists, where μ is a probability measure on the space Ω.

2) *The random field ξ on Y with values in S and probability distribution μ is translation invariant, time reversible, and Markov.*

Statement 1) can be proved using the cluster expansion for the measures μ_Λ (see [26]). The field properties stated in 2) follow easily from our assumptions regarding the interaction potential $\{R_n\}$.

C. Examples of Euclidean (boson) fields on a discrete space. Here we present some examples of Euclidean fields of the above type that are frequently encountered in the literature.

I. Spin fields on the lattice $Z^{\nu+1}$. In this case $X = Z^\nu$ is a ν-dimensional lattice, $S = \{1, -1\}$, and ν_0 is a uniform distribution on S: $\nu_0(1) = \nu_0(-1) = 1/2$. The field variables are the values of the field,

$$\xi_y = \pm 1, \qquad y \in Z^{\nu+1}.$$

Evidently, every monomial (50) is of the form

$$\xi_B = \prod_{y \in B} \xi_y,$$

where $B \subset Y$ is a finite set, and thus the action of the spin field is given by

$$V_\Lambda = \sum_{B \subseteq \Lambda} R_B \xi_B,$$

where the interaction potential $\{R_B\}$ is defined on finite subsets B of the lattice $Z^{\nu+1}$ and satisfies the analogs of hypotheses a)–d) above.

For the simplest and best-studied spin field model (the *Ising model*), the potential R_B is nonzero only for subsets B that consist of a single point and for subsets of cardinality two, consisting of a pair of neighboring points in $Z^{\nu+1}$.

II. Rotator model. As before, we take $X = Z^\nu$ (i.e., $Y = Z^{\nu+1}$), $S = T^1$ is the circle, and $d\nu_0 = d\xi/2\pi$, where $d\xi$ is Lebesgue measure on T^1. The field variables are

$$\Phi_{\pm;y} = \exp\{\pm i\xi_y\}.$$

The action V_Λ is given by

$$\begin{aligned} V_\Lambda &= \varepsilon \sum_{\substack{y_1,y_2\in\Lambda \\ |y_1-y_2|=1}} \cos(\xi_{y_1} - \xi_{y_2}) \\ &= \varepsilon \sum_{\substack{y_1,y_2\in\Lambda \\ |y_1-y_2|=1}} (\Phi_{+,y_1}(\xi)\Phi_{-,y_2}(\xi) + \Phi_{-,y_1}(\xi)\Phi_{+,y_2}(\xi)). \end{aligned} \tag{53^a}$$

Here the metric $\rho(y_1, y_2) \equiv |y_1 - y_2|$ on $Z^{\nu+1}$ is given by the formula

$$|y_1 - y_2| = \sum_{i=0}^{\nu} |x_1^{(i)} - x_2^{(i)}|,$$
$$y_k = (x_k^{(0)}, x_k^{(1)}, \ldots, x_k^{(\nu)}), \qquad k = 1, 2.$$

III. Yang Mills lattice gauge field. Here $Y = E^{\nu+1}$ is the set of edges of the lattice $Z^{\nu+1}$. This set can be expressed in the form

$$E^{\nu+1} = \widetilde{E}^\nu \times Z^1,$$

where the time coordinate is explicitly indicated. Here $\widetilde{E}^\nu$ is the set of edges $y = (x_1, x_2)$, such that one vertex lies in the zeroth slice $Z^\nu \subset Z^{\nu+1}$ of the lattice $Z^{\nu+1}$: $\{x_1 = (x_1^0, x_1), x_1^0 = 0\}$, and the other vertex lies either in the zeroth slice or in the first slice: $\{x_2 = (x_2^0, x_2), x_2^0 = 0, 1\}$.

Here the space $S = G$ is a compact group, ν^0 the normalized Haar measure on G. The action of the field V_Λ, $\Lambda \subset E^{\nu+1}$ is given by

$$V_\Lambda = \sum_{p:\partial p\subset\Lambda} \operatorname{Re}\chi_0(g_p), \tag{54}$$

where the sum is over all two-dimensional faces p of $Z^{\nu+1}$ with boundary $\partial p = \{y_1, y_2, y_3, y_4\} \subset \Lambda$; the edges y_i in ∂p are given in the order in which they are encountered in going around the face, and g_p is given by

$$g_p = g_{y_1}^{\varepsilon_1} g_{y_2}^{\varepsilon_2} g_{y_3}^{\varepsilon_3} g_{y_4}^{\varepsilon_4},$$

where $\varepsilon_i = 1$ if upon traversing the boundary of p in the given orientation, the edge y_i is traversed in the direction of the corresponding unit vector $(e_s,\ s = 0, \ldots, \nu)$ of the lattice $Z^{\nu+1}$, and $\varepsilon_i = -1$ if y_i is traversed in the opposite direction. Now let χ_0 be the character of some "fundamental" irreducible representation $g \to T_g^0$ of the group G (see [20]). It is easy to show that the value of $\operatorname{Re}\chi_0(g_p)$ depends neither on the orientation of ∂p nor on its origin, i.e., it is uniquely determined by the face p itself. The representation $g \to T_g^0$ is "fundamental" if and only if every irreducible representation $g \to T_g$ of G is contained in the tensor product

$(T_g^0)^{\otimes n_1} \otimes (\overline{T}_g^0)^{\otimes n_2}$, for suitable n_1 and n_2, of tensor powers of the representation $g \to T_g^0$ and its contragradient $g \to \overline{T}_g^0$ (see [20]).

We write $\varphi^0_{\alpha_1,\alpha_2}(g) = (T_g^0\eta_{\alpha_1}, \eta_{\alpha_2})$ for the matrix elements of the representation $g \to T_g^0$ in some fixed orthonormal basis $\{\eta_\alpha\}$ of the space on which the representation acts (then the matrix elements for the contragradient representation $g \to \overline{T}_g^0$ are given by $\overline{\varphi}^0_{\alpha_1\alpha_2}(g)$). We take the field variables to be the functions

$$\Phi^+_{\alpha_1,\alpha_2,y} = \varphi^0_{\alpha_1,\alpha_2}(g_y), \qquad \Phi^-_{\alpha_1,\alpha_2,y} = \overline{\varphi}^0_{\alpha_1,\alpha_2}(g_y), \quad y \in E^{\nu+1}.$$

It is readily verified that the above model satisfies all the above hypotheses a)–d) and hence for β in (52) sufficiently small, there exists a random field on $E^{\nu+1}$ obtained by taking the limit (53). It is called the Yang-Mills gauge field.

The action (54) and hence also the limit field are invariant under the group $J = G^{Z^{\nu+1}}$ of gauge transformations

$$g_y \to \gamma(s_1) g_y \gamma^{-1}(s_2), \qquad y = (s_1, s_2),$$

where $\gamma = \{\gamma(s), s \in Z^{\nu+1}\}$ is a function on $Z^{\nu+1}$ with values in the group G.

We write $L_2^{\text{gauge}}(\Omega, \Sigma, \mu) \subset L_2(\Omega, \Sigma, \mu)$ for the set of all gauge-invariant functionals in $L_2(\Omega, \Sigma, \mu)$ of the field $g = \{g_y\}$, and $\mathcal{H}^{\text{gauge}}_{\text{phys}} = \mathcal{H}_{\text{phys}} \cap L_2^{\text{gauge}}(\Omega, \Sigma, \mu)$.

The space $\mathcal{H}^{\text{gauge}}_{\text{phys}}$ is evidently invariant under the transfer matrix $\mathcal{J}$ of the gauge field, and one checks easily that $\mathcal{J}$ acts trivially on the orthogonal complement

$$\mathcal{H}_{\text{phys}} \ominus \mathcal{H}^{\text{gauge}}_{\text{phys}}.$$

Thus we need only study the part

$$\mathcal{J}^{\text{gauge}}_{\text{phys}} = \mathcal{J}_\tau / \mathcal{H}^{\text{gauge}}_{\text{phys}}$$

of $\mathcal{J}$ acting on the space $\mathcal{H}^{\text{gauge}}_{\text{phys}}$.

Remark concerning the radial gauge. A somewhat different (but equivalent) definition of the space $\mathcal{H}^{\text{gauge}}_{\text{phys}}$ and transfer matrix $\mathcal{J}^{\text{gauge}}_{\text{phys}}$ is also often used. Note that the functionals $f \in L_2^{\text{gauge}}(\Omega, \Sigma, \mu)$ are constant on the orbits of the gauge group J in the configuration space Ω of the gauge field. We may therefore limit our attention to a smaller class $\Omega' \subset \Omega$ of field configurations $g = \{g_y\}$ and replace J by the smaller group J', which acts on Ω' in such a way that the set of orbits remains the same. This can be accomplished, for instance, by setting

$$g_y = e \quad \text{(the identity of } G\text{)} \tag{55}$$

for all edges y parallel to the time ("vertical") axis $e_0 \in Z^{\nu+1}$ (radial gauge). The gauge field $g = \{g_y\}$ is thus in fact specified only on the set $\widehat{Y}$ of "horizontal" edges:

$$\widehat{Y} = E^\nu \times Z^1,$$

where E^ν is the set of edges of the lattice Z^ν. With (55), we see that the action of the field is the same as before, and there exists a limit distribution μ' on the configuration space Ω' for the new field. The gauge group J' now consists of only those transformations $\{\gamma(s)\}$ that do not violate the condition (55) (i.e., the function $\gamma(s)$ does not depend on the time coordinate $s^{(0)}$ of the point $s \in Z^{\nu+1}$).

It is easily verified that the space $\mathcal{H}^{\text{gauge}}_{\text{phys}} \subset \mathcal{H}_{\text{phys}}$, and also the gauge-invariant part $\mathcal{J}^{\text{gauge}}_{\text{phys}}$ of the transfer matrix in this space, remain unchanged. However, in contrast to the preceding case, the transfer matrix now acts nontrivially on the orthogonal complement $\mathcal{H}_{\text{phys}} \ominus \mathcal{H}^{\text{gauge}}_{\text{phys}}$.

C. Fermion fields on a lattice. It is usually convenient to specify a fermion field on the lattice $\widetilde{Z}^{\nu+1} = Z^\nu \times \widetilde{Z}^1$, where $\widetilde{Z}^1$ is the lattice of half-integers $\{\pm 1/2, \pm 3/2, \dots\}$. The field Grassmann algebra $\mathcal{E}_F$ is generated by the elements $\{\psi_\alpha(x), \overline{\psi}_\alpha(x), x \in \widetilde{Z}^{\nu+1}, \alpha \in M\}$, where M is a finite index set. The elements of $\mathcal{E}_F$ are series of the form

$$A = \sum C_Q \varphi_Q, \tag{56}$$

where ψ_Q is a monomial in the variables $\psi_\alpha(x)$, $\overline{\psi}_\alpha(x)$, which are given in lexicographic order in the set $\widetilde{Z}^{\nu+1} \times M$, with all the $\overline{\psi}_\alpha(x)$ appearing to the right of the $\psi_\alpha(x)$. We assume that the coefficients C_Q of the series (56) satisfy the condition

$$\sum_Q |C_Q| r^{|Q|} = \|A\|_r < \infty \tag{57}$$

for some fixed $r > 1$, and $|Q|$ is the number of factors in the monomial ψ_Q. Equipped with the norm (57), $\mathcal{E}_F$ becomes a Banach algebra, with

$$\|A_1 A_2\| \le \|A_1\|_r \|A_2\|_r.$$

All the concepts and constructions introduced above for the continuous case the quasistate $\langle\cdot\rangle$, involution Θ, Θ-invariance of $\langle\cdot\rangle$, OS-positivity, definition of conditional expectation, reversible Markov property, construction of the physical Hilbert space $\mathcal{H}_{\text{phys}}$, and transfer matrix $\mathcal{J}_\tau$ carry over without change to the discrete case. We merely note that the physical Hilbert space is constructed in the standard way (see above) via the quotient $\mathcal{E}_{1/2}/N_0$ of the algebra $\mathcal{E}_{1/2}$ generated by the set

$$\{\psi_\alpha(x), \overline{\psi}_\alpha(x), \alpha \in M, x \in Y_{1/2}\},$$

where $Y_{1/2} = \{x\colon x^0 = 1/2\}$ and $N_0 \subset \mathcal{E}_{1/2}$ is the subspace of elements with zero norm $(\langle \Theta F_1 F_2\rangle)^{1/2}$, $F_1, F_2 \in \mathcal{E}_{1/2}$ (see above).

The quasistates $\langle\cdot\rangle$ found in the various examples are usually Gibbs modifications of independent or Gaussian quasistates (see [26]). In the next chapter, we will study the transfer matrix for the Gibbs modification of an independent Gaussian quasistate $\langle\cdot\rangle_0$ on $\mathcal{E}_F$, defined by the conditions

$$\begin{gathered} \langle \psi_\alpha(x)\psi_\beta(x')\rangle_0 = \langle \overline{\psi}_\alpha(x)\overline{\psi}_\beta(x')\rangle_0 = 0, \\ \langle \psi_\alpha(x)\overline{\psi}_\beta(x')\rangle_0 = \delta_{\alpha,\beta}\delta_{x,x'}. \end{gathered} \tag{58}$$

The Euclidean action generating this modification is given by

$$V_\Lambda = \lambda \sum_{Q:\operatorname{supp} Q \subset \Lambda} R_Q \psi_Q, \tag{58^a}$$

where the ψ_Q are even monomials and the coefficients R_Q satisfy the analogs of conditions a)–d) introduced above. We write $\operatorname{supp} Q \subset \widetilde{Z}^{\nu+1}$ for the set of points $x \in \widetilde{Z}^{\nu+1}$ for which the generator $\psi_\alpha(x)$ or $\overline{\psi}_\alpha(x)$ lies in ψ_Q for some $\alpha \in M$. In addition, for every set $\Lambda \subset \widetilde{Z}^{\nu+1}$ symmetric with respect to the plane $x^{(0)} = 0$, we postulate that:

1) the action V_Λ is invariant under the involution Θ:

$$\Theta V_\Lambda = V_\Lambda, \tag{59}$$

2) the part

$$\sum_{\operatorname{supp} Q \subset Y_{-1/2} \cup Y_{1/2}} R_Q \psi_Q$$

of the sum in V_Λ involving the monomials ψ_Q that couple the "past" and the "future" can be expressed in the form

$$\sum_k c_k F_k \Theta F_k, \tag{60}$$

where the $F_k \in \mathcal{E}_{1/2}$ are homogeneous elements of $\mathcal{E}_{1/2}$ (i.e., each F_k is either even or odd), and $c_k \geq 0$.

LEMMA 14. *Let $\langle\cdot\rangle$ be an independent Gaussian quasistate, and suppose that β is sufficiently small. Then if the action (58^a) satisfies all of the above conditions, it induces a Gibbs modification $\langle\cdot\rangle_\beta$ of $\langle\cdot\rangle$; $\langle\cdot\rangle_\beta$ is translationally invariant and has the reversible Markov property on the algebra $\mathcal{E}_F$. Moreover, $\langle\cdot\rangle_\beta$ is Θ-invariant and OS-positive, and in addition it admits a cluster expansion.*

The cluster expansion of the quasistate $\langle\cdot\rangle_\beta$ follows from the general constructions set forth in [26]. From this and the properties of the coefficients R_Q, we see that $\langle\cdot\rangle_\beta$ has the Markov property. The invariance under Θ and the OS positivity follow from (59) and (60). The translation invariance follows from the translation invariance of the coefficients R_Q.

E. General case. One often also encounters fields on the full field algebra $\mathfrak{A} = \mathcal{E}_B \otimes \mathcal{E}_F$, which includes both fermion and boson variables.

As an example, we consider a Dirac fermion field on $\widetilde{Z}^4$ interacting with a gauge field with gauge group $U(1) = \{e^{i\Theta}, 0 \leq \Theta < 2\pi\}$ (lattice model of Euclidean quantum electrodynamics). In this model, the field algebra is generated by the Grassmann algebra $\mathcal{E}_F$ with generators $\{\psi_\alpha(x), \overline{\psi}_\alpha(x), x \in \widetilde{Z}^4, \alpha = 0,1,2,3\}$ and by the field algebra $\{g_b\}$ of the gauge field (taken with the radial gauge). The Euclidean action for this field is

$$V_\Lambda = m \sum_{x\in\Lambda,\alpha} \psi_\alpha(x)\overline{\psi}_\alpha(x) + \kappa \sum_{b=\langle x,y\rangle,\alpha,\beta} \psi_\alpha(x)\Gamma_{\alpha\beta}(b) g_b \overline{\psi}_\beta(y) + \lambda \sum_p \cos(g_p), \tag{61}$$

where $\Lambda \subset \widetilde{Z}^4$ is a finite set and m, κ, λ are parameters; the summation in the second sum in (61) is over all (ordered) pairs of neighboring points (x, y), and $b = \langle x, y\rangle$ is the edge starting at x and ending at y; $\{\Gamma_{\alpha,\beta}(b)\} = E = \varepsilon(b)\gamma_\mu$, where $\varepsilon(b) = +1$ if the edge b is directed along the unit vector e_μ, $\mu = 0,1,2,3$, and $\varepsilon(b) = -1$ otherwise; the γ_μ are the four Euclidean Dirac matrices (see $(47')$). The sum $\sum_p \cos(g(p))$ coincides with the Euclidean (Wilson) action for the Yang-Mills gauge field with gauge group $U(1)$ (see (54)). For each finite $\Lambda \subset \widetilde{Z}^4$ we can define the quasistate

$$\langle F\rangle_\Lambda = \frac{1}{Z_\Lambda} \int F e^{-V_\Lambda} \prod_{x,\alpha} d\psi_\alpha(x) \prod_{\alpha,x} d\overline{\psi}_\alpha(x) \prod_b dg_b \tag{62}$$

on the field algebra $\mathfrak{A}_\Lambda$ generated by $\{\psi_\alpha(x), \overline{\psi}_\alpha(x), x \in \Lambda, \alpha = 0,1,2,3, g_b, b = \langle x,y\rangle \subset \Lambda\}$. In (62), Z_Λ is a normalization factor, $\int \cdots \prod_{\alpha,x} d\psi_\alpha(x), \prod_{\alpha,x} d\overline{\psi}_\alpha(x)$ is the Berezin integral, and $\prod_b dg_b$ is a product of Haar measures on the group $U(1)$ (since $g_b = g^{-1}(b)$ and we are using a radial gauge (see (55)), the product $\prod_b$ is taken over the set of unoriented "horizontal" edges in Λ).

For symmetric about the plane $x_0 = 0$, the quasistate $\langle\cdot\rangle_\Lambda$ is reversible-Markov and OS-positive, provided the matrix defining the involution Θ is equal to $\gamma_0 \colon \varepsilon = \gamma_0$, and $\kappa > 0$. For small enough κ and β, the thermodynamic limit $\langle\cdot\rangle = \lim\langle\cdot\rangle_\Lambda$ exists; it is a reversible-Markov, OS-positive, and translation-invariant quasistate on the full field algebra.

This field is invariant under gauge transformations (homomorphisms) of the algebra $\mathfrak{A}$, which act on the generators by the formula

$$\gamma g_b = \gamma^{-1}(x) g_b \gamma(y), \qquad b = \langle x,y\rangle,$$
$$\gamma\psi_\alpha(x) = \gamma(x)\psi_\alpha(x), \qquad \gamma\psi_\alpha(x) = \gamma^{-1}(x)\overline{\psi}_\alpha(x),$$

where $\gamma = \{\gamma(x)\}$ is a function on $\widetilde{Z}^4$ with values in $U(1)$.

§6. Euclidean fields for temperature states. The modular operator

Given an "infinite" dynamics α_t on a C^*-algebra, one can also study the temperature states by using the Euclidean Markov approach. Another possible ("algebraic") method is based on a deep theorem of Tomita and Takesaki and uses the notion of modular operator. We will briefly describe both approaches. This information is given only for completeness and will not be used elsewhere in the book.

1. Euclidean temperature fields.

Case of a finite system. The constructions given below are once again based on the Feynman-Kac formula, which we have already discussed several times: the kernel of the operator $\exp\{-\beta H\}(x,y)$, $x,y \in S$, where $H = H_0 + V$, H_0 is the infinitesimal generator of a Markov process $\{\xi_t\}$ with values in S, and V is a function on S, can be expressed as an average over the trajectories of the process on a time interval,

$$\exp\{-\beta H\}(x,y) = \left\langle \exp\left\{ -\int_0^\beta V(\xi_\tau)\, d\tau \right\} \right\rangle_{\xi_0 = x, \xi_\beta = y}$$

(see 5.0 for more details).

We now consider a temperature state defined on the algebra $\mathfrak{B}(L_2(S,\nu_0))$ of operators acting on the Hilbert space $L_2(S,\nu_0)$, where ν_0 is a stationary measure for the process $\{\xi_t\}$, by the formula

$$\langle A\rangle_\beta = \frac{1}{Z}\operatorname{Tr}(Ae^{-\beta H}) \tag{2}$$

($Z = \operatorname{Tr} e^{-\beta H}$). When $A = \widehat{F}$ is the operator given by multiplication by a function F defined on S, the numerator in (2) is equal to

$$\begin{aligned} \operatorname{Tr}\widehat{F}e^{-\beta H} &= \int_S d\nu_0(x) \left\langle \exp\left\{ -\int_0^\beta V(\xi_\tau)\, d\tau \right\} \right\rangle_{\xi_0 = \xi_\beta = x} F(x) \\ &= \left\langle F(\xi(\beta)) \exp\left\{ -\int_0^\beta V(\xi_\tau)\, d\tau \right\} \right\rangle_{\xi_0 = \xi_\beta}, \end{aligned} \tag{3}$$

and the denominator is

$$Z = \left\langle \exp\left\{ -\int_0^\beta V(\xi_\tau)\, d\tau \right\} \right\rangle_{\xi_0=\xi_\beta}, \tag{4}$$

where in both formulas (3) and (4), $\langle\cdot\rangle_{\xi_0=\xi_\beta}$ denotes an (unnormalized) average over the set of "periodic" trajectories of the process $\{\xi_t\}$, i.e., trajectories taking the same values at the endpoints of the interval $[0, \beta]$.

The original Markov process $\{\xi_t\}$ can be used to define a new Markov stationary "periodic" process $\{\eta_t^0, t \in T_\beta\}$, i.e., a process on the circle $T_\beta = [0, \beta]$. Thus equation (2) takes the form

$$\langle \widehat{F} \rangle_\beta = \frac{\langle F(\eta_\tau^0) \exp\{-\int_0^\beta V(\eta_\tau^0)\, d\tau\rangle_{\mu_0}}{\langle \exp\{-\int_0^\beta V(\eta_\tau^0)\, d\tau\rangle_{\mu_0}}, \tag{5}$$

where the mean $\langle\cdot\rangle_{\mu_0}$ denotes an average over the distribution μ_0 on the space of trajectories of the process $\{\eta_t^0, t \in T_\beta\}$. We will show below how to construct the periodic process $\{\eta_t^0\}$. Here we merely observe that the Markov property for $\{\eta_t^0\}$ means that for any two points $t_1, t_2 \in T_\beta$ and any two values s_1, s_2 of the process η_t^0 at these points, the values of η_t on the two arcs bounded by the points t_1 and t_2 on the circle T_β are independent with respect to the conditional distribution generated by the conditions

$$\eta_{t_1} = s_1, \qquad \eta_{t_2} = s_2.$$

The stationarity of $\{\eta_t^0\}$ means that the distribution μ_0 on the space of trajectories does not change under rotations of T_β.

The last step now is to introduce the Gibbs modification of the process $\{\eta_t^0\}$, i.e., define a new distribution μ on the space of trajectories by the formula

$$\frac{d\mu}{d\mu_0} = \frac{\exp\{-\int_0^\beta V(\eta_\tau)\, d\tau\}}{\langle \exp\{-\int_0^\beta V(\eta_\tau^0)\, d\tau\}\rangle_{\mu_0}},$$

then (5) becomes

$$\langle \widehat{F} \rangle_\beta = \langle F(\eta_{\tau=\beta}) \rangle_\mu, \tag{6}$$

where $\langle\cdot\rangle_\mu$ denotes the mean with respect to the measure μ. It is easy to show that the process $\{\eta_t, t \in T_\beta\}$ with probability distribution μ is also Markov and stationary. We have already seen that for a finite system, say one contained in a finite region $\Lambda \subset R^\nu$ and described by an explicit Hamiltonian $H = H_\Lambda$, a temperature state $\langle\cdot\rangle_\beta$ is expressible in the form (6). But for infinite systems (obtained, e.g., by taking the thermodynamic limit $\Lambda \uparrow R^\nu$), (6) can again serve as the definition of a temperature state and its associated Hamiltonian H, exactly as was explained in the case of the ground state (see also below). We will shortly describe the corresponding infinite Euclidean fields on the set $T_\beta \times R^\nu$, but first we explain how to construct the periodic Markov process $\{\eta_t^0\}$ mentioned above.

To simplify the discussion we take S to be a finite set. The finite-dimensional distributions of the process $\{\eta_t^0, t \in T_\beta\}$ are defined as follows. Let $0 = t_0 < t_1 < \cdots < t_n < \beta$; then

$$\Pr(\eta_{t_0} = s_0, \ldots, \eta_{t_n} = s_n) = Q^{-1}\nu_0(s_0)P_{t_1}(s_0, s_1)P_{t_2-t_1}(s_2, s_1)\ldots P_{\beta-t_n}(s_n, s_0),$$

where $Q = \sum_{s\in S} \nu_0(s)P_\beta(s,s)$, and the $P_t(s,s')$ are the transition probabilities for the stationary process $\{\xi_t, t \in R^1\}$. Note that the distribution of the values $\{\eta_t^0, t \in T_\beta\}$ at any point $t \in T$ is given by

$$\Pr(\eta_t = s) = Q^{-1}\nu_0(s)P_\beta(s,s).$$

Infinite Euclidean temperature fields. Here we consider general fields on the space $T_\beta \times R^\nu$. These are general objects defined in analogy with the definition of Euclidean fields for ground states given in §5. We will confine ourselves here to the case of a boson field.

As before, we will consider a (generalized) random field ξ_φ on a probability space (Ω, Σ, μ), where $\varphi = \varphi(t,x) \in S(T_\beta \times R^\nu)$, and $T_\beta \times R^\nu$ is the space of infinitely differentiable functions decaying sufficiently rapidly as $|x| \to \infty$. Averages over the distribution μ will be denoted by $\langle\cdot\rangle_\mu$. As above, the field algebra $\mathcal{E}$ is defined by

$$\mathcal{E} = \bigcap_{\infty > p \geq 1} L_p(\Omega, \Sigma, \mu).$$

The Schwinger functions $S_n(\varphi_1, \dots, \varphi_n) = \langle \xi_{\varphi_1} \dots \xi_{\varphi_n}\rangle_\mu$ are also defined as previously, and they are assumed to be continuous in the variables $\varphi_1, \dots, \varphi_n$. We further require the field to be translationally invariant, i.e., the mean $\langle\cdot\rangle_\mu$ is invariant under the homomorphisms $U(s,y)$, $(s,y) \in T_\beta \times R^\nu$ of the algebra $\mathcal{E}$, induced by the translations

$$U(s,y)\xi_\varphi = \xi_{\varphi(\cdot - (s,y))}$$

of the generators. This group of homomorphisms of $\mathcal{E}$ generates a group of unitary operators on $L_2(\Omega, \Sigma, \mu)$, which we will continue to denote by $U(s,y)$. The homomorphisms $U(s,0) = U(s)$ and $U(0,y) = U(y)$ will be called *time* and *space* translations, respectively.

The involution Θ induced by time reflection with respect to the zeroth slice $\{0\} \times R^\nu$ in the algebra $\mathcal{E}$, and the Θ-invariance of the mean $\langle\cdot\rangle_\mu$ are defined exactly as in §5. The definition of OS-positivity is defined similarly: one requires the positivity of the Hermitian form

$$(F,F) = \langle F\Theta F\rangle \geq 0 \tag{7}$$

for all elements $F \in \mathcal{E}_+ \subset \mathcal{E}$, where $\mathcal{E}_+$ is the subalgebra generated by the elements with $\operatorname{supp}\varphi[0, \beta/2] \times R^\nu$. The form (7) is used to construct the physical Hilbert space $\mathcal{H}_{\text{phys}}$ just as in §5.

Because the mean is invariant under the space translations $U(y)$, and since $U(y)\mathcal{E}_+ = \mathcal{E}_+$, the spatial translations induce a well-defined group of unitary operators on $\mathcal{H}_{\text{phys}}$, which will again be denoted by $U(y)$.

However, because the time translations $U(s)$, $s \in T_\beta$, do not take the algebra $\mathcal{E}_+$ into itself: $U(s)\mathcal{E}_+ \not\subset \mathcal{E}_+$, to construct the Hamiltonian H we must proceed somewhat differently than in §5.

For any $t \in [0, \beta/2]$ we consider the subalgebra $\mathcal{E}_t \subset \mathcal{E}_+$ generated by those ξ_φ for which $\operatorname{supp}\varphi \subset [0, \beta/2 - t] \times R^\nu$.

LEMMA 1. *Let the element $F \in \mathcal{E}_t$ be such that*

$$\|F\|^2_{\text{phys}} \equiv \langle F\Theta F\rangle = 0.$$

Then for every $0 \le s \le t$ *we have*

$$\|U(s)F\|_{\text{phys}} = 0. \tag{8}$$

PROOF. First suppose that $0 < s \le t/2$. Then

$$\begin{aligned}\langle U(s)F\Theta U(s)F\rangle &= \langle FU(-s)\Theta U(s)F\rangle \\ &= \langle F\Theta U(2s)F\rangle \le \|F\|_{\text{phys}}\|U(2s)F\|_{\text{phys}} = 0.\end{aligned}$$

Here we have used the translation invariance of the mean $\langle\cdot\rangle_\mu$ under the homomorphisms $U(s)$, the commutation relation

$$U(-s)\Theta = \Theta U(s),$$

the fact that $U(2s)F \in \mathcal{E}_+$, and the Cauchy-Schwarz inequality for the nonnegative form (7) (considered on the algebra $\mathcal{E}_+$). Next suppose that $t/2 < s < 3/4t$. Then $U(s)F = U(s_1)U(t/2)F$, where $s_1 = s - t/2$. We have $F_1 = U(t/2)F \in \mathcal{E}_{t/2}$, and by what has already been proved, its norm $\|F_1\|_{\text{phys}} = 0$. Since $s_1 < t/4$, application of the previous argument to F_1 leads to (8). The case $3t/4 < s < 7t/8$ is considered similarly, and so on. We have thus established the lemma for all $s < t$. For $s = t$ we must use the continuity of the form $\langle U(s)F\Theta U(s)F\rangle$ in the variable s, which follows from the continuity of the Schwinger functions and the estimate $|\langle F\Theta F\rangle| < \|F\|^2_{L_2(\Omega,\Sigma,\mu)}$. The lemma is proved.

For any $t \in [0, \beta/2]$ we define the (nonclosed) subspace $D_t \subset \mathcal{H}_{\text{phys}}$ as the image of $\mathcal{E}_t$ in the quotient $\mathcal{E}_+/I_0$ used to construct $\mathcal{H}_{\text{phys}}$ (see §5). By the lemma just proved, the operator

$$P_t\colon D_t \to \mathcal{H}_{\text{phys}}, \qquad P_t[A] = [U(t)A]$$

is well defined on the subspace $D_t \subset \mathcal{H}_{\text{phys}}$, where $[A] \in \mathcal{H}_{\text{phys}}$ denotes the image of $A \in \mathcal{E}_+$ in the quotient $\mathcal{E}_+/I_0$. The family of operators $\{P_t, 0 < t < \beta/2\}$ form an object called a *local symmetric semigroup.*

DEFINITION. A family of operators $\{P_t, D_t, 0 \le t < T\}$ acting on a Hilbert space $\mathcal{H}$, where D_t is the domain of P_t (in general, nonclosed), is called a *locally symmetric semigroup* if the following conditions are satisfied:

(i) The subspaces D_t are monotone decreasing (do not increase), and the largest subspace D_0 is dense in $\mathcal{H}$.

(ii) The family $\{P_t\}$ has the semigroup property in the following sense:

a) $P_0 = E$,

b) $P(s)D_t \subset D_{t-s}$ for $0 \le s \le t \le T$;

c) $P(t)P(s)f = P(t+s)$ for all $f \in D_{t+s}$, where $0 < s,\ t \le T$, $0 \le t + s \le T$.

(iii) Each operator P_t is symmetric on D_t:

$$(f, P_t g) = (P_t f, g), \qquad f, g \in D_t.$$

(iv) The family $\{P_t\}$ is weakly continuous: for $f \in D_s$, $s > t$, the quadratic form $(f, P_t f)$ depends continuously on $t \in [0, s]$.

One checks without difficulty that the operator family constructed above on the space $\mathcal{H}_{\text{phys}}$ satisfies all of these conditions. The main result in the theory of local symmetric semigroups is contained in the following:

THEOREM 2. *Let $\{P_t, D_t, 0 < t < T\}$ be a local symmetric semigroup on a Hilbert space $\mathcal{H}$. Then there exists a unique selfadjoint operator H on $\mathcal{H}$ with domain D_H such that $D_t \subset D_{\exp\{-tH\}}$ and P_t coincides with the restriction of the operator $\exp\{-tH\}$ to D_t for $0 \le t \le T$. Moreover, for any $0 < \tau \le T$ the subspace*

$$\widehat{D}_\tau = \bigcup_{0<t\le\tau} \bigcup_{0<s<t} P_s D_t$$

is contained in D_H and is a core (domain of essential selfadjointness) for H.

We refer to [53] for a proof of this theorem. The operator H for the local symmetric semigroup $\{P_t\}$ was constructed above, and it serves as the field Hamiltonian on $T_\beta \times R^\nu$.

REMARK. A field ξ_φ having the Markov property (in the sense explained above) is OS-positive, and the physical Hilbert space $\mathcal{H}_{\text{phys}}$ can be identified with $L_2(\Omega, \Sigma_{\{0\}\cup\{\beta/2\}}, \mu)$, where $\Sigma_{\{0\}\cup\{\beta/2\}} \subset \Sigma$ is the σ-algebra of events depending on the behavior of the field at times $t = 0$ and $t = \beta/2$ (i.e., on the set $\{0\} \times R^\nu \cup \{\beta/2\} \times R^\nu$). This assertion is verified in exactly the same way as in the case of Markov chains on $R^1 \times R^\nu$.

2. The modular theory of Tomita-Takesaki. We first discuss some abstract facts concerning von Neumann algebras, which will then be related to the Heisenberg dynamics

$$C \to \exp\{itH_{\text{GNS}}\} C \exp\{-itH_{\text{GNS}}\}, \qquad C \in \mathfrak{B}(\mathcal{H}_{\text{GNS}}),$$

which acts on operators in the Hilbert space $\mathcal{H}_{\text{GNS}}$ and is constructed from a C^*-algebra $\mathfrak{A}$ and a state $\langle\cdot\rangle$ on $\mathfrak{A}$ which is a β-KMS state relative to the dynamics $\alpha_t : \mathfrak{A} \to \mathfrak{A}$ on the algebra $\mathfrak{A}$.

I. Let $\mathfrak{M} \subset \mathfrak{B}(\mathcal{H})$ be a von Neumann operator algebra acting on the Hilbert space $\mathcal{H}$, and let $\mathfrak{M}' \subset \mathfrak{B}(\mathcal{H})$ be the commutant of $\mathfrak{M}$, i.e., the algebra of operators commuting with every operator in $\mathfrak{M}$. We assume that there is a vector $\Omega \in \mathcal{H}$ that is cyclic and separating for $\mathfrak{M}$. That is, the set $\{M\Omega, M \in \mathfrak{M}\}$ is dense in $\mathcal{H}$, and $M\Omega = 0$ for $M \in \mathfrak{M}$ implies that $M = 0$. Then it can be shown that Ω is also cyclic and separating for the algebra $\mathfrak{M}'$ (see [7]). We define antilinear operators

$$S(M\Omega) = M^*\Omega, \qquad M \in \mathfrak{M},$$
$$F(M'\Omega) = (M')^*\Omega, \qquad M' \in \mathfrak{M}'$$

on the dense sets $\{M\Omega, M \in \mathfrak{M}\}$ and $\{M'\Omega, M' \in \mathfrak{M}'\}$ (these are well defined because Ω is separating for both $\mathfrak{M}$ and $\mathfrak{M}'$).

It is proved in the Tomita-Takesaki theory that both these operators are closable, and their closures $\overline{S}$ and $\overline{F}$ have uniquely defined polar decompositions

$$\overline{S} = J\Delta^{1/2} \tag{9}$$

with an analogous decomposition for $\overline{F}$. Here Δ is a positive selfadjoint operator on $\mathcal{H}$ called the *modular operator,* and the antiunitary operator J is called the *modular involution* (an antiunitary operator is an invertible antilinear operator such that $(\mathcal{J}_\xi, \mathcal{J}_\eta) = ((\xi, \eta), \xi, \eta \in \mathcal{H})$. We have the relations

$$\Delta = FS, \qquad \Delta^{-1} = SF, \qquad J = J^{-1}$$

and

$$\Delta^{-1/2} = J\Delta^{1/2}J.$$

The main result of the Tomita-Takesaki theory is that

$$J\mathfrak{M}J = \mathfrak{M}'$$

and the unitary automorphisms of $\mathfrak{B}(\mathcal{H})$

$$\overline{\alpha}_t(C) = \Delta^{-it}C\Delta^{it} \tag{10}$$

take the algebra $\mathfrak{M}$ into itself.

Here for all $A, B \in \mathfrak{M}$ such that $A\Omega, B\Omega \in D_\Delta$, we have the relation

$$(\Delta A\Omega, B\Omega) = (JA^*\Omega, JB^*\Omega) = (B^*\Omega, A^*\Omega) \tag{11}$$

from which it follows readily that the state

$$\langle A\rangle = (A\Omega, \Omega), \qquad A \in \mathfrak{M} \tag{12}$$

on $\mathfrak{M}$ is a β-KMS state with respect to the dynamics (10), with $\beta = 1$.

We note that any operator satisfying (11) for all $A \in \mathfrak{M}$ and $B \in \mathfrak{M}$ must coincide with Δ.

II. Now let $\mathfrak{A}$ be a C^*-algebra with unit, and let $\langle\cdot\rangle$ be a faithful KMS state on $\mathfrak{A}$ with respect to the dynamics α_t (a state $\langle\cdot\rangle$ is said to be *faithful* if $\langle A\rangle > 0$ for every positive nonzero element $A \in \mathfrak{A}$). In this case, it is clear that $\mathcal{H}_{\mathrm{GNS}}$ is the closure of $\mathfrak{A}$ under the inner product

$$(A, B) = \langle B^*A\rangle \tag{13}$$

(see §2). We recall that under the GNS homomorphism $\pi\colon \mathfrak{A} \to \mathfrak{B}(\mathcal{H}_{\mathrm{GNS}})$, acting by the formula

$$\pi(A)B = AB \in \mathcal{H}_{\mathrm{GNS}}, \qquad A, B \in \mathfrak{A},$$

the dynamics α_t on $\mathfrak{A}$ goes over into the dynamics

$$\pi(\alpha_t A) = \exp\{itH\}\pi(A)\exp\{-itH\} \equiv \widetilde{\alpha}_t(\pi(A)) \tag{13$'$}$$

on $\mathfrak{B}(\mathcal{H}_{\mathrm{GNS}})$. Here the selfadjoint operator H is the generator for the unitary group on $\mathcal{H}_{\mathrm{GNS}}$ acting on elements $A \in \mathcal{H}_{\mathrm{GNS}}$ by

$$U_t A = \alpha_t(A)$$

(see 2.1). The vector $\Omega = 1$ is cyclic and separating for the algebra $\pi(\mathfrak{A}) \subset \mathfrak{B}(\mathcal{H}_{\mathrm{GNS}})$ and satisfies the condition

$$\exp\{itH\}\Omega = \Omega.$$

We can rewrite (13) as

$$(\pi(A)\Omega, \pi(B)\Omega) = \langle B^*A\rangle.$$

In order to exploit the KMS condition, we write

$$F_{A_1,A_2}(t) = \langle A_1\alpha_t(A_2)\rangle = (\pi(\alpha_t(A_2))\Omega, \pi(A_1^*)\Omega) = (e^{itH}\pi(A_2)\Omega, \pi(A_1^*)\Omega)$$

for any A_1 and A_2 in $\mathfrak{A}$; hence, under the hypothesis that $\pi(A_2)\Omega \in D_{\exp\{-\beta H\}}$, the domain of the operator $\exp\{-\beta H\}$, the KMS condition implies that

$$(e^{itH}e^{\beta H}\pi(A_2)\Omega, \pi(A_1^*)\Omega) = (\pi(A_1)\Omega, e^{itH}\pi(A_2^*)\Omega)$$

and for $t = 0$ we find that

$$(e^{\beta H}\pi(A_2)\Omega, \pi(A_1^*)\Omega) = (\pi(A_1)\Omega, \pi(A_2^*)\Omega). \tag{14}$$

Now let $\mathfrak{M}$ be the smallest von Neumann algebra containing the algebra $\pi(\mathfrak{A})$. We can rewrite (14) for any $C, D \in \mathfrak{M}(C, D \in D_{\exp\{-\beta H\}})$ as

$$(e^{\beta H}C\Omega, D\Omega) = (D^*\Omega, C^*\Omega)$$

and thus $\exp\{-\beta H\} = \Delta$, where Δ is the modular operator for the algebra $\mathfrak{M}$ (with separating and cyclic vector Ω). Consequently, the group of automorphisms (13′) coincides with the modular group (10),

$$\Delta^{-it/\beta}B\Delta^{it/\beta}, \tag{15}$$

up to a change of parameter.

This conclusion has the following important consequence relevant to the practical construction of the operator H_{GNS} (and the dynamics) for a temperature state.

Assume that for every finite system described by a Hamiltonian H_Λ on a Hilbert space $\mathcal{H}_\Lambda$ we are given a Heisenberg dynamics

$$\alpha_\Lambda(t)A = \exp\{itH_\Lambda\}A\exp\{-itH_\Lambda\}, \qquad A \in \mathfrak{B}(\mathcal{H}_\Lambda)$$

and an invariant β-KMS (Gibbs) state

$$\langle A\rangle_\Lambda = \frac{\operatorname{Tr}\exp\{-\beta H_\Lambda\}A}{\operatorname{Tr}\exp\{-\beta H_\Lambda\}}.$$

Passing to the infinite system by taking the thermodynamic limit Λ, we must construct both the thermodynamic limit

$$\langle A\rangle = \lim_{\Lambda\uparrow}\langle A\rangle_\Lambda$$

of the states, in terms of a state on a suitable algebra $\mathfrak{A}$, and also the limit dynamics

$$\alpha_t(A) = \lim_{\Lambda\uparrow}\alpha_t^\Lambda(A)$$

on $\mathfrak{A}$ (or some extension of it), in such a way that the KMS condition is satisfied.

Under the assumption that the first limit exists and induces a faithful state on $\mathfrak{A}$, the Tomita-Takesaki theory makes it unnecessary to establish the existence of the limiting dynamics, because the limit state can be used directly to construct the space $\mathcal{H}_{\mathrm{GNS}}$, the homomorphism $\pi\colon \mathfrak{A} \to \mathfrak{B}(\mathcal{H}_{\mathrm{GNS}})$, the smallest von Neumann algebra $\mathfrak{M}$ containing $\pi(\mathfrak{A})$ with cyclic and separating vector $\Omega = 1 \in \mathfrak{A}$, and the modular operator Δ. The required dynamics is then defined by the group (15), and the Hamiltonian is given by

$$H = \frac{1}{\beta}\ln\Delta.$$

For a Euclidean field theory on $T_\beta \times R^\nu$, the associated modular theory is described in [53]. The von Neumann algebra $\mathfrak{M}$ is taken to be the algebra generated by all operators of the form $\exp\{itH\}\widehat{f}\exp\{-itH\}$, where the operator $\widehat{f}$ acts on $\mathcal{H}_{\text{phys}} = L_2(\Omega, \Sigma_{\{0\}\cup\{\beta/2\}}, \mu)$ (see above) by the formula

$$\widehat{f}F = fF,$$

where $f \in L_\infty(\Omega, \Sigma_{\{0\}}, \mu)$ is an essentially bounded function that depends on the configurations in the zeroth slice $\{0\} \times R^\nu$. The modular involution coincides with the antilinear involution given by reflection in the point $\beta/2$ (and it takes the subspace $L_2(\Omega, \Sigma_{\{0\}}, \mu) \subset L_2(\Omega, \Sigma_{\{0\}\cup\{\beta/2\}}, \mu)$ into the subspace $L_2(\Omega, \Sigma_{\{\beta/2\}}, \mu)$); the modular operator Δ turns out to be $\exp\{-\beta H\}$. We refer to [53] for more details.

SPECTRAL ANALYSIS OF THE EUCLIDEAN FIELD TRANSFER MATRIX

In the previous chapter we have seen that in many cases, the study of the Hamiltonian H_{GNS} of an infinite quantum system in its ground state can be reduced to the analysis of the transfer matrix $\mathcal{J}_t$ (a semigroup of stochastic operators) of some suitable Euclidean (Markov) field. As we have seen, this problem is also of independent interest in the study of quantum Euclidean fields.

In this chapter we consider the case of Euclidean fields obtained as Gibbs modifications (see 5.2) of certain simple "free" fields (e.g., independent fields), when the interaction generating these modifications is small, i.e., the Euclidean fields studied here are small perturbations of these "free" fields. This makes it possible to study them by use of cluster expansions.

For this case we develop here a general method for constructing and analyzing the "highest branches" of the spectrum of the transfer matrix $\mathcal{J}_t$, i.e., the largest (in absolute value) elements in the spectrum of $\mathcal{J}_t$, when $\mathcal{J}_t$ is restricted to certain special invariant subspaces (because of the equality $\mathcal{J}_t = \exp\{-tH\}$, where H is the Hamiltonian of the field, the highest branches of the spectrum of $\mathcal{J}_t$ correspond to the lower branches of the spectrum of H).

As already indicated in Chapter 0, the highest invariant subspaces of $\mathcal{J}_t$ to be constructed below in a certain natural sense describe one-particle, two-particle, ... , n-particle states of the field, i.e., in the case considered the usual physical ideas regarding the corpuscular picture for elementary excitations of the ground state are indeed borne out.

In our treatment we focus on ordinary random fields defined on a discrete space-time. Fermion fields (also on the lattice $Z^{\nu+1}$) are then analyzed; finally, we briefly consider the case of fields with discrete space and continuous time.

Thus, we begin with the spectral analysis of transfer matrices for Gibbs modifications of an independent field on the space $X = E \times Z^1$ (here the "space" E is a countable set

(graph), and Z^1 is the "time"). We first recall the basic hypotheses regarding these fields introduced in §5.

Let $\{\xi_x,\ x = (y,t) \in E \times Z^1\}$ be an independent random field labeled by the points $x \in E \times Z^1$ and taking values in a spin space S equipped with a probability measure ν. The distribution of the field $\xi = \{\xi_x,\ x \in E \times Z^1\}$ is assumed to coincide with the measure $\mu_0 = \prod_{x \in E \times Z^1} \nu_x$, an infinite product of identical copies of the measure ν, and is defined on the space $\Omega = S^{E \times Z^\nu}$ of all realizations of the field. Let further $\Phi = \{\varphi_1, \dots, \varphi_n\}$ be a finite set of bounded functions on S, which without loss of generality can be taken to be orthonormal with respect to ν and orthogonal to the constants. For simplicity we consider only the case when the action V_Λ defining the Gibbs modification μ of the measure μ_0 is of the form

$$V_\Lambda = \sum_{\substack{\alpha,\beta \\ x,y,\rho(x,y)=1}} R^{x,y}_{\alpha,\beta} \varphi_\alpha(\xi_x) \overline{\varphi}_\beta(\xi_y), \qquad \Lambda \subset Z, \tag{0}$$

where $\Lambda \subset Z^\nu$ is a finite set and the coefficients $R^{x,y}_{\alpha,\beta}$ satisfy all the conditions a)–d) in part 5 of Chapter 2, §5. Interactions of the general type (50.5.11) will be discussed separately. Thus, in this chapter we will study the transfer matrix $\mathcal{J}$ for a Gibbs field with the interaction (0), although the methods developed here carry over to more general interactions.

§1. Cluster expansion of the transfer matrix

The analysis of the spectral properties of the transfer matrix $\mathcal{J}$ for a random field is based on a special expansion of the operator $\mathcal{J}$ called the *cluster expansion.*

The first step in constructing this expansion is to construct a basis in $\mathcal{H}_{\text{phys}}$ having a particular multiplicative form. The matrix elements of $\mathcal{J}$ in this basis will be seen to admit a cluster expansion.

1. Multiplicative basis. Let $\{\varphi_\gamma,\ \gamma \in \mathcal{N}\}$ be an orthonormal basis in $L_2(S,\nu)$ consisting of polynomials in the field variables φ_α and containing φ_α and 1. We assume that all the φ_γ are jointly bounded.

We take the index set $\mathcal{N}$ to be a connected graph with a distinguished vertex θ; $\varphi_\theta = 1$, and the degrees of the polynomials φ_γ and $\varphi_{\gamma'}$ at two adjacent vertices of $\mathcal{N}$ differ by at most 1. The distance from a vertex γ to the initial vertex θ (i.e., the length of the shortest path in $\mathcal{N}$ from θ to γ) is called the *rank* $N(\gamma)$ of γ.

It is assumed that the product $\varphi_{\gamma_1}\varphi_{\gamma_2}$ of two polynomials can be expanded as

$$\varphi_{\gamma_1}\varphi_{\gamma_2} = \sum_{\gamma:|N(\gamma_1)-N(\gamma_2)| \le N(\gamma) \le N(\gamma_1)+N(\gamma_2)} C^\gamma_{\gamma_1,\gamma_2} \varphi_\gamma, \tag{1}$$

where the constants $C^\gamma_{\gamma_1,\gamma_2}$ are all uniformly bounded. It follows from (1) that the number of polynomials with rank $N(\gamma) \le n$ is bounded by c^n for some absolute constant c.

We denote the tth time slice by $Y_t = E \times \{t\}$. As was observed in subsection 5 of §5 in Chapter 2, the Hilbert space $\mathcal{H}_{\text{phys}}$ can be identified with $L_2(S^{Y_0}, \mu|_{Y_0})$, where $\mu|_{Y_0}$ is the restriction of the measure μ to the space S^{Y_0}. The functions

$$\varphi_\Gamma(\xi) = \prod_{x \in Y_0} \varphi^{(x)}_{\gamma(x)}(\xi), \qquad \Gamma = \{\gamma(x), x \in Y_0\}, \qquad \xi \in S^{Y_0} \tag{2}$$

form an orthonormal basis in the space $L_2(S^{Y_0}, \mu_0|_{Y_0})$, where μ_0 is a free (independent) measure. Here $\varphi^{(x)}_{\gamma(x)}(\xi_x) = \varphi_{\gamma(x)}(\xi)$, and $\Gamma = \{\gamma(x), x \in Y_0\}$ is a multi-index, i.e., a finite

function on Y_0 with values in $\mathcal{N}$ (the finiteness of Γ means that only finitely many $\gamma(x)$ differ from θ). In the sequel we will construct an orthonormal basis in the true space $L_2(S^{Y_0}, \mu|_{Y_0})$ having a form

$$\Psi_\Gamma = \prod_{x \in Y_0} \psi^{(x)}_{\gamma(x)} \tag{3}$$

analogous to (2). Here $\Gamma = \{\gamma(x), x \in Y_0\}$ is a multi-index, and $\{\psi^{(x)}_\gamma, x \in Y_0\}$ is a family of bounded quasilocal functions defined on S^{Y_0}, which we now describe.

We introduce an order on the slice $Y_0 (= E)$ and for any point $x \in Y_0$ denote by $V_x \subset Y_0$ its "past":

$$V_x = \{y \in Y_0 : y < x\}. \tag{4}$$

For each fixed configuration $\overline{\xi}^{(x)} = \{\overline{\xi}_y, y \in V_x\}$, $\overline{\xi}_y \in S$ defined on the set V_x, we define functions of the values of the field ξ_x at the point x by

$$\widetilde{\varphi}^{(x)}_\gamma(\xi_x; \overline{\xi}^{(x)}) = \begin{cases} 1, & \gamma = \theta, \\ \varphi_\gamma(\xi_x) - \langle \varphi_\gamma(\xi_x) | \overline{\xi}^{(x)} \rangle, & \gamma \neq \theta, \end{cases} \tag{5}$$

where $\langle \cdot \mid \overline{\xi}^{(x)} \rangle$ to the conditional distribution $\mu(\cdot / \xi|_{V_x} = \overline{\xi}^{(x)})$ of the field, subject to the condition that the values of ξ on V_x are fixed and coincide with $\xi^{(x)}$. We next introduce the conditional Gramm matrix of functions $\{\widetilde{\varphi}^{(x)}_\gamma, \gamma \in \mathcal{N}\}$:

$$G_x = G_x(\overline{\xi}^{(x)}) = \{\langle \widetilde{\varphi}^{(x)}_{\gamma_1} \overline{\widetilde{\varphi}^{(x)}_{\gamma_2}} | \xi^{(x)} \rangle_{\gamma_1, \gamma_2 \in \mathcal{N}}\} \tag{6}$$

and orthogonalize $\{\widetilde{\varphi}^{(x)}_\gamma\}$ (for a fixed configuration $\overline{\xi}^{(x)}$) with respect to the conditional distribution $\mu(\cdot / \xi|_{V_x} = \overline{\xi}^{(x)})$. That is, we set

$$\widetilde{\psi}^{(x)}_\gamma(\xi_x; \overline{\xi}^{(x)}) = \sum_{\gamma'} m^{(x)}_{\gamma, \gamma'}(\overline{\xi}^{(x)}) \widetilde{\varphi}^{(x)}_{\gamma'}(\xi_x; \overline{\xi}^{(x)}), \tag{7}$$

where the matrix $M_x = \{m^{(x)}_{\gamma, \gamma'}(\overline{\xi}^{(x)})\}$ is equal to

$$M_x = (G_x)^{-1/2}.$$

(We will show below that the matrix $(G_x)^{-1/2}$ exists and has an explicit series expansion.)

For any $\gamma \in \mathcal{N}$ and $x \in Y_0$ we now define functions on S^{Y_0} by

$$\psi^{(x)}_\gamma(\xi) = \widetilde{\psi}^{(x)}_\gamma(\xi_x; \xi|_{V_x}), \qquad \xi \in S^{Y_0}. \tag{8}$$

The definitions (5), (6), and (7) imply that

$$\psi^{(x)}_\theta \equiv 1.$$

The system of functions $\{\Psi_\Gamma\}$ in (3) is defined using the functions in (8).

LEMMA 1. *The system* $\{\Psi_\Gamma\}$ *forms an orthonormal basis in* $\mathcal{H}_{\text{phys}}$.

PROOF. First we show that $\{\Psi_\Gamma\}$ is orthonormal in $\mathcal{H}_{\text{phys}}$. Let $x(\Gamma) \in \operatorname{supp}\Gamma$ be a maximal point in the set $\operatorname{supp}\Gamma = \{x \in Y_0 \colon \gamma(x) \neq \theta\}$. Let Γ and Γ' be two multi-indices. Then two cases are possible: a) $x(\Gamma) \neq x(\Gamma')$, and b) $x(\Gamma) = x(\Gamma')$. In case a) (supposing for definiteness that $x(\Gamma) > x(\Gamma')$), we find that

$$
\begin{aligned}
(\Psi_\Gamma, \Psi_{\Gamma'}) &= \left\langle \prod_x \psi^{(x)}_{\gamma(x)} \prod_x \overline{\psi}^{(x)}_{\gamma'(x)} \right\rangle \\
&= \left\langle \prod_{x \neq x(\Gamma)} \psi^{(x)}_{\gamma(x)} \prod_x \overline{\psi}^{(x)}_{\gamma'(x)} \ \langle \psi^{(x(\Gamma))}_{\gamma(x(\Gamma))} | \xi|_{V_x} - \text{fixed} \rangle \right\rangle = 0.
\end{aligned} \tag{9}
$$

Here we have used the fact that for $x < x(\Gamma)$, $\psi^{(x)}_{\gamma(x)}$ depends only on the values of the field on $V_{x(\Gamma)}$, and $\langle \psi^{(x)}_\gamma | \xi|_{V_x}$ is fixed$\rangle = 0$ for $\gamma \neq \theta$ by the construction of $\psi^{(x)}_\gamma$.

In case b), with $x(\Gamma) = x(\Gamma') = x$, a similar calculation leads to the equality

$$
(\Psi_\Gamma, \Psi_{\Gamma'}) = (\Psi_{\widehat{\Gamma}}, \Psi_{\widehat{\Gamma}'}) \delta_{\gamma(x), \gamma'(x)}, \tag{10}
$$

where the multi-indices $\widehat{\Gamma}$ and $\widehat{\Gamma}'$ are obtained from Γ and Γ' by replacing the values $\gamma(x)$ and $\gamma'(x)$ by θ. Applying this repeatedly, we get from (9) and (10) that

$$
(\Psi_\Gamma, \Psi_{\Gamma'}) = \delta_{\Gamma, \Gamma'}. \tag{11}
$$

To prove that the system $\{\Psi_\Gamma\}$ is complete we will need the following expansion of the function $\psi^{(x)}_\gamma$.

LEMMA 2. *For* $\gamma \neq \theta$ *we have the equality*

$$
\psi^{(x)}_\gamma = \varphi^{(x)}_\gamma + \sum_{\substack{\overline{\gamma} \in \mathcal{N} \\ \Gamma : \operatorname{supp}\Gamma \subset V_x}} B^{(x), \gamma}_{\Gamma, \overline{\gamma}} \varphi_\Gamma \varphi^{(x)}_{\overline{\gamma}}, \tag{12}
$$

where the coefficients $B^{(x),\gamma}_{\Gamma,\overline{\gamma}}$ *satisfy the bound*

$$
|B^{(x),\gamma}_{\Gamma,\overline{\gamma}}| < L(C\beta)^{d_{\Gamma,\gamma,\overline{\gamma},x}}. \tag{13}
$$

Here L *and* C *are absolute constants (depending on the geometry of the graph* E*, the basis* $\{\varphi_\gamma\}$*, and the coefficients* $R^{x,y}_{\alpha,\beta}$ *in formula* (0) *at the beginning of this chapter, but not on the parameter* β*). The exponent* $d_{\Gamma,\gamma,\overline{\gamma},x}$*, where* $\gamma, \overline{\gamma} \in \mathcal{N}$*,* $\Gamma = \{\gamma(y), y \in V_x\}$*, is given by*

$$
d_{\Gamma,\gamma,\overline{\gamma},x} = \min_{n = \{n(\tau)\}} \left\{ \sum_\tau n(\tau) \right\}, \tag{14}
$$

where the minimum is taken over all nonnegative finite integer-valued functions $n = \{n(\tau)\}$ *which are defined on the edges* $\tau = (x_1, x_2)$ *of the graph* X *and satisfy the following conditions*:

1) $\operatorname{supp} n$ *is a connected set of edges*;

2) *for every* $y \in V_x$ *we have*

$$\sum_{\tau: y \in \tau} n(\tau) \geq N(\gamma(y)) \quad (\textit{rank of } \gamma(y));$$

3) $\sum_{\tau: x \in \tau} n(\tau) \geq |N(\gamma) - N(\overline{\gamma})|$;

4) *no edge* $\tau \in \operatorname{supp} n$ *lies completely in* V_x.

REMARKS. I. In the calculations to follow, it will be helpful to have the estimate

$$d_{\Gamma,\gamma,\overline{\gamma},x} \geq \frac{1}{2}\left\{ d_{\operatorname{supp}\Gamma \cup \{x\}} + \sum_{y \in \operatorname{supp}\Gamma} N(\gamma(y)) + |N(\gamma) - N(\overline{\gamma})| \right\}, \tag{15}$$

where for $B \subset X$, d_B is the length of the smallest connected set of edges of X such that B is contained in the union of their vertices.

II. We will sometimes write equation (12) more compactly in the form

$$\psi_\gamma^{(x)} = \sum_{\Gamma: \operatorname{supp}\Gamma \subset V_x \cup \{x\}} \overline{B}_\Gamma^{(x),\gamma} \varphi_\Gamma, \tag{15^a}$$

where the $\overline{B}_\Gamma^{(x),\gamma}$ are easily expressed in terms of the coefficients $B_{\Gamma,\overline{\gamma}}^{(x),\gamma}$.

PROOF. We first derive the expression

$$\langle \psi_\gamma^{(x)} | \overline{\xi}^{(x)} \rangle = \sum_{\Gamma: \operatorname{supp}\Gamma \subset V_x} \widetilde{B}_\Gamma^{(x),\gamma} \varphi_\Gamma(\overline{\xi}^{(x)}), \tag{16}$$

for the conditional mean, where the coefficients $\widetilde{B}_\Gamma^{(x),\gamma}$ satisfy the inequalities

$$|\widetilde{B}_\Gamma^{(x),\gamma}| \leq L'(C'\beta)^{d_{\Gamma,\gamma,x}}, \tag{17}$$

L' and C' are absolute constants, and $d_{\Gamma,\gamma,x} = d_{\Gamma,\gamma,\theta,x}$. We now use the well-known expansion of the mean over a Gibbs field distribution as a series in the semi-invariants (see [26]). In our case, this expansion takes the form

$$\langle \psi_\gamma^{(x)} | \overline{\xi}^{(x)} \rangle = \sum_{n=0}^{\infty} \frac{(-\beta)^n}{n!} \sum \langle \psi_\gamma^{(x)}, \psi_{\alpha_1}^{(x_1)} \overline{\psi}_{\beta_1}^{(x_1')}, \ldots, \psi_{\alpha_n}^{(x_n)} \overline{\psi}_{\beta_n}^{(x_n')} \rangle_0 \prod_{i=1}^{n} R_{\alpha_i,\beta_i}^{x_i,x_i'}, \tag{18}$$

where the semi-invariants $\langle \cdot, \cdot, \ldots, \cdot \rangle_0$ are calculated relative to the free measure μ_0, and the sum is over the ordered sets of edges $\{\tau_1, \ldots, \tau_n\}$, $\tau_i = (x_i, x_i')$ such that $\tau_u \not\subset V_x$ and over the pairs of indices $\{(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$. The sum on the right in (18) can be rewritten as

$$\sum_n \prod_{\tau,\alpha,\beta} \frac{(-\beta)^{n_{\alpha,\beta}(\tau)}}{n_{\alpha,\beta}(\tau)!} \left\langle \varphi_\gamma^{(x)}, \prod{}' \varphi'^{n_{\alpha,\beta}(\tau)}_{\alpha,\beta}(\tau) \right\rangle, \tag{19}$$

where we write $\varphi_{\alpha,\beta}(\tau) = \varphi_\alpha^{(y)} \overline{\varphi}_\beta^{(y')}$, $\tau = (y, y')$ an oriented edge of X, and the sum is over all nonnegative integer-valued finite functions $n = \{n_{\alpha,\beta}(\tau)\}$ defined on triples (α, β, τ) such

that $n_{\alpha,\beta}(\tau) = 0$ for every edge $\tau \subset V_x$. Furthermore, each semi-invariant in the sum (19) can be written in the form

$$\left\langle \varphi_\gamma^{(x)}, \prod{}' {\varphi'}_{\alpha,\beta}^{n_{\alpha,\beta}(\tau)}(\tau) \right\rangle_0 = \prod_{\substack{(\alpha,\beta,\tau)\\ \tau=(y,y'),y\in V_x}} (\varphi_\alpha^{(y)})^{n_{\alpha,\beta}(\tau)} \times \prod_{\substack{(\alpha,\beta,\tau)\\ \tau=(y,y'),y'\in V_x}} (\overline{\varphi}_\beta^{(y')})^{n_{\alpha,\beta}(\tau)} \left\langle \varphi_\gamma^{(x)}, \prod_{\substack{(\alpha,\beta,\tau)\\ \tau\cap V_x=\emptyset}}{}' {\varphi'}_{\alpha,\beta}^{n_{\alpha,\beta}(\tau)}(\tau), \prod_{\substack{(\alpha,\beta,\tau)\\ \tau=(y,y'),y\in V_x}}{}' (\overline{\varphi}_\beta^{(y')})'^{n_{\alpha,\beta}(\tau)}, \prod_{\substack{(\alpha,\beta,\tau)\\ \tau=(y,y'),y'\in V_x}}{}' (\varphi_\alpha^{(y')})'^{n_{\alpha,\beta}(\tau)} \right\rangle_0. \tag{20}$$

In view of (1), we can express the product of the field variables in (20) as

$$\prod_{\substack{(\alpha,\beta,\tau)\\ \tau=(y,y'),y\in V_x}} (\varphi_\alpha^{(y)})^{n_{\alpha,\beta}(\tau)} \prod_{\substack{(\alpha,\beta,\tau)\\ \tau=(y,y'),y'\in V_x}} (\overline{\varphi}_\beta^{(y')})^{n_{\alpha,\beta}(\tau)} = \sum_{\Gamma:\operatorname{supp}\Gamma\subset V_x} D_\Gamma \varphi_\Gamma, \tag{21}$$

where the sum is over the multi-indices $\Gamma = \{\gamma(y), y \in V_x\}$ such that for every $y \in V_x$

$$\sum_{(\alpha,\beta,\tau):\tau\cap V_x=\{y\}} n_{\alpha,\beta}(\tau) = K(y) \geq N(\gamma(y)), \tag{22}$$

and the coefficients D_Γ do not exceed $\prod_{y\in\operatorname{supp}\Gamma} m^{K(y)}$, where m is an absolute constant (depending on the coefficients $C_{\gamma_1,\gamma_2}^{\gamma}$ in equation (1), and also on the number of field variables). We note further that the semi-invariant in (20) is nonzero only for those $n = \{n_{\alpha,\beta}(\tau)\}$ for which the set of edges $\{\tau\colon n_{\alpha,\beta}(\tau) \neq 0 \text{ for some pair } \alpha,\beta\}$ is connected, and

$$\sum_{(\alpha,\beta,\tau):x\in\tau} n_{\alpha,\beta}(\tau) \geq N(\gamma). \tag{23}$$

The expansions (19) and (21) lead to equation (16), while (17) follows from inequalities (22), (23), and the bound on the coefficients D_Γ, together with general bounds for semi-invariants discussed in [26] (see, e.g., the estimate (8.7.II) for partially independent variables).

Repeating the previous arguments as applied to the mean $\langle \varphi_{\gamma_1}^{(x)} \varphi_{\gamma_2}^{(x)} \mid \xi^{(x)} \rangle$, and using (16) and equation (1) to expand the product of the means $\langle \varphi_{\gamma_1}^{(x)} \mid \overline{\xi}^{(x)} \rangle \langle \varphi_{\gamma_2}^{(x)} \mid \overline{\xi}^{(x)} \rangle$ in terms of the monomials φ_Γ, we find that the Gramm matrix $G_x = \{g_{\gamma_1,\gamma_2}^{(x)}\}$ has the form

$$g_{\gamma_1,\gamma_2}^{(x)} = \delta_{\gamma_1,\gamma_2} + \sum_{\Gamma:\operatorname{supp}\Gamma\subset V_x} D_\Gamma^{(x),\gamma_1,\gamma_2} \varphi_\Gamma(\overline{\xi}^{(x)}), \tag{24}$$

where the coefficients $D_\Gamma^{(x),\gamma_1,\gamma_2}$ satisfy an estimate analogous to (13). From the representation $G_x = E + D_x$ we obtain that

$$(G_x)^{-1/2} = E + \sum_{k=1}^{\infty} \alpha_k (D_x)^k \tag{25}$$

where the α_k are the binomial coefficients. The last assertion of Lemma 2 follows from this by use of (24) and equations (1) and (7).

It follows from Lemma 2 that the functions $\psi_\gamma^{(x)}$ are jointly bounded by

$$|\psi_\gamma^{(x)}| < C$$

for sufficiently small β, where $C > 0$ is an absolute constant.

Upon expanding the matrix $G_x^{1/2}$ as

$$(G_x)^{1/2} = E + \sum_{k=1}^{\infty} \beta_k (D_x)^k, \tag{26}$$

we find that the function $\varphi_\gamma^{(x)}$ is expressible in the form

$$\varphi_\gamma^{(x)} = \sum_{\overline{\gamma}} (G_x)_{\gamma,\overline{\gamma}}^{1/2} \psi_\gamma^{(x)} = \sum_{\overline{\gamma}} R_{\gamma,\overline{\gamma}}^{(1)} \psi_{\overline{\gamma}}^{(x)} + \sum_{\overline{\gamma},\Gamma:\neq \operatorname{supp}\Gamma \subset V_x} F_{\Gamma,\overline{\gamma}}^{(1),(x),\gamma} \varphi_\Gamma \psi_{\widetilde{\gamma}}^{(x)}, \tag{27}$$

where the coefficients $F_{\Gamma,\overline{\gamma}}^{(1),(x),\gamma}$ satisfy an estimate similar to (13).

To prove the completeness of the system $\{\Psi_\Gamma\}$ it suffices to show that every monomial φ_Γ can be uniformly approximated by sums of the type

$$\sum_{\Gamma} R_\Gamma \Psi_\Gamma. \tag{28}$$

For simplicity, we consider a "one-point" monomial $\varphi_\gamma^{(x)}$ and write each monomial φ_Γ appearing in the second summand in (27) as $\varphi_\Gamma = \varphi_{\gamma_1}^{(x_1)} \varphi_{\widehat{\Gamma}}$, where $x_1 = x(\Gamma)$, $\gamma_1 = \gamma(x_1)$, and $\widehat{\Gamma}$ is the multi-index obtained from Γ by replacing the value γ_1 at the point x_1 by θ. Again expressing $\varphi_{\gamma_1}^{(x)}$ by an expansion of the form (27) and using relation (1), we find that

$$\varphi_\gamma^{(x)} = \sum_{\Gamma} R_{\gamma,\Gamma}^{(2)} \Psi_\Gamma + \sum_{\Gamma,\Gamma'} F_{\Gamma,\Gamma'}^{(2),(x),\gamma} \Psi_\Gamma \varphi_{\Gamma'}, \tag{29}$$

where the sum $\sum_\Gamma$ is over all multi-indices Γ such that $|\operatorname{supp}\Gamma| \le 2$, and the sum $\sum_{\Gamma,\Gamma'}$ is over all pairs (Γ,Γ') of multi-indices, where Γ is as above, $\operatorname{supp}\Gamma' \neq \emptyset$, and $x(\Gamma') < \operatorname{supp}\Gamma$. (This means that the inequality $x(\Gamma') < x'$ holds for each $x' \in \operatorname{supp}\Gamma$.) Repeating this procedure n times, we obtain the expansion

$$\varphi_\gamma^{(x)} = \sum_{\Gamma} R_{\gamma,\Gamma}^{(n)} \Psi_\Gamma + \sum_{\Gamma,\Gamma'} F_{\Gamma,\Gamma'}^{(n),(x),\gamma} \Psi_\Gamma \varphi_{\Gamma'}. \tag{30}$$

The sum $\sum_\Gamma$ is over all multi-indices Γ with $\operatorname{supp}\Gamma = \{x_1, x_2, \dots, x_k\}$, $x_i \le x$, $i = 1, \dots, k$, $k \le n$, and the sum $\sum_{\Gamma,\Gamma'}$ is over the pairs (Γ,Γ') with $\Gamma = \{x_n < x_{n-1} < \cdots < x_1\}$, $x_i \le x$, with $\operatorname{supp}\Gamma' \neq \emptyset$ and $\operatorname{supp}\Gamma' < x_n$.

By induction on n, we get the following estimate for the $F_{\Gamma,\Gamma'}^{(n),(x),\gamma}$

$$|F_{\Gamma,\Gamma'}^{(n),(x),\gamma}| < L(C\beta)^n (C\beta)^{\frac{1}{2} d_{\operatorname{supp}\Gamma \cup \operatorname{supp}\Gamma' \cup \{x\}}} (C\beta)^{\frac{1}{2}[N(\Gamma)+N(\Gamma')]}, \tag{31}$$

where $N(\Gamma) = \sum_x N(\gamma(x))$, $\Gamma = \{\gamma(x), x \in Y_0\}$, and $C > 0$ and $L > 0$ are absolute constants. From (31) and the uniform boundedness of the $\varphi_\gamma^{(x)}$ and $\psi_\gamma^{(x)}$ we find that for

sufficiently large n, the last summand in (30) becomes uniformly small, so that the function $\varphi_\gamma^{(x)}$ is approximated by sums of the form (28). Similar arguments show that the same is true for any monomial φ_Γ.

This completes the proof of the lemma.

REMARK. The above proof that the functions $\{\Psi_\Gamma\}$ are complete in $\mathcal{H}_{\text{phys}}$ relies on several special properties of the cluster expansion of the measure μ (the expansions (12) and (24), the smallness of β, etc.), and in this sense it is a bit rough. For instance, it seems likely that for the basis (3) described here, completeness should always hold when the measure μ is ergodic.

Under the assumption that our ordering $<$ on Y_0 is preserved under all spatial translations τ_s, the system of functions $\{\Psi_\Gamma\}$ is also invariant under the group $(U_s,\ s \in Z^\nu)$ of spatial translations in $\mathcal{H}_{\text{phys}}$, and

$$U_s\Psi_\Gamma = \Psi_{\Gamma+s}, \tag{31^a}$$

where $\Gamma + s = \{\widetilde{\gamma}(x),\ x \in Y_0\}$ is the translation of the multi-index $\Gamma = \{\gamma(x), x \in Y_0\}$ by s,

$$\widetilde{\gamma}(x) = \gamma(\tau_s^{-1}x),$$

and τ_s is the action of the group Z^ν on E.

2. The cluster expansion. We begin with some conventions concerning terminology and notation. For any subset $Y \subset X$, we write $\mathfrak{M}(Y)$ for the set of multi-indices $\Gamma = \{\gamma(x),\ x \in X\}$ with nonempty supports $\emptyset \neq \operatorname{supp}\Gamma \subset Y$; we write $\mathfrak{M}(Y_0) = \mathfrak{M}_0$. A set of multi-indices $\{\Gamma_1, \dots, \Gamma_n\}$ is said to be *disjoint,* if $\operatorname{supp}\Gamma_i \cap \operatorname{supp}\Gamma_j = \emptyset$ whenever $i \neq j$, and *consistent* if

$$\Gamma_i|_{\operatorname{supp}\Gamma_i \cap \operatorname{supp}\Gamma_j} = \Gamma_j|_{\operatorname{supp}\Gamma_i \cap \operatorname{supp}\Gamma_j}$$

for all pairs $i \neq j$ (any disjoint set is clearly consistent). Evidently, given any consistent set of multi-indices $\{\Gamma_1, \dots, \Gamma_n\}$ there exists a multi-index Γ such that $\operatorname{supp}\Gamma = \bigcup_i \operatorname{supp}\Gamma_i$ and $\Gamma|_{\operatorname{supp}\Gamma_i} = \Gamma_i|_{\operatorname{supp}\Gamma_i}$ for all $i = 1, \dots, n$. We call Γ the *sum* of the set $\{\Gamma_1, \dots, \Gamma_n\}$ and write $\Gamma = \bigvee_{i=1}^n \Gamma_i$. Further, any representation of a multi-index Γ as a sum $\Gamma = \bigvee_{i=1}^n \Gamma_i$ of a disjoint set of multi-indices $\{\Gamma_1, \dots, \Gamma_n\}$ will be called a *partition* of Γ. Finally, if two disjoint sets $\tau = \{\Gamma_1, \dots, \Gamma_n\}$ and $\tau' = \{\Gamma'_1, \dots, \Gamma'_n\}$ are partitions of the same multi-index $\Gamma = \bigvee_{i=1}^n \Gamma_i = \bigvee_{j=1}^{n'} \Gamma'_j$ and for every $i = 1, 2, \dots, n$ we have $\operatorname{supp}\Gamma_i \subset \operatorname{supp}\Gamma'_j$ for some $j = 1, 2, \dots, n'$, then we write $\tau < \tau'$ and call τ' a refinement of τ. The set of partitions $\{\Gamma_i\}$ of a fixed multi-index $\Gamma = \mathfrak{M}(X)$ clearly forms a chain with respect to the relation $<$; we will denote it by $\mathfrak{A}_\Gamma$.

Let $\mathcal{H}'_{\text{phys}} = \mathcal{H}_{\text{phys}} \ominus \{\text{const}\}$ be the set of functions $\Phi \in \mathcal{H}_{\text{phys}}$ such that $\langle\Phi\rangle_\mu = 0$. Evidently, $\mathcal{H}'_{\text{phys}}$ is invariant under the transfer matrix $\mathcal{J}$ and the group $\{U_s\}$, and in the sequel we will study the action of these operators on $\mathcal{H}'_{\text{phys}}$ without indicating this specifically (we will continue to write $\mathcal{J}$ and U_s). The functions $\{\Psi_\Gamma, \Gamma \in \mathfrak{M}_0 \equiv \mathfrak{M}(Y_0)\}$ clearly form a basis in $\mathcal{H}'_{\text{phys}}$. For any $\Gamma, \Gamma' \in \mathfrak{M}_0$, we write

$$a_{\Gamma,\Gamma'} = (\Psi_\Gamma, \mathcal{J}\Psi_{\Gamma'}) = \langle\Psi_\Gamma\overline{\widehat{\Psi}_{\Gamma'}}\rangle \tag{32}$$

for the matrix elements of the transfer matrix $\mathcal{J}$ in the basis $\{\Psi_\Gamma\}$. Here $\widehat{\Psi}_{\Gamma'}$ is the field function

$$\widehat{\Psi}_{\Gamma'} = U_{e_0}\Psi_{\Gamma'} = \prod \psi^{\{x\}}_{\gamma'(x)},$$
$$\widehat{\psi}^{(x)}_{\gamma'(x)} = U_{e_0}\psi^{(x)}_{\gamma(x')},$$

where U_{e_0} is the operator on $L_2(S^X, \mu)$ corresponding to the unit translation along the positive time direction.

If we expand the function $f \in \mathcal{H}'_{\text{phys}}$ in terms of the basis $\{\Psi_\Gamma\}$:

$$f = \sum_{\Gamma \in \mathfrak{M}_0} f(\Gamma)\Psi_\Gamma,$$

then the coordinates $\{f(\Gamma)\}$ of f transform under the action of $\mathcal{J}$ by the formula

$$(\mathcal{J}f)(\Gamma) = \sum_{\Gamma'} a_{\Gamma,\Gamma'} f(\Gamma'). \tag{33}$$

Because the function $\widehat{\Psi}_\Gamma$ depends on the values of the field ξ on the slice Y_1, it will sometimes be convenient to regard the second multi-index Γ' in (32) as "translated by the slice Y_1", i.e., to regard $\operatorname{supp}\Gamma'$ as lying in Y_1, so that every pair $(\Gamma, \Gamma') \in \mathfrak{M}_0 \times \mathfrak{M}_0$ can be represented as a single multi-index $\widetilde{\Gamma} = \Gamma \vee \Gamma' \subset \mathfrak{M}(Y_0 \cup Y_1)$. Any $\widetilde{\Gamma} \in \mathfrak{M}(Y_0 \cup Y_1)$ (i.e., such that $\operatorname{supp}\widetilde{\Gamma} \cap Y_0 \neq \emptyset$ and $\operatorname{supp}\widetilde{\Gamma} \cap Y_1 \neq \emptyset$) will be called a *standard* multi-index.

Using the well-known expression for the moments of random variables in terms of their semi-invariants (see [26]), we obtain that

$$a_{\Gamma,\Gamma'} \equiv a_{\widetilde{\Gamma}} = \sum_{\tau = (\widetilde{\Gamma}_1, \ldots, \widetilde{\Gamma}_n)} \omega_{\widetilde{\Gamma}_1} \omega_{\widetilde{\Gamma}_2} \ldots \omega_{\widetilde{\Gamma}_n}, \tag{34}$$

where the sum is over all partitions of the standard multi-index $\widetilde{\Gamma}$ into disjoint unordered sets $\tau = \{\widetilde{\Gamma}_1, \ldots, \widetilde{\Gamma}_n\}$ of standard multi-indices. Here the quantities $\omega_{\widetilde{\Gamma}}$ are the semi-invariants:

$$\omega_{\widetilde{\Gamma}} = \omega_{\Gamma,\Gamma'} = \left\langle \prod_{y \in \operatorname{supp}\Gamma}{}' \psi^{(x)}_{\gamma(x)}, \prod_{x \in \operatorname{supp}\Gamma}{}' \overline{\widehat{\psi}}^{(x)}_{\gamma'(x)} \right\rangle. \tag{35}$$

In deriving (35) we have used the fact that for any $\Gamma \in \mathfrak{M}_0$ the semi-invariants satisfy

$$\left\langle \prod_{x \in \operatorname{supp}\Gamma}{}' \psi^{(x)}_{\gamma(x)} \right\rangle = \left\langle \prod_{x \in \operatorname{supp}\Gamma}{}' \overline{\widehat{\psi}}^{(x)}_{\gamma(x)} \right\rangle = 0, \tag{36}$$

as follows from the equalities $\langle \overline{\Psi}_\Gamma \rangle = \langle \widehat{\Psi}_\Gamma \rangle = 0$ for all $\Gamma \in \mathfrak{M}_0$ and the formula expressing the semi-invariants in terms of the moments (see [26]).

LEMMA 3 (cluster estimate). *The semi-invariant $\omega_{\widetilde{\Gamma}}$ satisfy the estimate*

$$|\omega_{\widetilde{\Gamma}}| < L(C\beta)^{d_{\widetilde{\Gamma}}}, \tag{37}$$

where $L > 0$, $C > 0$ are absolute constants (depending on the quantities $R^{x,y}_{\alpha,\beta}$, the geometry of the graph X, and the system of functions φ_γ, but independent of β). Moreover, the quantities $d_{\widetilde{\Gamma}}$ are defined in analogy with $d_{\Gamma,\gamma,\overline{\gamma},\overline{x}}$ in equation (13):

$$d_{\widetilde{\Gamma}} = \min_n \sum_\tau n(\tau), \qquad \widetilde{\Gamma} = \{\widetilde{\gamma}(x), x \in Y_0 \cup Y_1\}, \tag{38}$$

where $n = \{n(\tau)\}$ is an integer-valued nonnegative finite function on the edges of X, and the minimum in (38) *is taken over all functions whose support is a connected set of edges and such that for every $x \in \operatorname{supp}\widetilde{\Gamma}$ we have*

$$\sum_{\tau: x\in\tau} n(\tau) \geq N(\widetilde{\gamma}(x)). \tag{39}$$

Proof. The estimate (37) can be derived by the arguments used in [26, Chapter 6, §2] to estimate the semi-invariants of bounded quasilocal functions (the analytic method). Specifically, using the cluster expansion for a suitable Gibbs distribution, one can show that the semi-invariant $\omega_{\widetilde{\Gamma}}$ is analytic in the parameter β ($|\beta| < \beta_0$), and then conclude that $\omega_{\widetilde{\Gamma}}$ is of order $(C\beta)^{d_{\widetilde{\Gamma}}}$ for small β. This assertion is obtained from the cluster estimates for the semi-invariants by using equation (15^a), expanding the semi-invariant $\omega_{\widetilde{\Gamma}}$ as a sum of semi-invariants of the form ($\widetilde{\Gamma} = (\Gamma, \Gamma')$)

$$\prod_{x\in\operatorname{supp}\Gamma} B^{(x),\gamma(x)}_{\Gamma_x} \prod_{x\in\operatorname{supp}\Gamma} B^{(x),\gamma'(x)}_{\Gamma'_x} \left\langle \prod_x{}' \overline{\varphi}_{\Gamma_x}, \prod_x{}' \widehat{\varphi}_{\Gamma'_x} \right\rangle$$

and then expanding the expectation $\langle \prod'_x \overline{\varphi}_{\Gamma_x}, \prod'_x \widehat{\varphi}_{\Gamma_x} \rangle$, in terms of the semi-invariants of an independent field, as in (18) in the preceding paragraph; see [26]. The estimate (37) now follows readily.

Formula (34) for the matrix elements $a_{\Gamma,\Gamma'}$ together with the cluster estimate (37) is called the *cluster expansion* of the transfer matrix.

We note that since the transfer matrix $\mathcal{J}$ commutes with the group $\{U_s\}$, (31^a) implies that

$$a_{\Gamma+s,\Gamma'+s} = a_{\Gamma,\Gamma'} \tag{40}$$

and also that

$$\omega_{\Gamma+s,\Gamma'+s} = \omega_{\Gamma,\Gamma'}. \tag{41}$$

Remark. We note that our constructions remain valid for a field with an action of the form

$$V_\Lambda = \sum R_k \varphi^k, \tag{42}$$

where $k = \{k_\alpha(y)\}$, $\varphi_k = \prod_{y,\alpha} \varphi_\alpha^{h_\alpha(y)}(\xi(y))$, and R_k are the coefficients. In this case, to define the quantity d appearing in the estimate (37) for the semi-invariant $\omega_{\widetilde{\Gamma}}$, one must minimize the sum

$$\sum_\Delta n(\Delta),$$

where $n = \{n(\Delta)\}$ is an integer-valued function defined on $\Delta = \operatorname{supp} k$, the support of the potential V_Λ in (42). The minimum is over the set of functions $n = \{n(\Delta)\}$ for which the set $\{\Delta\colon n(\Delta) \neq 0\}$ is connected, and for every $y \in \operatorname{supp}\widetilde{\Gamma}$ the estimate

$$\sum_{\Delta: y\in\Delta} n(\Delta) k_\Delta(y) > N(\widetilde{\gamma}(y)), \qquad \widetilde{\Gamma} = \{\widetilde{\gamma}(x)\}$$

holds. Here $k_{\Delta(y)}$ is the smallest power of the field variables $\varphi_\alpha(\xi(y))$ appearing in the monomials φ^k, and in (42) $\operatorname{supp} k = \Delta$.

§2. Cluster operators. Definition and basic properties

Upon introducing a basis $\{\psi_\Gamma, \Gamma \in \mathfrak{M}_0\}$ in $\mathcal{H}_{\text{phys}}$, we can identify $\mathcal{H}_{\text{phys}}$ with the space $l_2(\mathfrak{M}_0)$ of functions $f = \{f(\Gamma)\}$ on the set $\mathfrak{M}_0$ such that

$$\|f\| = \sum_{\Gamma \in \mathfrak{M}_0} |f(\Gamma)|^2 < \infty. \tag{1}$$

Under this identification, the group $\{U_s\}$ acts by

$$(U_s f)(\Gamma) = f(\Gamma - s), \tag{2}$$

and the operator $\mathcal{J}$ by the formula (33.1). We now introduce a more general class of operators acting on $l_2(\mathfrak{M}_0)$ and called *cluster operators*; they include, in particular, the transfer matrices of Gibbs fields.

Let the operator A be defined by the formula

$$(Af)(\Gamma) = \sum_{\Gamma'} a_{\Gamma,\Gamma'} f(\Gamma') \tag{3}$$

and write $a_{\Gamma,\Gamma'} \equiv a_{\widetilde{\Gamma}}$ for its matrix elements, where $\widetilde{\Gamma}$ is the standard multi-index generated by the pair (Γ, Γ'):

$$\widetilde{\Gamma} = \Gamma \bigvee \Gamma'$$

(the multi-index Γ' is regarded as "translated" by the slice Y_1, see above). The matrix elements are assumed to admit a decomposition

$$a_{\widetilde{\Gamma}} = \sum_{\tau = (\widetilde{\Gamma}_1, \dots, \widetilde{\Gamma}_n)} \omega(\widetilde{\Gamma}_1, \dots, \widetilde{\Gamma}_n), \tag{4}$$

where the sum is over all disjoint partitions of $\widetilde{\Gamma}$ into sets of standard multi-indices $\widetilde{\Gamma}_1, \dots, \widetilde{\Gamma}_n$; and $\omega(\Gamma_1, \dots, \Gamma_n)$ is a function defined on the collection of all unordered sets $\tau = (\widetilde{\Gamma}_1, \dots, \widetilde{\Gamma}_n)$ of standard (in general, nondisjoint) multi-indices, and satisfies the following conditions:

1. *Independence of clusters*: for every set of standard multi-indices $(\widetilde{\Gamma}_1, \dots, \widetilde{\Gamma}_n)$ and every n-tuple of vectors $\{s_1, \dots, s_n\}$, $s_i \in Z^\nu$, we have

$$\omega(\widetilde{\Gamma}_1 + s_1, \dots, \widetilde{\Gamma}_n + s_n) = \omega(\Gamma_1, \dots, \Gamma_n). \tag{5}$$

2. *Cluster estimate*: there exists a number λ, $0 < \lambda < 1$ (the cluster parameter) and a constant L such that for every set $(\widetilde{\Gamma}_1, \dots, \widetilde{\Gamma}_n)$ we have the estimate

$$|\omega(\widetilde{\Gamma}_1, \dots, \widetilde{\Gamma}_n)| < L \prod_{i=1}^{n} \lambda^{d_{\widetilde{\Gamma}_i}}. \tag{6}$$

The function $\omega(\widetilde{\Gamma}_1, \dots, \widetilde{\Gamma}_n)$ appearing in (4) is called the *cluster* function. We now make some remarks concerning the above definition of cluster operators.

I. Condition (5) expresses the basic physical idea of the cluster property, that is, a multicomponent system breaks apart into several "noninteracting" subsystems (clusters) that move independently of one another in space.

We note that the expansion (4) involves the values of the cluster function $\omega(\widetilde{\Gamma}_1,\dots,\widetilde{\Gamma}_n)$ on disjoint sets. In essence, condition (5) provides a way of defining $\omega(\widetilde{\Gamma}_1,\dots,\widetilde{\Gamma}_n)$ on all sets $(\widetilde{\Gamma}_1,\dots,\widetilde{\Gamma}_n)$ of standard multi-indices.

II. The cluster estimate (6) shows that the main contribution to the matrix element $a_{\Gamma,\Gamma'}$ comes from those partitions $\tau=(\widetilde{\Gamma}_1,\dots,\widetilde{\Gamma}_n)$ of the multi-index $\widetilde{\Gamma}$ whose blocks $\widetilde{\Gamma}_i$ are not too "spread out" in space (i.e., the quantities $d_{\operatorname{supp}\widetilde{\Gamma}_i}$ are as small as possible). Here the matrix elements $a_{\widetilde{\Gamma}}$ themselves are small for multi-indices $\widetilde{\Gamma}=\{\widetilde{\gamma}(x)\}$ with sufficiently large powers $N(\widetilde{\Gamma})=\operatorname{supp}_x N(\widetilde{\gamma}(x))$.

The cluster expansion (6) chosen here is quite convenient in applications and agrees well with the cluster estimate (37.1) for the semi-invariants $\omega_{\widetilde{\Gamma}}$; of course, this estimate could be replaced by various others of similar type. In particular, in the study of the transfer matrices for Gibbs Euclidean fields defined by an action of the general type (42.1), one must change the definition of the $d_{\widetilde{\Gamma}}$ in (6) as was explained in the remark at the end of the preceding paragraph.

We now establish some general properties of cluster operators.

LEMMA 1. *The cluster function is uniquely determined by the cluster operator.*

In other words, it is impossible to find two distinct cluster expansions (4) for the matrix elements of the cluster operator, with distinct cluster functions ω_1 and ω_2, respectively, satisfying conditions 1 and 2.

PROOF. Let $\tau^0=(\widetilde{\Gamma}_1,\dots,\widetilde{\Gamma}_n)$ be a disjoint set of standard multi-indices. Note that the expansion (4) together with conditions (5) and (6) implies that

$$\lim_{\substack{|s_i-s_j|\to\infty\\ i\neq j}} a_{\bigvee_{i=1}^n(\Gamma_i+s_i)} \equiv B(\tau_0)=\sum_{\tau\le\tau_0}\omega(\tau), \tag{7}$$

where the sum is over all partitions τ refining τ_0. We write $\mathfrak{A}_{\widetilde{\Gamma}}$ for the structure[3] of partitions of the standard multi-index $\widetilde{\Gamma}$ into standard multi-indices. We see from equation (7) that the sums

$$\sum_{\tau<\overline{\tau}}\omega(\tau)\equiv B(\overline{\tau})$$

are uniquely defined by the operator itself. Applying the Möbius inversion formula, we obtain that

$$\omega(\tau)=\sum_{\overline{\tau}:\tau\le\overline{\tau}}\mu_{\mathfrak{A}_{\widetilde{\Gamma}}}(\tau,\overline{\tau})B(\overline{\tau}),$$

where $\mu_{\mathfrak{A}_{\widetilde{\Gamma}}}$ is the Möbius function for the structure $\mathfrak{A}_{\widetilde{\Gamma}}$. The lemma is proved.

Let $\mathcal{W}_\lambda$ be the class of cluster operators acting on $l_2(\mathfrak{M}_0)$, where they satisfy the cluster estimate (6) with a fixed cluster parameter λ. Evidently, $\mathcal{W}_\lambda$ is a linear space which is complete with respect to the norm

$$\|\!|A|\!\|_\lambda=\sup_\tau|\omega(\tau)|\prod_{\widetilde{\Gamma}\in\tau}\lambda^{-d_{\widetilde{\Gamma}}},\qquad A\in\mathcal{W}_\lambda, \tag{8}$$

where the supremum is over all disjoint sets $\tau=\{\widetilde{\Gamma}\}$ of standard multi-indices.

[3]Translator's note: That is, directed set with respect to the partial order given by refinement.

LEMMA 2. *For λ sufficiently small, every cluster operator $A \in \mathcal{W}_\lambda$ is a bounded operator on $l_2(\mathfrak{M}_0)$, and*

$$\|A\|_{l_2(\mathfrak{M}_0)} < C\lambda |||A|||_\lambda, \tag{9}$$

where $C > 0$ is an absolute constant.

PROOF. It follows easily from the general formula

$$\|A\|_{l_2(\mathfrak{M}_0)} = \sup_{\substack{f,g \in l_2(\mathfrak{M}_0) \\ \|f\|=\|g\|=1}} (Af, g)$$

that the norm $\|A\|_{l_2(\mathfrak{M}_0)}$ satisfies the estimate

$$\|A\|_{l_2(\mathfrak{M}_0)} \le \frac{1}{2}\left(\sup_\Gamma \sum_{\Gamma'} |a_{\Gamma,\Gamma'}| + \sup_{\Gamma'} \sum_\Gamma |a_{\Gamma,\Gamma'}| \right), \tag{10}$$

where the $a_{\Gamma,\Gamma'}$ are the matrix elements of A. Using (4) and the bound (6), we compute easily that for small λ

$$\sum_{\Gamma'} |a_{\Gamma,\Gamma'}| < (C\lambda)^{N(\Gamma)} |||A|||_\lambda, \tag{11}$$

where C is an absolute constant; a similar estimate holds for $\sum_\Gamma |a_{\Gamma,\Gamma'}|$. The result (9) now follows from these estimates and (10).

The set of cluster operators (with a sufficiently small cluster parameter) turns out to form an algebra. More precisely, we have

LEMMA 3. *There exists an absolute constant $\overline{C}$ such that for every sufficiently small λ and every set of cluster operators $A_0, A_1, \dots, A_n$ in $\mathcal{W}_\lambda$, the product*

$$B = A_0 A_1 \dots A_n \tag{12}$$

belongs to the class $\mathcal{W}_{\overline{C}\lambda}$, and

$$|||B|||_{\overline{C}\lambda} < (\overline{C}\lambda)^n \prod_{i=0}^{n} |||A_i|||_\lambda. \tag{13}$$

PROOF. Clearly, the matrix elements $a^B_{\Gamma,\Gamma'}$ of the operator B are equal to

$$a^B_{\Gamma,\Gamma'} = \sum_{(\Gamma_1,\Gamma_2,\dots,\Gamma_n)} a^{A_0}_{\Gamma,\Gamma_1} a^{A_1}_{\Gamma_1,\Gamma_2} \dots a^{A_n}_{\Gamma_n,\Gamma'}. \tag{14}$$

In what follows we will regard each multi-index Γ_i as lying in the slice Y_i ($\operatorname{supp}\Gamma_i \subset Y_i$) and define standard multi-indices

$$\widetilde{\Gamma}_0 = (\Gamma, \Gamma_1), \widetilde{\Gamma}_1 = (\Gamma_1, \Gamma_2), \dots, \widetilde{\Gamma}_n = (\Gamma_n, \Gamma')$$

with supports $\operatorname{supp}\Gamma_i \subset Y_i \cup Y_{i+1}$ ($\operatorname{supp}\widetilde{\Gamma}_i \cap Y_i \neq \emptyset$, $\operatorname{supp}\widetilde{\Gamma}_i \cap Y_{i+1} \neq \emptyset$), such that the set of multi-indices $(\widetilde{\Gamma}_0, \widetilde{\Gamma}_1, \dots, \widetilde{\Gamma}_n)$ is consistent, so that the multi-index

$$\widehat{\Gamma} = \bigvee \widetilde{\Gamma}_i, \qquad \operatorname{supp}\widehat{\Gamma} \subset \bigcup_{i=0}^{n+1} Y_i,$$

is defined, and $\operatorname{supp}\widetilde{\Gamma} \cap Y_i \neq \emptyset$, $i = 0, 1, \dots, n+1$.

For every unordered set

$$\alpha = \{\widetilde{\Gamma}_i^j, \quad i = 0, 1, \dots, n, \quad j = 1, \dots, \overline{j}(i)\}$$

of standard multi-indices such that

$$\operatorname{supp}\widetilde{\Gamma}_i^j \subset Y_i \cup Y_{i+1}$$

we define $\omega(\alpha)$ by

$$\omega(\alpha) = \omega^{A_0}(\widetilde{\Gamma}_0^1, \dots, \widetilde{\Gamma}_0^{\overline{j}(0)})\omega^{A_1}(\widetilde{\Gamma}_1^1, \dots, \widetilde{\Gamma}_1^{\overline{j}(1)}) \dots \omega^{A_n}(\widetilde{\Gamma}_n^1, \dots, \widetilde{\Gamma}_n^{\overline{j}(n)}). \tag{15}$$

The set $\alpha = \{\widetilde{\Gamma}_i^j,\ i = 0, 1, \dots, n,\ j = 1, \dots, \overline{j}(i)\}$ is said to be *admissible* if:

1) for every $i = 0, 1, \dots, n$ the sets $\{\widetilde{\Gamma}_j^1, \dots, \widetilde{\Gamma}_i^{j(i)}\}$ are disjoint;

2) the entire set $\{\widetilde{\Gamma}_i^j\}$ is consistent, i.e.,

$$\bigvee_j \widetilde{\Gamma}_i^j|_{Y_i} = \bigvee_j \widetilde{\Gamma}_{i-1}^j|_{Y_i}, \qquad i = 1, \dots, n.$$

For every admissible set $\{\widetilde{\Gamma}_i^j\}$ we introduce the following multi-indices

$$\Gamma(\alpha) = \bigvee_{i,j} \widetilde{\Gamma}_i^j, \qquad \Gamma_0(\alpha), \qquad \Gamma_{n+1}(\alpha)$$

which are such that

$$\begin{aligned} \operatorname{supp}\Gamma_0(\alpha) \subset Y_0, \qquad & \operatorname{supp}\Gamma_{n+1}(\alpha) \subset Y_{n+1}, \\ \Gamma_0(\alpha)|_{Y_0} = \Gamma(\alpha)|_{Y_0}, \qquad & \Gamma_{n+1}(\alpha)|_{Y_{n+1}} = \Gamma(\alpha)|_{Y_{n+1}}. \end{aligned}$$

¿From (14), (4), and the definition (15) we get the formula

$$a^B_{\Gamma,\Gamma'} = \sum_{\alpha:\Gamma_0(\alpha)=\Gamma,\Gamma_{n+1}(\alpha)=\Gamma'} \omega_\alpha \tag{16}$$

the sum being over all admissible sets α of standard multi-indices with fixed first and last multi-indices $\Gamma_0(\alpha)$ and $\Gamma_{n+1}(\alpha)$.

An admissible set $\alpha = \{\widetilde{\Gamma}_i^j\}$ is called a *bond* if the union of the supports $\{\operatorname{supp}\widetilde{\Gamma}_i^j\}$ is connected. A set of bonds $\beta = \{\alpha_1, \dots, \alpha_m\}$ is said to be *regular* if the collection of multi-indices $\Gamma(\alpha_1), \dots, \Gamma(\alpha_m)$ is disjoint. Since every admissible set $\alpha = \{\widetilde{\Gamma}_i^j\}$ of standard

multi-indices can be uniquely decomposed into a regular set of bonds $\beta = \{\alpha_1, \dots, \alpha_m\}$, where the α_k are the connected components of α, we can write (16) in the form

$$a^B_{\Gamma,\Gamma'} = \sum_{\beta=\{\alpha_1,\dots,\alpha_m\}} \omega(\beta), \tag{17}$$

where the sum is over the regular sets of bonds $\beta = \{\alpha_1, \dots, \alpha_m\}$ such that

$$\bigvee_{k=1}^{m} \Gamma_0(\alpha_k) = \Gamma, \qquad \bigvee_{k=1}^{m} \Gamma_{n+1}(\alpha_k) = \Gamma'. \tag{18}$$

Here, for every set of bonds $\beta = \{\alpha_k,\, k = 1, \dots, n\}$ we define $\omega(\beta) = \omega(\alpha(\beta))$, where $\alpha(\beta) = \{\widetilde{\Gamma}^j_i(k), i, j, k\}$ is the set of standard multi-indices appearing in the bonds α_k, $k = 1, \dots, m$, each multi-index in $\alpha(\beta)$ being repeated with a multiplicity equal to the number of bonds α_k containing it. We say further that the set of bonds $\beta = \{\alpha_1, \dots, \alpha_m\}$ is *connected* if the system of subsets

$$\operatorname{supp}\Gamma(\alpha_1), \operatorname{supp}\Gamma(\alpha_2), \dots, \operatorname{supp}\Gamma(\alpha_m)$$

is connected. We now define the cluster functions ω^B of the operator ω^B by the formula

$$\omega^B((\Gamma,\Gamma'), \dots, (\Gamma_s, \Gamma'_s)) = \sum_{\{\beta_1,\dots,\beta_s\}} D(\beta_1) \dots D(\beta_s)\omega(\beta_1 \cup \beta_2 \cup \dots \cup \beta_s), \tag{19}$$

where the sum is over all unordered sequences $\{\beta_1, \dots, \beta_s\}$ of connected sets of pairwise-distinct bonds $\beta_l = \{\alpha^l_k, k = 1, \dots, m_l\}$ such that the following condition holds: for every $l = 1, \dots, s$ the sets of multi-indices

$$\begin{gathered}\{\Gamma_0(\alpha^l_1), \dots, \Gamma_0(\alpha^l_{m(l)})\},\\ \{\Gamma_{n+1}(\alpha^l_1), \dots, \Gamma_{n+1}(\alpha^l_{m(l)})\}\end{gathered} \tag{19^a}$$

taken individually are disjoint, and

$$\bigvee_{k=1}^{m(l)} \Gamma_0(\alpha^l_k) = \Gamma_l, \qquad \bigvee_{k=1}^{m(l)} \Gamma_{n+1}(\alpha^l_k) = \Gamma'_l, \tag{19^b}$$

where $\beta_1 \cup \dots \cup \beta_s = \beta$ is the set of bonds appearing in the sets β_i (and repeated in β with multiplicity equal to the number of sets β_i containing them). For every $\beta = \{\alpha_1, \dots, \alpha_m\}$ we define the quantity $D(\beta)$ as follows. Let G_β be the graph with vertices labeled by $1, \dots, m$ and such that the edge $(i, j) \in G_\beta$ if and only if

$$\operatorname{supp}\Gamma(\alpha_i) \cap \operatorname{supp}\Gamma(\alpha_j) \neq \emptyset.$$

Let $\mathfrak{A}_\beta$ be the structure[4] formed by the partitions of G_β into connected subgraphs (see [26]) and let $\mu_{\mathfrak{A}_\beta}(\cdot,\cdot)$ be its Möbius function. We set

$$D(\beta) = \mu_{\mathfrak{A}_\beta}(\mathbf{0}, \mathbf{1}), \tag{20}$$

where $\mathbf{0}$ is the smallest partition (the partition of G_β into vertices) and $\mathbf{1}$ is the largest (the partition into the connected components of G_β).

[4]Translator's note: That is, the directed set with respect to refinement.

LEMMA 4. *Let $B = \{\alpha_1, \dots, \alpha_m\}$ be a fixed set of bonds. Then*

$$S(\beta) = \sum_{\{\beta_1,\dots,\beta_s\},\beta=\beta_1\cup\beta_2\cup\dots\cup\beta_s} D(\beta_1)\dots D(\beta_s) = \begin{cases} 1 & \text{if } \beta \text{ is regular,} \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

The sum is over all partitions $(\beta_1, \dots, \beta_s)$ of β into connected subsets: $\beta = \beta_1 \cup \beta_2 \cup \dots \cup \beta_s$.

PROOF. Let $G_1 = G_{\beta_1}, G_2 = G_{\beta_2}, \dots, G_k = G_{\beta_k}$ be the connected components of the graph G_β. Evidently,

$$S(\beta) = S(\beta_1)\dots S(\beta_k). \tag{22}$$

On the other hand, for any connected graph $G = G_\beta$ and any partition $\varepsilon = (\beta_1, \dots, \beta_s)$ into connected subgraphs G_{β_i}, we have

$$D(\beta_1)\dots D(\beta_k) = \mu_G(0, \varepsilon). \tag{23}$$

It follows from the definition of the Möbius function that

$$\sum_{\varepsilon\in\mathfrak{A}_G} \mu_G(0,\varepsilon) = \begin{cases} 1 & \text{if } \mathbf{0} = \mathbf{1}, \\ 0 & \text{if } \mathbf{0} < \mathbf{1}. \end{cases}$$

¿From this and (22), (23) we see that $S(\beta) = 1$ only for a completely disconnected graph, i.e., for a regular set of bonds β; $S(\beta) = 0$ in all other cases. We observe that if the multi-index sets $\{\Gamma_1, \dots, \Gamma_s\}$ and $\{\Gamma'_1, \dots, \Gamma'_s\}$ in (19) are separately disjoint, then for every set $(\beta_1, \dots, \beta_s)$ on the right in (19), the union $\beta = \beta_1 \cup \dots \cup \beta_s = \{\alpha_k^l\}$ contains only pairwise disjoint bonds, and the sets of their first and last multi-indices $\Gamma_0(\alpha_k^l)$ and $\Gamma_{n+1}(\alpha_k^l)\}$ are disjoint. If we write $\Gamma_0(\beta) = \bigvee_{l,k} \Gamma_0(\alpha_k^l)$ and $\Gamma_{n+1}(\beta) = \bigvee_{l,k} \Gamma_{n+1}(\alpha_k^l)$, then it is plain that every partition $\beta = \overline{\beta}_1 \dots \overline{\beta}_s$ into connected subsets $\overline{\beta}_i$ gives rise to a disjoint decomposition

$$(\overline{\Gamma}_1, \overline{\Gamma'}_1), \dots, (\overline{\Gamma}_{\overline{s}}, \overline{\Gamma'}_{\overline{s}})$$

of the pair $\Gamma_0(\beta)$, $\Gamma_{n+1}(\beta)$ satisfying (19^a) and (19^b). From this remark, Lemma 4, expression (17), and equation (19) it follows that the matrix elements $a^B_{\Gamma,\Gamma'}$ of the operator B admit a cluster expansion (4) with cluster function ω^B. Property 1), expressing the independence of the clusters, now follows directly from the definition (19). The cluster estimate for ω^B and the bound on the norm $\|\|B\|\|_{c\lambda}$ (where C is some absolute constant) now follow from (19), the cluster estimates for the functions ω^{A_i}, and the following bound on $D(\beta)$, where $\beta = \{\alpha_k\}_1^m$ is a set of pairwise distinct bonds:

$$|D(\beta)| \le \prod_{k=1}^{m} \widetilde{C}^{d_{\operatorname{supp}\Gamma(\alpha_k)}} \tag{24}$$

here $\widetilde{C}$ is an absolute constant and d_B for $B \subset X$ is defined in Remark I, §1. The bound (24) is derived in [26, Chapter 2, §6]. We also need the inequality

$$\sum_{i,j,k} d_{\Gamma^i_j(k)} \ge n + d_{\widetilde{\Gamma}},$$

which is valid for any connected set of bonds $\beta = \{\alpha_k, k = 1, \dots, m\}$, $\alpha_k = \{\Gamma_j^i(k)\}$, for which (19^a) is satisfied and

$$\bigvee_k (\Gamma_0(\alpha_k)) = \Gamma, \qquad \bigvee_k (\Gamma_{n+1}(\alpha_k)) = \Gamma'.$$

Here $\widetilde{\Gamma} = (\Gamma, \Gamma')$ denotes the standard multi-index determined by the pair (Γ, Γ') (we regard Γ' as a multi-index with support in the slice Y_1). This completes the proof of the lemma.

Now let $A_0, A_1, \dots, A_n$ be a fixed set of cluster operators, $A_i \in \mathcal{W}_\lambda$, $i = 0, 1, \dots, n$, where the parameter λ is sufficiently small: $\lambda < \lambda_0$. Let $\mathfrak{B} = \mathfrak{B}(A_0, A_1, \dots, A_n)$ be the set of operators expressible in the form

$$B = \sum_{\{i_1, \dots, i_k\}} x_{i_1, \dots, i_k} A_{i_1} \dots A_{i_k} \equiv T(A_0, A_1, \dots, A_n), \tag{25}$$

where the sum is over all finite ordered pairs $\{i_1, \dots, i_k\}$, $i_s = 0, 1, \dots, n$, $s = 1, \dots, k$, $k = 1, 2, \dots$, and the coefficients $x_{i_1, \dots, i_n}$ are such that

$$|T(A_0, A_1, \dots, A_n)| \equiv \sum_{\{i_1, \dots, i_k\}} |x_{i_1, \dots, i_k}| (\overline{C}\lambda)^{k-1} \prod_{s=1}^{k} |||A_{i_s}|||_\lambda < \infty. \tag{26}$$

The lemma just proved shows that $\mathfrak{B}(A_0, A_1, \dots, A_n) \subseteq \mathcal{W}_{\overline{C}\lambda}$ and that it is an operator algebra. Defining a norm on $\mathfrak{B}$ by

$$|B|_\lambda = \inf_T |T(A_0, A_1, \dots, A_n)|,$$

where the infimum is over all representations of $B \in \mathfrak{B}$ as a series (25), we have the estimate

$$|||B|||_{\overline{C}\lambda} \leq |B|_\lambda$$

and moreover,

$$|B_1 B_2|_\lambda \leq \overline{C}\lambda |B_1|_\lambda |B_2|_\lambda, \qquad B_1, B_2 \in \mathfrak{B}. \tag{27}$$

Let A be a cluster operator with cluster function ω; then its adjoint A^* is again a cluster operator, and its cluster function ω^* is given by

$$\omega^*(\tau) = \overline{\omega}(\tau'), \tag{28}$$

where τ' is obtained from the set of pairs $\tau = \{(\Gamma_1, \Gamma_1'), \dots, (\Gamma_s, \Gamma_s')\}$ by transposing each pair, i.e.,

$$\tau' = \{(\Gamma_1', \Gamma_1), \dots, (\Gamma_s', \Gamma_s)\}.$$

A cluster operator A whose cluster function ω is a product

$$\begin{aligned} \omega(\tau) &= \omega\{(\Gamma_1, \Gamma_1'), \dots, (\Gamma_s, \Gamma_s')\}, \\ \tau &= \{(\Gamma_1, \Gamma_1'), \dots, (\Gamma_s, \Gamma_s')\}, \end{aligned} \tag{29}$$

will be called a *multiplicative* cluster operator; here $\omega\{(\Gamma,\Gamma')\}$ is the value of the cluster function on sets consisting of the single pair (Γ,Γ'). We see from (34.1) that the transfer matrix for a Euclidean field is precisely of this type (for small β). It follows further from equation (19) that the product of two multiplicative cluster operators is again a multiplicative cluster operator. Moreover, if $A\in\mathcal{W}_\lambda$ is a cluster operator and T is a diagonal operator of the form

$$(Tf)(\Gamma)=\kappa^{N(\Gamma)}f(\Gamma),$$

where $N(\Gamma)=\sum N(\gamma(x))$ is the rank of the multi-index Γ, $0<\kappa<1$, then TA and AT are again cluster operators, and

$$|||TA|||_\lambda\le\kappa|||A|||_\lambda \tag{30}$$

with a similar estimate for $|||AT|||_\lambda$. Let $\mathfrak{M}_0^k\subset\mathfrak{M}_0$, for $k>0$ an integer, denote the set of multi-indices $\Gamma=\{\gamma(x),\,x\in Y_0\}\subset\mathfrak{M}_0$ with rank

$$N(\Gamma)=k \tag{31}$$

and for any subset R of the natural numbers denote by $\mathfrak{M}_0^R$ the union

$$\mathfrak{M}_0^R=\bigcup_{k\in R}\mathfrak{M}_0^k.$$

An operator $A\colon l_2^{R_1}\to l_2^{R_2}$ acting by the formula

$$(Af)(\Gamma)=\sum_{\Gamma'\in\mathfrak{M}_0^{R_1}}a_{\Gamma,\Gamma'}f(\Gamma'),\quad \Gamma\in\mathfrak{M}_0^{R_2} \tag{32}$$

will also be said to be a *cluster* operator if its matrix elements are expressible in the cluster form (4). Here $l_2^R=l_2(\mathfrak{M}_0^R)$ and R_1, R_2 can be any subsets of the natural numbers. The cluster function ω^A of A, which as before satisfies conditions (5) and (6), is defined only for the sets $\{(\Gamma_1,\Gamma_1'),\dots,(\Gamma_s,\Gamma_s')\}$ for which

$$\sum_i N(\Gamma_i)\in R_2,\qquad \sum_i N(\Gamma_i')\in R_1. \tag{33}$$

Extending the definition of the cluster function ω^A so that it is zero for all other sets, we can extend A to a cluster operator acting on the whole space $l_2(\mathfrak{M}_0)$ and will continue to denote it by A. We then have

$$l_2^{\mathcal{N}\setminus R_2}\subseteq\operatorname{Ker}A,\qquad \operatorname{Im}A\subseteq l_2^{R_2}. \tag{34}$$

Let $R\subset\mathcal{N}$ be a finite set of natural numbers, with $M=\max\limits_{n\in R}n$, and let $T\colon l_2^R\to l_2^R$ be the diagonal operator on l_2^R,

$$(Tf)(\Gamma)=\kappa^{N(\Gamma)}f(\Gamma),\qquad \Gamma\in\mathfrak{M}_0^R,$$

where $\kappa>1$. Then for every cluster operator $A\in\mathcal{W}_\lambda$ such that $\operatorname{Im}A\subset l_2^R$, the operator TA is again cluster and

$$|||TA|||_\lambda<\kappa^M|||A|||_\lambda. \tag{35}$$

For every cluster operator A such that $\operatorname{Ker}A\supseteq l_2^{\mathcal{N}\setminus R}$, AT is again a cluster operator and its norm $|||AT|||$ satisfies the estimate (35).

§3. Invariant k-particle subspaces of a cluster operator

A cluster operator $A\colon l_2^{\{k\}} \to l_2^{\{k\}}$, k is an integer, is called a k-particle cluster operator. Now let A act on a Hilbert space $\mathcal{H}$ and commute with a unitary representation $\{U_s, s \in Z^\nu\}$ of the group Z^ν on $\mathcal{H}$. A subspace $\mathcal{W} \subset \mathcal{H}$ invariant under the operators A and $\{U_s, s \in Z^\nu\}$ will be called a *cluster k-particle* invariant subspace of A with respect to (E, N), where E is our initial "space" and $\mathcal{N}$ is the graph of the indices γ (described in §1), provided there exists a unitary transformation of $\mathcal{W}$ onto the space $l_2^{(k)}(\mathfrak{M}_0)$ which takes $A|_{\mathcal{W}}$ (the part of A acting on the invariant subspace $\mathcal{W}$) into a cluster k-particle operator, and the U_s into the operators (2.2).

It turns out that under quite general conditions, a selfadjoint cluster operator A with a small enough cluster parameter λ admits k-particle invariant subspaces $\mathcal{H}_k$, $k = 1, 2, \ldots, N$ up to a certain $N = N(\lambda)$ depending on λ. We will now state these conditions and construct the associated subspaces.

To this end, for any cluster operator A we define its principal part A^0 to be the cluster operator whose cluster function ω is defined as follows. We see from the definition (38.1) that for any pair of multi-indices $\widetilde{\Gamma} = (\Gamma, \Gamma')$ the quantities $d_{\widetilde{\Gamma}} \equiv d_{\Gamma,\Gamma'}$ satisfy

$$d_{\widetilde{\Gamma}} \geq \frac{1}{2}(N(\Gamma) + N(\Gamma')), \tag{1}$$

where equality $d_{\Gamma,\Gamma'} = (N(\Gamma) + N(\Gamma'))/2$ is clearly possible only if $N(\Gamma)$ and $N(\Gamma')$ have the same parity. A pair of multi-indices (Γ, Γ') for which

$$d_{\Gamma,\Gamma'} = \frac{1}{2}(N(\Gamma) + N(\Gamma')) \tag{2}$$

will be called *minimal*. We set

$$\omega^{A^0}(\tau) = \begin{cases} \omega^A(\tau) & \text{if } \tau = \{(\Gamma_i, \Gamma_i')\} \text{ consists only of minimal pairs,} \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

For any two subsets R_1, R_2 of natural numbers we write $A^{R_1 R_2}$

$$A^{R_1 R_2} = Q_{R_1} A Q_{R_2}, \tag{4}$$

where Q_R is the projection operator onto the subspace l_2^R of $l_2(\mathfrak{M}_0)$.

When $R_1 = R_2 = \{1, \ldots, k\} \equiv R_k$ we will write A_k for $A^{R_k R_k}$. We now assume that for some integer N the cluster operator A, with cluster parameter λ and norm $|||A||| \geq 1$, satisfies the following conditions (the *N-particle separability conditions*):

i) All the operators A_k^0, acting on the spaces l_2^R for $k = 1, \ldots, N$, are invertible.

ii) Each of the operators $(A_k^0)^{-1}$ (on $l_2^{R_k}$) is expressible as

$$(A_k^0)^{-1} = (T_k^{\mu_k^{(1)}})^{-1} \widetilde{A}_k^0 (T_k^{\mu_k^{(1)}})^{-1}, \tag{5}$$

where $\widetilde{A}_k^0$ is a cluster operator acting on $l_2^{R_k}$ with cluster parameter λ^{β_k}, where $\beta_k = \frac{1}{2(4k+12)}$, and with norm $|||\widetilde{A}_k^0|||_{\lambda^{\beta_k}} \geq \lambda^{-\varepsilon}$, where $\varepsilon = 1/4$, and T_k^μ is the restriction to $l_2^{R_k}$ of the diagonal operator T^μ on $l_2(\mathfrak{M}_0)$:

$$(T^\mu f)(\Gamma) = (\lambda^\mu)^{N(\Gamma)} f(\Gamma) \tag{6}$$

and

$$\mu_k^{(1)} = \frac{1 + \beta_k}{2}. \tag{7}$$

THEOREM 1. *For any integer N there exists $\lambda_0 = \lambda_0(N)$ such that if A is a selfadjoint cluster operator with cluster parameter $\lambda < \lambda_0$ and norm $\|\|A\|\| \le 1$ satisfying the N-particle separability conditions, then there exist N invariant mutually orthogonal subspaces $\mathcal{H}_1, \dots, \mathcal{H}_N$, where $\mathcal{H}_k$ is a k-particle cluster subspace, $k = 1, \dots, N$. The spectrum $\sigma(A|_{\mathcal{H}_k})$ of A restricted to $\mathcal{H}_k$, is bounded by*

$$C_2\lambda^{k+3/8-\Delta_k^{(2)}} < \inf_{z\in\sigma(A|_{\mathcal{H}_k})} |z| \le \sup_{\sigma(A|_{\mathcal{H}_k})} |z| < C_1\lambda^{k-5/8+\Delta_k^{(1)}}, \tag{8}$$

where C_1, C_2 are absolute constants and

$$\Delta_k^{(1)} = \frac{1}{k+2}, \qquad \Delta_k^{(2)} = \frac{1}{2(k+3)}. \tag{9}$$

The spectrum of A on the orthogonal complement $\widetilde{\mathcal{H}}_N = (\bigoplus_{h=1}^N \mathcal{H}_h)^\perp$ to the sum of the subspaces $\mathcal{H}_k$ satisfies the bound

$$\sup_{\sigma(A|_{\mathcal{H}_N})} |z| < C\lambda^{N+\frac{3}{8}+\frac{5}{8(N+3)}}, \tag{10}$$

where $C > 0$ is an absolute constant.

REMARK. According to (8) and (10), the spectra of A on the subspaces $\mathcal{H}_1, \dots, \mathcal{H}_N$, $\widetilde{\mathcal{H}}_N$ are pairwise disjoint.

Before proving this general theorem, we consider the cases $N = 1$ and $N = 2$ in more detail.

Case $N = 1$ (one-particle subspace). For any minimal pair (Γ, Γ') such that $N(\Gamma) = N(\Gamma') = 1$, the supports of Γ and Γ' coincide and consist of a single point:

$$\operatorname{supp}\Gamma = \operatorname{supp}\Gamma' = \{x\}, \tag{10^a}$$

and $N(\gamma(x)) = N(\gamma'(x)) = 1$. Thus, the matrix elements of the operator $A^0_{k=1}$ on the space $l_2^{\{1\}}$ are given by

$$a^0_{\Gamma,\Gamma'} = \omega((\Gamma,\Gamma')) = b^{(1)}_{\alpha,\alpha'}(s(x))\delta_{x,x'}, \tag{11}$$

where $\{x\} = \operatorname{supp}\Gamma$, $\{x'\} = \operatorname{supp}\Gamma'$, $\alpha = \gamma(x)$, $\alpha' = \gamma(x')$ are the indices of the field variables, and $s(x)$ is the orbit of the point $x \in Y_0$ under the action of the group of translations in E $(= Y_0)$. The numbers $b^{(1)}_{\alpha,\alpha'}(s)$ are less than λ:

$$|b^{(1)}_{\alpha,\alpha'}(s)| < \lambda \tag{12}$$

and we will denote the matrix $\{b^{(1)}_{\alpha,\alpha'}(s)\}$ by $B^{(1)}(s)$. Thus, $A^0_{k=1}$ has an inverse if and only if all the matrices $\{B^{(1)}(s),\ s \in E/Z^\nu\}$ are invertible, and in this case the matrix elements of $(A^0_{k=1})^{-1}$ have the same form as before:

$$a^{0,-1}_{\Gamma,\Gamma'} = b^{(1),-1}_{\alpha,\alpha'}(s(x))\delta_{x,x'}, \tag{13}$$

where $\{b^{(1),-1}_{\alpha,\alpha'}(s)\} = (B^{(1)}(s))^{-1}$. When the estimate

$$|b^{(1),-1}_{\alpha,\alpha'}(s)| < \frac{1}{\lambda^{1+1/4}} \tag{14}$$

holds, the representation (5) is valid for the operator $(A^0_{k=1})^{-1}$. Thus if the cluster operator A has a small enough cluster parameter λ and conditions (13) and (14) are fulfilled, then A possesses a one-particle subspace $\mathcal{H}_1$. It follows further from the estimates (12) and (14) that the eigenvalues $\{E_\alpha(s),\ \alpha \in M,\ s \in E/Z^\nu\}$ of the matrices $B^{(1)}(s)$ lie in the intervals

$$k^{-1}\lambda^{1+1/4} < E_\alpha(s) < k\lambda, \tag{15}$$

where k is the number of field variables. The next theorem gives more precise information concerning the location of the spectrum of A on the invariant one-particle subspace $\mathcal{H}_1$.

THEOREM 2. *When conditions* (13) *and* (14) *are satisfied, the spectrum of* $A|_{\mathcal{H}_1}$ *is contained in a* $C\lambda^2$*-neighborhood of the eigenvalues* $E_\alpha(s)$ *of the matrices* $B(s)$, $s \in E/Z^\nu$, *where* C *is an absolute constant.*

The case $N = 2$ (one- and two-particle invariant subspaces exist). We consider more closely the form of the operator $A^0_{N=2}$ on the subspace $l_2^{R_2}$, where $R_2 = \{1, 2\}$. Every minimal pair of multi-indices (Γ, Γ') for which $N(\Gamma) \le 2$, $N(\Gamma') \le 2$ is either of the form (10^a), with $N(\Gamma) = N(\Gamma') = 1$, or of the form

$$\operatorname{supp}\Gamma = \operatorname{supp}\Gamma' = \{x\}, \qquad N(\gamma(x)) = N(\gamma'(x)) = 2. \tag{16}$$

Hence we find that the nonzero matrix elements $a^0_{\Gamma,\Gamma'}$ must have one of the following three forms. Either they are given by (11) with $N(\gamma(x)) = N(\gamma'(x)) = 1$; or they are of the form

$$\begin{gathered} a^0_{\Gamma,\Gamma'} = b^{(1,1)}_{\alpha_1,\alpha_2,\alpha'_1,\alpha'_2}(s(x_1), s(x_2))\delta_{\operatorname{supp}\Gamma,\operatorname{supp}\Gamma'}, \\ \operatorname{supp}\Gamma = \{x_1, x_2\}, \end{gathered} \tag{17}$$

where $x_1 < x_2$ for a suitable ordering on E, and α_i and α'_i index the field variables:

$$\alpha_i = \gamma(x_i), \qquad \alpha'_i = \gamma'(x_i), \qquad \alpha_i, \alpha'_i \in M, \qquad i = 1, 2$$

or, finally, they are of the form

$$\begin{gathered} a^0_{\Gamma,\Gamma'} = b^{(2)}_{\gamma,\gamma'}(s(x))\delta_{x,x'}, \\ \operatorname{supp}\Gamma = \operatorname{supp}\Gamma' = \{x\}, \qquad N(\gamma(x)) = N(\gamma'(x)) = 2. \end{gathered} \tag{18}$$

The numbers $\{b^{(1,1)}_{\alpha_1,\alpha_2,\alpha'_1,\alpha'_2}(s_1, s_2)\} \equiv B^{(1,1)}(s_1, s_2)$ and $\{b^{(2)}_{\gamma,\gamma'}\} \equiv B^{(2)}(s)$ are bounded by

$$\max\{|b^{(1,1)}_{\alpha_1,\alpha_2,\alpha'_1,\alpha'_2}(s_1, s_2)|, |b^{(2)}_{\gamma,\gamma'}(s)|\} < \lambda^2. \tag{19}$$

It is clear from (17) and (18) that the inverse operator $(A^0_{N=2})^{-1}$ exists if and only if the matrices $B^{(1)}(s)$, $B^{(1,1)}(s_1, s_2)$, and $B^{(2)}(s)$ are invertible, and in this case the matrix elements $a^{0,-1}_{\Gamma,\Gamma'}$ of the inverse are given either by (13), or by equations (17) and (18), in which

the matrix elements of the matrices $B^{(1,1)}_{(s_1,s_2)}$ and $B^{(2)}(s)$ are replaced by the elements of the inverses:

$$(B^{(1,1)}(s_1,s_2))^{-1} = \{b^{(1,1),-1}_{\alpha_1,\alpha_2,\alpha_1',\alpha_2'}(s_1,s_2)\},$$
$$(B^{(2)}(s))^{-1} = \{b^{(2),-1}_{\gamma,\gamma'}(s)\}. \tag{20}$$

Finally, the two-particle separability condition is equivalent to the estimate (14), together with the estimates

$$|b^{(1,1),-1}_{\alpha_1,\alpha_2,\alpha_1',\alpha_2'}(s_1,s_2)|, |b^{(2),-1}_{\gamma,\gamma'}(s)| < \frac{1}{\lambda^{2+1/4}}. \tag{21}$$

Thus if (14) and (21) are satisfied, then there exist two invariant cluster subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ (one-and two-particle subspaces, respectively). We now assume that the cluster function ω of the operator A is nonzero only for sets $\{(\Gamma_1,\Gamma_1'),\dots,(\Gamma_s,\Gamma_s')\}$ such that for each pair (Γ_i,Γ_i') of multi-indices the ranks $N(\Gamma)$ and $N(\Gamma')$ have the same parity:

$$N(\Gamma) = N(\Gamma') \pmod 2. \tag{22}$$

In this case also the matrix elements $a^0_{\Gamma,\Gamma'}$ are nonzero only when (22) holds. The location of the spectrum of A on $\mathcal{H}_2$ is given more precisely by the following theorem. According to (19) and (21), the eigenvalues $\{E^{1,1}_{\alpha_1,\alpha_2}(s_1,s_2)\}$ and $\{E^{(2)}_\gamma(s)\}$ of the matrices $B^{(1,1)}(s_1,s_2)$ and $B^{(2)}(s)$ lie in the intervals

$$C_2\lambda^{2+1/4} < |E^{1,1}_{\alpha_1,\alpha_2}(s_1,s_2)| < C_1\lambda^2, \tag{23}$$

where C_1 and C_2 are absolute constants; similar estimates hold for $\{E^{(2)}_\gamma(s)\}$.

THEOREM 3. *Under the above hypotheses* (21) *and* (22), *the spectrum of* $A|_{\mathcal{H}_2}$ *is contained in a* $C\lambda^3$*-neighborhood of the eigenvalues of the matrices* $B^{(1,1)}(s_1,s_2)$, $B^{(2)}(s)$ (C *is an absolute constant*).

The verification of the separability conditions for $N > 2$ is more complicated than for $N = 1$ or 2, because A^0_k may contain off-diagonal elements when $k > 2$ (due to nondiagonal minimal pairs (Γ,Γ') of the form $\operatorname{supp}\Gamma = (x_1,x_2)$, where x_1, x_2 are adjacent points and $N(\gamma(x_1)) = 2$, $N(\gamma(x_1)) = 1$, $\operatorname{supp}\Gamma' = \{x, N(\gamma(x)) = 1\}$).

PROOF OF THEOREM 1. Let $A \in \mathcal{J}_\lambda$ be a selfadjoint cluster operator satisfying the hypotheses of the theorem. We begin by constructing an increasing chain of subspaces

$$\mathcal{L}_1 \subset \mathcal{L}_2 \subset \cdots \subset \mathcal{L}_N \subset l_2(\mathfrak{M}_0) \tag{24}$$

invariant under A and the group $\{U_s,\ s \in Z^\lambda\}$. The subspaces $\mathcal{H}_k$ will then be defined by taking successive orthogonal complements:

$$\mathcal{H}_1 = \mathcal{L}_1, \qquad \mathcal{H}_2 = \mathcal{L}_2 \ominus \mathcal{L}_1, \qquad \mathcal{H}_N = \mathcal{L}_N \ominus \mathcal{L}_{N-1}. \tag{25}$$

Construction of the subspaces $\mathcal{L}_k$, $k = 1,\dots,N$. The direct sum decomposition

$$l_2^{R_k} \oplus l_2^{\mathcal{N}\setminus R_k}$$

of $l_2(\mathfrak{M}_0)$ induces a matrix decomposition

$$A \sim \begin{pmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_{22}^{(k)} \end{pmatrix}$$

of the operator A, where $A_{11}^{(k)}\colon l_2^{R_k} \to l_2^{R_k}$, $A_{12}^{(k)}\colon l_2^{\mathcal{N}\setminus R_k} \to l_2^{R_k}$, and so on. Here $A_{11}^{(k)}$ is expressible in the form

$$A_{11}^{(k)} = A_k^{(0)} + A_k^{(1)}, \tag{26}$$

where $A_k^{(0)}$ is the restriction of the principal part $A^{(0)}$ of A to $l_2^{R_k}$, and $A_k^{(1)}$ is expressible as

$$A_k^{(1)} = T_k^{\mu_k^{(1)}} \widetilde{A}_k^{(1)} T_k^{\mu_k^{(2)}}, \tag{27}$$

where $\widetilde{A}_k^{(1)}$ is a cluster operator with cluster parameter λ^{β_k} and norm

$$|||\widetilde{A}_k^{(1)}|||_{\lambda^{\beta_k}} < \lambda^{\mu_k^{(2)}}, \qquad \mu_k^{(2)} = \frac{1-\beta_k}{2}. \tag{28}$$

LEMMA 4. *Assume that the separability condition* (5) *is satisfied. Then the operator* $A_{11}^{(k)}$ *is invertible and its inverse* $(A_{11}^{(k)})^{-1}$ *is of the form*

$$(A_{11}^{(k)})^{-1} = (T^{\mu_k^{(1)}})^{-1} \widetilde{A}_{11}^{(k)} (T^{\mu_k^{(1)}})^{-1}, \tag{29}$$

where $\widetilde{A}_{11}^{(k)}$ *is a cluster algebra in the algebra* $\mathfrak{B}(\widetilde{A}_k^{(0)}, \widetilde{A}_k^{(1)}, T_k^{\mu_k^{(2)}-\mu_k^{(1)}})$ *of series of the form* (25.2), *and*

$$|\widetilde{A}_{11}^{(k)}|_{\lambda^{\beta_k}} < 2/\lambda^{1/4}. \tag{30}$$

PROOF. Using the decomposition (26) and expressions (5) and (27), we can write the operator $(A_{11}^{(l)})^{-1}$ as

$$\begin{aligned}
(A_{11}^{(k)})^{-1} &= (A_k^{(0)})^{-1} - (A_k^{(0)})^{-1} A_k^{(1)} (A_k^{(0)})^{-1} + \cdots \\
&\quad + \underbrace{(-1)^n (A_k^{(0)})^{-1} A_k^{(1)} (A_k^{(0)})^{-1} \dots A_k^{(1)} (A_k^{(0)})^{-1}}_{n \text{ times}} + \cdots \\
&= (T_k^{\mu_k^{(1)}})^{-1} [\widetilde{A}_k^{(0)} - \widetilde{A}_k^{(0)} (T_k^{\mu_k^{(1)}-\mu_k^{(2)}})^{-1} \widetilde{A}_k^{(1)} (T_k^{\mu_k^{(1)}-\mu_k^{(2)}})^{-1} \widetilde{A}_k^{(0)} + \cdots \\
&\quad + (-1)^n \widetilde{A}_k^{(0)} (T_k^{\mu_k^{(1)}-\mu_k^{(2)}})^{-1} \widetilde{A}_k^{(1)} (T_k^{\mu_k^{(1)}-\mu_k^{(2)}})^{-1} \widetilde{A}_k^{(0)} \cdots \\
&\quad \cdots \widetilde{A}_k^{(1)} (T_k^{\mu_k^{(1)}-\mu_k^{(2)}})^{-1} \widetilde{A}_k^{(0)} + \cdots] (T_k^{\mu_k^{(2)}})^{-1}.
\end{aligned} \tag{31}$$

We take $\widetilde{A}_{11}^{(k)}$ to be the sum of the series in square brackets in the last part of equation (31). The norm $|\widetilde{A}_{11}^{(k)}|_{\lambda^{\beta_k}}$ in the algebra $\mathfrak{B}(\widetilde{A}_k^{(0)}, \widetilde{A}_k^{(1)}, T_k^{\mu_k^{(1)}-\mu_k^{(2)}})$ is at most

$$\begin{aligned}
&|||\widetilde{A}_k^{(0)}|||_{\lambda^{\beta_k}} + \sum_{n=1}^{\infty} |||\widetilde{A}_k^{(0)}|||_{\lambda^{\beta_k}}^{n+1} |T_k^{\mu_k^{(1)}-\mu_k^{(2)}}|^{-n} |||\widetilde{A}_k^{(1)}|||_{\lambda^{\beta_k}} (C\lambda^{\beta_k})^n \\
&\qquad < \frac{1}{\lambda^{\varepsilon}} \left(1 + \sum_{1}^{\infty} (\lambda^{1/2-\varepsilon-(k+1/2)\beta_k})^n C^{2n} \right) < \frac{2}{\lambda^{1/4}}
\end{aligned}$$

if $1/2 - \varepsilon - (k+1/2)\beta_k > 0$ and λ is sufficiently small. This completes the proof of the lemma.

Now we want to find each subspace $\mathcal{L}_k$ as the graph

$$\mathcal{L}_k = \{f + S^{(k)}f, \quad f \in l_2^{R_k}\} \tag{32}$$

of some operator $S^{(k)}\colon l_2^{R_k} \to l_2^{\mathcal{N}\setminus R_k}$. The invariance of $\mathcal{L}_k$ under A is equivalent to

$$A_{21}^{(k)} + A_{22}^{(k)}S^{(k)} = S^{(k)}(A_{11}^{(k)} + A_{12}^{(k)}S^{(k)}),$$

which in view of the preceding lemma can be recast in the form

$$S^{(k)} = A_{21}^{(k)}(A_{11}^{(k)})^{-1} + A_{22}^{(k)}S^{(k)}(A_{11}^{(k)})^{-1} - S^{(k)}A_{12}^{(k)}S^{(k)}(A_{11}^{(k)})^{-1}. \tag{33}$$

We take this as the equation defining the operator $S^{(k)}$. Note that the operators $A_{12}^{(k)}$ and $A_{21}^{(k)}$ are expressible as

$$A_{21}^{(k)} = \widetilde{A}_{21}^{(k)}T_k^{\mu_k^{(2)}}, \qquad A_{12}^{(k)} = T_k^{\mu_k^{(2)}}\widetilde{A}_{12}^{(k)}, \tag{34}$$

where $\widetilde{A}_{21}^{(k)}$ and $\widetilde{A}_{12}^{(k)}$ are cluster operators with cluster parameter λ^{β_k} and norms

$$|||\widetilde{A}_{12}^{(k)}|||_{\lambda^{\beta_k}}, \qquad |||\widetilde{A}_{21}^{(k)}|||_{\lambda^{\beta_k}} < \lambda^{\mu_k^{(2)}(k+1)}. \tag{35}$$

We can view $A_{22}^{(k)}$ as a cluster operator with cluster parameter λ^{β_k} and norm

$$|||A_{22}^{(k)}|||_{\lambda^{\beta_k}} < \lambda^{2\mu_k^{(2)}(k+1)}. \tag{36}$$

LEMMA 5. *Equation* (33) *has a solution* $S^{(k)}$ *belonging to the algebra*

$$\mathfrak{B}(\widetilde{A}_k^{(0)}, A_k^{(1)}, \widetilde{A}_{12}^{(k)}, \widetilde{A}_{21}^{(k)}, \widetilde{A}_{22}^{(k)}, (T_k^{\mu_k})^{-1})$$

and its norm $|S^{(k)}|_{\lambda^{\beta_k}}$ *in this algebra is bounded by*

$$|S^{(k)}|_{\lambda^{\beta_k}} < C\lambda^{\frac{1}{4(12k+4)}}, \tag{37}$$

where C *is an absolute constant. A similar result holds for the adjoint operator* $(S^{(k)})^*$.

PROOF. We first expand the solution of equation (33) as a series in the operators $(A_{11}^{(k)})^{-1}$, $A_{12}^{(k)}$, $A_{21}^{(k)}$, $A_{22}^{(k)}$. For any pair $\alpha = (s,q)$ of integers with $s \geq 0$, $q \geq 0$ we introduce the operator

$$B_\alpha = (A_{22}^{(k)})^s A_{21}^{(k)}(A_{11}^{(k)})^{-(q+1)} \tag{38}$$

and for a sequence $\{\alpha_1, \ldots, \alpha_r\}$, $\alpha_i = (s_i, q_i)$ we set

$$B_{\alpha_1,\ldots,\alpha_r} = B_{\alpha_1}A_{12}^{(k)}B_{\alpha_2}A_{12}^{(k)}\ldots A_{12}^{(k)}B_{\alpha_r}. \tag{39}$$

The sequence $(\alpha_1, \dots, \alpha_r)$ will be called *regular* if the sum

$$S_i = s_1 + s_2 + \cdots + s_i, \qquad Q_i = q_1 + \cdots + q_i$$

satisfies the conditions

$$\begin{gathered} Q_i \geq S_i + (i-1), \qquad i = 1, \dots, r-1, \\ Q_r = S_r + (r-1) \end{gathered} \tag{40}$$

and $q_2 > 1$ for $r > 1$. We show that equation (33) has a solution expressible as a series

$$S^{(k)} = \sum_{(\alpha_1, \dots, \alpha_r)} x_{\alpha_1, \dots, \alpha_r} B_{\alpha_1, \dots, \alpha_r}, \tag{41}$$

where the sum is over all regular sequences $(\alpha_1, \dots, \alpha_r)$. Substituting $S^{(k)}$ as in (41) into equation (33), we obtain the following recursion relations for the coefficients $x_{\alpha_1, \dots, \alpha_r}$:

1) for $r = 1$,

$$\begin{gathered} x_{(s,q)} = x_{(s-1,q-1)}, \qquad s > 0, \qquad q > 0, \\ x_{0,0} = 1, \qquad x_{0,q} = x_{s,0} = 0, \qquad s > 0, \quad q > 0, \end{gathered} \tag{42}$$

2) for $r > 1$ and $\alpha_1 \neq (0, q)$,

$$x_{\alpha_1, \dots, \alpha_r} = x_{\widehat{\alpha}_1, \alpha_2, \dots, \overline{\alpha}_r} - \sum_{p=1}^{r-1} x_{\alpha_1, \dots, \alpha_p} x_{\alpha_{p+1}, \dots, \widetilde{\alpha}_r}, \tag{43}$$

where for $\alpha = (s, q)$, $\widehat{\alpha} = (s-1, q)$ (for $s > 0$) and $\widetilde{\alpha} = (s, q-1)$ (for $q > 0$). When $\alpha_1 = (0, q)$ the term $x_{\overline{\alpha}_1, \alpha_2, \dots, \overline{\alpha}_r}$ does not appear in the left-hand side of (43). We note that if the sequence $(\alpha_1, \dots, \alpha_r)$ is not regular, then neither is the sequence $(\overline{\alpha}_1, \dots, \overline{\alpha}_r)$, and for any $p = 1, \dots, r-1$ one of the sequences $(\alpha_1, \dots, \alpha_p)$ or $(\alpha_{r+1}, \dots, \widetilde{\alpha}_r)$ must be nonregular. Thus, our requirement that $x_{\alpha_1, \dots, \alpha_r} = 0$ for all nonregular sequences is consistent with the relations (43). It follows from (42) that when $r = 1$,

$$x_{s,q} = \delta_{s,q},$$

while the coefficients $x_{\alpha_1, \dots, \alpha_r}$ for the regular sequences $(\alpha_1, \dots, \alpha_r)$ can be recovered uniquely from the recursion relations (43). Let us show that

$$\sum |x_{\alpha_1, \dots, \alpha_r}| u^{S_r} v^r < \infty \tag{44}$$

for sufficiently small u and v. For this purpose we introduce the coefficients $y_{\alpha_1, \dots, \alpha_r}$ (where $(\alpha_1, \dots, \alpha_r)$ is a regular sequence) satisfying the recursion relations

$$\begin{aligned} y_{(s,q)} &= \delta_{s,q}, r = 1, \\ y_{\alpha_1, \dots, \alpha_r} &= y_{\widehat{\alpha}_1, \dots, \widetilde{\alpha}_r} + \sum_p y_{\alpha_1, \dots, \alpha_p} y_{\alpha_{p+1}, \dots, \widetilde{\alpha}_r} \end{aligned} \tag{45}$$

for $r > 0$ and $\alpha_1 \neq (0, q)$ (if $\alpha_1 = (0, q)$ then the first term $y_{\widehat{\alpha}_1, \alpha_2, \dots, \widetilde{\alpha}_r}$ does not appear in the right-hand side of (45)). Evidently, we have $y_{\alpha_1, \dots, \alpha_r} \geq 0$ and

$$|x_{\alpha_1, \dots, \alpha_r}| \leq y_{\alpha_1, \dots, \alpha_r}. \tag{46}$$

Defining

$$Y_{s,r} = \sum_{\substack{(\alpha_1,\dots,\alpha_r):\\ S_\tau = S > 0}} y_{\alpha_1,\dots,\alpha_r}. \tag{47}$$

we easily get from (45) that for $r > 1$ and $S > 0$,

$$Y_{S,r} = Y_{S-1,r} + \sum_{\substack{S_1+S_2=S\\ 1\le p\le r-1}} Y_{S_1,p} Y_{S_2,r-p}, \tag{48}$$

and for $r > 1$ and $S = 0$

$$Y_{0,r} = \sum_{p=1}^{r-1} Y_{0,p} Y_{0,p-r}. \tag{49}$$

Finally,

$$Y_{S,1} = 1. \tag{50}$$

¿From this we readily conclude that the function

$$w(u,v) = \sum_{\substack{S\ge 0\\ \tau>0}} Y_{S,r} u^s v^r \tag{51}$$

satisfies the quadratic equation

$$w = v + uw + w^2, \qquad w(u,0) = 0. \tag{52}$$

For small u, v the solution of (52) vanishing at $v = 0$ is analytic in the variables u and v. Thus the series (43), and hence also (44), converges in a small neighborhood of the point $u = v = 0$.

Inserting expressions (29) and (34) for the operators $(A_{11}^{(k)})^{-1}$ and $A_{21}^{(k)}$, $A_{12}^{(k)}$ into (38), we obtain

$$B_\alpha = (A_{22}^{(k)})^s (\widetilde{A}_{21}^{(k)})(T_k^{\mu_1^{(k)}-\mu_2^{(k)}})^{-1} \underbrace{\widetilde{A}_{11}^{(k)}(T_k^{\mu_1^{(k)}})^{-2}\widetilde{A}_{11}^{(k)}(T_k^{\mu_1^{(k)}})^{-2}\dots\widetilde{A}_{11}^{(k)}(T_k^{\mu_1^{(k)}})^{-1}}_{(q+1)\text{ times}} \tag{53}$$

Thus, again using expression (34) for $A_{12}^{(k)}$, we find that

$$\begin{aligned} B_{\alpha_1,\dots,\alpha_r} =& (A_{22}^{(k)})^{s_1}(\widetilde{A}_{21}^{(k)})(T_k^{\mu_2^{(k)}-\mu_1^{(k)}})^{-1}\\ &\times \underbrace{\widetilde{A}_{11}^{(k)}(T_k^{\mu_1^{(k)}})^{-2}\dots\widetilde{A}_{11}^{(k)}}_{(q_1+1)\text{ times}}(T_k^{\mu_1^{(k)}-\mu_2^{(k)}})^{-1}(\widetilde{A}_{12}^{(k)})(A_{22}^{(k)})^{s_2}\\ &\times \widetilde{A}_{21}^{(k)}(T_k^{\mu_2^{(k)}-\mu_1^{(k)}})^{-1}\underbrace{\widetilde{A}_{11}^{(k)}(T_k^{\mu_2^{(k)}})^{-2}\dots\widetilde{A}_{11}^{(k)}(T_k^{\mu_1^{(k)}-\mu_2^{(k)}})^{-1}}_{(q_2+1)\text{ times}}\\ &\times \widetilde{A}_{12}^{(k)}(A_{22}^{(k)})^{s_3}\dots(A_{22}^{(k)})^{s_2}\widetilde{A}_{21}^{(k)}(T_k^{\mu_1^{(k)}-\mu_2^{(k)}})^{-1}\\ &\times \underbrace{\widetilde{A}_{11}^{(k)}(T_k^{\mu_2^{(k)}})^{-2}\dots\widetilde{A}_{11}^{(k)}(T_k^{\mu_1^{(k)}})^{-1}}_{(q_2+1)\text{ times}}. \end{aligned} \tag{54}$$

Since $\widetilde{A}_{11}^{(k)}$ is given as a power series in the operators $\widetilde{A}_k^{(0)}, \widetilde{A}_k^{(1)}$, and $(T_k^{\mu_1^{(k)}-\mu_2^{(k)}})^{-1}$, we see that (41) does indeed imply that $S^{(k)}$ belongs to the algebra

$$\mathfrak{B}(\widetilde{A}_k^{(0)}, \widetilde{A}_k^{(0)}, (T_k^{\mu_2^{(k)}})^{-1}, T_k^{\mu_1^{(k)}}, A_{22}^{(k)}, \widetilde{A}_{12}^{(k)}, \widetilde{A}_{21}^{(k)}) \equiv \mathfrak{B}_k,$$

and in view of (30), (35), and (36) the norm $|S^{(k)}|_{\lambda^{\beta_k}}$ is bounded by

$$\begin{aligned}
|S^{(k)}|_{\lambda^{\beta_k}} < & \sum_{\alpha_1,\dots,\alpha_r} |x_{\alpha_1,\dots,\alpha_r}| \cdot |||A_{22}^{(k)}|||_{\lambda^{\beta_k}}^{S_r} \\
& \times |||\widetilde{A}_{21}^{(k)}|||_{\lambda^{\beta_k}}^{r} |||\widetilde{A}_{12}^{(k)}|||_{\lambda^{\beta_k}}^{r-1} |||(T^{\mu_2^{(k)}-\mu_1^{(k)}})^{-1}|||_{\lambda^{\beta_k}}^{2r-1} \\
& \times |||\widetilde{A}_{11}^{(k)}|||_{\lambda^{\beta_k}}^{Q_r+r} |||(T^{\mu_1^{(k)}})^{-1}|||_{\lambda^{\beta_k}}^{2Q_r+1} (C\lambda^{\beta_k})^{S_r+Q_r+3r-2} \\
< & \lambda^{[(k+1)(1-\beta_k)S_r + \frac{1-\beta_k}{2}(k+1)(2r-1) - \beta_k k(2r-1)]} \\
& \times \lambda^{[-\varepsilon(Q_r+r) - \frac{1+\beta_k}{2}k(2Q_r+1) + \beta_k(S_r+Q_r+3r-2)]} \\
& \times 2^{Q_r+r} C^{S_r+Q_r-3r-2} \\
< & \frac{1}{2C^3} [2C^2 \lambda^{1-(2k-1)\beta_k-\varepsilon}]^{S_r} [4C^4 \lambda^{1-(4k+3)\beta_k-2\varepsilon}]^r \lambda^{-\frac{1}{2}+\varepsilon+(2k-\frac{5}{2})\beta_k}.
\end{aligned}$$

Here we have made use of the regularity condition (40). Thus, provided that $1-(2k-1)\beta_k-\varepsilon > 0$, $1-(4k+3)\beta_k - 2\varepsilon > 0$, we see that the series (41) converges and the norm $|S^{(k)}|_{\lambda^{\beta_k}}$ satisfies the estimate

$$|S^{(k)}|_{\lambda^{\beta_k}} < C_1 \lambda^{\frac{1}{2}-\varepsilon-(2k-\frac{11}{2})\beta_k} = C\lambda^{\frac{1}{4(12k+4)}},$$

for absolute constants $C_1, C > 0$. Thus, $S^{(k)}$ is a cluster operator with cluster parameter $C_0\lambda^{\beta_k}$, and for $\varepsilon = 1/4$, λ sufficiently small, and

$$\frac{1}{2} - \varepsilon - \left(2k - \frac{11}{2}\right)\beta_k = \frac{1}{4(12k+4)}$$

its norm $|||S^{(k)}|||_{C_0\lambda^{\beta_k}}$ is very small.

The operator $(S^{(k)})^*$ is treated similarly. This completes the proof of the lemma.

We have thus constructed a subspace $\mathcal{L}_k$ of the form (32) which is invariant under A. Moreover, $\mathcal{L}_k$ is also invariant under the translation group $\{U_s,\ s \in Z^\nu\}$, since it is plain from the construction of $S^{(k)}$ that it commutes with every U_s. We now derive bounds for the spectrum $\sigma(A|_{\mathcal{L}_k})$ of the operator A on $\mathcal{L}_k$ and on the orthogonal complement $\mathcal{L}_k^\perp$.

We first observe that the cluster operator $(S^{(k)})^* S^{(k)}$ maps the subspace $l_2^{R_k}$ into itself, and both its norm $|||(S^{(k)})^* S^{(k)}|||_{C_0\lambda^{\beta_k}}$ and the operator norm $\|(S^{(k)})^* S^{(k)}\|$ are small. This implies, in particular, that we have a well-defined map

$$\begin{aligned}
V_k : l_2^{R_k} \to \mathcal{L}_k : f \to g = & (E_k + (S^{(k)})^*(S^{(k)}))^{-1/2} f \\
& + S^{(k)}(E_k + (S^{(k)})^*(S^{(k)}))^{-1/2} f, \quad f \in l_2^{R_k},
\end{aligned} \tag{55}$$

where E_k is the identity operator on $l_2^{R_k}$ and V_k maps $l_2^{R_k}$ unitarily onto all of $\mathcal{L}_k$. The quadratic form (Ag, g), $g \in \mathcal{L}_k$ is equal to

$$(Ag, g) = (B_k f, f), \qquad f = V_k^{-1} g \in l_2^{R_k}, \tag{56}$$

while the operator B_k acting on $l_2^{R_k}$ and unitarily equivalent to $A|_{\mathcal{L}_k}$ is given by

$$\begin{aligned} B_k =&(E_k + (S^{(k)})^*(S^{(k)}))^{-1/2}[A_{11}^{(k)} + A_{12}^{(k)}S^{(k)} \\ &+ (S^{(k)})^*A_{21}^{(k)} + (S^{(k)})^*A_{22}^{(k)}](E_k + (S^{(k)})^*(S^{(k)}))^{-1/2} \\ =&(E_k + (S^{(k)})^*(S^{(k)}))^{-1/2}(A_{11}^{(k)} + A_{12}^{(k)}S^{(k)})(E_k + (S^{(k)})^*(S^{(k)}))^{-1/2}. \end{aligned} \tag{57}$$

Here we have used equation (33).

We have further

$$A_{11}^{(k)} + A_{12}^{(k)}S^{(k)} = A_{11}^{(k)}[E_k + (A_{11}^{(k)})^{-1}A_{12}^{(k)}S^{(k)}] \tag{58}$$

and using (29) and (34) we find that

$$(A_{11}^{(k)})^{-1}A_{12}^{(k)}S^{(k)} = (T_k^{\mu_1^{(k)}})^{-1}\widetilde{A}_{11}^{(k)}(T_k^{\mu_1^{(k)}-\mu_2^{(k)}})^{-1}\widetilde{A}_{12}^{(k)}S^{(k)}$$

and the norm of this operator in the algebra $\mathfrak{B}_k$ is at most

$$\begin{aligned} &C_1\lambda^{-(\frac{1+\beta_k}{2})k-\varepsilon-\beta_k k+(\frac{1-\beta_k}{2})(k+1)-\varepsilon-(2k+\frac{5}{2})\beta_k} \\ &\qquad = C_1\lambda^{1-2\varepsilon-(4k+3)\beta_k} < C\lambda^{\frac{9}{2(4k+12)}}, \end{aligned}$$

where C_1 is an absolute constant. The norm of $(A_{11}^{(k)})^{-1}A_{12}^{(k)}S^{(k)}$ is thus small. From (57) and (58) we find that B_k is invertible, and by (30) its norm satisfies

$$\|B_k^{-1}\| < \|(A_{11}^{(k)})^{-1}\| < C\lambda^{-(1+\beta_k)+\beta_k-1/4}. \tag{59}$$

¿From this we see that the spectrum $\sigma(B_k) = \sigma(A|_{\mathcal{L}_h})$ is bounded from below:

$$x_k \equiv \inf_{z\in\sigma(A|_{\mathcal{L}_k})}\{|z|\} > \overline{C}\lambda^{k+(k-1)\beta_k+1/4}, \tag{60}$$

where $\overline{C}$ is an absolute constant independent of k and λ.

We next note that the orthogonal complement $\mathcal{L}_k^\perp$ of the subspace $\mathcal{L}_k$ is of the form

$$\mathcal{L}_k^\perp = \{f - (S^{(k)})^*f, f \in l_2^{\mathcal{N}\setminus R_k}\}$$

and the map

$$\begin{aligned} \overline{V}_k : l_2^{\mathcal{N}-R_k} \to \mathcal{L}_k^\perp : f \to g = &(E_{>k} + S^{(k)}(S^{(k)})^*)^{-1/2}f \\ &- (S^{(k)})^*(E_{>k} + S^{(k)}(S^{(k)})^*)^{-1/2}f, \quad f \in l_2^{\mathcal{N}\setminus R_k} \end{aligned} \tag{61}$$

takes $l_2^{\mathcal{N}\setminus R_k}$ unitarily into $\mathcal{L}_k^\perp$($E_{>k}$ is the identity operator on $l_2^{\mathcal{N}\setminus R_k}$). Arguing as above, we find that $A|_{\mathcal{L}_k^\perp}$ is unitarily equivalent under this map to an operator $\overline{B}_k$ on $l_2^{\mathcal{N}\setminus R_k}$ of the form

$$\overline{B}_k = (E_{>k} + S^{(k)}(S^{(k)})^*)^{1/2}(A_{22}^{(k)} - S^{(k)}A_{12}^{(k)})(E_{>k} + S^{(k)}(S^{(k)})^*)^{-1/2}.$$

¿From the expansion (41) of $S^{(k)}$ and representation (34) for A_{12}^k it follows that the norm $|S^{(k)}A_{12}^{(k)}|_{\lambda^{\beta_k}}$ is less than $C_1\lambda^{(k+1)-(3k+4)\beta_k}\lambda^{-1/4}$ and thus $\|S^{(k)}A_{12}^{(k)}\| < \overline{C}\lambda^{(k+1)-(3k+4)\beta_k-1/4}$.

Using the obvious estimate $\|A_{22}^{(k)}\| < \overline{C}\lambda^{k+1}$ we find finally that $\|\overline{B}_k\| < \overline{C}\lambda^{(k+1)-(3k+4)\beta_k-1/4}$ and thus,

$$\overline{x}_k \equiv \operatorname{supp}\sigma(A|_{\mathcal{L}_k^\perp}) < \overline{C}\lambda^{(k+1)-(3k+4)\beta_k-1/4}. \tag{62}$$

Since $1-(4k+3)\beta_k-1/2=\frac{9}{2(4k+12)}>0$, we have $x_k>\overline{x}_k$ for sufficiently small λ, i.e., the spectrum of $A|_{\mathcal{L}_k}$ is separated from the spectrum of A on $\mathcal{L}_k^\perp$.

It follows from the above that $\mathcal{L}_k = \mathcal{H}(-\infty,-x_k)\oplus\mathcal{H}(x_k,\infty)$ and $\mathcal{L}_k^\perp = \mathcal{H}(-\overline{x}_k,\overline{x}_k)$, where $\{\mathcal{H}(\Delta),\ \Delta\subset R^1\}$ is a spectral family of invariant subspaces for the selfadjoint operator A.

We use the procedure described above to construct subspaces $\mathcal{L}_k$ for every $k=1,2,\dots,N$. For $k>1$ we see from estimates (6) and (62) that $x_{k-1}>\overline{x}_k$, so that $\mathcal{L}_{k-1}\subset\mathcal{L}_k$. The subspaces are thus well defined by (25). They are invariant under A and $\{U_s\}$ and mutually orthogonal. We find from (60) and (62) that

$$\begin{aligned}\overline{C}_1\lambda^{k+3/8-\frac{7}{8(k+3)}} < x_k &= \inf_{z\in\sigma(A|_{\mathcal{H}_k})}|z| < \sup_{z\in\sigma(A|_{\mathcal{H}_k})}|z| \\ &\le \overline{x}_{k-1} < \overline{C}_2\lambda^{k-5/8+\frac{5}{8(k+2)}},\end{aligned} \tag{63}$$

whence we obtain (8).

We now show that each of the invariant subspaces constructed above is a cluster k-particle subspace. For $k=1$, the unitary transformation $V_1\colon l_2^{R_1}\to\mathcal{L}_1=\mathcal{H}_1$ takes $A|_{\mathcal{H}_1}$ into the operator B_1 given by (57) with $k=1$:

$$B_1 = (E_1+(S^{(1)})^*S^{(1)})^{-1/2}(A_{11}^{(1)}+A_{12}^{(1)}S^{(1)})(E_1+(S^{(1)})^*S^{(1)})^{-1/2}. \tag{64}$$

Since $(E_1+(S^{(1)})^*S^{(1)})^{\pm1/2}=E_1+D_1^\pm$, where the $D_1^\pm$ are cluster operators, we see from (64) that B_1 is a cluster operator belonging to the algebra $\mathfrak{B}_1$.

In the case when $k>1$, we construct a unitary map $W_k\colon l_2^{\{k\}}\to\mathcal{H}_k$. We first define the transformation

$$\widetilde{W}_k\colon l_2^{(k)}\to\mathcal{H}_k : f\to g=P_{\mathcal{L}_{k-1}^\perp}(f+S^{(k)}f)\in\mathcal{H}_k,$$

$f\in l_2^{\{k\}}$, where $P_{\mathcal{L}_{k-1}^\perp}$ is the orthogonal projection onto the space $\mathcal{L}_{k-1}^\perp$ orthogonal to $\mathcal{L}_{k-1}$. To find P explicitly, we note that for every $\varphi\in l_2(\mathfrak{M}_0)$

$$\begin{aligned}P_{\mathcal{L}_{k-1}}\varphi &= \psi+S^{(k-1)}\psi, \qquad \psi\in l_2^{R_k},\\ P_{\mathcal{L}_{k-1}^\perp}\varphi &= \xi-(S^{(k-1)})^*\xi, \qquad \xi\in l_2^{\mathcal{N}\setminus R_k}.\end{aligned}$$

Hence we have

$$\begin{aligned}Q_{l_2^{R_{k-1}}}\varphi\equiv\varphi_1 &= \psi-(S^{(k-1)})^*\xi,\\ Q_{l_2^{\mathcal{N}\setminus R_k}}\varphi\equiv\varphi_2 &= \xi+S^{(k-1)}\psi,\end{aligned}$$

where $Q_{l_2^{R_{k-1}}}$, $Q_{l_2^{\mathcal{N}\setminus R_k}}$ are the projections onto the corresponding spaces. Eliminating ψ from these relations, we get

$$\xi=(E_{>(k-1)}+S^{(k-1)}(S^{(k-1)})^*)^{-1}(\varphi_2-S^{(k-1)}\varphi_1),$$

where $E_{>(k-1)}$ is the identity operator on $l_2^{\mathcal{N}\setminus R_k}$ and the inverse operator $(E_{>(k-1)} + S^{(k-1)}(S^{(k-1)})^*)^{-1}$ is taken on this space. We thus have

$$P_{\mathcal{L}_{k-1}^{\perp}}\varphi = (E_{>(k-1)} + (S^{(k-1)})^*)(E_{>(k-1)} + S^{(k-1)}(S^{(k-1)})^*)^{-1}(\varphi_2 - S^{(k-1)}\varphi_1).$$

Recalling that $Q_{l_2^{R_{k-1}}} f = (f + S^{(k)}f) = 0$ for $f \in l_2^{\{k\}}$, we find finally that

$$\widetilde{W}_k f = (E_{>(k-1)} + (S^{(k-1)})^*)(E_{>(k-1)} + S^{(k-1)}(S^{(k-1)})^*)^{-1}(f + S^{(k)}f). \tag{65}$$

¿From this we find after some straightforward algebra that

$$(\widetilde{W}_k f, \widetilde{W}_k f) = ((E_{\{k\}} + D_k)f, f),$$

where $E_{\{k\}}$ is the identity operator on $l_2^{\{k\}}$ and D_k is a selfadjoint cluster operator acting on $l_2^{\{k\}}$ and belonging to the algebra $\widetilde{\mathfrak{B}}_k$ of series with generators in the algebras $\mathfrak{B}_k$ and $\mathfrak{B}_{k-1}$; moreover, the cluster norm of D_k is small. We have thus defined a map

$$W_k f = \widetilde{W}(E_{\{k\}} + D_k)^{-1/2}, \qquad f \in l_2^{\{k\}}, \tag{66}$$

that takes $l_2^{\{k\}}$ unitarily onto $\mathcal{H}_k$. The operator $B_k = W_k^* A W_k$ acting on $l_2^{\{k\}}$ is then unitarily equivalent to $A|_{\mathcal{H}_k}$ and is a cluster operator, as is easily seen from equations (65) and (66). This concludes the proof of Theorem 1.

Proof of Theorem 2. We note that the eigenvalues $\{E_\alpha(s)\}$ of the matrices $B^{(1)}(s)$ comprise the spectrum of the operator $A_1^{(0)}$ (which is of infinite multiplicity). Since the norm of $A_{11}^{(1)}$ is less than $C\lambda^2$ (as follows from Lemma 2.2), the spectrum of $A_{11}^{(1)}$ is contained in a $\overline{C}\lambda^2$ neighborhood of the eigenvalues $\{E_\alpha(s)\}$, where $\overline{C} > C > 0$ are absolute constants. For $k = 1$, equation (33) reads

$$S^{(1)} = A_{21}^{(1)}(A_{11}^{(1)})^{-1} + A_{22}^{(1)}S^{(1)}(A_{11}^{(1)})^{-1} - S^{(1)}A_{12}^{(1)}S^{(1)}(A_{11}^{(1)})^{-1}. \tag{67}$$

The right-hand side can be regarded as a map T of the space of operators $S \in \mathfrak{A}_{l_2^{\{1\}}, l_2^{\mathcal{N}\setminus\{1\}}}$, $S\colon l_2^{\{1\}} \to l_2^{\mathcal{N}\setminus\{1\}}$, into itself. Since

$$\|A_{21}^{(1)}\|, \|A_{22}^{(1)}\|, \|A_{12}^{(1)}\| < C_0\lambda^2, \tag{68}$$

where C_0 is an absolute constant, and $\|(A_{11}^{(1)})^{-1}\| < \lambda^{-5/4}$, as follows readily from the separability condition for $A_0^{(1)}$ and the estimate $C\lambda^2$, we see that there exists a constant $\overline{C} > 0$ such that T takes the ball

$$\{S : \|S\| < \overline{C}\lambda^{3/4}\} \tag{69}$$

in $\mathfrak{A}_{l_2^{\{1\}}, l_2^{\mathcal{N}\setminus\{1\}}}$ into itself and acts as a contraction. Hence equation (67) has a solution $S^{(1)}$ with norm at most (69), which is easily seen to coincide with the series (41). Using the estimates (68) and (69), and also equation (57) with $k = 1$, we obtain that the operator B_1 on $l_2^{\{1\}}$ unitarily equivalent to $A|_{\mathcal{H}_1}$ is of the form

$$B_1 = A_{11}^{(1)} + G_1,$$

where $\|G\| < C\lambda^{5/2}$. Theorem 2 now follows.

Theorem 3 can be proved similarly if we observe that when condition (21) holds, the subspaces $l_2^{\text{even}} = l_2^{\mathcal{N}_{\text{even}}}$ and $l_2^{\text{odd}} = l_2^{\mathcal{N}_{\text{odd}}}$, where $\mathcal{N}_{\text{even}}$ and $\mathcal{N}_{\text{odd}}$ denote the even and odd positive integers, are invariant under A and the group U_s, and the subspace $\mathcal{H}_2 \subset l_2^{\text{even}}$ is the "highest" invariant subspace in l_2^{even} (i.e., the absolute value of the spectrum of A is greatest for the subspace $\mathcal{H}_2$).

REMARK. It follows from the computations given in the proof of Theorem 1 (see, e.g., equation (57)) that the operator $A|_{\mathcal{L}_k}$ restricted to the invariant subspace $\mathcal{L}_k$ is unitarily equivalent to $A_{11}^{(k)} + A_{12}^{(k)} S^{(k)}$. On the subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$, which coincide with the corresponding subspaces $\mathcal{L}_1$ and $\mathcal{L}_2$ (under the hypotheses of Theorem 3), the spectrum of A coincides with the spectra of the operators $A_{11}^{(1)} + A_{12}^{(1)} S^{(1)}$ and $A_{11}^{(2)} + A_{12}^{(2)} S^{(2)}$, respectively.

§4. Some examples

We consider here some applications of the theory developed above to some specific models that were mentioned in Chapter 2, §5.

1. High-temperature Ising model on the lattice $Z^{\nu+1}$, $\nu > 0$. This model was described in Chapter 2, §5. We recall that the space of spins is $S = \{-1, 1\}$, and the measure ν_0 is given by $\nu_0(\{1\}) = \nu_0(\{-1\}) = 1/2$. An unperturbed basis for S consists of the functions

$$\varphi_0(\sigma) = 1, \qquad \varphi_1(\sigma) = \sigma, \qquad \sigma = \pm 1.$$

The set of multi-indices in this case coincides with the set C_{Z^ν} of all finite subsets of the lattice Z^ν. The results of the previous paragraphs specialized to this model show that the transfer matrix of the Ising field is unitarily equivalent to a multiplicative cluster operator A acting on $l_2(C_{Z^\nu})$ with cluster parameter $\lambda = C_0\beta$, where $C_0 > 0$ is an absolute constant. It is easy to check that the minimal pairs $x, x' \in C_{Z^\nu}$ (see §3) for A are of the form

$$X = X' = \{x\}, \quad \text{where } x \in Z^\nu, \tag{1}$$

i.e., they consist of the same one-point subsets, and the operator $A^{(0)}$ (the principal part of A, see §3) is diagonal with matrix elements

$$a^0_{X,X'} = \prod_{x \in X} \omega(\{x\}, \{x\}) \delta_{X,X'}, \tag{2}$$

where $\omega(X, Y)$ is the cluster function for A. A simple estimate of this semi-invariant shows that

$$\omega(\{x\}, \{x\}) = C_1\beta + O(\beta^2), \tag{3}$$

where C_1 is an absolute constant. It follows that for small enough $\beta < \beta_0(N)$, the N-particle separability condition is satisfied for the operator A, and consequently the transfer matrix for the Ising model possesses N invariant cluster subspaces $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_N$ on which A has the cluster form (for a suitable choice of basis in $\mathcal{H}_k$). In particular, on the space $\mathcal{H}_1$ we find that $A|_{\mathcal{H}_1} = A_1$ acts by convolution:

$$A_1 h_x = \sum_{y \in Z^\nu} a^{(1)}_{x-y} h_y \tag{4}$$

in terms of the basis $\{h_x, x \in Z^\nu\}$, where $h_x = W_1 h_x^0$, $h_x^0(T) = \delta_{\{x\},T}$, and the operator W_1 was defined in the previous paragraph. In (4) the function a_ξ, $\xi \in Z^\nu$ can be estimated by

$$|a_\xi| < L(C_0\beta)^{|\xi|}, \tag{5}$$

as follows from the estimates and formulas in the preceding paragraph. Moreover,

$$|a_0| > D\beta, \tag{6}$$

where L, C_0, and D are absolute constants. The bounds (5) and (6) imply in particular that the inverse operator A_1^{-1} exists and acts by the formula

$$(A_1^{-1})h_x = \sum_y b_{x-y}^{-1} h_y, \tag{7}$$

where

$$|b_\xi^{-1}| < \frac{L}{\beta}(C_0\beta)^{|\xi|}, \tag{8}$$

where L is an absolute constant.

Let us examine the operator $A_2 = A|_{\mathcal{H}_2}$ in more detail. Because A_2 is a cluster operator, with respect to the basis $\{h_{x_1,x_2}, x_1, x_2 \in Z^\nu, h_{x_1,x_2} = W_2 h_{x_2,x_2}^0,\ h_{x_1,x_2}^0(T) = \delta_{T,\{x_1,x_2\}}\}$ its matrix elements $a_{T,T'}$ admit the cluster expansion

$$a_{T,T'} = \omega((x_1,x_1'),(x_2,x_2'))\omega((x_1,x_2'),(x_2,x_1')) + \omega(T,T'), \tag{9}$$

where $T = (x_1, x_2)$, $T' = (x_1', x_2')$, and the cluster function ω satisfies the cluster bounds (6.2) with parameter $\lambda_2 = (C_0\beta)^{\beta_2}$, C_0 is a constant, and $\beta_2 = 1/40$ (see the previous subsection).

THEOREM 1. *There exists an orthonormal basis $\{\widehat{h}_{x_1,x_2}\}$ in $\mathcal{H}_2$ such that the cluster expansion* (9) *continues to hold for the matrix elements of A_2 in this basis*; *furthermore,*

$$\omega(\{x_1,x_1'\},\{x_2,x_2'\}) = a^{(1)}_{x_1-x_1'} a^{(1)}_{x_2-x_2'}, \tag{10}$$

where a_ξ is the function defined in (4).

PROOF. We will assume as usual that $\mathcal{H}_1$ and $\mathcal{H}_2$ are subspaces of $l_2(C_{Z^\nu})$. Then it follows easily from the constructions in the preceding paragraph that the functions $\{h_x(T), x \in Z^\nu\}$ have the form

$$h_x(T) = \begin{cases} \delta_{xy} + \widetilde{h}_x(\{y\}), & T = \{y\}, \\ \widetilde{h}_x(T), & |T| \geq 2, \end{cases} \tag{10^a}$$

where $\widetilde{h}_x(T)$ satisfies the estimate

$$|\widetilde{h}_x(T)| < L(C\beta)^{d_{\{x\}\cup T}}. \tag{11}$$

To construct the required basis $\widehat{h}_{x_1,x_2}$ in $\mathcal{H}_2$, we consider the system $\{\widetilde{h}_{x_1,x_2}\}$ of functions in $l_2(C_{Z^\nu})$ defined by

$$\widetilde{h}_{x_1,x_2}(T) = \begin{cases} 0, & |T| = 1, \\ \sum\limits_{\substack{T_1 \cup T_2 = T \\ T_1 \cap T_2 = \emptyset}} h_{x_1}(T_1) h_{x_2}(T_2), & |T| > 1. \end{cases}$$

Using (11) and the fact that A is a multiplicative cluster operator, one can show that

$$\begin{aligned} (A\widetilde{h}_{x_1,x_2})(T) &= \sum_{\substack{T_1 \cup T_2 = T \\ T_1 \cap T_2 = \emptyset}} (Ah_{x_1})(T_1)(Ah_{x_2})(T_2) + g_{x_1,x_2}(T) \\ &= \sum_{\substack{T_1 \cup T_2 = T \\ T_1 \cap T_2 = \emptyset}} a_{x_1 - x_1'} h_{x_1'}(T_1) a_{x_2 - x_2'} h_{x_2'}(T_2) + g_{x_1,x_2}(T) \\ &= \sum_{\{x_1', x_2'\}} (a_{x_1 - x_1'} a_{x_2 - x_2'} + a_{x_1 - x_2'} a_{x_2 - x_1'}) \widetilde{h}_{x_1', x_2'}(T) + \widetilde{g}_{x_1,x_2}(T), \end{aligned} \tag{12}$$

where $\widetilde{g}_{x_1,x_2}(T)$ satisfies the estimate

$$|\widetilde{g}_{x_1,x_2}(T)| = L(C\lambda_2)^{d_{\{x_1,x_2\} \cup T}}. \tag{12^a}$$

We introduce the two families of functions

$$\begin{aligned} \delta^{(1)}_{x_1,x_2} &= P_{\mathcal{H}^\perp} \widetilde{h}_{x_1,x_2}, \\ \widehat{g}^{(1)}_{x_1,x_2} &= P_{\mathcal{H}^\perp} \widetilde{g}_{x_1,x_2}, \end{aligned} \tag{13}$$

where $\mathcal{H}^\perp$ is the orthogonal complement to the direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ and $P_{\mathcal{H}^\perp}$ is the projection onto $\mathcal{H}^\perp$. We obtain from (12) that

$$(A\delta^{(1)}_{x_1,x_2})(T) - \sum_{x_1', x_2'} b_{(x_1,x_2),(x_1',x_2')} \delta^{(1)}_{x_1',x_2'}(T) = \widehat{g}^{(1)}_{(x_1,x_2)}(T),$$

where

$$b_{(x_1,x_2),(x_1',x_2')} = a_{x_1 - x_1'} a_{x_2 - x_2'} + a_{x_1 - x_2'} a_{x_2 - x_1'}. \tag{14^a}$$

We will have occasion to use the following result, whose proof will be deferred.

LEMMA 2. *Let A_0 be the cluster operator on $l_2(C^{(2)}_{Z^\nu})$ (where $C^{(2)}_{Z^\nu}$ is the set of two-point subsets of Z^ν) defined by*

$$(A_0 f)(T) = \sum_{T' \in C^{(2)}_{Z^\nu}} b_{T,T'} f(T'), \qquad T \in C^{(2)}_{Z^\nu}, \tag{14^b}$$

where $b_{T,T'}$ is given by equation (14). *Then A_0 is invertible on $l_2(C^{(2)}_{Z^\nu})$ and its inverse A_0^{-1} is again a cluster operator with cluster parameter $C\lambda_2$ and norm*

$$|||A_0^{-1}||| < L\beta^{-2}, \tag{15}$$

where C and L are constants.

We denote the matrix elements of A_0^{-1} by $b^{-1}_{\{x_1,x_2\},\{x_1',x_2]\}}$ and rewrite (14^a) as

$$\sum_{x_1',x_2'} b^{-1}_{\{x_1,x_2\},\{x_1',x_2'\}}(A^\perp \delta^{(1)}_{x_1',x_2'})(T) - \delta^{(1)}_{x_1,x_2}(T) = \sum_{x_1',x_2'} b^{-1}_{\{x_1,x_2\},\{x_1',x_2'\}} \widehat{g}^{(1)}_{x_1',x_2'}(T), \tag{16}$$

where $A^\perp = A|_{\mathcal{H}^\perp}$.

Now consider the space $\mathcal{K}$ of families of functions $\{\delta_{x_1,x_2}(T),\ (x_1,x_2) \in C^{(2)}_{Z^\nu}\}$ such that $\delta_{x_1,x_2} \in \mathcal{H}^\perp$ for all $\{x_1,x_2\}$ and the estimate

$$|\delta_{x_1,x_2}(T)| < L(C\lambda_2)^{d_{\{x_1,x_2\}\cup T}} \tag{17}$$

is satisfied, where L is a constant. The norm in $\mathcal{K}$ is defined in the usual way:

$$\|\delta\|_{\mathcal{K}} = \inf L,$$

where the infimum is taken over all L for which (17) holds. Using the results in the previous subsection, one can show that

$$\|\{A^\perp \delta_{x_1,x_2}\}\|_{\mathcal{K}} \le C\beta^3 \|\delta\|_{\mathcal{K}}.$$

Thus, in view of estimate (15) the map

$$\mathcal{K} \to \mathcal{K} : \delta = \{\delta_{x_1,x_2}\} \to \left\{ \sum_{x_1',x_2'} b^{-1}_{\{x_1,x_2\},\{x_1',x_2'\}} A\delta_{x_1',x_2'}, \{x_1,x_2\} \in C^{(2)}_{Z^\nu} \right\}$$

is a contraction. From this and estimates (15) and (12) we see that the family $\{\delta^{(1)}_{x_1,x_2}\}$ has a finite norm (of the order of a constant). We next consider

$$\delta^{(2)}_{x_1,x_2} = P_{\mathcal{H}_1} \widetilde{h}_{x_1,x_2} = \sum_y C^y_{x_1,x_2} hy,$$

where

$$C^y_{x_1,x_2} = (h_y, \widetilde{h}_{x_1,x_2}).$$

It is easily established from the estimates for h_{x_1,x_2} that

$$|C^y_{x_1,x_2}| < L(C\lambda_2)^{d_{\{x_1,x_2\}\cup\{y\}}}$$

so that it follows that the family $\{\delta^{(2)}_{x_1,x_2}\}$ satisfies the bound (17). Now set

$$\widetilde{\widetilde{h}}_{x_1,x_2} = \widetilde{h}_{x_1,x_2} - \delta^{(1)}_{x_1,x_2} - \delta^{(2)}_{x_1,x_2} \in \mathcal{H}_2. \tag{18}$$

The Gramm matrix $G_{\{x_1,x_2\},\{x_1',x_2'\}}$ for this family is given by

$$G_{\{x_1,x_2\},\{x_1',x_2'\}} = (\widetilde{\widetilde{h}}_{x_1,x_2}, \widetilde{\widetilde{h}}_{x_1',x_2'}) = \delta_{\{x_1,x_2\},\{x_1',x_2'\}} + D_{\{x_1,x_2\},\{x_1',x_2'\}}, \tag{19}$$

where $D_{\{x_1,x_2\},\{x_1',x_2'\}}$ satisfies the bound

$$|D_{\{x_1,x_2\},\{x_1',x_2'\}}| < L(C\lambda_2)^{d_{\{x_1,x_2\}\cup\{x_1',x_2'\}}}. \tag{20}$$

It follows that the matrix $G^{-1/2}$ is also of the form (19), and hence that the orthonormalized system of functions

$$\widehat{h}_{x_1,x_2} = G^{-1/2}\widetilde{\widetilde{h}}_{x_1,x_2}$$

is of the form

$$\widehat{h}_{x_1,x_2} = \widetilde{\widetilde{h}}_{x_1,x_2} + \widehat{\delta}_{x_1,x_2},$$

where $\widehat{\delta}_{x_1,x_2}$ satisfies (17). From expression (10^a) for h_x it is easily shown that $\widehat{h}_{x_1,x_2}$ gives a basis in $\mathcal{H}_2$. The matrix elements of A_2 with respect to this basis admit as before a cluster expansion (9), as follows from Lemma 3.2, and (10) follows in view of equation (12) and the uniqueness of the cluster expansion. The theorem is proved.

REMARK. Equality (10) means that on $\mathcal{H}_1 \oplus \mathcal{H}_2$ the operator A has a multiplicative cluster form. This plainly remains true for any N: in each of the subspaces $\mathcal{H}_k$, $k = 1, \ldots, N$ one can find a basis $\{h_T,\ T \in C^{(k)}_{Z^\nu}\}$ ($C^{(k)}_{Z^\nu}$ is the set of all subsets of Z^ν consisting of k points) with the property that the operator A on $\bigoplus_{k=1}^N \mathcal{H}_k$ has the multiplicative cluster form.

PROOF OF LEMMA 2. The space $l_2(C^{(2)}_{Z^\nu})$ can be identified with $\widetilde{l}_2^{\,\mathrm{sym}}(Z^\nu \times Z^\nu) \subset l_2^{\mathrm{sym}}(Z^\nu \times Z^\nu)$, the space of symmetric functions $F(x_1, x_2)$ of two variables $x_1, x_2 \in Z^\nu$ such that

$$f(x, x) = 0.$$

Taking the Fourier transform

$$f(x_1, x_2) \to \widetilde{f}(x_1, x_2) = \sum_{x_1,x_2} e^{i[(\lambda_1,x_1)+(\lambda_2,x_2)]} f(x_1, x_2)$$

(where $\lambda_1, \lambda_2 \in T^\nu$ are points on the-dimensional torus), we find that A_0 in (14^b) goes into the operator $\widetilde{A}^0$ acting on $\widetilde{l}_2^{\,\mathrm{sym}}(T^\nu \times T^\nu)$ by the formula

$$(\widetilde{A}_0\widetilde{f})(\lambda_1, \lambda_2) = \widetilde{\omega}(\lambda_1, \lambda_2)\widetilde{f}(\lambda_1, \lambda_2) - \int_{T^\nu \times T^\nu} \widetilde{\omega}(\mu_1, \mu_2)\delta(\lambda_1 + \lambda_2 - \mu_1 - \mu_2)\widetilde{f}(\mu_1, \mu_2)d\mu_1 d\mu_2,$$

where $\widetilde{L}_2^{\mathrm{sym}}(T^\nu \times T^\nu) \subset L_2^{\mathrm{sym}}(T^\nu \times T^\nu)$ is the space of symmetric functions $\widetilde{f}(\lambda_1, \lambda_2)$ of two variables $\lambda_1, \lambda_2 \in T^\nu$, such that

$$\int_{T^\nu \times T^\nu} f(\lambda_1, \lambda_2)\delta(\lambda_1 + \lambda_2 - \Lambda)d\lambda_1 d\lambda_2 = 0$$

for any $\Lambda \in T^\nu$ ($d\lambda$ is the normalized Haar measure on the torus), and $\widetilde{\omega}(\lambda_1, \lambda_2)$ is the Fourier transform of the function $a_{\xi_1}a_{\xi_2}$. An easy calculation shows that the inverse $\widetilde{A}_0^{-1}$ of the operator $\widetilde{A}_0$ is given by

$$(\widetilde{A}_0^{-1}\widetilde{f}) = \frac{1}{\widetilde{\omega}(\lambda_1, \lambda_2)}\left[\widetilde{f}(\lambda_1, \lambda_2) - \left(\int_{\mu_1+\mu_2=\lambda_1+\lambda_2} (\widetilde{\omega}(\mu_1, \mu_2))^{-1}d\mu_1 d\mu_2)^{-1} \times \int_{\mu_1+\mu_2=\lambda_1+\lambda_2} f(\mu_1, \mu_2)\widetilde{\omega}^{-1}(\mu_1, \mu_2)d\mu_1 d\mu_2\right)\right].$$

Passing again to functions $f \in \widetilde{l}_2^{\mathrm{sym}}(Z^\nu \times Z^\nu)$, we find that A_0^{-1} acts by the formula

$$(A_0^{-1}f)(x_1,x_2) = \sum_{(x_1',x_2')} b^{-1}_{x_1-x_1'} b^{-1}_{x_2-x_2'} f(x_1',x_2') - \sum_{(x_1',x_2')} K(x_1,x_2;x_1',x_2') f(x_1',x_2'),$$

where the kernel $K(x_1,x_2;x_1',x_2')$ is given by

$$K(x_1,x_2;x_1',x_2') = \sum_{z,t} b^{-1}_{x_1-z} b^{-1}_{x_2-z} p(t-z) b^{-1}_{t-x_1'} b^{-1}_{t-x_2'},$$

and

$$p(\xi) = \int_{T^\nu} e^{i(\Lambda,\xi)} \biggl(\int_{\mu_1+\mu_2=\Lambda} \omega^{-1}(\mu_1,\mu_2)\, d\mu_1\, d\mu_2 \biggr)^{-1} d\Lambda.$$

Note that by (5), (6) the function $\omega(\lambda_1,\lambda_2)^{-1}$ is analytic in a strip

$$|\operatorname{Im}\lambda_i| < \overline{C}_1 \beta^{-1}, \qquad i=1,2,$$

where $\overline{C}_1$ is an absolute constant. Using the same arguments, we find that the function

$$\biggl(\int\limits_{\mu_1+\mu_2=\Lambda} \omega^{-1}(\mu_1,\mu_2)\, d\mu_1\, d\mu_2 \biggr)^{-1}$$

depends analytically on Λ in the strip

$$|\operatorname{Im}\Lambda| < \overline{C}_2 \beta^{-1}.$$

¿From this and the estimates (5) and (6), it follows that $p(\xi)$ satisfies the bound

$$|p(\xi)| < L\beta^2 (C\beta)^{|\xi|},$$

where L and C are absolute constants.

Thus, the kernel $K(x_1,x_2;x_1',x_2')$ is bounded by

$$|K(x_1,x_2;x_1',x_2')| < \frac{L}{\beta^2} (C\beta)^{d_{\{x_1,x_2\}\cup\{x_1',x_2'\}}}.$$

Lemma 2 now follows from this and estimate (8).

2. The rotator model. We recall (see Chapter 2, §5) that in this case $S = T^1$ is the circle, the field variables are $\varphi_1 = e^{i\theta}$ and $\varphi_2 = e^{-i\theta}$, $\theta \in T^1$, and the basis in $L_2(T^1, d\Theta)$ is given by

$$\{\varphi_n = e^{in\theta}, \quad n = 0, \pm 1, \dots\}.$$

The set of multi-indices $\mathfrak{M}_0$ thus coincides with the set of all finite-support, integer-valued functions on Z^ν. If $\Gamma = \{n(x),\, x \in Z^\nu\}$ is such a function, we have $N(\Gamma) = \sum_x |n(x)|$. For small β, the transfer matrix for the rotator model is unitarily equivalent to a multiplicative cluster operator A on $l_2(\mathfrak{M}_0)$ with cluster parameter $\lambda_0 = C_0\beta$, where C_0 is an absolute constant.

We note that two symmetry groups preserving the interaction (53^a.5.2) act on the field configuration space $\Omega = S^{Z^\nu}$:

1) the one-parameter group

$$G\colon \theta(x) \to \theta(x) + \alpha, \qquad \alpha \in T^1, \qquad x \in Z^\nu,$$

2) the involution

$$J\colon \theta(x) \to -\theta(x).$$

These symmetries generate unitary operators $\{U_\alpha, \lambda \in T^1\}$ and U_J acting on $\mathcal{H}_{\text{phys}}$ and commuting with the transfer matrix $\mathcal{J}$ and the group of translations $\{U_s, s \in Z^\nu\}$. From the procedure described in §1 for constructing the multiplicative basis $\{\Psi_\Gamma\}$, under the unitary equivalence the group U_α acts on $l_2(\mathfrak{M}_0)$ by the formula

$$(U_\alpha f)(\Gamma) = e^{in(\Gamma)\alpha} f(\Gamma), \tag{21}$$

where $n(\Gamma) = \sum_x n(x)$, while the action of U_J is given by

$$(U_J f)(\Gamma) = f(-\Gamma). \tag{22}$$

Clearly, the desired invariant subspaces of the transfer matrix must be required to be invariant under the symmetry group U_α. Applying the previous constructions to the present model, we obtain the following result concerning the one- and two-particle invariant subspaces for the transfer matrix $\mathcal{J}$.

THEOREM 3. 1. *There exist two one-particle subspaces $\mathcal{H}_1^+$ and $\mathcal{H}_1^-$ which are mutually orthogonal and invariant under $\mathcal{J}$, the translation group $\{U_s\}$, and the group of symmetries $\{U_\alpha\}$. For every $f \in \mathcal{H}_1^\pm$ we have*

$$U_\alpha f = e^{\pm i\alpha} f$$

and

$$U_J \mathcal{H}_1^+ = \mathcal{H}_1^-, \qquad U_J \mathcal{H}_1^- = \mathcal{H}_1^+. \tag{23}$$

Here, as usual, the transfer matrix $\mathcal{J}$ on each of the spaces $\mathcal{H}_1^\pm$ is unitarily equivalent to convolution with the functions $a^\pm(\xi)$; and by (23) *$a^+(\xi) = a^-(\xi) = a(\xi)$, and the function $a(\xi)$ satisfies the estimates* (5) *and* (6). *The spectrum of the transfer matrix on the spaces $\mathcal{H}_1^\pm$ is contained in a $C\beta^2$-neighborhood of the values $\varepsilon\beta$, where ε is the interaction constant in* (53^a.5.2).

2. *There exist three mutually orthogonal two-particle subspaces $\mathcal{H}_2^\pm$, $\mathcal{H}_2^0$ invariant under the transfer matrix, translation group, and symmetry group U_α. For $f \in \mathcal{H}_2^\pm$ we have*

$$U_\alpha f = e^{\pm 2i\alpha} f$$

and for $f \in \mathcal{H}_2^0$

$$U_\alpha f = f, \qquad U_J \mathcal{H}_2^\pm = \mathcal{H}_2^\mp, \qquad U_J \mathcal{H}_2^0 = \mathcal{H}_2^0. \tag{24}$$

There exist bases $\{h_\Gamma^{\pm,0}\}$ in $\mathcal{H}_2^{\pm,0}$ labeled by multi-indices $\Gamma = \{n(x),\ x \in Z^\nu\}$ such that $N(\Gamma) = 2$ and $n(\Gamma) = \pm 2, 0$ (in $\mathcal{H}_2^\pm$ and $\mathcal{H}_2^0$, respectively), with the property that the matrix elements of the transfer matrix in these bases possess multiplicative cluster expansions, i.e., for any pair of multi-indices with one-point supports the function $\omega(\{\Gamma_1, \Gamma_1'\}, \{\Gamma_2, \Gamma_2'\})$ is equal to the product

$$\omega(\{\Gamma_1, \Gamma_1'\}, \{\Gamma_2, \Gamma_2'\}) = \omega(\{\Gamma_1, \Gamma_1'\})\omega(\{\Gamma_2, \Gamma_2'\}), \tag{25}$$

with $\omega(\{\Gamma,\Gamma'\}) = a(x-x')$, *where* $\{x\} = \operatorname{supp}\Gamma'$, $\{x'\} = \operatorname{supp}\Gamma$.

The spectrum of $\mathcal{J}$ *on all three spaces* $\mathcal{H}_2^{\pm}$, $\mathcal{H}_2^0$ *lies in a* $(C\beta)^3$*-neighborhood of the point* $\varepsilon\beta^2$, *where* C *is an absolute constant. By* (24), *the spectrum of* $\mathcal{J}$ *on* $\mathcal{H}_2^+$ *coincides with the spectrum on* $\mathcal{H}_2^-$.

Almost all the statements of the theorem follow immediately from Theorems 1.3, 2.3, and 3.3. The proof of the assertion concerning the multiplicative cluster property and the cluster function is similar to the proof of the corresponding assertion for the Ising model.

REMARK. It would be of interest to determine whether there exist three-, four-, and higher k-particle invariant subspaces of the transfer matrix for the rotator model. This problem is nontrivial because nondiagonal minimal pairs exist for this model (cf. the remark after Theorem 1 in §3).

3. Yang-Mills gauge field. We start by considering a gauge field with abelian group $U(1)$ (the complex numbers z such that $|z| = 1$). Introducing a radial gauge as explained in 5.2 (i.e., setting $g(b) = 1$ on all "vertical" edges of $Z^{\nu+1}$), we shall consider a field defined only on the "horizontal" edges of the lattice, and the zeroth slice Y_0 coincides with the set E^ν of edges of the lattice Z^ν. The set $S = U(1) = \{e^{i\theta},\ 0 \le \theta < 2\pi\}$ coincides with the circle, which we take with the standard normalized Haar measure $d\theta$. As initial basis we choose the functions $\{\varphi_n = e^{in\theta},\ n = 0, \pm1, \pm2, \dots\}$. We consider well-oriented edges, i.e., ones directed along the unit vectors e_μ of Z^ν. Writing each such edge in the form $b = \{x_0, e_\mu\}$, where x_0 is the initial vertex of b, we can introduce a lexicographic ordering in E^ν. Then as described in §1, we construct a basis in $\mathcal{H}_{\text{phys}}$ of the form

$$\Psi_\Gamma = \prod \psi^b_{n(b)}, \tag{26}$$

where $\Gamma = \{n(b)\}$ is an integer-valued function on E^ν with finite support. From the construction of the functionals $\psi^b_{n(b)}$ described in §1, we conclude easily that under a gauge transformation $\{e^{i\alpha(x)}\}$ of the field they transform according to the formula

$$\psi^b_{n(b)} \to \exp\{in(b)(\alpha(x_0) - \alpha(x_1))\}\psi^b_{n(b)}, \tag{27}$$

where x_1 and x_0 are the endpoint and initial point of the edge b. It follows that the elements of the basis $\{\Psi_\Gamma\}$ transform as

$$\Psi_\Gamma \to \prod_{x\in\operatorname{supp}\partial\Gamma} \exp\{\partial\Gamma)(x)\alpha(x)\} \tag{28}$$

under gauge transformations. Here $\partial\Gamma$ is the function on Z^ν defined by

$$(\partial\Gamma)(x) = \sum_{b\in\operatorname{supp}\Gamma} n(b)(-1)^{\varepsilon(b,x)}, \tag{29}$$

where the sum is over all edges in $\operatorname{supp}\Gamma$ incident on the point x; $\varepsilon(b,x) = 0$ if the edge b leaves x, and $\varepsilon(b,x) = 1$ if b enters x.

According to (28), the matrix elements $(\mathcal{J}\Psi_\Gamma, \Psi_{\Gamma'})$ of the transfer matrix $\mathcal{J}$ in the basis $\{\Psi_\Gamma\}$ are nonzero only if

$$\partial\Gamma = \partial\Gamma'. \tag{30}$$

They admit the cluster expansion (34.1)

$$(30^a) \qquad a_{\Gamma,\Gamma'} = \sum_{\{(\Gamma_1,\Gamma_1'),\dots,(\Gamma_k,\Gamma_k')\}} \prod_{i=1}^{k} \omega(\Gamma_i, \Gamma_i'),$$

where the sum is over all partitions of (Γ, Γ') into pairs $(\{\Gamma_i, \Gamma_i'\})$ for which condition (30) is satisfied. Moreover, by a remark made in §1, in the definition (38.1) and (39.1) of the quantity $d_{\Gamma,\Gamma'} = d_{\tilde{\Gamma}}$ appearing in the cluster estimate for the semi-invariants $\omega(\Gamma, \Gamma')$, the function $n = \{n(p)\}$ should be regarded as defined on the two-dimensional faces (plaquettes) of the lattice Z^ν. It follows from (28) that the vectors $\{\Psi_\Gamma\}$ that belong to the space $\mathcal{H}_{\text{phys}}$ of gauge-invariant functionals are precisely the ones for which

$$(31) \qquad \partial\Gamma \equiv 0.$$

Simple estimates show that if a multi-index Γ satisfies (31) and the matrix elements $(\mathcal{J}\Psi_\Gamma, \Psi_\Gamma)$ are maximal, then we must have $\operatorname{supp}\Gamma = \partial p = (b_1, b_2, b_3, b_4)$, where p is a plaquette (a two-dimensional face of the lattice), and the values $n(b_i)$ of the multi-index Γ on the edges b_i are equal to either $+1$ or -1 (the same choice holding for all four edges). All the plaquettes in Z^ν can now be divided into $k = \nu(\nu-1)/2$ classes $\sigma_1, \dots, \sigma_k$ consisting of the orbits of the action of the translation group. Thus, upon fixing a representative p_{σ_i} in each class σ_i, we see that every multi-index Γ for which $(\mathcal{J}\Psi_\Gamma, \Psi_\Gamma)$ is maximal is given by a triple (σ, x, μ), where σ is the class of the plaquette p, x is a vector taking the chosen representative p_σ into p, and $\mu = \pm 1$ are the values of Γ on the boundary of p. Using the expansion (29) and the cluster estimates, we verify easily that for such a multi-index $\Gamma = (\sigma, x, \mu)$

$$(\mathcal{J}\Psi_\Gamma, \Psi_\Gamma) \sim \beta^4.$$

Moreover, if (Γ, Γ') is any pair of multi-indices such that either $\Gamma \neq \Gamma'$ or else $\Gamma = \Gamma'$ but $(\mathcal{J}\Psi_\Gamma, \Psi_\Gamma)$ is not maximal. Then we have

$$(\mathcal{J}\Psi_\Gamma, \Psi_{\Gamma'}) \ll \beta^5.$$

An analog of the one-particle separability condition thus holds here, and the following theorem is valid.

THEOREM 4. *For a gauge field with gauge group $U(1)$ on the lattice $Z^{\nu+1}$ there exist $\nu(\nu-1)$ one-particle subspaces $\mathcal{H}_1^{\pm,\sigma}$ invariant under the transfer matrix $\mathcal{J}$ and the group of translations $\{U_s\}$. In each of these subspaces one can choose an orthonormal basis $\{h_x^{\pm,\sigma}, x \in Z^\nu\}$ such that*

$$(32) \qquad \mathcal{J} h_x^{\pm,\sigma} = \sum_{x'} a_{x-x'}^{\pm,\sigma} h_{x'}^{\pm,\sigma},$$

where the functions $a_\xi^{\pm,\sigma} = a_\xi$ coincide for all the subspaces $\mathcal{H}_1^{\pm,\sigma}$ and a_ξ is of the form

$$a_\xi = \beta^4 \delta_{\xi,0} + O(\beta^8).$$

The subspaces $\mathcal{H}_1^{\pm,\sigma}$ introduced above describe excitations of the gauge field whose energy $4 \ln \beta$ is minimal (these are the so-called *ground-state gluons*). These are followed by the

lowest *excited gluons,* with energy $6 \ln \beta$. The spaces describing these states are obtained by slightly perturbing the spaces spanned by the vectors Ψ_Γ, where $\operatorname{supp} \Gamma$ is a loop consisting of six edges $b_1, \dots, b_6$ bounding two adjacent plaquettes, and $n(b_i) = \pm 1$.

Similar arguments can also be used to treat the more general case of nonabelian, compact gauge groups. Let us consider the case $SU(2)$ (the 2×2 unitary matrices u with $\det u = 1$). As a basis on $SU(2)$ we choose the functions

$$\varphi_\gamma(u) = \sqrt{2l+1} a^l_{m,n}(u), \quad \gamma = (l, m, n),$$

where the $a^l_{m,n}(u)$ are the matrix elements of the irreducible representation $u \to T^l_u$ of $SU(2)$ in the "canonical" basis $\{\xi^l_m,\ m = -l, \dots, l\}$ of $SU(2)$. The number l is called the weight of the irreducible representation (it can assume any integer or half-integer value), and $m, n = -l, \dots, l$ (see [9] for more details). As above, we can choose a multiplicative basis in $\mathcal{H}_{\text{phys}}$ given by

$$\Psi_\Gamma = \prod_b \Psi^b_{\gamma(b)}.$$

The action of the gauge transformations $u_0 = \{u_0(x),\ x \in Z^\nu\}$ on the vector $\Psi^b_{\gamma(b)}$ is given by

$$U_{u_0} \Psi^b_{\gamma(b)} = \sum_{\gamma' : l(\gamma') = l(\gamma)} C^b_{\gamma,\gamma'} \Psi^b_{\gamma'}, \tag{33}$$

where $C^b_{\gamma,\gamma'} = a^l_{m,m'}(u_0(x_0)) a^l_{n,n'}(u_0(x_1))$, with $\gamma = (l, m, n)$, $\gamma' = (l, m', n')$; x_0 and x_1 are the origin and endpoint of the edge b.

Formula (33) can be used to construct an orthonormal basis $\{\Psi^{\{\lambda(x)\}}_{\Delta,L}\}$ in $\mathcal{H}_{\text{phys}}$ by taking linear combinations of the vectors Ψ_Γ. Specifically, consider a multi-index $\Gamma = \{l(b), m(b), n(b)\}$, b an oriented edge of the lattice $Z^\nu\}$. We will say that Γ is *regular* if for every point $x \in Z^\nu$ we have the conditions

1) the equality

$$\sum_{\substack{b \in \operatorname{supp} \Gamma \\ b \text{ enters } x}} m(b) = \sum_{\substack{b \in \operatorname{supp} \Gamma \\ b \text{ leaves } x}} n(b) \tag{34}$$

holds;

2) the tensor product

$$\bigotimes_{\substack{b \in \operatorname{supp} \Gamma \\ b \text{ incident on } x}} T^{l(b)}_u \tag{35}$$

of representations contains the identity (trivial) representation (possibly with multiplicity > 1) as an irreducible component.

Let Δ denote a connected set of oriented edges b and let $L = \{l(b),\ b \in \Delta\}$ be a function on Δ taking integer and half-integer values, and such that condition (35) holds for every point $x \in \partial\Delta$. Choose a vector

$$\Psi^{\{\lambda(x), x \in \partial\Delta\}}_{L,\Delta} = \sum_\Gamma \prod_{x \in \Delta} C(\Gamma_x) \Psi_\Gamma,$$

where the sum is over all regular multi-indices $\Gamma = \{l(b), m(b), n(b),\ b \in \Delta\}$ such that $\operatorname{supp}\Gamma = \Delta$, the function $\{l(b),\ l \in \Delta\}$ is equal to L, and $\Gamma_x = \{l(b), m(b), n(b),\ b \in \operatorname{supp}\Gamma,$ b incident to $x\}$ is a regular multi-index.

The numbers $C^{\lambda, l_1, \ldots, l_k}_{m_1, \ldots, m_s, n_1, \ldots, n_{k-s}}$ are the coefficients of one of the invariant vectors η^λ (the superscript λ distinguishing these vectors) in the tensor product of representations

$$\bigotimes T_u^{l_i}$$

in the basis

$$\left(\bigotimes_{i=1}^{s} \xi_{m_i}^{l_i}\right) \otimes \left(\bigotimes_{i=1}^{k-s} \xi_{n_i}^{l_{s+i}}\right).$$

For an arbitrary (nonconnected) set Δ we define the functions $\Psi_{L,\Delta}^{\{\lambda(x), x \in \partial\Delta\}}$ by

$$\Psi_{L,\Delta}^{\{\lambda(x), x \in \partial\Delta\}} = \prod_{i=1}^{k} \Psi_{L_i, \Delta_i}^{\{\lambda_i(x), x \in \partial\Delta_i\}}, \tag{36}$$

where $\Delta_1, \ldots, \Delta_k$ are the connected components of Δ, L_i is the restriction of the function L to Δ_i, and $\{\lambda_i(x)\}$ is defined similarly.

One can show that the vectors (36) form an orthonormal basis in $\mathcal{H}_{\text{phys}}$ in terms of which the matrix elements of $\mathcal{J}$ admit a natural cluster expansion. Using this expansion and some precise estimates for the leading matrix elements of $\mathcal{J}$ in this basis, one can prove the following analog of Theorem 4.

THEOREM 5. *Let $\mathcal{J}$ be the transfer matrix for a Yang-Mills field with gauge group $SU(2)$, and suppose that β is sufficiently small. Then there exist $\nu(\nu-1)/2$ invariant one-particle subspaces $\mathcal{H}_1^\sigma$, where the superscript σ labels the classes (orbits under the translation group, see above) of the plaquettes in Z^ν. The spectra of $\mathcal{J}$ on all the spaces $\mathcal{H}_1^\sigma$ coincide and lie within a $(C\beta)^8$-neighborhood of the values β^4 ($C > 0$ is an absolute constant).*

REMARKS. 1. The number of highest one-particle subspaces for the gauge group $SU(2)$ is seen to be one-half the number for the group $U(1)$. This comes about because every irreducible representation of $SU(2)$ is selfadjoint (i.e., coincides with its contragredient representation, see [9]).

2. As in the case of a $U(1)$ gauge field, so also for $SU(2)$ there exist a series of one-particle subspaces which are invariant under the transfer matrix $\mathcal{J}$. The spectrum of $\mathcal{J}$ restricted to these subspaces is $\sim \beta^6$, and they describe excited gluons.

4. Random walk of a particle in a stochastic medium. Here we will illustrate the techniques discussed above for studying stochastic operators by considering the random walk of a particle along the lattice Z^ν, on which a Markov field is given, and the particle interacts with the field. Let $x_t \in Z^\nu$, $t = 0, 1, \ldots$ be the position of the particle at time t and let $\xi_t = \{\xi_t(x),\ x \in Z^\nu\}$ be the field configuration, taking the values $\xi_t(x) = \pm 1$ at time t.

The transition probabilities for the particle-field random system are given by

$$\begin{aligned} &\Pr(\xi_t \in A, x_t = x | \xi_{t-1} = \eta, x_{t-1} = z) \\ &\qquad = \Pr(\xi_t \in A | \xi_{t-1} = \eta, x_{t-1} = z) \Pr(x_t = x | \xi_t = \eta, x_{t-1} = z), \end{aligned} \tag{37}$$

where $A \subset \{-1, 1\}^{Z^\nu}$ is an arbitrary (measurable) subset of configurations and $\eta = \{\eta(y),$ $y \in Z^\nu\}$ is a fixed field configuration on Z^ν. In other words, the changes in position of the

particle and the field configuration for fixed η and z are conditionally independent. We now introduce the following hypotheses regarding the transition probabilities of the particle and field:

1. We have

$$\Pr(x_t = x | \xi_{t-1} = \eta, x_{t-1} = z) = p_0(x - z) + c(x - z; \eta(z)). \tag{38}$$

Here $p_0(x-z)$ is the probability that the particle will go from z to x on the lattice: $\Pr(z \to x) = p_0(x-z)$, $c(u,s), u \in Z^\nu$, $s = \pm 1$, is a function on $Z^\nu \times \{-1, 1\}$ such that

$$\sum_{u \in Z^\nu} c(u,s) = 0, \qquad s = \pm 1,$$

and $p_0(u) + c(u,s) \geq 0$ for all $u \in Z^\nu$ and $s = \pm 1$. An additional condition on $c(u,s)$ will be given later (see equation (46^a)).

2. We have

$$\Pr(\xi_t = \overline{\xi}(x), x \in \Lambda | \xi_{t-1} = \eta, x_{t-1} = z) \prod_{y \in \Lambda} q_y(\overline{\xi}(y), \eta(y); z), \tag{39}$$

where Λ is an arbitrary finite set of Z^ν, $\overline{\xi}$ is any field configuration in Λ, and

$$q_y(s, s'; z) = \begin{cases} q_0(s, s'), & y \neq z, \\ q_1(s, s'), & y = z, \end{cases}$$

where the two stochastic matrices q_0, q_1 define ergodic Markov chains in the state space $\{-1, 1\}$. We will henceforth assume that the field interacts weakly with the particle, i.e.,

$$\max_{u,s} |c(u,s)| \equiv \varepsilon_0 \ll 1$$

and

$$\max_{s,s'} |q_0(s,s') - q_1(s,s')| \equiv \varepsilon_1 \ll 1.$$

In addition, the functions $p^0(u)$ and $c(u,s)$ will be assumed to have finite support:

$$p_0(u) = 0 \quad \text{and } c(u,s) = 0 \quad \text{for } |u| > R.$$

We will now prove the next result.

THEOREM 6. *For sufficiently small ε_0 and ε_1 and for any initial distribution Π of the values of the field ξ_0, the probability for the particle to move a distance $x_t - x_0 = u$ from its initial position is given asymptotically for large t by*

$$\Pr(x_t - x_0 = u) = \frac{D^{1/2}}{\sqrt{(2\pi t)^\nu}} \exp\left\{ -\frac{1}{2t}(A(u - bt)\right\}\{1 + o(1)\}, \tag{40}$$

where $b \in R^\nu$ is a ν-dimensional vector, $A(\cdot)$ is a positive definite quadratic form on R^ν, and D is its determinant. The vector b and form $A(\cdot)$ do not depend on the initial distribution Π. The asymptotic formula (40) is valid in the interval $|u - bt| < t^{1/2+\varepsilon}$, where ε is sufficiently small ($\varepsilon < 1/6$).

PROOF. We introduce the space $C(\Omega \times Z^\nu)$ of functionals $\Phi(\xi, x)$, $\xi \in \Omega = \{-1, 1\}$, $x \in Z^\nu$ depending on the coordinates of the random system, continuous in ξ (Ω is endowed with the Tikhonov topology), and tending rapidly to zero as $x \to \infty$. Define on $C(\Omega \times Z^\nu)$ the operator

$$(\mathcal{J}\Phi)(\eta, z) = \int_{\Omega \times Z^\nu} \Phi(\xi, x) dP(\xi, x|\eta, z) \tag{41}$$

(this is the stochastic operator, or the transfer matrix, for the random system).

It is easy to check that $\mathcal{J}$ does indeed take $C(\Omega \times Z^\nu)$ into itself. We fix the initial position of the system: $x_0 = y$, $\xi_0 = \eta$, and for any functional $\Phi \in C(\Omega \times Z^\nu)$ consider the conditional mean

$$\langle \Phi(\xi_t, x_t) | \xi_0 = \eta, x_0 = y \rangle,$$

where (ξ_t, x_t) is the position of the system at time t. Then it is clear that

$$\langle \Phi(\xi_t, x_t) | \xi_0 = \eta, x_0 = y \rangle = (\mathcal{J}^t \Phi)(\eta, y).$$

Hence in particular,

$$\Pr(x_t = \overline{x} | x_0 = y) = \int_\Omega (\mathcal{J}^t \Phi_{\overline{x}})(\eta, y) d\Pi(\eta), \tag{42}$$

where $\Phi_x(\xi, x) = \delta_{x,\overline{x}}$.

Let $\{\pi_0(s),\ s = \pm 1\}$ be the stationary distribution of a Markov chain with transition probability matrix $q_0(s, s')$. Consider the initial distribution $\Pi_0 = (\pi_0)^{Z^\nu}$ and write $\mathcal{H} = L_2(\Omega, \Pi_0) \otimes l_2(Z^\nu)$. Now set $e_0(s) \equiv 1$ and let e_1 be a normalized eigenvector of the stochastic matrix

$$(Qe_1)(s') = \sum_{s=\pm 1} q_0(s, s') e_1(s)$$

with eigenvalue μ, $|\mu| < 1$. We choose the basis

$$\Psi_{Q,z}(\xi, x) = \prod_{y \in Q} e_1(\xi(y)) \delta_{z,x}, \tag{43}$$

in $\mathcal{H}$, where Q is a finite subset of Z^ν, $z \in Z^\nu$.

In view of the obvious relation

$$\sum_s \pi_0(s) e_1(s) = 0$$

and the condition

$$\sum_s \pi_0(s) |e_1(s)|^2 = 1$$

the basis $\{\Psi_{Q,z}\}$ is orthonormal in $\mathcal{H}$. Any element $\Phi \in \mathcal{H}$ has an expansion

$$\Phi = \sum_{(Q,z)} f(Q, z) \Psi_{Q,z} \tag{43^a}$$

which gives an isomorphism $\mathcal{H} \to l_2(C_{Z^\nu} \times Z^\nu)$, where C_{Z^ν} is the set of all finite subsets of Z^ν. The transfer matrix $\mathcal{J}$ defines a bounded operator on $\mathcal{H}$ which under the above isomorphism has the form

$$\begin{aligned}(\mathcal{J}f)(Q,z) = &\sum_{z'} R_{Q-z}(z'-z)f(Q,z') \\ &+ (1-\chi(Q-z))\sum_{z'} G_{(Q-z)\cup\{0\}}(z'-z)f(Q\cup\{z\},z') \\ &+ \chi(Q-z)\sum_{z'} H_{(Q-z)\cup\{0\}}(z'-z)f(Q\setminus\{z\},z').\end{aligned} \tag{44}$$

Here we have written

$$\chi_0(Q) = \begin{cases} 1 & \text{if } 0 \in Q, \\ 0 & \text{otherwise.}\end{cases}$$

Next, for $Q \neq \emptyset$ we have

$$\begin{aligned}R_Q(u) &= \mu^{|Q|}p_0(u) + \chi_0(Q)\mu^{|Q|-1}(\alpha_1 p_0(0) + c_1(u))(\mu b_1 + b_1\alpha_1 + \alpha_0), \\ G_Q(u) &= \mu^{|Q|-1}\{\alpha_0 p_0(u) + b_0\mu c_1(u) + \alpha_0 b_0 c_1(u)\}, \\ H_Q(u) &= \mu^{|Q|}c_1(u).\end{aligned} \tag{45}$$

For $Q = \emptyset$,

$$R_\emptyset(u) = p_0(u), \qquad G_\emptyset(u) = 0, \qquad H_\emptyset(u) = c_1(u). \tag{46}$$

Here the constants α_0 and α_1, b_0, b_1, and the function $c_1(u)$ are defined as follows:

$$\sum_s (q_1(s,s') - q_0(s,s'))e_1(s) = \alpha_0 + \alpha_1 e_1(s'),$$

$$e_1(s)e_1(s) = b_0 + b_1 e_1(s),$$

$$c(u,s) = c_0(u) + c_1(u)e_1(s).$$

Assuming further that

$$\sum_s c(u,s)\pi_0(s) = 0 \tag{46^a}$$

(as can always be arranged by modifying p_0), then $c_0(u) \equiv 0$. We note that the representation $\{U_\nu,\ \nu \in Z^\nu\}$ of the group of lattice translations, acting on $\mathcal{H}$ by the formula

$$(U_v\Phi)(\xi,x) = \Phi(\xi - v, x - v),$$

where $\xi - v$ denotes the translate of the configuration ξ by the vector v, goes over into the operators

$$(U_v f)(Q,x) = f(Q - v, z - v)$$

on the space $l_2(C_{Z^\nu} \times Z^\nu)$.

We note further that any function on $C_{Z^\nu} \times Z^\nu$ of the form

$$f_x(Q,z) = \varphi(Q-z)e^{i(x,z)},$$

where $\varphi(Q) \in l_2(C_{Z^\nu})$ and $\kappa \in T^\nu$ (the ν-dimensional torus), is a generalized eigenfunction in the continuous spectrum of the operators U_v (see [10]) with eigenvalue $\exp\{i(\kappa, v)\}$. We denote the space of such functions by $\mathcal{H}(\kappa)$. It is evidently isomorphic to $l_2(C_{Z^\nu})$, and every function $f \in l_2(C_{Z^\nu} \times Z^\nu)$ has a unique integral decomposition

$$f(Q, z) = \int_{T^\nu} \varphi_\kappa(Q - z) \exp\{i(\kappa, z)\} d\kappa,$$

where $d\kappa$ is the normalized Haar measure on T^ν. Since $\mathcal{J}$ commutes with the group $\{U_v\}$, it generates a family of operators $\{\mathcal{J}(\kappa), \kappa \in T^\nu\}$, each acting on the space $\mathcal{H}(\kappa)$ or, in view of the above isomorphism, on $l_2(C_{Z^\nu})$. We have

$$(\mathcal{J}f)(Q, z) = \int_{T^\nu} (\mathcal{J}(\kappa)\varphi_\kappa)(Q - z) \exp\{i(\kappa, z)\} d\kappa,$$

and the operator $\mathcal{J}(\kappa)$ acts on $l_2(C_{Z^\nu})$ by the formula

$$\begin{aligned}(\mathcal{J}(\kappa)\varphi)(Q) = \sum_{u \in Z^\nu} [&R_Q(u)\varphi(Q - u) \\ &+ (1 - \chi_0(Q))G_{Q \cup \{0\}}(u)\varphi((Q - u) \cup \{-u\}) \\ &+ \chi_0(Q)H_{Q \setminus \{0\}}(u)\varphi((Q - u) \cup \{-u\})] e^{-i(\kappa, u)},\end{aligned} \tag{47}$$

where R_Q, G_Q, and H_Q are defined by (45) and (46).

We now consider the case of an "unperturbed" random walk: $c(u, s) = 0$ and $q_0 = q_1$ so that the particle and field evolve independently. In this case the operators $\mathcal{J}^0(\kappa)$ act by the formula

$$(\mathcal{J}^0(\kappa)\varphi)(Q) = \mu^{|Q|} \sum_{u \in Z^\nu} p_0(u)\varphi(Q - u)e^{-i(\kappa, u)}. \tag{48}$$

We observe that the space $l_2(C_{Z^\nu})$ splits as a direct orthogonal sum of the subspaces $l_2(C^n_{Z^\nu})$:

$$l_2(C_{Z^\nu}) = \bigoplus_{n=0}^{\infty} l_2(C^n_{Z^\nu}),$$

where $C^n_{Z^\nu}$ is the set of n-point subsets of Z^ν ($l_2(C^0_{Z^\nu})$ is a one-dimensional space). Each $l_2(C^n_{Z^\nu})$ is invariant under the operators $\mathcal{J}^0(\kappa)$. In particular, the one-dimensional subspace $l_2(C^0_{Z^\nu}) = \{B\delta_Q\}$ is an eigenspace for $\mathcal{J}^0(\kappa)$ with eigenvalue

$$\widetilde{p}_0(\kappa) = \sum_{\xi \in Z^\nu} p_0(\xi)e^{i(\kappa, \xi)}.$$

The norm of $\mathcal{J}^0(\kappa)$ on the orthogonal complement

$$l_2^\perp(C_{Z^\nu}) = \bigoplus_{n=1}^{\infty} l_2(C^n_{Z^\nu})$$

is bounded by

$$\|\mathcal{J}^0(\kappa)|_{l_2^\perp(C_{Z^\nu})}\| < |\mu|.$$

This is readily shown from the explicit formula (48). Now consider a neighborhood V_δ of $0 \in T^\nu$ such that

$$\min_{\kappa \in V_\delta} |\widetilde{p}_0(\kappa)| > |\mu| + \delta,$$

where $0 < \delta < 1 - |\mu|$ is a constant. Thus, the eigenvalue $\widetilde{p}_0(\kappa)$ for $\kappa \in V_\delta$ is separated from the rest of the spectrum of $\mathcal{J}^0(\kappa)$ by a gap of width at least δ. Since for small ε_0, ε_1 the operator $\mathcal{J}(\kappa)$ is nearly equal to $\mathcal{J}^0(\kappa)$, for each $\kappa \in V_\delta$ it has an eigenvector $\chi_\kappa \in l_2(C_{Z^\nu})$, normalized by the condition $\chi_\kappa(\emptyset) = 1$, with eigenvalue $\widetilde{p}(\kappa)$ very close to $\widetilde{p}_0(\kappa)$. Both χ_κ and $\widetilde{p}(\kappa)$ depend analytically on κ for κ varying in some complex neighborhood of V_δ. In addition, it is not difficult to verify that

$$|\widetilde{p}(\kappa)| \leq 1, \qquad \widetilde{p}(0) = 1, \tag{49}$$

and the real part of the quadratic form satisfies

$$\operatorname{Re} \sum_{i<j} \frac{\partial^2 \widetilde{p}}{\partial \kappa_i \partial \kappa_j} \Big|_{\kappa=0}^{\Delta\kappa_i \Delta\kappa_j} > 0. \tag{50}$$

Consider the subspace $\mathcal{H}_\delta$ of elements of the form

$$f(Q, z) = \int_{V_\delta} \chi_\kappa(Q - z) \varphi(\kappa) \exp\{-i(\kappa, z)\} d\kappa, \qquad \varphi \in L_2(V_\delta, d\kappa). \tag{51}$$

Clearly, $\mathcal{H}_\delta$ is a subspace of $l_2(C_{Z^\nu} \times Z^\nu)$ invariant under $\mathcal{J}$ and $\{U_\nu, \nu \in Z^\nu\}$. Now consider the operator $\mathcal{J}^*$ on $\mathcal{H}$ adjoint to $\mathcal{J}$. Arguing as above, we construct an invariant subspace $\mathcal{H}_\delta^*$ for $\mathcal{J}^*$ in the same way that $\mathcal{H}_\delta$ was constructed. The orthogonal complement

$$\overline{\mathcal{H}}_\delta = (\mathcal{H}_\delta^*)^\perp$$

is plainly invariant under $\mathcal{J}$, and the space $l_2(C_{Z^\nu} \times Z^\nu)$ splits as a direct sum

$$l_2(C_{Z^\nu} \times Z^\nu) = \mathcal{H}_\delta + \overline{\mathcal{H}}_\delta.$$

A similar splitting holds for the space $\mathcal{H}$:

$$\mathcal{H} = \widehat{\mathcal{H}}_\delta + \widehat{\overline{\mathcal{H}}}_\delta,$$

where $\widehat{\mathcal{H}}_\delta \subset \mathcal{H}$, $\widehat{\overline{\mathcal{H}}}_\delta \subset \mathcal{H}$ are the inverse images of the spaces $\mathcal{H}_\delta$ and $\overline{\mathcal{H}}_\delta$ under the isomorphism (43^a).

We check further that under the decomposition

$$\Phi_x = \Phi_x^{(\delta)} + \overline{\Phi}_x^{(\delta)}, \qquad \Phi_x^{(\delta)} \in \widehat{\mathcal{H}}_\delta, \qquad \overline{\Phi}_x^{(\delta)} \in \widehat{\overline{\mathcal{H}}}_\delta$$

of the functional $\Phi_x(\xi, z) = \delta_{zx}$, both components satisfy

$$\Phi_x^{(\delta)}, \overline{\Phi}_x^{(\delta)} \in C(\Omega \times Z^\nu)$$

and we have

$$\|\mathcal{J}^t \overline{\Phi}_x^{(\delta)}\|_{C(\Omega \times Z^\nu)} = O((|\mu| + \delta)^t).$$

It follows that the computation of the asymptotics of the probability (42) reduces to evaluating the integral

$$\begin{aligned}\int_\Omega (\mathcal{J}^t\Phi_x^{(\delta)})(\eta,y)d\Pi(\eta) &= \int_\Omega (U_x\mathcal{J}^t\Phi_0^{(\delta)})(\eta,y)d\Pi(\eta)\\ &= \int_{V_\delta} (\widetilde{p}(x))^t\varphi(\kappa,y)e^{i((y-x),\kappa)}d\kappa,\\ \varphi(\kappa,y) &= \varphi^0(\kappa)\sum_Q \chi_\kappa(Q-y)a_Q,\end{aligned}$$

where $a_Q = \int_\Omega \Phi_Q(\eta)\,d\Pi(\eta)$, and $\varphi^0(\kappa)$ is the function appearing in (51) for $f_0^\delta(Q,z)$, the coefficients of Φ_0^Δ with respect to the basis $\Psi_{Q,z}$. Applying the method of steepest descent, we obtain the asymptotic expression (40) where the quadratic form $A(\cdot)$ is defined by the inverse of the matrix of second derivatives

$$\left\{\frac{\partial^2\widetilde{p}(\kappa)}{\partial\kappa_i\partial\kappa_j}\right\}\Big|_{\kappa=0}, \tag{52}$$

and the vector $b = \{i\frac{\partial\widetilde{p}}{\partial k_j}\big|_{k=0}\}$. We see from (49) that b is a real vector, and since expression (40) represents a probability, the matrix (52) is positive. This concludes the proof of the theorem.

§5. Spectral analysis of the transfer matrix for a fermion field

In this section we generalize the methods for analyzing the spectrum of the transfer matrix for ordinary random (boson) fields to the case when the field algebra contains Grassmann variables (fermion field). We will begin by studying the case of pure fermion fields, and then illustrate the general mixed case by means of an example.

We assume that a Gibbs modification $\langle\cdot\rangle$ of an independent Gaussian quasistate $\langle\cdot\rangle_0$ has been defined on the normed algebra $\mathcal{E}_F$ by means of the Euclidean action described in 5.2 (with small parameter λ); here $\mathcal{E}_F$ is generated by the set $\{\psi_\alpha(x), \overline{\psi}_\alpha(x),\ x \in \widetilde{Z}^{\nu+1} = Z^\nu \times \widetilde{Z}^1,\ \alpha \in M\}$, where M is an index set. We recall that the perturbed quasistate $\langle\cdot\rangle$ on $\mathcal{E}_F$ is invertibly Markov (see 5.3).

The general approach to analyzing the transfer matrix $\mathcal{J}$ for such a field is similar to the one described above for boson (probability) fields. However, there are some technical differences stemming from the fact that the inner product $(\cdot,\cdot)$ on the algebra $\mathcal{E}_{1/2}$ (generated by the elements $\psi_\alpha(x)$ and $\overline{\psi}_\alpha(x)$ "living" on the time slice $Y_{1/2} = Z^\nu \times \{1/2\}$) is defined using the involution θ, which takes $\mathcal{E}_{1/2}$ into the different algebra $\mathcal{E}_{-1/2}$. In order to preserve the previous construction for the multiplicative basis $\{\Psi_\Gamma\}$, we define a Hermitian bilinear form on $\mathcal{E}_{1/2}$ by the formula

$$[F_1,F_2] = \langle F_1, \vartheta F_2\rangle, F_1, F_2 \in \mathcal{E}_{1/2}, \tag{1}$$

where ϑ is the antilinear involution on $\mathcal{E}_{1/2}$ acting on the generators by

$$\begin{aligned}\vartheta\psi_\alpha(x) &= \sum_{\alpha'\in M} \varepsilon_{\alpha,\alpha'}\overline{\psi}_{\alpha'}(x),\\ \vartheta\overline{\psi}_\alpha(x) &= \sum_{\alpha'\in M} \overline{\varepsilon}_{\alpha,\alpha'}\psi_{\alpha'}(x).\end{aligned} \tag{2}$$

Here $\varepsilon = \{\varepsilon_{\alpha,\alpha'}\}$ is the matrix defining θ in $\mathcal{E}$ (see 5.2, paragraph 2). We can then extend ϑ to all of $\mathcal{E}_{1/2}$ by the analog of relation (17″) in 5.2:

$$\vartheta(A_1A_2) = \vartheta(A_2)\vartheta(A_1).$$

The Hermiticity of the form (1) follows from the ϑ-invariance of the state $\langle\cdot\rangle$ on the algebra $\mathcal{E}_F$:

$$\langle\vartheta F\rangle = \langle\overline{F}\rangle,$$

which in turn follows from the representation

$$\vartheta F = U_{e_0}\Theta F. \tag{3}$$

Without loss of generality, we may suppose that the matrix e is diagonal,

$$\varepsilon_{\alpha,\alpha'} = \varepsilon_\alpha\delta_{\alpha,\alpha'}. \tag{4}$$

We now construct a multiplicative basis $\{\Psi_\Gamma\}$ in $\mathcal{E}_{1/2}$ analogous to the basis (3.1) for the case of ordinary random fields (the multi-indices Γ will be described below), orthonormal with respect to the form (1):

$$[\Psi_{\Gamma_1}, \Psi_{\Gamma_2}] = (-1)^{\alpha(\Gamma)}\delta_{\alpha,\alpha'}, \tag{5}$$

where the function $\alpha(\Gamma)$ that determines the sign will be given below. We first consider the case of an independent Gaussian state $\langle\cdot\rangle_0$ (unperturbed field) and construct the basis for it. For any point $x_0 \in Y_{1/2}$ we write $\mathfrak{A}(x_0) \subset \mathcal{E}_{1/2}$ for the Grassmann algebra generated by $\{\psi_\alpha(x_0), \overline{\psi}_\alpha(x_0),\ \alpha \in M\}$. For each $\alpha \in M$ we denote by $\eta_\alpha^\pm(x_0)$, $\eta_\alpha^0(x_0)$, $\eta_\alpha^{\overline{0}}(x_0)$ the elements

$$\eta_\alpha^+(x_0) = \psi_\alpha(x_0), \qquad \eta_\alpha^-(x_0) = \overline{\psi}_\alpha(x_0), \qquad \eta_\alpha^0(x_0) = 1,$$
$$\eta_\alpha^{\overline{0}}(x_0) = \psi_\alpha(x_0)\overline{\psi}_\alpha(x_0) - 1.$$

Consider the monomials

$$\eta_s(x_0) = \prod_{\alpha\in M}\eta_\alpha^{s(\alpha)}(x_0), \tag{6}$$

in $\mathfrak{A}(x_0)$, where $s = \{s(\alpha), s(\alpha) = 0, \overline{0}, \pm\}$ is a multi-index, and the factors in (6) appear in order of increasing α (we assume that M has been ordered in some way). We easily compute that these monomials are orthonormal with respect to the form $[\cdot,\cdot]_0$

$$[\eta_{s_1}(x_0), \eta_{s_2}(x_0)]_0 = \langle\eta_{s_1}(x_0)\vartheta\eta_{s_2}(x_0)\rangle_0 = \delta_{s_1,s_2}(-1)^{\sigma(s_1)}, \tag{7}$$

where $\sigma(s) = \sum_{\alpha: s(\alpha)\neq\overline{0}}\varepsilon(\alpha)s\{\alpha\colon \alpha = \overline{0}\}$. We further introduce the multiplicative basis

$$\Phi_\Gamma^0 = \prod_x \eta_{s(x)}(x) \tag{8}$$

in $\mathcal{E}_{1/2}$, where the factors are given in lexicographic order and the multi-index $\Gamma = \{s(x) = \{s(x,\alpha)\}\}$ is orthonormal with respect to $[\cdot,\cdot]_0$:

$$[\Phi^0_\Gamma, \Phi^0_{\Gamma'}]_0 = \delta_{\Gamma,\Gamma'}(-1)^{\sigma(\Gamma)}, \tag{9}$$

where $\sigma(\Gamma) = \sum_x \sigma(s(x))$, and is orthogonal to the unit element:

$$[\Phi^0_\Gamma, 1] = 0.$$

We will now construct a basis Φ_Γ for the perturbed field $\langle\cdot\rangle$.

As before, for each $x \in Y_{1/2}$ we define the elements

$$\widetilde{\eta}_s(x) = \eta_s(x) - \langle \eta_s(x) | \mathcal{E}^{<x}_{1/2}\rangle,$$

where $\langle F \mid \mathcal{E}^{<x}_{1/2}\rangle \in \mathcal{E}^{<x}_{1/2}$, $F \in \mathcal{E}_{1/2}$, denotes the conditional expectation relative to the subalgebra $\mathcal{E}^{<x}_{1/2} \subset \mathcal{E}_{1/2}$ generated by $\{\psi_\alpha(y), \overline{\psi}_\alpha(y),\ y < x,\ \alpha \in M\}$, and we use the lexicographic ordering on Z^ν. Consider the conditional Gramm matrix

$$D_{s,s'} = \langle \widetilde{\eta}_s(x) \vartheta \widetilde{\eta}_{s'}(x) | \mathcal{E}^{<x}_{1/2}\rangle \in \mathcal{E}^{<x}_{1/2},$$

whose entries are homogeneous elements of the superalgebra $\mathcal{E}^{<x}_{1/2}$. Applying as before the cluster expansions for the conditional means with respect to the quasistate $\langle\cdot\rangle$ (see [26]), we obtain that

$$D_{s,s'} = \delta_{s,s'}(-1)^{\sigma(s)} + m_{s,s'}(x), \tag{9^a}$$

where the $m_{s,s'}(x) \in \mathcal{E}^{<x}_{1/2}$ are "small" and can be expanded as a series (8) in the monomials Φ^0_Γ:

$$m_{s,s'}(x) = \sum_{\Gamma:\operatorname{supp}\Gamma<x} C^{x,s,s'}_\Gamma \Phi^0_\Gamma. \tag{9^b}$$

The coefficients $C^{x,s,s'}_\Gamma$ in this expansion satisfy the cluster estimates

$$|C^{x,s,s'}_\Gamma| < (C\lambda)^{d_{\{x\}\cup\operatorname{supp}\Gamma}}, \tag{10}$$

where d_B for $B \subset \widetilde{Z}^\nu$ is the cardinality of the smallest connected collection $\Delta_1, \dots, \Delta_k$ of the supports of the potential in (58^a.5.2) covering the set B.

We now observe that if we pass from the basis $\widetilde{\eta}_s(x)$ to the basis

$$h_s(x) = \sum_{s'} B_{s,s'}(x)\widetilde{\eta}_s(x), \tag{11}$$

where $B_{s,s'}(x) = \mathcal{E}^{<x}_{1/2}$, then the Gramm matrix $\{D_{s,s'}(x)\}$ for the new basis is given by

$$\widehat{D} = B(x)D(x)B^*(x),$$

where $\{B_{s,s'}(x)\}$, and $B^*(x) = \{B^*_{s,s'}(x)\}$, where $B^*_{s,s'} = \vartheta B_{s,s'}(x)$. Applying successive transformations of the type (11), we now reduce $D(x)$ to the diagonal matrix

$$\{(-1)^{\sigma(s)}\delta_{s,s'}\}. \tag{11^a}$$

The set N of values of the multi-indices can now be divided into the two subsets $N_\pm = \{s\colon (-1)^{\sigma(s)} = \pm 1\}$. Then the Gramm matrix D can be written in block form as

$$D = \begin{pmatrix} D_{+,+} & D_{+,-} \\ D_{-,+} & D_{-,-} \end{pmatrix}, \tag{12}$$
$$D^*_{+,+} = D_{+,+}, \qquad D^*_{-,-} = D_{-,-}, \qquad D^*_{+,-} = D_{-,+},$$

where $D_{+,+} = \{D_{s,s'}, s, s' \in N_+\}$, $D_{+,-} = \{D_{s,s'}, s \in N_+, s' \in N_-\}$, and so on. By (9^a), $D_{+,+} = E + M_{+,+}$, $D_{-,-} = -E + M_{-,-}$, $D_{+,-} = M_{+,-}$, $D_{-,+} = M_{-,+}$, where the matrices $M_{+,+} = \{m_{s,s'},\ s, s' \in N_+\}$, etc., are "small". Applying a similarity transform using the matrix

$$B_1 = \begin{pmatrix} (E + M_{+,+})^{-1/2} & 0 \\ 0 & (E - M_{-,-})^{-1/2} \end{pmatrix},$$

we arrive at the Gramm matrix

$$\begin{pmatrix} E & \widehat{D}_{+,-} \\ \widehat{D}_{-,+} & -E \end{pmatrix},$$

where $\widehat{D}_{-,+} = \widehat{D}^*_{+,-}$, and both $\widehat{D}_{+,-}$ and $\widehat{D}_{-,+}$ are small with entries admitting an expansion (9^b) with cluster estimates (10) for the coefficients.

Transformation by the triangular matrix B_2,

$$B_2 = \begin{pmatrix} (E + D_{+,-}D^*_{+,-})^{-1/2} & (E + D_{+,-}D^*_{+,-})^{-1/2} D_{+,-} \\ 0 & -E \end{pmatrix}$$

takes the matrix (12) into (11^a). Thus, for each $x \in Y_{1/2}$ we have constructed a conditionally orthonormal basis $\{h_s(x)\}$ in $\mathcal{E}^{\le x}_{1/2}$ (the algebra $\mathcal{E}^{\le x}_{1/2} \subset \mathcal{E}_{1/2}$ being defined analogously to $\mathcal{E}^{<x}_{1/2}$):

$$\langle h_s(x) \vartheta h_{s'}(x) | \mathcal{E}^{<x}_{1/2} \rangle = (-1)^{\sigma(s)} \delta_{s,s'}. \tag{13}$$

The basis

$$\Psi_\Gamma = \prod_x h_{s(x)}(x)$$

is easily verified to be orthonormal with respect to the form $[\cdot, \cdot]$:

$$[\Psi_\Gamma, \Psi_{\Gamma'}] = \delta_{\Gamma,\Gamma'} (-1)^{\sigma(\Gamma)}. \tag{14^a}$$

Here $\Gamma = \{s(x),\ x \in Y_{1/2}\}$ is a finite multi-index, and the factors in the product in (14) are taken in lexicographic order on $Y_{1/2}$.

Just as we did above for the case of ordinary fields, we can show that for all sufficiently large $r > 1$, the system of elements (14) is complete in $\mathcal{E}_{1/2}$ with respect to the norm $\| \cdot \|_r$ defined by equation (57.5.2). We also note that the basis $\{\Psi_\Gamma\}$ is invariant under spatial translations:

$$U_v \Psi_\Gamma = \Psi_{\Gamma + v},$$

where $\Gamma + v$ is the multi-index Γ translated by the vector $v \in Z^\nu$.

Let $\mathcal{E}^0_{1/2} \subset \mathcal{E}_{1/2}$ be the linear span of the basis elements Ψ_Γ. Then for $F \in \mathcal{E}^0_{1/2}$ we have the expansion

$$F = \sum_\Gamma f_\Gamma \Psi_\Gamma,$$

where the sum is over some finite set of indices Γ. This induces a map $\mathcal{E}^0_{1/2} \to l_{\text{phys}}(\mathfrak{M}_0)$ into the set of finite-support functions on $\mathfrak{M}_0$ (where $\mathfrak{M}_0$ is the set of multi-indices Γ). Given two elements $F, G \in \mathcal{E}^0_{1/2}$ we have

$$(F, G)_{\text{phys}} = \langle F \Theta G \rangle = \sum_{\Gamma, \Gamma'} D_{\Gamma, \Gamma'} f_\Gamma \overline{g}_{\Gamma'},$$

where $\{D_{\Gamma,\Gamma'}\}$ is the (transpose) of the Gramm matrix for the basis $\{\Psi_\Gamma\}$:

$$D_{\Gamma,\Gamma'} = (\Psi_{\Gamma'}, \Psi_\Gamma)_{\text{phys}} = \langle \Psi_{\Gamma'} \theta \Psi_\Gamma \rangle. \tag{15}$$

We will describe the physical Hilbert space $\mathcal{H}_{\text{phys}}$ in terms of the functions $f = \{f_\Gamma, \Gamma \in \mathfrak{M}_0\}$ on the set $\mathfrak{M}_0$.

Again as in the case of an ordinary random field, we find from (15) that the matrix elements $D_{\Gamma,\Gamma'}$ have a cluster expansion

$$D_{\Gamma,\Gamma'} = \sum_{\tau = \{(\Gamma_1, \Gamma'_1), \dots, (\Gamma_k, \Gamma'_k)\}} (-1)^{\pi(\tau)} \omega(\Gamma, \Gamma') \dots \omega(\Gamma_k, \Gamma'_k), \tag{16}$$

where as in (34.1) the sum is over all partitions $\{(\Gamma, \Gamma')\}$ of a pair of multi-indices into disjoint (nonempty) pairs of multi-indices, and

$$\omega(\Gamma, \Gamma') = \left\langle \prod_{x \in \operatorname{supp} \Gamma}{}' h_{s(x)}(x), \prod_{x' \in \operatorname{supp} \Gamma'} \theta h_{s'(x')}(x') \right\rangle \tag{17}$$

are the semi-invariants of the two sets of elements

$$\{h_{s(x)}(x), x \in \operatorname{supp} \Gamma, \qquad h_{s'(x')}(x'), x' \in \operatorname{supp} \Gamma'\},$$

arranged in a suitable order (lexicographic order in the first set, antilexicographic order in the second); the sign $(-1)^{\pi(\tau)}$ depends on the parity of the permutation induced by the partition τ (see [26] for more details).

Using once again the techniques discussed in detail at the beginning of this chapter (and also in [26]), we obtain that when λ in (58^a.5.2) is small, the semi-invariants $\omega(\Gamma, \Gamma')$ satisfy the cluster estimate (10):

$$|\omega(\Gamma, \Gamma')| < (C\lambda)^{d_{\operatorname{supp} \Gamma \cup \vartheta \operatorname{supp} \Gamma'}}, \tag{18}$$

where d_B, $B \subset \widetilde{Z}^{\nu+1}$ was defined above, C is an absolute constant, and ϑ is time reflection.

It follows from this estimate that the matrix $D = \{D(\Gamma, \Gamma')\}$ defines a bounded selfadjoint operator

$$(Df)(\Gamma) = \sum_{\Gamma' \in \mathfrak{M}} D(\Gamma, \Gamma') f(\Gamma'), \qquad f \in l_2(\mathfrak{M}_0),$$

on the space $l_2(\mathfrak{M}_0)$, with moreover the property that

$$(Df, g)_{l_2(\mathfrak{M}_0)} = \sum_{(\Gamma, \Gamma')} D(\Gamma, \Gamma') f(\Gamma') \overline{g}(\Gamma) = (F, G)_{\text{phys}},$$
$$F, G \in \mathcal{E}^0_{1/2}, \qquad F = \sum f(\Gamma) \Psi_\Gamma, \qquad G = \sum g(\Gamma) \Psi_\Gamma,$$

for all finite-support functions f, g in $l_2(\mathfrak{M}_0)$. This means that we can enlarge the algebra $\mathcal{E}_{1/2}$ to the Hilbert space $\overline{\mathcal{E}}_{1/2}$ of elements of the form $\sum_\Gamma f(\Gamma)\Psi_\Gamma$, where $f = \{f(\Gamma)\} \in l_2(\mathfrak{M}_0)$. This Hilbert space is isomorphic to $l_2(\mathfrak{M}_0)$, and as before the bilinear form

$$(F, G)_{\text{phys}} = \langle F\theta G\rangle$$

is well defined and continuous on $\overline{\mathcal{E}}_{1/2}$. According to Lemma 11.5.2, the physical Hilbert space $\mathcal{H}_{\text{phys}}$ is equal to

$$\mathcal{H}_{\text{phys}} = \overline{(\overline{\mathcal{E}}_{1/2}/\overline{N})}, \tag{19}$$

where $\overline{N} \subset \overline{\mathcal{E}}_{1/2}$ is the subspace of $\overline{\mathcal{E}}_{1/2}$ consisting of elements with zero norm ($\|F\|^2_{\text{phys}} = (F, F)_{\text{phys}} = 0$), and the completion is with respect to the inner product $(\cdot, \cdot)_{\text{phys}}$.

Let $K_0 \subset l_2(\mathfrak{M}_0)$ be the subspace

$$K_0 = \operatorname{Ker} D$$

(it is isomorphic to $\overline{N}$) and let K_1 be its orthogonal complement:

$$K_1 = \operatorname{Im} D = K_0^\perp \subset l_2(\mathfrak{M}_0),$$

which can be identified with $l_2(\mathfrak{M}_0)/K_0$. The bilinear form

$$(Df, g)_{l_2(\mathfrak{M}_0)} \equiv (f, g)_{\text{phys}}$$

on K_1 induces the inner product $(\cdot, \cdot)_{\text{phys}}$, and hence the space $\mathcal{H}_{\text{phys}}$ is isomorphic to the completion of K_1 with respect to $(\cdot, \cdot)_{\text{phys}}$, which we will denote by $\overline{K}_1$.

For each element of the basis $\{\Psi_\Gamma\}$ we now consider the expansion

$$\langle U_{e_0}\Psi_\Gamma|\mathcal{E}_{1/2}\rangle = \sum a_{\Gamma,\Gamma'}\Psi_{\Gamma'}. \tag{20}$$

¿From equations (14^a), (1), (3), and (33^a.5.2), we find that

$$\begin{aligned} a_{\Gamma,\Gamma'} &= (-1)^{\sigma(\Gamma')}[\langle U_{e_0}\Psi_\Gamma|\mathcal{E}_{1/2}\rangle, \Psi_{\Gamma'}] = \langle\langle U_{e_0}\Psi_\Gamma|\mathcal{E}_{1/2}\rangle\vartheta\Psi_{\Gamma'}\rangle(-1)^{\sigma(\Gamma')} \\ &= \langle\langle U_{e_0}\Psi_\Gamma, \vartheta\Psi_{\Gamma'}\rangle\rangle(-1)^{\sigma(\Gamma')} = (-1)^{\sigma(\Gamma')}\langle\Psi_\Gamma\theta\Psi_{\Gamma'}\rangle = (-1)^{\sigma(\Gamma')}D_{\Gamma,\Gamma'}. \end{aligned} \tag{21}$$

For every $F \in \overline{\mathcal{E}}$ we have

$$F = \sum_\Gamma f(\Gamma)\Psi_\Gamma, \qquad \langle U_{e_0}F|\mathcal{E}_{1/2}\rangle = \sum_\Gamma \widetilde{f}(\Gamma)\Psi_\Gamma,$$

where

$$\widetilde{f}(\Gamma) = \sum a_{\Gamma,\Gamma'}f = (-1)^{\sigma(\Gamma)}D_{\Gamma,\Gamma'}f(\Gamma').$$

Thus the conditional expectation operator $\langle U_{e_0}F \mid \mathcal{E}_{1/2}\rangle$ extends to a bounded operator on the space $\overline{\mathcal{E}}_{1/2}$, which under the isomorphism of $\overline{\mathcal{E}}_{1/2}$ with $l_2(\mathfrak{M}_0)$ goes into the cluster operator

$$A = JD, \tag{22}$$

where J is the diagonal operator

$$J(f)(\Gamma) = (-1)^{\sigma(\Gamma)} f(\Gamma).$$

Note that $\operatorname{Ker} A = \operatorname{Ker} D = K_0$. Together with equations (46^a.5.2) and (19), this implies that under the identification of $\mathcal{H}_{\text{phys}}$ with $\overline{K}_1$, the transfer matrix $\mathcal{J}$ of our field coincides with the closure (with respect to the norm $\|\cdot\|_{\text{phys}}$ in $\overline{K}_1$) of the operator

$$P_{K_1} J D_1 \tag{23}$$

acting on K_1, where D_1 is the part of D acting on the invariant subspace K_1 and P_{K_1} is the projection of $l_2(\mathfrak{M}_0)$ onto the subspace K_1.

Note that the map

$$\operatorname{Im} D_1^{1/2} \to K_1 : f \to D_1^{-1/2} f, \qquad f = \operatorname{Im} D_1^{1/2} \subset K_1$$

extends to a unitary transformation

$$U \colon K_1 \to \overline{K}_1,$$

where we take K_1 and $\overline{K}_1$ with the inner products $(\cdot,\cdot)_{l_2(\mathfrak{M}_0)}$ and $(\cdot,\cdot)_{\text{phys}}$, respectively. We easily check that under this transformation the operator $\mathcal{J}$ acting on $\overline{K}_1$ goes into the unitarily equivalent selfadjoint operator

$$\mathcal{J}' = D_1^{1/2} J_1 D_1^{1/2}, \tag{24}$$

acting on K_1, where $J_1 = P_{K_1} J P_{K_1}$. We remark that (24) is the restriction to the invariant subspace K_1 of the operator $D^{1/2} J D^{1/2}$ acting on $l_2(\mathfrak{M}_0)$.

Although useful in some respects, equation (24) is not well suited for direct calculations with the operator $\mathcal{J}$, because no effective methods are available for obtaining a cluster expansion for the square root of the positive cluster operator D (or even for establishing that the square root has the cluster property).

It is therefore more convenient to deal directly with the cluster operator A. We have the following simple lemma.

LEMMA 1. *Let $\mathcal{L} \subset l_2(\mathfrak{M}_0)$ be an invariant subspace for the operator A on $l_2(\mathfrak{M}_0)$ such that for all $f \in \mathcal{L}$*

$$\|Af\|_{l_2(\mathfrak{M}_0)} > C \|f\|_{l_2(\mathfrak{M}_0)}, \tag{25}$$

for some constant $C > 0$. Then:

1) *the quadratic form*

$$(Df, f) = (f, f)_{\text{phys}} > \overline{C} \|f\|_{l_2(\mathfrak{M}_0)}, \qquad f \in \mathcal{L}, \tag{25^a}$$

where $\overline{C} > 0$ is a constant;

2) *the space $\mathcal{L}$ is complete with respect to the norm $\|\cdot\|_{\text{phys}}$.*

3) *The map*

$$j \colon l_2(\mathfrak{M}_0) \to l_2(\mathfrak{M}_0)/D_0 \subset \mathcal{H}_{\text{phys}}$$

takes the space $\mathcal{L}$ *into a* (*closed in* $\mathcal{H}_{\text{phys}}$) *invariant subspace* $\widehat{\mathcal{L}}$ *of the transfer matrix* $\mathcal{J}$, *and* $j\colon \mathcal{L} \to \widehat{\mathcal{L}}$ *is unitary* (*with respect to the norm* $\|\cdot\|_{\text{phys}}$ *on* $\mathcal{L}$).

4) *The operator* A *on* $\mathcal{L}$ *is unitarily equivalent to the transfer matrix* $\mathcal{J}$ *on* $\widehat{\mathcal{L}}$:

$$j(A|_{\widetilde{\mathcal{L}}})j^{-1} = \mathcal{J}|_{\widehat{\mathcal{L}}}.$$

The first statement of the lemma follows from the estimate

$$\begin{aligned}(Af, Af) = \|Df\|^2 = (D^2 f, f)\\ \leq (Df, f)^{1/2}(D^2 f, Df)^{1/2} < (Df, f)^{1/2}\|D\|^{1/2}\|Df\|.\end{aligned}$$

Hence $\|Af\|_{l_2(\mathfrak{M}_0)} = \|Df\|_{l_2(\mathfrak{M}_0)} < \|D\|^{1/2}(Df, f)^{1/2}$ and (25^a) follows from (25). The remaining assertions of the lemma are obvious.

Constructing the invariant subspaces for the transfer matrix $\mathcal{J}$ in $\mathcal{H}_{\text{phys}}$ thus reduces to constructing invariant subspaces for A in $l_2(\mathfrak{M}_0)$ satisfying condition (25).

REMARK. If the invariant subspace $\mathcal{L}$ for A in $l_2(\mathfrak{M}_0)$ constructed above does not satisfy (25), so that (26) may also be violated, then one must consider the space $\mathcal{L}_1 = \mathcal{L} \ominus (\mathcal{L} \cap D_0)$ (the orthogonal complement being taken with respect to the inner product in $l_2(\mathfrak{M}_0)$) and the operator $\widehat{D}_1 = P_{\mathcal{L}_1} D_1 P_{\mathcal{L}_1}$, defining the inner product $(\cdot, \cdot)_{\text{phys}}$ in $\mathcal{L}_1$. The space $\mathcal{L}_1$ is then unitarily equivalent to an invariant subspace $\widehat{\mathcal{L}} \subset \mathcal{H}_{\text{phys}}$ of the transfer matrix $\mathcal{J}$, and the restriction $\mathcal{J}|_{\widehat{\mathcal{L}}}$ is unitarily equivalent to the operator $\widehat{D}_1^{1/2} J \widehat{D}_1^{1/2}$ on $\mathcal{L}_1$ (cf. equation (24)). In some simple cases (for one-particle subspaces, say) the operator $\widehat{D}^{1/2}$ is not hard to compute.

THEOREM 2. *Let the coefficient* λ *in the action* $(58^a.5.2)$ *modifying the free Gibbs field* (58.5.2) *be sufficiently small, and assume that the* N*-particle separability conditions* (*see Theorem* 1.3) *are satisfied for the operator* $A = JD$ *on* $l_2(\mathfrak{M}_0)$. *Then* $l_2(\mathfrak{M}_0)$ *contains* N *invariant cluster subspaces* $\mathcal{H}_1, \ldots, \mathcal{H}_N$ *for* A, *on which the spectra of* A *are disjoint and lie in the intervals given in Theorem* 1.3 (*and in particular, they satisfy condition* (10.3)). *These subspaces are mutually orthogonal with respect to the inner product* $(\cdot, \cdot)_{\text{phys}}$.

The proof follows along the lines of the general Theorem 1.3 for a selfadjoint cluster operator. The fact that A is not selfadjoint relative to the inner product $(\cdot, \cdot)_{l_2(\mathfrak{M}_0)}$ (but is selfadjoint only with respect to the form $(Df, f)_{l_2(\mathfrak{M}_0)} = (\cdot, \cdot)_{\text{phys}}$) leads to some additional complications which we will now indicate. We first note that the N-particle separability condition can be used to construct N invariant subspaces $\mathcal{L}_1, \ldots, \mathcal{L}_N$ for A just as was done in the proof of the theorem. These spaces are constructed by using the operators $S^{(1)}, \ldots, S^{(N)}$, which map the spaces $l_2^{R_k}(\mathfrak{M}_0)$, $k = 1, \ldots, N$, respectively, into their orthogonal complements $l_2^{N \setminus R_k}(\mathfrak{M}_0)$. Using the estimates given above in Theorem 1.3 for the operators $S^{(k)}$, we can easily obtain the lower bound

$$\inf_{f \in \mathcal{L}_k, \|f\|_{l_2(\mathfrak{M}_0)} = 1} \|Af\|_{l_2(\mathfrak{M}_0)} = c_k,$$

for the norms, where c_k is equal to x_k (see (60.3)).

We now consider the adjoint $A^* = DJ$ of A. This also satisfies the N-particle separability condition and has N invariant subspaces

$$\widehat{\mathcal{L}}_1, \ldots, \widehat{\mathcal{L}}_N$$

defined by the operators $\widehat{S}^{(k)}$, $l_2^{R_k} \to (l_2^{R_k})^\perp$. We observe that the orthogonal complements $\widehat{\mathcal{L}}_k^\perp$ to these spaces are invariant under A, and $\mathcal{L}_k \cap \widehat{\mathcal{L}}_k^\perp = (0)$, the trivial element, so that their sum is direct, and we have

$$\mathcal{L}_k + \widehat{\mathcal{L}}_k^\perp = l_2(\mathfrak{M}_0). \tag{26}$$

This is easily proved by use of the norm estimates for $S^{(k)}$ and $\widehat{S}^{(k)}$. Furthermore, as in the proof of Theorem 1.3, we obtain the upper bound

$$\inf_{f \in \mathcal{L}_k^\perp, \|f\|_{l_2(\mathfrak{M}_0)}=1} \|Af\|_{l_2(\mathfrak{M}_0)} = b_k,$$

where $b_k = \overline{x}_k$, see equation (62.3).

As above, we note that

$$c_k > b_k > c_{k+1} \tag{26^a}$$

for all $k = 1, \dots, N$. From this, together with (25) and the selfadjointness of A relative to the inner product $(\cdot, \cdot)_{\text{phys}}$, we conclude that

$$\mathcal{L}_1 \subset \mathcal{L}_2 \subset \cdots \subset \mathcal{L}_N$$

and consequently,

$$\widehat{\mathcal{L}}_1^\perp \supset \widehat{\mathcal{L}}_2^\perp \supset \cdots \supset \widehat{\mathcal{L}}_N^\perp.$$

We now define

$$\mathcal{H}_k = \mathcal{L}_k \cap \widehat{\mathcal{L}}_{k-1}^\perp.$$

Clearly, $\mathcal{H}_k$ is an invariant subspace and the spectrum of $A|_{\mathcal{H}_k}$ is contained in the intervals (8.3). Since by (26) the subspaces $\mathcal{L}_k$ and $\mathcal{L}_k^\perp$ are orthogonal with respect to the inner product $(\cdot, \cdot)$ for all k, the same is true of the subspaces $\mathcal{H}_1, \dots, \mathcal{H}_N$. We must now prove that $A|_{\mathcal{H}_k}$ is a cluster operator, i.e., find in each $\mathcal{H}_k$ a basis $\{\widehat{h}_\Gamma,\ \Gamma \in \mathfrak{M}_0^{(k)}\}$ (in general, nonorthogonal) such that the matrix elements $b_{\Gamma,\Gamma'}$ of $A|_{\mathcal{H}_k}$ in this basis:

$$A\widehat{h}_\Gamma = \sum_{\Gamma} b_{\Gamma,\Gamma'} \widehat{h}_{\Gamma'} \tag{27}$$

admit a cluster expansion (4.2) and, in addition,

$$U_v \widehat{h}_\Gamma = \widehat{h}_{\Gamma+v}, \qquad v \in Z^\nu.$$

We note that the orthogonal projection $P_{\mathcal{H}_k}$ onto the subspace $\mathcal{H}_k$ of $l_2(\mathfrak{M}_0)$ is constructed using the cluster operators $S^{(k)}$ and $\widehat{S}^{(k)}$ in exactly the same way as in the proof of Theorem 1.3, and is of the form

$$P_{\mathcal{H}_k} = P_{l_2^{(k)}} + D^{(k)}, \tag{28}$$

where $D^{(k)}$ is a cluster operator of small norm and $P_{l_2^{(k)}}$ is the projection onto the subspace $l_2^{(k)}$. Consider the basis $h_\Gamma = P_{\mathcal{H}_k} e_\Gamma$ in $\mathcal{H}_k$, where $e_\Gamma(\Gamma') = \delta_{\Gamma,\Gamma'}$ is the standard basis in $l^{(k)}$. The Gramm matrix relating the two bases is given by

$$B^{(k)}_{\Gamma,\Gamma'} = (e_\Gamma, h_{\Gamma'}) = \delta_{\Gamma,\Gamma'} + D^{(k)}_{\Gamma,\Gamma'}, \tag{28^a}$$

where the $D^{(k)}_{\Gamma,\Gamma'}$ are the matrix elements for the cluster operator $D^{(k)}$ in the basis $\{e_\Gamma\}$. Thus if we pass to the bases

$$\widehat{e}_\Gamma = \sum_{\Gamma'} S_{\Gamma,\Gamma'} e_{\Gamma'} = e_\Gamma + \sum_{\Gamma'} U_{\Gamma,\Gamma'} e_{\Gamma'}$$

and

$$\widehat{h}_\Gamma = \sum_{\Gamma'} S_{\Gamma,\Gamma'} h_{\Gamma'} = h_\Gamma + \sum_{\Gamma'} U_{\Gamma,\Gamma'} h_{\Gamma'}, \tag{29}$$

where $S_{\Gamma,\Gamma'} = (E + D^{(k)})^{1/2}_{\Gamma,\Gamma'} = \delta_{\Gamma,\Gamma'} + U_{\Gamma,\Gamma'}$, and $\{U_{\Gamma,\Gamma'}\}$ is a cluster operator with small norm, these bases will be biorthogonal:

$$(\widehat{e}_\Gamma, \widehat{h}_{\Gamma'}) = \delta_{\Gamma,\Gamma'}, \Gamma, \Gamma' \in \mathfrak{M}^{(k)}_0. \tag{30}$$

We conclude from (27) and (30) that the matrix elements of $A|_{\mathcal{H}_k}$ with respect to the basis $\{\widehat{h}_\Gamma\}$ are $b_{\Gamma,\Gamma'} = (A\widehat{h}_\Gamma, \widehat{e}_{\Gamma'})$. ¿From this and equations (28), (28^a), (29) we find that the $b_{\Gamma,\Gamma'}$ admit the cluster expansion (4.2). This proves the theorem.

As an application of the results established in this and the preceding subsections, we consider the lattice model for quantum Euclidean electrodynamics on $\widetilde{Z}^{3+1}$, which was introduced in 5.2 and describes a Dirac fermion field interacting with a gauge field (with gauge group $U(1)$). Recall that this model contains three parameters: the fermion mass m (which we take equal to 1); the interaction parameter $\kappa > 0$ for the interaction of the fermion with the gauge field; and the parameter β (the coefficient in the Wilson action for the gauge field). The results described below are valid for sufficiently small κ and β.

We note that the rotations of the lattice Z^3 act differently on the invariant subspaces of the transfer matrix. We therefore consider the algebra homomorphisms $T_g\colon \mathcal{E}_{1/2} \to \mathcal{E}_{1/2}$ that correspond to the representation $g \to T_g$ of the orthogonal group $O(Z^3)$ of rotations of Z^3 (i.e., the subgroup of the rotation group $O(R^3)$ of three-dimensional space taking the lattice into itself). On the generators of the algebra $\mathcal{E}_{1/2}$ the homomorphism T_g acts by

$$T_g\psi_\alpha(x) = \sum s_{\alpha\beta}(g)\psi_\beta(g^{-1}x),$$
$$T_g\overline{\psi}_\alpha(x) = \sum \overline{s}_{\alpha\beta}(g)\overline{\psi}_\beta(g^{-1}x),$$

where $s(g) = \{s_{\alpha\beta}(g)\}$ and $g \in O(Z^3)$ is the matrix for the spinless representation of $O(Z^3)$ (see [9]),

$$T_g u(b) = u(g^{-1}b).$$

Consider for each $x \in Y_{1/2}$ the following quadratic forms in the generators:

$$\Phi_{\mathrm{sc}}(x) = \sum_\alpha \psi_\alpha(x)\overline{\psi}_\alpha(x),$$
$$\Phi_{\mathrm{psc}}(x) = \sum_{\alpha,\beta} \psi_\alpha(x)(\gamma_5)_{\alpha,\beta}\overline{\psi}_\beta(x),$$
$$\Phi^\mu_{\mathrm{vect}}(x) = \sum_{\alpha,\beta} \psi_\alpha(x)(\gamma_\mu)_{\alpha,\beta}\overline{\psi}_\beta(x), \qquad \mu = 1,2,3,$$
$$\Phi^\mu_{\mathrm{pvect}}(x) = \sum_{\alpha,\beta} \psi_\alpha(x)(\gamma_\mu\gamma_5)_{\alpha,\beta}\overline{\psi}_\beta(x), \qquad \mu = 1,2,3,$$

where $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$.

These formulas are gauge-invariant, and under translations $x \to x + s$, $s \in Z^\nu$, $x \in Y_{1/2}$, we have

$$\Phi_{\mathrm{sc}}(x) \to \Phi_{\mathrm{sc}}(x + s)$$

and similarly for the other forms. Under the action T_g of the rotation group on $\mathcal{E}_{1/2}$, they transform as scalar, pseudoscalar, vector, and pseudovector fields, respectively:

$$\begin{aligned} T_g\Phi_{\mathrm{sc}}(x) &= \Phi_{\mathrm{sc}}(g^{-1}x), \\ T_g\Phi_{\mathrm{psc}}(x) &= \det g\Phi_{\mathrm{psc}}(g^{-1}x), \\ T_g\Phi^\mu_{\mathrm{vect}}(x) &= \sum_{\mu'} g_{\mu,\mu'}\Phi^{\mu'}_{\mathrm{vect}}(g^{-1}x), \\ T_g\Phi^\mu_{\mathrm{pvect}}(x) &= \det g \sum_{\mu'} g_{\mu,\mu'}\Phi^{\mu'}_{\mathrm{pvect}}(g^{-1}x). \end{aligned}$$

We now consider the linear spans of the corresponding forms:

$$\begin{aligned} \mathcal{L}_{\mathrm{sc}} &= \sum_x c_x\Phi_{\mathrm{sc}}(x), \\ \mathcal{L}_{\mathrm{vect}} &= \sum_x c_x\Phi^\mu_{\mathrm{vect}}(x) \end{aligned}$$

and so on, where almost all the coefficients c_x are zero.

The subspaces $\mathcal{L}_{\mathrm{sc}}$, $\mathcal{L}_{\mathrm{psc}}$, $\mathcal{L}_{\mathrm{vect}}$, $\mathcal{L}_{\mathrm{pvect}}$ constructed above are invariant under gauge transformations, translations, and rotations. One finds that for sufficiently small $\kappa > 0$ and $\beta > 0$ such that $\kappa > c|\beta|^{1/2}$ ($c > 0$ some absolute constant), there exist two highest $\mathcal{J}$-invariant subspaces $\mathcal{H}_{\mathrm{psc}}$ and $\mathcal{H}_{\mathrm{vect}}$ close to the spaces $\mathcal{L}_{\mathrm{psc}}$ and $\mathcal{L}_{\mathrm{vect}}$, respectively. On these subspaces, the spectrum of the transfer matrix $\mathcal{J}$ is $\sim \kappa^2$. More precisely, we can choose orthonormal bases $\{h^{\mathrm{psc}}_x,\ x \in Y_{1/2}\}$ and $\{h^{\mathrm{vect}}_{x,\mu},\ x \in Y_{1/2},\ \mu = 1,2,3\}$ in $\mathcal{H}_{\mathrm{psc}}$ and $\mathcal{H}_{\mathrm{vect}}$ which transform under translations and rotations in the same way as $\{\Psi_{\mathrm{psc}}(x)\}$ and $\{\Phi^\mu_{\mathrm{vect}}(x)\}$, and on which the transfer matrix acts by the formulas

$$\begin{aligned} \mathcal{J}h^{\mathrm{psc}}_x &= \sum_y a^{\mathrm{psc}}(x - y)h^{\mathrm{psc}}_y, \\ \mathcal{J}h^{\mathrm{vect}}_{x,\mu} &= \sum_y a^{\mathrm{vect}}(x - y)h^{\mathrm{vect}}_{y,\mu}, \end{aligned}$$

where

$$\begin{aligned} a^{\mathrm{psc}}(\xi) &= 4\kappa^2\delta_{\xi,0} + O(\kappa^4), \\ a^{\mathrm{vect}}(\xi) &= 4\kappa^2\delta_{\xi,0} + O(\kappa^4). \end{aligned}$$

The spaces $\mathcal{H}_{\mathrm{psc}}$ and $\mathcal{H}_{\mathrm{vect}}$ describe pseudoscalar and vector "meson" states in our model. On the other hand, the quadratic form $\langle F\Theta F\rangle$, $F \in \mathcal{L}_{\mathrm{sc,pvect}}$, is seen to be of order $\kappa^2\beta$ when restricted to the spaces $\mathcal{L}_{\mathrm{sc}}$ and $\mathcal{L}_{\mathrm{pvect}}$, and it is therefore more difficult to construct the corresponding invariant subspaces (scalar and pseudoscalar "mesons") for them. This problem remains unsolved.

§6. Continuous-time models

Here we will consider Gibbs fields defined on the "space-time" $Z^\nu \times R^1$, where Z^ν is the space and R^1 the continuous time; the fields will take values in some finite set S. Let an ergodic Markov process $\eta_0 = \{\eta_0(t),\ t \in R^1\}$ be defined with values in S and let the measure ν_0 be stationary. The "free field" on $Z^\nu \times R^1$ is defined as the set of identical copies $\{\eta_x = \{\eta_x(t),\ t \in R^1\},\ x \in Z^\nu\}$ of the process η_0, one for each point x in Z^ν. We assume as above that a finite number of "field variables" $\{\varphi_i, i = 1, \dots, k\}$ are defined on S, and that we have an orthonormal basis $\{\varphi_\gamma,\ \gamma \in N\}$ in $l_2(s, \nu_0)$ consisting of polynomials in the φ_i (including the functions φ_i themselves) which satisfies condition (1.1).

We will assume that the infinitesimal matrix $a(s, s')$ for the transition probabilities for the process η_0, defined by the relations

$$\begin{aligned} \Pr(s \to s' \ \text{ during time } \Delta t) &= a(s, s')\Delta t + o(\Delta t), \qquad s \neq s', \\ \Pr(s \to s \ \text{ during time } \Delta t) &= 1 - a(s, s)\Delta t + o(\Delta t), \end{aligned} \tag{1}$$

where

$$a(s, s) = -\sum_{s \neq s'} a(s, s')$$

is symmetric, so that the process $\eta(t)$ and hence also the free field $\{\eta_x(t)\}$ are invariant under time-reversal.

For any finite set $\Lambda \subset Z^\nu$ and time interval $T \subset R^1$, we define the action

$$S_{\Lambda,T} = \beta \sum_{\substack{i,j \\ x,y \in \Lambda \\ |x-y|=1}} \int_T \Phi^{i,j}_{x,y} \varphi_i(\eta_x(t))\varphi_j(\eta_y(t))\, dt \tag{2}$$

which we use to get a finite Gibbs modification $\mu_{\Lambda,T}$ of the measure μ_0 (the "free field" distribution).

As was shown in [26], for β small enough this measure admits a cluster expansion. Specifically, the expansion reduces to the study of a discrete Gibbs field on the lattice $Z^\nu \times Z^1_a$, where Z^1_a is the one-dimensional lattice with spacing $a = |\ln \beta|$, and the cluster expansion for this auxiliary lattice field is used to get a corresponding expansion for the original field (the small cluster parameter λ in this expansion is given by $\lambda = C\beta|\ln\beta|$ (see [26]), where $C > 0$ is an absolute constant). The resulting cluster expansion implies, in particular, the existence of the thermodynamic limit

$$\mu = \lim_{\Lambda \uparrow Z^\nu} \mu_{\Lambda,T}$$

of the measures $\mu_{\Lambda,T}$. This limit field is a Markov process with state space S^{Z^ν}. Writing Ω for the space of field configurations $\eta = \{\eta_x(t),\ (x,t) \in Z^\nu \times R^1\}$, let $\mathcal{H}_{\text{phys}} \subset L_2(\Omega, \mu)$ be the subspace of functionals depending only on the values $\eta_0 = \{\eta_x(t = 0),\ x \in Z^\nu\}$ of the configuration at zero time. As above, one gets a contraction semigroup of stochastic operators $\mathcal{J}_t$, $t > 0$, which are selfadjoint by virtue of the symmetry of the matrix $a(s, s')$ and the form (2) of the action. Representing this semigroup in the form

$$\mathcal{J}_t = \exp\{-tH\} \tag{3}$$

we get a nonnegative selfadjoint operator $H \geq 0$, which is the Hamiltonian of our field. As above, for the case of continuous time we can also construct a multiplicative orthonormal basis $\{\Psi_\Gamma\}$ in $\mathcal{H}_{\text{phys}}$ of the form

$$\Psi_\Gamma = \prod_x \psi^x_{\gamma(x)}, \qquad \Gamma = \{\gamma(x)\},$$

where the quasilocal functionals $\psi^x_{\gamma(x)}$ admit an expansion of the form (12.1). The matrix elements $a^t_{\Gamma,\Gamma'} = (\mathcal{J}_t\Psi_\Gamma, \Psi_{\Gamma'})$ thus have a multiplicative cluster expansion of the form (34.1):

$$a^t_{\Gamma,\Gamma'} = \sum_{(\{\Gamma,\Gamma'\})} \prod_i \omega^t(\Gamma_i, \Gamma'_i) \tag{4}$$

and the semi-invariants ω^t satisfy the cluster bound

$$|\omega^t(\Gamma,\Gamma')| < (C\lambda)^{d_{\operatorname{supp}\Gamma\cup\operatorname{supp}(\Gamma+t)}},$$

where d_B, $B \subset Z^\nu \times R^1$ is the length of the smallest connected graph with vertices lying in the finite set $B \subset Z^\nu \times R^1$. Here the metric on $Z^\nu \times R^1$ is defined by the formula

$$\rho(x,t)(x',t') = \rho_{Z^\nu}(x,x') + \left[\frac{|t-t'|}{a}\right],$$

where $[t]$ is the largest integer $\leq t$.

We thus see that for $t = a$ the operator $\mathcal{J}_a$ is a selfadjoint cluster operator, and if the k-particle separability conditions are fulfilled, then $\mathcal{J}_a$ has k invariant spaces $\mathcal{H}_1, \ldots, \mathcal{H}_k$, as described in Theorem 1.3. These subspaces are also invariant under the Hamiltonian H.

Moreover, the cluster expansion (4), valid for all $t > 0$, implies that the matrix elements $b_{\Gamma,\Gamma'}$ of H in the basis $\{\Psi_\Gamma\}$, which are equal to

$$b_{\Gamma,\Gamma'} = \frac{d}{dt} a^t_{\Gamma,\Gamma'}\bigg|_{t=0}, \tag{5}$$

have the form

$$b_{\Gamma,\Gamma'} = \sum_{\substack{(\overline{\Gamma},\overline{\Gamma'})\subset(\Gamma,\Gamma') \\ \Gamma\setminus\overline{\Gamma}=\Gamma'\setminus\overline{\Gamma'}}} W(\overline{\Gamma},\overline{\Gamma'}), \tag{6}$$

where

$$W(\overline{\Gamma},\overline{\Gamma'}) = \frac{d}{dt}\omega^t(\overline{\Gamma},\overline{\Gamma'})\bigg|_{t=0}, \tag{7}$$

and the sum is over all pairs of multi-indices $(\overline{\Gamma},\overline{\Gamma}')$ subordinate to the pair (Γ,Γ'), and such that the restrictions

$$\Gamma|_{\operatorname{supp}\Gamma\setminus\operatorname{supp}\overline{\Gamma}} \equiv \Gamma\setminus\overline{\Gamma}, \qquad \Gamma'|_{\operatorname{supp}\Gamma'\setminus\operatorname{supp}\overline{\Gamma'}} \equiv \Gamma'\setminus\overline{\Gamma'}$$

coincide. The formula is easily derived from the representation (4) by observing that $a_{\Gamma,\Gamma'} \to \delta_{\Gamma,\Gamma'}$ as $t \to 0$.

As an example, we consider the *quantum Ising model*. Here $S = \{1, -1\}$ and the transition probability matrix defining the unperturbed process is given by

$$a_{s,s'} = 1, \qquad s \neq s', \qquad a_{s,s} = -1.$$

The invariant distribution ν_0 on S is given by $\nu_0(1) = \nu_0(-1) = 1/2$. The action $S_{\Lambda,T}$ is

$$S_{\Lambda,T} = \beta \sum_{\substack{x,y\in\Lambda \\ |x-y|=1}} \int s_x(t)s_y(t)\,dt.$$

If β is small enough, the N-particle separability condition holds for the transfer matrix $\mathcal{J}_t$ of the limit field in this model (for $t \geq a = |\ln\beta|$), and hence it also holds for $\mathcal{J}_a$. Thus, there exist k-particle cluster invariant subspaces $\mathcal{H}_1, \dots, \mathcal{H}_N$ for the entire semigroup $\mathcal{J}_t$.

Now the operator $H|_{\mathcal{H}_1} = H_1$ is given by convolution

$$H_1 h_x = \sum_y a_{x-y} h_y \tag{8}$$

with respect to a suitable orthonormal basis $\{h_x, x \in Z^\nu\}$, and $|a_\xi| < (C\lambda)^{|\xi|}$, $\xi \in Z^\nu$. In the two-particle subspace $\mathcal{H}_2$ we can choose a basis $\{h_{x,y},\ x, y \in Z^\nu\}$ in terms of which $H_2 = H|_{\mathcal{H}_2}$ has the form

$$H_2 h_{x,y} = \sum_{x'\in Z^\nu} a_{x-x'} h_{x',y} + \sum_{y'\in Z^\nu} a_{y-y'} h_{x,y'} + \sum_{x',y'\in Z^\nu} B_{x,y,x',y'} h_{x',y'}, \tag{9}$$

where the function a_ξ is the same as in (8), and the kernel $B_{x,y,x',y'}$ satisfies the estimate

$$|B_{x,y,x',y'}| < (C\lambda)^{d_{\{x,y;x',y'\}}}$$

(where both pairs (x, y) and (x', y') lie in the zero time slice).

The proof that the functions a_ξ in equations (8) and (9) coincide is similar to the proof of the multiplicative cluster property for the operator $\mathcal{J}$ in the discrete-time Ising model.

§7. Spectral analysis of k-particle cluster operators

For a more detailed analysis of the spectrum of k-particle cluster operators ($k < \infty$), it is helpful to use the Fourier transform in the space $l_2^{(k)}$. For this purpose we introduce the following class of operators, which will be called *cluster operators* in the p-representation. Let $L_2(T^\nu, dp, \mathcal{N}) = L_2(T^\nu, dp) \times l_2(\mathcal{N}) = L_2$ be the Hilbert space of functions $f(p, \gamma)$ defined on the Cartesian product $T^\nu \times \mathcal{N}$, where T^ν is the ν-dimensional torus, dp is the normalized Haar measure on T^ν, and $\mathcal{N}$ is a finite or countable set. Let $\mathcal{F}_s(L_2)$ be the Fock space on L_2, i.e., the space of sequences of functions

$$F = \{f_0, f_1(p, \gamma), f_2((p_1, \gamma_1), (p_2, \gamma_2)), \dots, f_n((p_1, \gamma_1), \dots, (p_n, \gamma_n)) \dots\}, \tag{1}$$

symmetric under all permutations of the pairs $(p_i, \gamma_i) \in N^\nu \times \mathcal{N}$.

We write $\widetilde{\mathcal{F}} \subset \mathcal{F}_s(L_2)$ for the subspace of sequences (1) whose members satisfy the condition

$$\int f_n((p_1, \gamma_1), \dots, (p_n, \gamma_n))h(p_i + p_j)dp_i dp_j \equiv 0 \tag{2}$$

for all $n, i, j = 1, \dots, n$, $i \neq j$ and all functions $h \in L_2(T^\nu)$.

Let $\widetilde{\mathcal{F}}_k \subset \widetilde{\mathcal{F}}$ be the subspace of sequences (1) for which $f_n = 0$ for all $n \neq k$. Clearly,

$$\widetilde{\mathcal{F}} = \bigoplus_{k=0}^{\infty} \widetilde{\mathcal{F}}_k \tag{3}$$

and every operator A on $\widetilde{\mathcal{F}}$ can be expressed as an operator matrix $A = \{A_{m,n}\}$, where $A_{m,n} \colon \widetilde{\mathcal{F}}_n \to \widetilde{\mathcal{F}}_m$.

Definition of a cluster operator in the p-representation. Consider an operator A on $\mathcal{F}$ such that each block $A_{m,n}$ is given by a (distribution) kernel

$$K_{m,n}((p_1, \gamma_1), \dots, (p_m, \gamma_m); (p'_1, \gamma'_1), \dots, (p'_n, \gamma'_n)) = \sum_{\varepsilon} a_\varepsilon \delta_\varepsilon, \tag{4}$$

where the sum is over all partitions of the pair of sets $M = \{1, \dots, m\}$, $M' = \{1, \dots, n\}$ into pairs $\{(\beta_1, \beta'_1), \dots, (\beta_k, \beta'_k)\}$ of nonempty subsets $\beta_i \subseteq M$, $\beta'_i \subseteq M'$, and

$$\delta_\varepsilon = \prod_i \delta(P_{\beta_i} - P'_{\beta'_i}).$$

Here, given $(\beta_i, \beta'_i) \subseteq (M, M')$ and any two sets $(p_1, \dots, p_m)$, $(p'_1, \dots, p'_n)$ we have written

$$P_\beta = \sum_{k \in \beta} p_k, \qquad P'_{\beta'} = \sum_{k \in \beta'} p'_k, \tag{5}$$

and $\delta(\cdot)$ is the usual δ-function on the torus T^ν. Moreover, a_ε in (4) is a smooth (analytic) function defined on $\Gamma_\varepsilon \times N^{m+n}$, where Γ_ε is the submanifold of $(T^\nu)^{m+n}$ defined by

$$\Gamma_\varepsilon = \{(p_1, \dots, p_m); (p'_1, \dots, p'_n) : p_{\beta_i} = p'_{\beta'_i}, \quad i = 1, \dots, k\}. \tag{6}$$

The kernels $K_{m,n}$ in (4) are assumed to be symmetric both in the variables $\{p_i, \gamma_i\}$ and in $\{p'_i, \gamma'_i\}$.

As previously, let $\mathfrak{M}_0$ be the space of multi-indices $\Gamma = \{\gamma(x)\}$. By convention we write each multi-index Γ as an unordered sequence of pairs $\{x_i, \gamma(x_i)\}$ for some enumeration of the points in $\operatorname{supp} \Gamma$.

The space $l_2(\mathfrak{M}_0)$ can thus be identified with the subspace of the Fock space $\mathcal{F}_s(l_2(Z^\nu \times \mathcal{N}))$ consisting of the sequences

$$\Psi = \{\psi_0, \psi_1((x_1, \gamma_1)), \dots, \psi_n((x_1, \gamma_1), \dots, (x_n, \gamma_n)), \dots\} \in \mathcal{F}_s(l_2(Z^\nu \times \mathcal{N})), \tag{7}$$

for which

$$\psi_n((x_1, \gamma_1), \dots, (x_n, \gamma_n)) = 0,$$

whenever $x_i = x_j$ for some $i \neq j$. Evidently, the Fourier transform

$$\Psi \to \{f_0, f_1(p_1, \gamma_1), \dots, f_n((x_1, \gamma_1), \dots, (x_n, \gamma_n)), \dots\},$$

where

$$\begin{aligned} &f_n((p_1, \gamma_1), \dots, (p_n, \gamma_n)) \\ &\quad = \sum_{\{x_1, \dots, x_n\}} \prod_{k=1}^{n} \exp\{i(p_k, x_k)\} \psi_n((x_1, \gamma_1), \dots, (x_n, \gamma_n)) \end{aligned} \tag{8}$$

takes $\mathcal{F}_s(l_2(Z^\nu \times \mathcal{N}))$ into $\mathcal{F}_s(l_2(T^\nu \times \mathcal{N}))$, and $l_2(\mathfrak{M}_0)$ goes into the space $\widetilde{\mathcal{F}}$.

Under Fourier transformation, the translation group $\{U_s, s \in Z^\nu\}$ acting on $l_2(\mathfrak{M}_0)$ by equation (31^a.1) goes into a group acting on $\widetilde{\mathcal{F}}$ by the formula

$$\begin{aligned} &(U_s F)_n((p_1, \gamma_1), \dots, (p_n, \gamma_n)) \\ &\qquad = \exp\{i(s, p_1 + \dots + p_n)\} f_n((p_1, \gamma_1), \dots, (p_n, \gamma_n)). \end{aligned} \tag{9}$$

LEMMA 1. *Let A be a cluster operator on $l_2(\mathfrak{M}_0)$ with cluster parameter λ. Then under Fourier transformation* (8) *A becomes a cluster operator in the p-representation. The functions a_ε in* (4) *are analytic in a complex neighborhood of the manifold Γ_ε and are bounded there by*

$$|a_\varepsilon((p_1,\gamma_1),\dots,(p_m,\gamma_m));((p'_1,\gamma'_1),\dots,(p'_n,\gamma'_n))| < \prod_{l=1}^{k}\prod_{i\in\beta_l}\lambda^{N(\gamma_i)/2}\prod_{i\in\beta'_l}\lambda^{N(\gamma'_i)/2}, \tag{10}$$

where $\varepsilon=\{(\beta_1,\beta'_1),\dots,(\beta_k,\beta'_k)\}$.

PROOF. We have

$$\begin{aligned} &K_{n,m}((p_1,\gamma_1),\dots,(p_n,\gamma_n));((p'_1,\gamma'_1),\dots,(p'_m,\gamma'_m)) \\ &\quad= \sum_{\substack{(x_1,\dots,x_n)\\(x'_1,\dots,x'_m)}}{}' a((x_1,\gamma_1),\dots,(x_n,\gamma_n));((x'_1,\gamma'_1),\dots,(x'_m,\gamma'_m)) \\ &\quad\quad\times\prod_{j=1}^{n}\exp\{i(x_j,p_j)\}\prod_{j=1}^{m}\exp\{-i(x'_{j'},p'_{j'})\}, \end{aligned} \tag{11}$$

where $\sum'$ indicates a summation over sets $(x_1,\dots,x_n)$, $(x'_1,\dots,x'_m)$ with pairwise distinct elements in each set. We remark that the function $K_{n,m}$ is separately symmetric in the pairs $\{(p_i,\gamma_i)\}$ and $\{(p'_i,\gamma'_i)\}$. Since in (11) we sum over $x_1,\dots,x_n,x'_1,\dots,x'_m$ for fixed $\gamma_1,\dots,\gamma_n$, $\gamma'_1,\dots,\gamma'_m$, in what follows we will suppress these variables so as not to burden the notation. Using the cluster expansion (4.2) for the function a, we can rewrite (11) as

$$\sum_k\sum_{\varepsilon=\{(\beta_l,\beta'_l),l=1,\dots,k\}}\sum_{\substack{(x_1,\dots,x_n)\\(x'_1,\dots,x'_n)}}{}'\omega(\{x_j,j\in\beta_l\},\{x'_{j'},j'\in\beta'_l\},l=1,\dots,k) \times\prod_{l=1}^{k}\prod_{j\in\beta_l}\exp\{i(x_j,p_j)\}\prod_{j'\in\beta'_l}\exp\{-i(x'_{j'},p'_{j'})\}. \tag{11^a}$$

Further, for a fixed partition ε we can write

$$\sum_{\substack{(x_1,\dots,x_n)\\(x'_1,\dots,x'_m)}}{}' = \sum_{\substack{\varepsilon\\(x_1,\dots,x_n)\\(x'_1,\dots,x'_n)}}{}' - \sum_{\substack{\varepsilon\\(x_1,\dots,x_n)\\(x'_1,\dots,x'_n)}}{}''$$

where $\sum'$ denotes a sum over sets $(x_1,\dots,x_n)$, $(x'_1,\dots,x'_n)$ in which elements with indices in the same block β_l (or β'_l), $l=1,\dots,k$, are distinct; $\sum''$ denotes a sum over pairs of sets $(x_1,\dots,x_n)$, $(x'_1,\dots,x'_m)$ with the same property, but at least one of which contains two or more elements that coincide but have indices in different blocks β_{l_1} and β_{l_2} (or β'_{l_1} and β'_{l_2}), $l_1\neq l_2$. Recall that the cluster function ω is still defined for such sets, because the clusters are independent (property (5.2)).

Now consider the first sum $\sum'_\varepsilon$ and pick the smallest element s_l in each block β_l. We note further that (again because the clusters are independent) this sum can be rewritten in

the form

$$
\begin{aligned}
&\sum\nolimits'_{\varepsilon}\omega(\{x_j - x_{s_l}, j\in\beta_l\},\{x'_{j'} - x_{s_l}, j'\in\beta'_l\}, l=1,\dots,k)\\
&\quad\times\prod_{l=1}^{k}\Bigg(\prod_{j\in\beta_l}\exp\{i(x_j - x_{s_l}, p_j)\}\\
&\qquad\qquad\times\prod_{j'\in\beta'_l}\exp\{-i(x'_{j'} - x_{s_l}, p'_{j'})\}\exp\{i(x_{s_l},(P_{\beta_l} - P_{\beta'_l}))\}\Bigg).
\end{aligned}
\tag{12}
$$

Having fixed the x_{s_1}, we sum over the variables $y_j = x_j - x_s$, $y'_{j'} = x'_{j'} - x_{s_1}$. By the cluster estimate (6.2), the sum (12) converges and represents an analytic function of the variables $\{p_j\}_{j=1,\dots,n}$, $\{p'_{j'}\}_{j'=1,\dots,m}$ and satisfies the bound (10). Summing over the x_{s_l} leads to additional factors $\prod_{l=1}^{k}\delta(P_{\beta_l} - P_{\beta'_l})$.

Turning now to the sum $\sum''_{\varepsilon}$, we observe that to each pair $\{x_j\}_{j=1,\dots,n}$, $\{x'_{j'}\}_{j'=1,\dots,m}$ appearing in the sum we may associate the pair $(y_1,\dots,y'_n)$, $(y_1,\dots,y'_m)$ consisting of the distinct elements in $\{x_j\}$ and $\{x'_{j'}\}$. To each y_j, $j=1,\dots,n'$ (and consequently to each $y'_{j'}$, $j'=1,\dots,m'$) we associate the set $\alpha_j\subseteq[1,\dots,n]$ (or $\alpha'_j\subseteq[1,\dots,m]$) such that $x_i = y_j$ if and only if $i\in\alpha_j$ (similarly, $x'_{i'} = y'_{j'}$ if and only if $i'\in\alpha'_{j'}$). We now associate to the two sets $\{y_j\}_{j=1,\dots,n'}$ and $\{y'_{j'}\}_{j'=1,\dots,m'}$ the partition $\widetilde{\varepsilon} = \{(\widetilde{\beta}_1,\widetilde{\beta}'_1),\dots,(\widetilde{\beta}_p,\widetilde{\beta}'_{p'})\}$ of the pair of sets $(1,\dots,n')$, $(1,\dots,m')$ into pairs of subsets $(\widetilde{\beta}_i,\widetilde{\beta}'_i)$ which is obtained by "gluing" the blocks of the original partition along the "intersecting blocks". More precisely, this means the following. From the blocks (β_l,β'_l) of the partition we form a graph in which edges join those pairs of blocks $(\beta_{l_1},\beta'_{l_1})$ and $(\beta_{l_2},\beta'_{l_2})$ for which either β_{l_1} and β_{l_2} both intersect the same set α_j, or else β'_{l_1} and β'_{l_2} both intersect one of the $\alpha'_{j'}$. We form the new partition of the pair of sets $(1,\dots,n)$, $(1,\dots,m)$ by combining into one block all the blocks in the original partition lying in the same connected component of our graph. Going now from the sets $\{x_j\}_{j=1,\dots,n}$ and $\{x'_{j'}\}_{j'=1,\dots,m}$ to the sets $\{y_j\}_{j=1,\dots,n'}$ and $\{y'_{j'}\}_{j'=1,\dots,m'}$, we get the desired partition. The number of blocks in $\widetilde{\varepsilon}$ is clearly less than k. We can thus write the sum $\sum''_{\varepsilon}$ in the form

$$
\begin{aligned}
&\sum_{n',m'}\sum_{\substack{(\alpha_1,\dots,\alpha'_n)\\(\alpha'_1,\dots,\alpha'_{m'})}}\sum_{p<k}\sum_{\substack{\widetilde{\varepsilon}=(\widetilde{\beta}_i,\widetilde{\beta}'_i)\\ i=1,\dots,p}}\\
&\quad\times\sum_{\substack{y_1,\dots,y_{n'}\\ y'_1,\dots,y'_{m'}}}\widetilde{\omega}(\{y_j, j\in\widetilde{\beta}_s\},\{y'_{j'}, j'\in\widetilde{\beta'}_s\}, s=1,\dots,p)\\
&\quad\times\prod_{s=1}^{p}\prod_{j\in\widetilde{\beta}_s}\exp\Bigg\{i\Bigg(y_j,\sum_{t\in\alpha_j}p_t\Bigg)\Bigg\}\prod_{j'\in\widetilde{\beta}'_s}\exp\Bigg\{-i\Bigg(y'_{j'},\sum_{t'\in\alpha'_{j'}}p'_{t'}\Bigg)\Bigg\}.
\end{aligned}
\tag{12^a}
$$

The function $\widetilde{\omega}$ of the sets of clusters $\{y_j,\ j\in\widetilde{\beta}_s\}$, $\{y'_{j'},\ j'\in\widetilde{\beta}'_s\}$, $s=1,\dots,p$, satisfy both the conditions (5.2) and (6.2) in the definition of cluster function.

For each partition in the sum (12^a) we must now repeat the preceding arguments and again separate the terms in (12^a), which gives rise to a product of p δ-functions in the expression for $K_{m,n}$. The remainder will again be a sum of the form (12^a), and the partitions involved in it contain fewer than p blocks. Repeating this procedure several times, we arrive at the expression (4). This proves the lemma.

Note that in the p-representation, a k-particle cluster operator on $l_2(\mathfrak{M}_0^{(k)})$ becomes an operator $\widetilde{\mathcal{F}}$ for which the kernels

$$K_{m,n}((p_1,\gamma_1),\dots,(p_m,\gamma_m);(p'_1,\gamma'_1),\dots,(p'_m,\gamma'_m))$$

are nonzero only when

$$\sum_{i=1}^{m} N(\gamma_i) = \sum_{j=1}^{n} N(\gamma'_j) = k$$

(and thus, $m,n \le k$).

We consider here the spectrum for a one- or two-particle cluster operator in more detail.

Case $k=1$. The Fourier transform takes the one-particle operator A acting on $l_2(\mathfrak{M}_0^{(1)})$ into an operator $\widetilde{A}$ on the space $L_2(T^\nu \times \mathcal{N}^{(1)})$, acting by the formula

$$(\widetilde{A}f)(p,\gamma) = \sum_{\gamma'\in\mathcal{N}^{(1)}} B_{\gamma,\gamma'}(p) f(p,\gamma'), \tag{13}$$

where $B(p) = \{B_{\gamma,\gamma'}(p),\ \gamma,\gamma' \in \mathcal{N}^{(1)}\}$ is a family of selfadjoint $r\times r$ matrices, $r = |\mathcal{N}^{(1)}|$ ($\mathcal{N}^{(1)} \subseteq \mathcal{N}$ is the set of indices γ for which $\mathcal{N}(\gamma) = k$). Let

$$\varepsilon_1(p) \ge \varepsilon_2(p) \ge \cdots \ge \varepsilon_r(p)$$

be the eigenvalues of the matrix $B(p)$. Clearly, the spectrum of A coincides with the union of the ranges of the functions ε_s:

$$\sigma(A) = \bigcup_{s=1}^{r} \operatorname{Im}\varepsilon_s(p).$$

If A is the one-particle part of the transfer matrix for a Gibbs field, then its eigenvectors (corresponding to the eigenvectors $e_s(p)$ of $B(p)$) can be interpreted physically as states of elementary particles (or "elementary excitations") of a field with quasimomentum $p \in T^\nu$. The energy of these particles is equal to $-\ln|\varepsilon_s(p)|$.

Case $k=2$. Here a cluster operator A on $l_2(\mathfrak{M}_0)$ goes into an operator acting on the L_2 space of pairs of functions

$$\{f_1(p,\gamma), f_2((p_1,\gamma_1),(p_2,\gamma_2))\} = F, \qquad \gamma\in\mathcal{N}^{(2)}, \quad \gamma_1,\gamma_2\in\mathcal{N}^{(1)}. \tag{14}$$

The function $f_2((p_1,\gamma_1),(p_2,\gamma_2))$ is symmetric in the pairs of variables (p_1,γ_1) and (p_2,γ_2) and satisfies the general condition (2),

$$\int f_2((p_1,\gamma_1),(p_2,\gamma_2))h(p_1+p_2)dp_1dp_2 = 0$$

for every function $h\in L_2(T^\nu)$. The action of $\widetilde{A}$ on F is given by the formula

$$\begin{aligned}
(\widetilde{A}F)_1(p,\gamma) &= \sum_{\gamma'} a^{(1,1)}_{\gamma,\gamma'}(p) f_1(p,\gamma') \\
&\quad + \sum_{\gamma'_1,\gamma'_2} \int\limits_{p'_1+p'_2=p} a^{(2,2)}_{\gamma;\gamma'_1,\gamma'_2}(p;p'_1,p'_2) f_2((p'_1,\gamma'_1),(p'_2,\gamma'_2))\, dp'_1\, dp'_2, \\
(\widetilde{A}F)_2((p_1,\gamma_1),(p_2,\gamma_2)) &= \sum_{\gamma'} \overline{a}^{(1,2)}_{\gamma';\gamma_1,\gamma_2}(p_1+p_2;p_1,p_2) f_1(p_1+p_2,\gamma') \\
&\quad + \sum_{\gamma'_1,\gamma'_2} a^{(2,2)}_{\gamma_1,\gamma_2;\gamma'_1,\gamma'_2}(p_1,p_2) f_2((p_1,\gamma'_1),(p_2,\gamma'_2)) \\
&\quad + \sum_{\gamma'_1,\gamma'_2} \int\limits_{p'_1+p'_2=p_1+p_2} b^{(2,2)}_{\gamma_1,\gamma_2;\gamma'_1,\gamma'_2}(p_1,p_2;p'_1,p'_2) f_2((p'_1,\gamma'_1),(p'_2,\gamma'_2))\, dp'_1\, dp'_2.
\end{aligned} \tag{15}$$

Switching to the variables $P = p_1 + p_2$ and writing $p = p_1$, we can express the space L_2 as a direct integral:

$$L_2 = \int_{T^\nu} L_2(P)\, dP \tag{16}$$

of the Hilbert spaces $L_2(P)$, consisting of pairs $F = \{f_1(\gamma), f_2(\gamma_1, \gamma_2, p)\}$, where for each pair $\gamma_1, \gamma_2 \in \mathfrak{M}_0^{(1)}$ we have $f_2(\gamma_1, \gamma_2, p) \in L_2(T^\nu)$, and in addition f_2 satisfies the conditions

$$\int_{T^\nu} f_2(\gamma_1, \gamma_2, p)\, dp = 0 \tag{17}$$

and

$$f_2(\gamma_1, \gamma_2, p) = f_2(\gamma_2, \gamma_1, P - p). \tag{18}$$

The operator $\widetilde{A}$ on L_2 can also be decomposed as a direct integral

$$\widetilde{A} = \int_{T^\nu} \widetilde{A}(P)\, dP \tag{19}$$

of operators $\widetilde{A}(P)$, each acting on $L_2(P)$ by the formulas

$$\begin{aligned}
(\widetilde{A}(P)F)_1(\gamma) &= \sum_{\gamma'} \widehat{a}^{(1,1)}_{\gamma,\gamma'}(P) f_1(\gamma') + \sum_{\gamma_1',\gamma_2'} \int_{T^\nu} \widehat{a}^{(1,2)}_{\gamma;\gamma_1',\gamma_2'}(P;p) f_2(\gamma_1', \gamma_2'; p)\, dp, \\
(\widetilde{A}(P)F)_2(\gamma_1, \gamma_2, p) &= \sum_{\gamma'} \overline{\widehat{a}}^{(1,2)}_{\gamma';\gamma_1,\gamma_2}(P;p) f_1(\gamma') + \sum_{\gamma_1',\gamma_2'} \widehat{a}^{(2,2)}_{\gamma_1,\gamma_2;\gamma_1',\gamma_2'}(P;p) f_2(\gamma_1', \gamma_2', p) \\
&\quad + \sum_{\gamma_1',\gamma_2'} \int_{T^\nu} \widehat{b}^{(2,2)}_{\gamma_1,\gamma_2;\gamma_1',\gamma_2'}(P;p,p') f_2(\gamma_1', \gamma_2', p')\, dp',
\end{aligned} \tag{20}$$

where the kernels $\widehat{a}^{(1,1)}, \widehat{a}^{(1,2)}$, and so on, admit a natural expression in terms of $a^{(1,1)}, a^{(1,2)}, \ldots$.

We can also express U_s as a direct integral of operators $U_s(P)$, where $U_s(P) = e^{i(s,P)} E_P$, and E_P is the identity operator on $L_2(P)$.

The operators (20), sometimes called generalized Friedrichs operators, have been studied by a number of authors (see [22, 41]). Applying those results to the present situation, we obtain the following information concerning the spectrum of $\widetilde{A}(P)$.

There exists a (unique) subspace $\mathcal{H}_0 \subset L_2(P)$ which is invariant under the operators $\widetilde{A}(P)$ and $U_s(P)$ and such that the restrictions $\widetilde{A}(P)|_{\mathcal{H}_0}$ and $U_s(P)|_{\mathcal{H}_0}$ to $\mathcal{H}_0$ are unitarily equivalent to the operators

$$\begin{aligned}
(\widehat{A}(P)f)(\gamma_1, \gamma_2, p) &= \sum_{\gamma_1',\gamma_2'} \widehat{a}^{(2,2)}_{\gamma_1,\gamma_2;\gamma_1',\gamma_2'}(P;p) f(\gamma_1', \gamma_2', p), \\
(U_s(P)f)(\gamma_1, \gamma_2, p) &= e^{i(P,s)} f(\gamma_1', \gamma_2', p),
\end{aligned}$$

which act on the space $\widetilde{L}_2^P(T^\nu \times \mathcal{N}' \times \mathcal{N}')$ of functions $f(\gamma_1, \gamma_2, p)$ satisfying (18). The orthogonal complement $L_2(P) \ominus \mathcal{H}_0$ is finite dimensional.

Denoting by

$$\varepsilon_1^{(P)}(p) \geq \varepsilon_2^{(P)}(p) \geq \cdots \geq \varepsilon_g^{(P)}(p), \qquad g = r^2 = |\mathcal{N}^{(1)}|^2, \tag{21}$$

the eigenvalues of the matrix $\{\widehat{a}^{(2,2)}_{\gamma_1,\gamma_2;\gamma_1',\gamma_2'}(P;p)\}$, we obtain that the spectrum of $A(P)$ consists of the union of the ranges of the functions $\{\varepsilon_i^{(P)}(p),\, p \in T^\nu\}$ plus, possibly, a finite number of eigenvalues

$$\mu_1(P) \geq \mu_2(P) \geq \dots \mu_s(P),$$

where $s = s(P)$.

Because the functions $\widehat{a}^{(1,1)}, \widehat{a}^{(1,2)}, \dots$ appearing in expression (20) for the operators $\widetilde{A}(P)$ are analytic in the variables p and P, we see that for all P, except possibly for a set S_{cr} of $P \in T^\nu$ consisting of finitely many manifolds of dimension less than ν, the eigenvalues $\mu_1(P), \dots, \mu_s(P)$ lie outside the continuous spectrum of $\widetilde{A}(P)$ and depend continuously on P. In other words, this means that there exist a finite number of regions $G_1, \dots, G_m$ on the torus T^ν and continuous functions $\{\mu_j(P), P \in G_j\}$, each defined on G_j, such that for every $P \in T^\nu$, $P \notin S_{\mathrm{cr}}$ contained in the regions $G_{j_1}, \dots, G_{j_s}$, the eigenvalues $\mu_1(P), \dots, \mu_s(P)$ of the operator $\widetilde{A}(P)$ coincide with the values of the functions $\mu_{j_1}(P), \dots, \mu_{j_s}(P)$.

Turning now to the complete operator A, we can state the following result.

THEOREM 2. *Let a two-particle cluster operator A be given on $l_2(\mathfrak{M}_0^{R_2})$. Then there exists a subspace $\mathcal{H}_0 \subset l_2(\mathfrak{M}_0^{R_2})$ which is invariant under A and the group $\{U_s\}$, such that the restrictions $A|_{\mathcal{H}_0}$ and $U_s|_{\mathcal{H}_0}$ are unitarily equivalent to the operators*

$$\begin{aligned} (\widehat{A}_0 f)((p_1,\gamma_1),(p_2,\gamma_2)) &= \sum_{\gamma_1',\gamma_2'} a^{(2,2)}_{\gamma_1,\gamma_2;\gamma_1',\gamma_2'}(p_1;p_2) f((p_1,\gamma_1'),(p_2,\gamma_2')), \\ (U_s f)((p_1,\gamma_1),(p_2,\gamma_2)) &= \exp\{i(s,(p_1+p_2))\} f((p_1,\gamma_1'),(p_2,\gamma_2')), \end{aligned} \tag{22}$$

which act on the Hilbert space $L_2(T^\nu \times T^\nu \times \mathcal{N}^{(1)} \times \mathcal{N}^{(1)})$ of functions that depend on two pairs of variables (p_1,γ_1), (p_2,γ_2) and are symmetric in them. Here the function $a^{(2,2)}_{\gamma_1,\gamma_2;\gamma_1',\gamma_2'}(p_1;p_2)$ is defined in (15).

The orthogonal complement $l_2(\mathfrak{M}_0^{R_2}) \ominus \mathcal{H}_0$ splits as an orthogonal direct sum of a finite number of subspaces $\mathcal{H}_1, \dots, \mathcal{H}_k$ invariant under A and U_s, and on which A and U_s are unitarily equivalent to the operators

$$\begin{aligned} (\widehat{A} f)(P) &= \mu_j(P) f(P), \qquad P \in G_j, \\ (\widehat{U}_s^{(i)} f)(P) &= e^{i(P,s)} f(P), \end{aligned}$$

which act respectively on the spaces $L_2(G_j)$, where $G_1, \dots, G_m$ are suitable regions on the torus T^ν and the $\mu_j(P)$ are continuous functions defined on G_j, respectively. For each $P \in T^\nu$, $P \in \bigcap_{k=1}^s G_{j_l}$, the values $\mu_{j_1}(P), \dots, \mu_{j_s}(P)$ lie outside the set consisting of the union of the eigenvalues $\varepsilon_1^{(P)}(p), \dots, \varepsilon_g^{(P)}(p)$ as p runs over the torus T^ν.

The (generalized) eigenvectors of $A|_{\mathcal{H}_0}$ correspond under the unitary isomorphism $\mathcal{H}_0 \xrightarrow{\sim} L_2(T^\nu \times T^\nu \times \mathcal{N}^{(1)} \times \mathcal{N}^{(1)})$ to the eigenvectors $l_m(p_1^0,p_2^0)\delta(p_1 - p_1^0)\delta(p_2 - p_2^0)$ of the operator (22), where $l_m(p_1^0,p_2^0)$ is an eigenvector of the matrix $\{a_{\gamma_1,\gamma_2;\gamma_1',\gamma_2'}(p_1^0,p_2^0)\}$ with eigenvalue $\varepsilon_m(p_1^0,p_2^0) = \varepsilon_m^{(p_1^0+p_2^0)}(p_1^0)$. According to the physical interpretation, these eigenvectors describe the "free" motion of two particles with quasimomenta $p_1^0, p_2^0 \in T^\nu$ and energy

$-\ln|\varepsilon_m(p_1^0,p_2^0)|$ ("scattering states"). The generalized eigenvectors of the operators $A|_{\mathcal{H}_j}$, $j=1,\dots,k$, corresponding to the eigenvectors $e_j(P_0)\delta(P-P_0)$ ($P_0\in G_j$) of the operators $\widehat{A}_j$ with eigenvalues $\mu_j(P_0)$, describe the motion of two bound particles with total quasimomentum $p_1+p_2=P_0$ and energy $-\ln|\mu_j(P_0)|$ ("bound states" of two particles).

Assume further that we are given a two-particle cluster operator (15) in the p-representation with a small cluster parameter λ. In this case, the "principal part" A_0 of the operator

$$(A_0F)_1(p,\gamma)=\sum_{\gamma'}a^{(1,1)}_{\gamma,\gamma'}(p)f_1(p,\gamma')$$
$$(A_0F)_2((p_1,\gamma_1),(p_2,\gamma_2))=\sum_{\gamma_1',\gamma_2'}a^{(2,2)}_{\gamma_1,\gamma_2;\gamma_1',\gamma_2'}(p_1;p_2)f_2((p_1,\gamma_1'),(p_2,\gamma_2'))$$

has lower order of smallness in λ than the additional terms in (15). If the matrix function $A=\{a_{\gamma_1,\gamma_2;\gamma_1',\gamma_2'}(p_1;p_2)\}$ is assumed to be a function of "general position" (see [3]), it is possible to effectively describe the number of possible "bound states" $\mathcal{H}_1,\dots,\mathcal{H}_k$, and also the regions $G_j\subseteq T^\nu$ of the total quasimomentum in which each of them exists. We will suppose for simplicity that the index γ takes only one value, and that the off-diagonal terms in (15) are zero. The operator (15) then splits into one- and two-particle parts, the latter being of the form

$$\begin{aligned}(Af)(p_1,p_2)=&\lambda^2\Big[a(p_1,p_2;\lambda)f(p_1,p_2)\\&-\int\limits_{p_1'+p_2'=p_1+p_2}a(p_1',p_2';\lambda)f(p_1',p_2')dp_1'dp_2'\\&+\lambda\int\limits_{p_1'+p_2'=p_1+p_2}B(p_1,p_2;p_1',p_2';\lambda)f(p_1',p_2')dp_1'dp_2'\Big],\\ f\in\widetilde{L}_2^{\mathrm{sym}}(T^\nu\times T^\nu)=&\left\{f:\int\limits_{p_1+p_2=P}f(p_1,p_2)dp_1dp_2=0\quad\text{for all }P\in T^\nu\right\},\end{aligned}$$

where λ is small, and for all λ the family $a(p_1,p_2;\lambda)$ consists of functions in general position, which stay away from zero by a finite distance independent of λ. Upon subtracting the total momentum $p_1+p_2=P$, we get the family of operators

$$(A_Pf)(p)=\lambda^2[a_P(p;\lambda)f(p)-\int_{T^\nu}a_P(p';\lambda)f(p')dp'+\lambda\int_{T^\nu}K_P(p,p';\lambda)f(p')dp'],$$

where $a_P(p;\lambda)=a(p,P-p;\lambda)$, $K_P(p,p';\lambda)=B(p,P-p,p',P-p';\lambda)$, each acting on the space $L_p(T^\nu)$ of functions satisfying the conditions

$$\int_{T^\nu}f(p)dp=0,\qquad f(p)=f(P-p).$$

We first consider the low-dimensional cases $\nu=1$ or $\nu=2$.

Case $\nu=1$. We observe that the function $a_P(p;\lambda)$ always has two critical points of the form $p^0=P/2$, $p^{0\prime}=P/2+\pi$, which we will call the *principal* critical points. All other (*supplementary*) critical points $p_{\mathrm{cr}}\in T^\nu$ occur in pairs p_{cr}, p_{cr}', where $p_{\mathrm{cr}}'=P-p_{\mathrm{cr}}\neq p_{\mathrm{cr}}$. Note that

$$a_P(p_{\mathrm{cr}};\lambda)=a_P(p_{\mathrm{cr}}';\lambda)$$

and

$$K_P(p_{\rm cr}, p_{\rm cr}; \lambda) = K_P(p_{\rm cr}, p'_{\rm cr}; \lambda) = K_P(p'_{\rm cr}, p_{\rm cr}; \lambda) = K_P(p'_{\rm cr}, p'_{\rm cr}; \lambda).$$

It will thus be convenient to regard a pair of supplementary critical points $(p_{\rm cr},\ p'_{\rm cr})$ as a single point. For the family of functions $a_P(p; \lambda = 0)$ (in general position) there can exist only finitely many caustic values $P_1^{\rm caust}, \dots, P_s^{\rm caust}$ of the parameter P, at which the function $a_{P_i^{\rm caust}}(p; 0)$ has a degenerate absolute extremum (which is taken either at one of the principal critical points, or at a pair of supplementary critical points). Moreover, for each $P_i^{\rm caust}$ the degeneracy at the extremal point is such that only the first three derivatives of the function $a_{P_i^{\rm caust}}(p; 0)$ vanish there. For all sufficiently small λ, this behavior is also preserved for the family $a_P(p; \lambda)$. For each caustic value $P_i^{\rm caust}|_{\lambda=0}$ we define

$$B_i = \frac{\partial^2 K_{P_i^{\rm caust}}}{\partial p \partial p'}\bigg|_{p=p'=p_{\rm cr}, \lambda=0},$$

which is always real since the kernel K_P is selfadjoint; here $p_{\rm cr}$ is the point at which the (degenerate) extremum $a_{P_i^{\rm caust}}(p; 0)$ is reached.

We now consider multiple values $P_1^{\rm mult}, \dots, P_m^{\rm mult}$ of the parameter P (of which there are also only finitely many), i.e., values such that the functions $a_{P_i^{\rm caust}}(p; \lambda = 0)$ have at least two critical points $p_{\rm cr}^{(1)}$, $p_{\rm cr}^{(2)}$, at both of which the function has either an absolute minimum or an absolute maximum (recall that paired critical points are regarded as being the same, i.e., the points $p_{\rm cr}^{(1)}$ and $p_{\rm cr}^{(2)}$ are not paired). We note that the number m of points $P_j^{\rm mult}$ does not change for small λ, and their positions (like those of the multiple critical points $p_{\rm cr}^{(1)}$ and $p_{\rm cr}^{(2)}$, vary only slightly for small changes in λ. For each value $P_j^{\rm mult}$, $j = 1, \dots, m$ we define the (real) quantity

$$\begin{aligned} B_j =& K_{P_j^{\rm mult}}(p_{\rm cr}^{(1)}, p_{\rm cr}^{(1)}, \lambda = 0) + K_{P_j^{\rm mult}}(p_{\rm cr}^{(2)}, p_{\rm cr}^{(2)}, \lambda = 0) \\ &- K_{P_j^{\rm mult}}(p_{\rm cr}^{(1)}, p_{\rm cr}^{(2)}, \lambda = 0) - K_{P_j^{\rm mult}}(p_{\rm cr}^{(2)}, p_{\rm cr}^{(1)}, \lambda = 0). \end{aligned}$$

A caustic or multiple value P will be called *regular* if the associated B_i is ≤ 0 and the degenerate or multiple extremum is a minimum, or if $B_i \ge 0$ and the extremum is a maximum. In all other cases we say that $P_i^{\rm caust}$ or $P_j^{\rm caust}$ is irregular. It turns out that for sufficiently small λ, bound states can occur only near regular caustics or regular multiple values P. More precisely, we have the following

THEOREM 3. *Let $P_0 = P_i^{\rm caust}$ or $P_0 = P_j^{\rm mult}$ be a regular value of the parameter P. Then for sufficiently small λ there exists an interval $G \subset T^1$, $P_0 \in G$ in a neighborhood of P_0 such that for all points $P \in G$ the operator A_P has a unique eigenvalue $\mu(P)$. This eigenvalue lies to the right (respectively, left) of the continuous spectrum if P_0 corresponds to a degenerate or multiple maximum (respectively, minimum) of the function a_{P_0} $(p, \lambda = 0)$. The distance from $\mu(P)$ to the nearest edge of the continuous spectrum (the gap width) is less than $C\lambda^4$ when $P_0 = P_i^{\rm caust}$ and less than $C\lambda^2$ when $P_0 = P_j^{\rm mult}$, where C is an absolute constant. For small λ, the only values of the parameter P for which a bound state can exist belong to the intervals G near the regular caustics or multiple values of P.*

We now turn to the case $\nu = 2$. Again, the function $a_P(p; 0) = a(p, P - p; 0)$, where p and $P \in T^2$, has four principal critical points

$$\begin{aligned} P_{\rm cr}^{0,0} &= \left(\frac{P_1}{2}, \frac{P_2}{2}\right), \qquad P_{\rm cr}^{0',0} = \left(\frac{P_1}{2} + \pi, \frac{P_2}{2}\right), \\ P_{\rm cr}^{0,0'} &= \left(\frac{P_1}{2}, \frac{P_2}{2} + \pi\right), \qquad P_{\rm cr}^{0',0'} = \left(\frac{P_1}{2} + \pi, \frac{P_2}{2} + \pi\right), \end{aligned}$$

$P = (P_1, P_2)$ and, possibly, some paired supplementary critical points $(p_{\rm cr}, p'_{\rm cr})$, where $p'_{\rm cr} = P - p_{\rm cr}$; we will again regard each such pair as a single critical point. Note that in contrast to the one-dimensional case, for the types of degeneracy possible for a family of functions in general position when $\nu = 2$, the degenerate critical points are not associated with bound states of the operator A; the latter arise only when the function $a_P(p; \lambda = 0)$ has multiple minima or maxima. Let $\Delta^{(h)}_{\rm min} \subset T^2$ be the set of $P \in T^2$ at which $a_P(p; \lambda = 0)$ has at least k distinct critical points corresponding to the absolute minimum. The set $\Delta^{(k)}_{\rm max}$ is defined similarly. For $\nu = 2$ and a function $a(p_1, p_2; \lambda = 0)$ in general position, only $\Delta^{(2)}_{\rm ext}$ and $\Delta^{(3)}_{\rm ext}$ can be nonempty (where ext = min or max); moreover, $\Delta^{(3)}_{\rm ext}$ is finite, and $\Delta^{(2)}_{\rm ext}$ decomposes into a finite number of arcs $\Delta^{(2)}_{{\rm ext},1}, \dots, \Delta^{(2)}_{{\rm ext},s}$ with endpoints at the points $P^{\rm triple} \in \Delta^{(3)}_{\rm ext}$, and exactly three arcs $\Delta^{(2)}_{{\rm ext},1}, \Delta^{(2)}_{{\rm ext},2}, \Delta^{(2)}_{{\rm ext},3}$ converge at each point $P^{\rm triple}$ in $\Delta^{(3)}_{\rm ext}$ (see Figure 1).

FIGURE 1

It turns out that bound states of the operator A can be present only in some neighborhood G of the set $\Delta^{(2)}_{\rm ext}$, and for each P in G the operator A_P has at most two eigenvalues $\mu(P)$. The following figures show all possible structures of the part of G containing the triple point $P^{\rm triple}$. Figure 2 shows two regions,

$$G_{\rm small} \subset G_{\rm big},$$

and for all $P \in G_{\rm small}$ there are two eigenvalues (here ext = min)

$$\mu_{\rm big}(P) < \mu_{\rm small}(P).$$

The eigenvalue $\mu_{\rm small}(P)$ vanishes on the boundary of the region $G_{\rm small}$, while $\mu_{\rm big}(P)$ remains the same throughout the region $G_{\rm big}$. The "width" of $G_{\rm big}$ (and also the diameter of $G_{\rm small}$) is of the order $\exp\{-\,{\rm const}\,|\lambda|^{-1}\}$, and $G_{\rm big}$ may be somewhat "elongated" along all the arcs $\Delta^{(2)}_{{\rm min},1}, \Delta^{(2)}_{{\rm min},2}, \Delta^{(2)}_{{\rm min},3}$ converging at $P^{\rm triple}$.

FIGURE 2

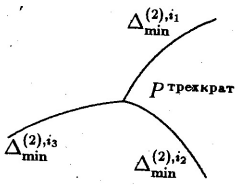

рис. 1

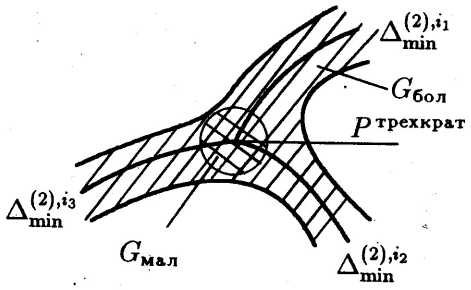

рис. 2

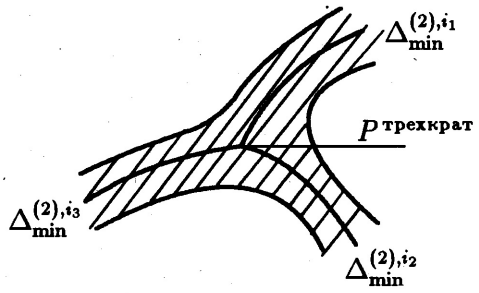

рис. 3

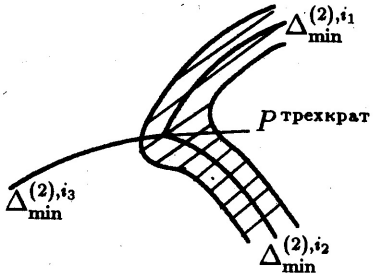

рис. 4

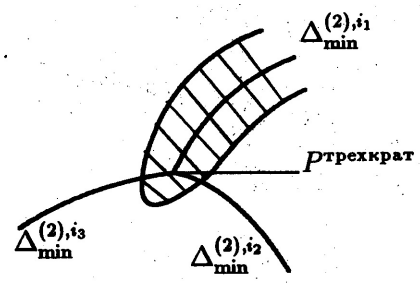

рис. 5

рис. 6

Figures 3–5 illustrate the cases when only one eigenvalue $\mu(P)$ exists in a neighborhood of P^{triple}. In all cases the width of the depicted region is $\sim \exp\{-\operatorname{const}|\lambda|^{-1}\}$, and it is elongated by a finite amount along one, two, or three of the arcs. Another case is also possible, in which the triple point P^{triple} does not lie in G; the case when G contains only interior points of one of the arcs $\Delta^{(2)}_{\text{ext}}$ can also occur (Figure 6).

FIGURE 3

FIGURE 4

FIGURE 5

In all these cases, the eigenvalues $\mu(P)$ lie to the left of the continuous spectrum of A_P by a distance $\sim \exp\{-\operatorname{const}|\lambda|^{-1}\}$.

FIGURE 6

A similar situation is found in a neighborhood of the set $\Delta^{(2)}_{\max}$, except that the eigenvalues $\mu(P)$ now lie to the right of the continuous spectrum of A_P.

There are some explicit recipes for determining which situation arises, but they are quite elaborate and will not be given here (see [28]). For dimensions $\nu > 2$, small λ, and a generic function $a(p_1, p_2)$, the operator A may fail to have any bound states at all.

A proof of all these results can be found in [28]. Let us note that in the case of an arbitrary k-particle cluster operator A, the structure of spectrum of A is qualitatively similar to that for a k-particle Schrödinger operator: each branch of the spectrum is determined by a partitioning of the k particles into bound groups (clusters), each of which moves "freely" (see [36] and [17] for more details).

§8. Asymptotic decay of the correlation functions for Gibbs fields

As an application of the results described above, we consider the asymptotic behavior of the correlation functions

$$\langle F_A, F_{A+x}\rangle \equiv \langle F_A F_{A+x}\rangle - \langle F_A\rangle^2 \tag{1}$$

as $|x| \to \infty$; here $\langle\cdot\rangle$ denotes an average over a translation-invariant Gibbs field on the lattice $Z^{\nu+1}$, F_A is a locally bounded functional of the field, and $F_{A+x} = U_x F_A$ is the translation of F_A by the vector $x \in Z^{\nu+1}$. For simplicity, we will examine only the case when $A \subset Y_0$ is contained in the zero slice of $Z^{\nu+1}$, and the vector $x = (t, 0)$ points along the "time" axis. Then

$$\langle F_A F_{A+t}\rangle = (\mathcal{J}_t F_A, F_A)_{\mathcal{H}_{\text{phys}}}, \tag{2}$$

where $\mathcal{J}_t = \mathcal{J}^t$, $\mathcal{J}$ is the transfer matrix.

Let us suppose that we have succeeded in finding the highest one-particle subspace $\mathcal{H}_1$ of the transfer matrix $\mathcal{J}$, such that the spectrum of $\mathcal{J}$ restricted to the orthogonal complement $\mathcal{H}_1^\perp$ of $\mathcal{H}_1$ is separated from the spectrum of $\mathcal{J}$ on $\mathcal{H}_1$:

$$\sup \sigma(\mathcal{J}|_{\mathcal{H}_1^\perp}) \equiv m_2 < \inf \sigma(\mathcal{J}|_{\mathcal{H}_1}). \tag{2'}$$

We have the orthogonal decomposition

$$F_A = F_A^0 + F_A^{(1)} + \widehat{F}_A^{(1)},$$

where $F_A^0 = \langle F_A\rangle$, $F_A^{(1)} \in \mathcal{H}_1$, $\widehat{F}_A^{(1)} \in \mathcal{H}_1^\perp$. Then (1) and (2) imply that

$$(\mathcal{J}^t F_A, F_A) = \langle F_A\rangle^2 + (\mathcal{J}^t F_A^{(1)}, F_A^{(1)}) + (\mathcal{J}^t \widehat{F}_A^{(1)}, \widehat{F}_A^{(1)}). \tag{3}$$

The last term is of order

$$O(m_2^t). \tag{3'}$$

Fourier transformation in the space $\mathcal{H}_1$ takes the element $F_1^{(1)}$ into the function $f = \{f(p, \gamma)\}$, and the second term in (3) becomes

$$(\mathcal{J}^t F_A^{(1)}, F_A^{(1)}) = \sum_{\gamma,\gamma'} \int_{T^\nu} b^t_{\gamma,\gamma'}(p) f(p, \gamma)\overline{f}(p, \gamma') dp, \tag{4}$$

where the $b^t_{\gamma,\gamma'}(p)$ are the matrix elements of $(B(\rho))^t$ (see (13.7)). Finally, diagonalizing the matrix $B(p)$ for each $p \in T^\nu$, we find that the expression on the right in (4) in equal to

$$\sum_k \int_{T^\nu} \varepsilon_k^t(p)|p_k(p)|^2 dp, \tag{5}$$

where the $f_k(p)$ are the components of the vector $\{f(\gamma,p)\}$ with respect to a basis of eigenvectors of $B(p)$ (with corresponding eigenvalues $\varepsilon_k(p)$). Let

$$m_1 \equiv \max_{p,k} |\varepsilon_k(p)| = \varepsilon_{k_0}(p_0) > 0,$$

where for simplicity we take the largest eigenvalue of $B(p_0)$ to be positive of multiplicity one. If we further assume that $f_{k_0}(p)$ is continuous at the point $p = p_0$, then the sum (5) behaves asymptotically as

$$C\frac{|f_{k_0}(p_0)|^2}{(2\pi t)^{\nu/2}} m_1^t (1 + o(1)). \tag{6}$$

In view of (2′) and (3′), the expression (6) does in fact give the asymptotics of the correlation function (1).

The above computation breaks down if the projection $F_A^{(1)}$ of the function F_A is equal to zero. This will be the case for the Ising field, e.g., whenever F is an even function (i.e., expressible as a sum of even monomials in the variables $\{\sigma_x, x \in Z^\nu\}$). Let us now suppose that in addition to the one-particle subspace $\mathcal{H}_1$ we have also constructed a two-particle invariant subspace $\mathcal{H}_2$ for the transfer matrix $\mathcal{J}$, and let

$$F = \langle F_A \rangle + F_A^{(2)} + \widehat{F}_A^{(2)}, \tag{7}$$

where $F_A^{(2)} \in \mathcal{H}_2$, $\widehat{F}_A^{(2)} \in (\mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2)^\perp \equiv \overline{\mathcal{H}}$.

Suppose further that the spectrum of $\mathcal{J}$ restricted to $\mathcal{H}_2$ is separated from the spectrum of $\mathcal{J}$ on $\overline{\mathcal{H}}$. Then as above, the asymptotic behavior of the correlation function (1) is given by

$$(\mathcal{J}_2^t F_A^{(2)}, F_A^{(2)}), \tag{8}$$

where $\mathcal{J}_2$ is the part of $\mathcal{J}$ on $\mathcal{H}_2$. Two cases may be distinguished:

1) The regular case

$$m_2 = \sup_{k,p_1,p_2} \varepsilon_k(p_1,p_2) > \sup_{l,p} \mu_l(p). \tag{9}$$

Here the $\varepsilon_k(p_1,p_2)$ are the eigenvalues of the matrix $\{a^{(2,2)}_{\gamma_1,\gamma_2;\gamma_1',\gamma_2'}(p_1,p_2)\}$ appearing in expression (15.7) for the two-particle cluster operator $\mathcal{J}_2$, and the $\mu_l(p)$ are the bound states for $\mathcal{J}_2$.

2) The nonregular case, for which (9) is replaced by the opposite inequality.

In the regular case, calculations involving the resolvent of $\mathcal{J}_2$ (see [18]) show that the asymptotic behavior of (8) is given as follows:

1) for $\nu = 1$

$$\frac{c(F_A)m_2^t}{t^2}(1+o(1)),$$

2) for $\nu = 2$

$$\frac{c(F_A)m_2^t}{t^2(\ln t)^2}(1+o(1)),$$

3) for $\nu \geq 3$

$$\frac{c(F_A)m_2^t}{t^\nu}(1+o(1)),$$

where $c(F_A)$ is a constant depending on F_A.

In the nonregular case, the large t behavior is given by (6) with m_1 replaced by $m_2 = \sup_{l,p}|\mu_l(p)|$:

$$\frac{c(F_A)m_2^t}{t^{\nu/2}}(1+o(1)).$$

In the case when the vector x in (1) is directed at an angle to the time axis, the large t behavior of (1) for $F_A^{(1)} \neq 0$ is of the form (6), where $m_1 = m_1(\alpha)$ depends on the direction α of the vector x.

CHAPTER IV

ASYMPTOTIC COMPLETENESS FOR INTERACTING FERMION SYSTEMS

In the previous chapters (see 4.0 and 3.1) we have considered the quantum dynamics of fermion systems defined on a CAR algebra $\mathfrak{A}(\mathcal{H})$, for an appropriate one-particle Hilbert space. Our treatment was most complete for a quasifree dynamics, generated by a Hamiltonian

$$H = d\Gamma(h),$$

where h is a one-particle Hamiltonian acting on $\mathcal{H}$ (see 3.1). The spectrum of the energy operator H_{GNS} in the GNS representation for such a dynamics relative to a quasifree KMS state was seen in 1.2 to consist of finitely many "branches" ("k-particle" branches), each describing the free motion of k "quasiparticles." If $\mathcal{H} = L_2(R^\nu)$ (or $\mathcal{H} = l_2(Z^\nu)$), i.e., the particles move in ν-dimensional space (or on a ν-dimensional lattice) and the dynamics $\alpha(t)$ on the CAR algebra is obtained from the free dynamics by adding a small interaction among the particles, then it behaves much like a free dynamics, provided that the dimension ν is not too small (usually $\nu \geq 3$ suffices, but in certain cases it may be necessary to require $\nu \geq 4$ or $\nu \geq 5$).

In precise language, this means that there exists a $*$-automorphism of the algebra $\mathfrak{A}(\mathcal{H})$ taking the dynamics $\tau(t)$ into a quasifree dynamics $\tau^0(t)$. This implies, in particular, that for a dynamics with interaction, the operator H_{GNS} generated by a representation with respect to some KMS state is unitarily equivalent to the operator H^0_{GNS} for the free dynamics (with respect to the corresponding quasifree KMS state). Among other things, this means that the spectrum of H_{GNS} still consists of branches that describe the motion of mutually independent quasiparticles.

We note that the assumption that the particles interact weakly (and that the dimension is large enough) is essential, because for strong interactions bound states can appear in the spectrum of H_{GNS} — these are spectral branches that describe the motion of a group (cluster) of finitely many mutually bound particles. For one- or two-dimensional particles, such branches can be present even for arbitrarily weak interactions. The similarity between $\tau_0(t)$ and $\tau(t)$ alluded to above can be proved by use of methods borrowed from the general theory of scattering in quantum mechanics, and it belongs to the circle of problems in scattering theory generally known as the *asymptotic completeness* problem (see [36] for more details).

We will begin by considering some cases involving an interaction in a pure fermion system. In the last section of this chapter we will analyze the analogous problem for a system consisting of a fermi gas interacting with another type of distinguished particle.

§1. Fermi systems with bounded interaction

1. Møller morphisms. Let $\mathcal{H}$ be a separable Hilbert space and let $\mathfrak{A}(\mathcal{H}) \subset \mathfrak{B}(\mathcal{F}_{as}(\mathcal{H}))$ be the CAR algebra for the bounded operators acting on the antisymmetric Fock space $\mathcal{F}_{as}(\mathcal{H})$ generated by the creation and annihilation operators $\{a(f),\ a^*(f),\ f \in \mathcal{H}\}$ in $\mathcal{F}_{as}(\mathcal{H})$. Let further h_0 be a selfadjoint operator on $\mathcal{H}$, $H_0 = d\Gamma(h_0)$ its second quantization (see 3.1), and let τ_t^0 be the group of $*$-automorphisms of $\mathfrak{A}(\mathcal{H})$ generated by H_0:

$$\tau_t^0(A) = \exp\{itH_0\}A\exp\{-itH_0\}, \qquad A \in \mathfrak{A}(\mathcal{H}). \tag{1}$$

As was indicated in 3.1 and 3.2, this is called the free dynamics on $\mathfrak{A}(\mathcal{H})$ generated by the one-particle operator h_0. The fact that $\tau_t^0(A) \in \mathfrak{A}(\mathcal{H})$ for all t and $A \in \mathfrak{A}(\mathcal{H})$ follows from the formula

$$\tau_t^0(a^\#(f)) = a^\#(e^{ith_0}f), \qquad f \in \mathcal{H}$$

(see (9^a.3.2)), where $a(f^\#)$ is a creation or annihilation operator.

Now let V be a selfadjoint element in $\mathfrak{A}(\mathcal{H})$ and

$$H = H_V = H_0 + V \tag{2}$$

be the "perturbed operator". The perturbed dynamics τ_t^V on $\mathfrak{A}(\mathcal{H})$ is defined by

$$\tau_t^V(A) = \exp\{itH_V\}A\exp\{-itH_V\}, \qquad A \in \mathfrak{A}(\mathcal{H}). \tag{2^a}$$

That $\tau_t^V(A) \in \mathfrak{A}(\mathcal{H})$ for every t follows from the series expansion

$$\tau_t^V(A) = \tau_t^0(A) + \sum (i)^n \int_{\Delta_n^t} [\tau_{s_1}^0(V)[\tau_{s_2}^0(V)[,\dots,[\tau_{s_n}^0(V), \tau_t^0(A)]]]]\, ds_1 \dots ds_n, \tag{3}$$

of the dynamics τ_t^V given previously in 4.0; here for $t > 0$ we have

$$\Delta_n^t = \{0 \le s_1 \le \dots \le s_n \le t\},$$

while for $t < 0$,

$$\Delta_n^t = \{t \le s_1 \le \dots \le s_n \le 0\}.$$

The series is easily shown to converge for all t with respect to the norm topology on $\mathfrak{A}(\mathcal{H})$.

We now introduce the general notion of Møller morphisms for an ordered pair $(\tau_t^{(2)}, \tau_t^{(1)})$ of dynamics acting on some $*$-algebra $\mathfrak{A}$.

Assume that the limits

$$\gamma_\pm^{\tau^{(2)},\tau^{(1)}}(A) = \lim_{t\to\pm\infty} \tau_{-t}^{(2)}(\tau_t^{(1)}(A)) \tag{4}$$

exist (in the norm on $\mathfrak{A}$) for all $A \in \mathfrak{A}$. The maps

$$\gamma_\pm^{\tau^{(2)},\tau^{(1)}} : \mathfrak{A} \to \mathfrak{A} : A \to \gamma_\pm^{\tau^{(2)},\tau^{(1)}}(A)$$

are called the Møller morphisms for the pair $(\tau_t^{(2)}, \tau_t^{(1)})$.

LEMMA 1 (intertwining property for Møller morphisms). *If $\tau_t^{(1)}$ and $\tau_t^{(2)}$ are two dynamics on the algebra $\mathfrak{A}$, we have the following*:

a) *if the Møller morphisms $\gamma_\pm^{\tau^{(2)},\tau^{(1)}}$ and $\gamma_\pm^{\tau^{(1)},\tau^{(2)}}$ exist, then*

$$\gamma_\pm^{\tau^{(2)},\tau^{(1)}} = [\gamma_\pm^{\tau^{(1)},\tau^{(2)}}]^{-1};$$

b) *if the Møller morphisms $\gamma_\pm^{\tau^{(2)},\tau^{(1)}}$ exist, then*

$$\gamma_\pm^{\tau^{(2)},\tau^{(1)}}\tau_t^{(1)} = \tau_t^{(2)}\gamma_\pm^{\tau^{(2)},\tau^{(1)}};$$

c) *if again both morphisms $\gamma_\pm^{\tau^{(2)},\tau^{(1)}}$ and $\gamma_\pm^{\tau^{(1)},\tau^{(2)}}$ exist, then*

$$\tau_t^{(2)} = \gamma_\pm^{\tau^{(2)},\tau^{(1)}}\tau_t^{(1)}\gamma_\pm^{\tau^{(1)},\tau^{(2)}}.$$

PROOF. Statement a) follows at once from the definition (4). Next,

$$\begin{aligned}(\tau_t^{(2)}\gamma_\pm^{\tau^{(2)},\tau^{(1)}})(A) &= \lim_{s\to\pm\infty}\tau_t^{(2)}(\tau_{-s}^{(2)}(\tau_s^{(1)}(A)))\\ &= \lim_{s\to\pm\infty}\tau_{t-s}^{(2)}(\tau_{s-t}^{(1)}(\tau_t^{(1)}(A))) = (\gamma_\pm^{\tau^{(2)},\tau^{(1)}}\tau_t^{(1)})(A)\end{aligned}$$

which is just b). Finally, c) follows from a) and b).

We now return to the pair of dynamics τ_t^0 and τ_t^V (free and perturbed) on the algebra $\mathfrak{A}(\mathcal{H})$. Then provided they exist, the Møller morphisms

$$\gamma_\pm^{\tau^{(V)},\tau^{(0)}}(A) = \lim_{t\to\pm\infty}\tau_{-t}^V(\tau_t^0(A)) = \gamma_\pm(A)$$

and

$$\gamma_\pm^{\tau^{(0)},\tau^{(V)}}(A) = \lim_{t\to\pm\infty}\tau_{-t}^0(\tau_t^V(A)) = \widehat{\gamma}_\pm(A)$$

are called the *forward* and *inverse* morphisms, respectively.

We now give a simple method which dates back to Cook [see (36)] for proving the existence of the morphisms $\gamma_\pm$.

LEMMA 2 (Cook's method). *Let τ_t^0 and τ_t^V be a free and perturbed dynamics on the algebra $\mathfrak{A}$. Suppose there exists a dense subset $\mathfrak{A}_0 \subseteq \mathfrak{A}$ such that*

$$\int_{-\infty}^{\infty}\|[\tau_t^0(V), A]\|dt < \infty, \qquad A \in \mathfrak{A}^0. \tag{5}$$

Then the forward morphisms $\gamma_\pm(A)$ exist.

PROOF. One checks readily that the element

$$A_t = \tau_{-t}^V(\tau_t^0(A)) = \exp\{-it(H_0+V)\}\exp\{iH_0t\}A\exp\{-iH_0t\}\exp\{i(H_0+V)t\}$$

satisfies the equation

$$\frac{dA_t}{dt} = i\tau_{-t}^V(\tau_t^0([\tau_{-t}^0(V), A])).$$

Upon integrating over t, it follows that

$$A_{t_1} - A_{t_2} = i \int_{t_1}^{t_2} \tau_{-t}^V(\tau_t^0([\tau_{-t}^0(V), A]))\, dt.$$

Since τ_t^V and τ_t^0 are norm-preserving, we have

$$\|A_{t_1} - A_{t_2}\| \leq \int_{t_1}^{t_2} \|[\tau_{-t}^0(V), A]\|\, dt$$

and by (5) we find that for $A \in \mathfrak{A}^0$ the limit $\lim_{t\to\pm\infty} A_t = \gamma_\pm(A)$ exists. It now follows easily that this limit exists for all $A \in \mathfrak{A}$ because the maps $\gamma_\pm$ are isometric: $\|\gamma_\pm(A)\| = \|A\|$.

2. Existence of the forward Møller morphisms for bounded perturbations of a free dynamics. In $\mathfrak{A}(\mathcal{H})$ we have the dense $*$-subalgebra $\mathfrak{A}^0(\mathcal{H})$, the set of all finite linear combinations of monomials in the creation/annihilation operators $a^\#(f)$ for functions $f \in L_2(R^\nu)$ whose Fourier transforms are in $C^\infty(R^\nu)$.

Consider the C^*-subalgebra $\mathfrak{A}_l(\mathcal{H})$ of $\mathfrak{A}(\mathcal{H})$ generated by the monomials in the creation and annihilation operators containing an even number of factors. The elements of the C^*-subalgebra $\mathfrak{A}_l(\mathcal{H})$ will be called *even.* The set $\mathfrak{A}_l^0(\mathcal{H}) = \mathfrak{A}^0(\mathcal{H}) \cap \mathfrak{A}_l(\mathcal{H})$ is a dense $*$-subalgebra of $\mathfrak{A}_l(\mathcal{H})$.

We introduce the following classes of bounded interactions. Let $V = V^*$ and V_i be given by

$$V = \sum_{i=1}^{d} V_i, \quad V_i = \sum_{k=1}^{M_i} c_k a^*(f_{i,1}^{(k)}) \dots a^*(f_{i,m_i}^{(k)}) a^*(f_{i,m_i+1}^{(k)}) \dots a^*(f_{i,m_i+k_i}^{(k)}), \tag{6}$$

where d and M_i are finite, $m_i + n_i > 0$, $\widetilde{f}_{i,j}^{(k)} \in C^\infty(R^\nu)$ for all i, j, k ($\widetilde{f}$ is the Fourier transform of f).

We say that $V \in A_l^0$ if the sums $m_i + n_i$ in (6) are even for all i, and that $V \in A^0$ if $m_i > 0$ and $n_i > 0$ for all i. Both classes A_l^0 and A^0 are evidently contained in $\mathfrak{A}^0(\mathcal{H})$, and $A_l^0 \subset \mathfrak{A}_l(\mathcal{H})$.

REMARK. All the results discussed below remain valid for less restricted classes A and A_l of smooth bounded interactions V, namely those for which $V = V^*$ is of the form

$$V = \sum_{i=1}^{d} \int V_i(x_1, \dots, x_{m_i}, x_{m_i+1}, \dots, x_{m_i+n_i}) a^*(x_1) \dots \\ a^*(x_{m_i}) a(x_{m_i+1}) \dots a(x_{m_i+n_i})\, dx_1 \dots\, dx_{m_i+n_i}, \tag{7}$$

with

$$V_i \subset S(R^{m_i+n_i}), \qquad m_i + n_i > 0.$$

If $m_i + n_i$ is even for all i, then $V \in A_l$. If on the other hand $m_i > 0$ and $n_i > 0$, then $V \in A$. The theorems in this section will be proved for the most part only for interactions in A^0 or A_l^0, and only brief comments will be made concerning the corresponding proofs for interactions in A, A_l.

In (6) and (7) we write, respectively,

$$m_{\max} = \max_i m_i = \max_i n_i,$$
$$m_{\min} = \min_i m_i = \min_i n_i$$

(that $\max_i m_i = \max_i n_i$ and $\min_i m_i = \min n_i$ follows from the selfadjointness of V).

We also introduce the following class $\mathbb{H}$ of one-particle Hamiltonians h on $L_2(R^\nu)$. We say that $h \in \mathbb{H}$ if the operator h (in the Fourier representation) is given by multiplication by a function $h(k)$, $h \in C^\infty(R^\nu)$, $k \in R^\nu$, and h and its first two derivatives are bounded by a polynomial on R^ν. In addition, we require

$$\operatorname{dist}(S_h, G_h) > 0,$$
$$\operatorname{mes} S_h = \operatorname{mes} G_h = 0,$$

where $S_h = \{k\colon \nabla h(k) = 0\}$ is the set of critical points of the function h, and G_h is the set of $k \in R$ for which the matrix of second derivatives of h is singular.

We note that the class $\mathbb{H}$ includes the nonrelativistic Hamiltonian $-\Delta + \mu$, as well as the relativistic Hamiltonian $(-\Delta + m^2)^{1/2}$, $m > 0$, both acting on $L_2(R^\nu)$.

We will henceforth assume that the free dynamics τ_t^0 is generated by a one-particle Hamiltonian h in the class $\mathbb{H}$.

THEOREM 3 (existence of forward Møller morphisms). *Let $\nu \geq 1$; then for $V = V^* \in A_l$ the morphisms*

$$\gamma_\pm(A) = \lim_{t \to \pm\infty} \tau_{-t}^V \tau_t^0(A), \qquad A \in \mathbb{H}(\mathcal{H})$$

exist.

PROOF. The cases $\nu = 1, 2$ differ slightly from the case $\nu \geq 3$, so let $\nu \geq 3$. By Cook's method, it is enough to exhibit a dense subset $\mathfrak{A}^0 \subseteq \mathfrak{A}(\mathcal{H})$ for which condition (5) is satisfied.

Let $\mathfrak{A}^0 \equiv \mathfrak{A}^0(\mathcal{H})$, i.e.,

$$\mathfrak{A}^0 = D\{a^*(f_1)\dots a^*(f_m)a(g_1)\dots a(g_n), \quad m, n \geq 0,$$
$$\widetilde{f}_i, \widetilde{g}_j \in C_0^\infty(R^\nu), \quad i = 1, \dots, m, \quad j = 1, \dots, n\}$$

where $D\{\cdot\}$ denotes the linear span of the set $\{\cdot\}$.

If we show that (5) holds for $A = a^\#(f)$, where $\widetilde{f} \in C_0^\infty(R^\nu)$ is an arbitrary function, then since $\tau_{-t}^V \tau_t^0$ is a $*$-automorphism for any fixed $t \in R^\nu$, it will follow that $\gamma_\pm(A)$ exists for all $A \in \mathfrak{A}^0$.

For $A = a(f)$ we have

$$\begin{aligned} \|[\tau_t^0(V), A]\| &= \|[\tau_{-t}^0(V), A]\| \leq \|[a(e^{ith}f), V]\| \\ &= \Bigg\| \sum_{i=1}^{d} \sum_{k=1}^{M_i} \sum_{j=1}^{m_i} (-1)^{j-1} (f_{i,j}^{(k)}, e^{-ith} f) \\ &\quad \times a^*(f_{1,j}^{(k)}) \dots \check{a}^*(f_{i,j}^{(k)}) \dots a^*(f_{i,m_i}^{(k)}) \\ &\quad \times a(f_{i,m_i+1}^{(k)}) \dots a(f_{i,m_i+n_i}^{(k)}) \Bigg\| < \frac{C(V,f)}{(1+|t|)^{\nu/2}} \end{aligned} \tag{8}$$

where the symbol $\check{}$ over $a^*(f_{i,j})$ means that this factor is to be omitted. Then by the method of stationary phase (see, e.g., [3]),

$$|(f_{i,j}^{(k)}, e^{ith} f)| < \frac{C(f, f_{i,j}^{(k)})}{(1+|t|)^{\nu/2}}, \tag{9}$$

where $C(f, f_{i,j}^{(k)})$ is a constant and

$$C(V,f) = \sum_{i=1}^{d} \sum_{k=1}^{M_i} \sum_{j=1}^{m_i} |c_k| C(f, f_{i,j}^{(k)}) \prod_{l \neq j}^{m_i+n_i} \|f_{i,l}^{(k)}\| < \infty. \tag{10}$$

Since the right-hand side of (8) is a function in $L_1(R^1)$, $\gamma_{\pm}(a(f))$ exists by Cook's criterion. The case $A = a^*(f)$ is treated similarly.

For $\nu = 1, 2$ we choose the dense subset $\mathfrak{A}^0$ in $\mathfrak{A}(\mathcal{H})$ to be

$$\mathfrak{A}^0 = D\{a^*(f_1) \dots a^*(f_m) a(g_1) \dots a(g_n), \quad m, n \geq 0, \quad \widetilde{f}_i, \widetilde{g}_j \in C_0^\infty(R^\nu \setminus S_h)\}.$$

For the function $\widetilde{f}_i \in C_0^\infty(R^\nu \setminus S_h)$ we integrate by parts q times for $|t| > 1$ to get the estimate

$$|(f, e^{ith} f_{i,j}^{(k)})| < \frac{C_q(f, f_{i,j}^{(k)})}{|t|^q}, \tag{11}$$

where the constant $C_q(f, f_{i,j}^{(k)})$ still depends on q. Taking $q \geq 2$, we get a function in $L_1(R \setminus [-1, 1])$ in (8).

This proves Theorem 3 for an interaction in A_l^0. To prove it for $V \in A_l$, we note that only the estimate (10) needs to be verified. For this, it suffices to write the interaction $V \in A_l$ in the following form:

$$V = \sum_{i=1}^{d} \sum_{N_1} \cdots \sum_{N_{m_i+n_i}} c_i(N_1, \dots, N_{m_i+n_i}) a^*(e_{N_1}) \dots a^*(e_{N_{m_i}}) \times a(e_{N_{m_i+1}}) \dots a(e_{N_{m_i+n_i}}), \tag{12}$$

where $\|e_{N_j}\| = 1$, $e_{N_j} \in C_0^\infty(R^\nu)$, and the N_j run over the countable set $\mathcal{N}$; moreover,

$$\sum_{i=1}^{d} \sum_{N_1} \cdots \sum_{N_{m_i+n_i}} |c_i(N_1, \dots, N_{m_i+n_i})| < \infty \tag{13}$$

and for all $N, N' \in \mathcal{N}$ and $f \in C_0^\infty(R^\nu)$ we have the bounds

$$|(e_N, e^{-ith} f)| < \frac{C(f)}{(1+|t|)^\delta}, \tag{14}$$

$$|(e_N, e^{-ith} e_{N'})| < \frac{C}{(1+|t|)^\delta}, \tag{15}$$

where $C(f)$ is a constant depending only on f, and C and δ are absolute constants with $\delta > 1$.

It is easy to see that (12)–(15) imply the bound (10).

Let us now prove that $V \in A_l$ can in fact be represented in the form (12). Assuming first that the Fourier transforms of the kernels V_i satisfy

$$\widetilde{V}_i \in C_0^\infty(R^{\nu(m_i+n_i)}),$$

we can choose L such that $\operatorname{supp} \widetilde{V}_i \subset [-L, L]^{\nu(m_i+n_i)}$ for every i. Let

$$e_n(k) = \frac{\kappa(k) \exp\{2\pi i n \frac{k}{A}\}}{d_n}, \tag{16}$$

where $\kappa(k) \in C_0^\infty(R)$ satisfies $0 \le \kappa(k) \le 1$ for all k and $\kappa(k) = 0$ for $|k| > B$. The constants d_n are chosen so that $\|e_n\| = 1$.

Choose A and B so that $L < B < 2L < A$. Then

$$\widetilde{V}_i = \sum_{\overline{N}} c_i(\overline{N}) \prod_{j=1}^{D_i} e_{n_j}(k_j), \tag{17}$$

where $\overline{N} = (n_1, \dots, n_D)$, $D_i = \nu(m_i + n_i)$. Clearly, for every q there exists a constant $C(q)$ such that

$$|c_{\overline{N}}| < \frac{C(q)}{|\overline{N}|^q}, \qquad |\overline{N}| = \sum_i |n_i|. \tag{18}$$

Plainly, for $q > \max_i D_i$ we have the estimate

$$\sum_{\overline{N}} |c_i(\overline{N})| < \infty.$$

For the functions

$$e_N = \prod_{j=1}^{\nu} e_{n_j}(k_j), \qquad e_{N'} = \prod_{j=1}^{\nu} e_{n'_j}(k_j)$$
$$N = (n_1, \dots, n_\nu), \qquad N' = (n'_1, \dots, n'_\nu)$$

we have the following estimates, uniformly in $N, N' \in Z^\nu$:

$$|(e_N, e^{ith} e_{N'})| < \frac{C}{(1+|t|)^{\nu/2-\delta'}}, \tag{19}$$

$$|(e_N, e^{ith} f)| < \frac{C(f)}{(1+|t|)^{\nu/2-\delta'}}, \tag{20}$$

where the constant $\delta' > 0$ can be chosen arbitrarily small, $C = C(\nu, \delta')$, $C(f) = C(f, \mu, \delta')$. Inequalities (14) and (15) now follow upon taking $\delta = \nu/2 - \delta' > 0$ for $\nu \ge 1$, $\delta' < 1/2$ in (19) and (20).

On the other hand, if $\widetilde{V}_i \in S(R^{(m_i+n_i)\nu})$ then we use a partition of unity

$$\sum_{\overline{N}} \alpha_{\overline{N}}(k) = 1, \tag{21}$$

where $\operatorname{diam} \operatorname{supp} \alpha_{\overline{N}} \le \text{const}$ uniformly in $\overline{N}$. We can then express the kernel $\widetilde{V}_i$ as a sum $\widetilde{V}_i \alpha_{\overline{N}}$ of kernels with compact support, and using the analog of (10) for the $\widetilde{V}_i \alpha_{\overline{N}}$, with suitable translations of the functions $e_{\overline{N}}$, we can repeat the previous proof.

This concludes the proof of Theorem 3.

3. Existence of inverse Møller morphisms for small bounded perturbations of the free dynamics. To prove the invertibility of the forward Møller morphisms, it suffices to prove the existence of the inverse Møller morphisms. As in the case of ordinary wave operators, the proof that the inverse Møller morphisms exist for k-particle quantum systems is considerably more difficult than the existence proof for the forward morphisms.

THEOREM 4 (invertibility of the forward Møller morphisms). *If $\nu \geq 3$ and $V = V^* \in A_l$, then there exists an $\varepsilon_0 > 0$ such that when $|\varepsilon| < \varepsilon_0$, $\varepsilon \in R^1$ the forward Møller morphisms*

$$\gamma_\pm(A) = \lim_{t \to \pm\infty} \tau_{-t}^{\varepsilon V} \tau_t^0(A), \qquad A \in \mathfrak{A}(\mathcal{H}),$$

exist and are invertible, and

$$\tau_t^{\varepsilon V} = \gamma_\pm \tau_t^0 \gamma_\pm^{-1}, \qquad t \in R. \tag{22}$$

COROLLARY. *Let $\nu \geq 3$. Then for $V = V^* \in A_l$ there exists an $\varepsilon_0 > 0$ such that for $|\varepsilon| < \mathcal{E}_0$, $\varepsilon \in R^1$, the C^*-dynamical systems $\{\mathfrak{A}(\mathcal{H}), \tau_t^0\}$ and $\{\mathfrak{A}(\mathcal{H}), \tau_i^{\varepsilon V}\}$ are equivalent.*

We preface the proof of Theorem 4 with two remarks.

REMARK 1. In contrast to the forward Møller morphisms, when $\nu = 1, 2$ the inverse Møller morphisms may not exist for any nonzero value of the coupling constant ε. An example may be constructed as follows.

EXAMPLE. Let $V = -a^*(f_0)a(f_0)$; then $\tau_t^{\varepsilon V} \in V$ is again a free dynamics, generated by the operator $h_\varepsilon = h + \varepsilon P_0$, where P_0 is the projection on the vector f_0. For $A = A^\#(f)$ the inverse Møller morphisms look like

$$\widehat{\gamma}_\pm(A) = \lim_{t \to \pm\infty} \tau_{-t}^0 \tau_t^{\varepsilon V}(a^\#(f)) = a^\#(W_\pm f), \tag{23}$$

where $\widehat{W}_\pm$ are the ordinary invertible wave operators. It is well known that if $h = -\Delta$ and the function f_0 is such that $\widetilde{f}_0 \in C_0^\infty(R^\nu)$ satisfies

$$\int_{R^\nu} \widetilde{f}_0(k)dk < \infty, \tag{24}$$

then the operator h_ε has an eigenvalue $\lambda_\varepsilon < 0$ (with eigenvector e_ε) for arbitrarily small ε. Therefore $\widehat{W}_\pm e_\varepsilon$ does not exist, and hence neither does $\widehat{\gamma}_\pm(a^\#(a_\varepsilon))$.

REMARK 2. In Theorem 4 with $\nu \geq 3$, the essential reason for requiring ε to be small is to rule out bound states, which may be present for large ε. The previous example with $\nu \geq 3$ and large $|\varepsilon|$ illustrates this phenomenon.

PROOF OF THEOREM 4. By Cook's criterion and the remark made in the proof of Theorem 3, to show that $\widehat{\gamma}_+$ exists it suffices to prove that

$$\|[\tau_t^{\varepsilon V}(V), a^\#(f)]\| \in L_1(R_+),$$

that is, that

$$\int_0^\infty \|[\tau_t^{\varepsilon V}(V), a^\#(f)]\| dt < \infty$$

for all $f \in C_0^\infty(R^\nu)$ for both values of $\#$.

Using equation (5.4.0), we obtain

$$\begin{aligned} \int_0^\infty \|[\tau_t^{\varepsilon V}(V), a^\#(f)]\|\, dt &\leq \int_0^\infty \|[\tau_t^0(V), a^\#(f)]\|\, dt \\ &+ \sum_{n=1}^\infty |\varepsilon|^n \int_0^\infty \int_{\Delta_n} \|[a^\#(f), [\tau_{s_1}^0(V)[\dots[\tau_{s_n}^0(V), \tau_t^0(V)]\dots]]\|\, ds_1 \dots ds_n\, dt. \end{aligned} \tag{25}$$

LEMMA 5. *There exist a constant $C = C(V, \nu)$ independent of f, and a constant $C(f, V)$ such that the following estimates hold*:

a)

$$\int_0^\infty \|[\tau_s^0(V), a^\#(f)]\|\, ds < C \cdot C(f, V); \tag{26}$$

b)

$$\int_0^\infty \int_{\Delta_n^s} \|[a^\#(f), [\tau_{s_1}^0, [\dots [\tau_{s_n}^0(V), \tau_s^0(V)] \dots]]\| ds_1 \dots ds_n ds < C^{n+1} C(f, V). \tag{27}$$

REMARK. Evidently, Theorem 4 follows from Lemma 5 with $\varepsilon_0 = C^{-1}$.

PROOF OF LEMMA 5. We first estimate the integrand in (27) by a sum:

$$\|[a^\#(f), [\tau_{s_1}^0(V), [\dots [\tau_{s_n}^0(V), \tau_s^0(V)] \dots]]\| \le \sum_G W_G(s_1, \dots, s_n, s), \tag{27^a}$$

where $\sum_G$ is taken over all admissible diagrams with weight W_G. These admissible diagrams G and their weights will be described below. We now estimate

$$\int_0^\infty \left(\int_{\Delta_n^s} \left[\sum_G W_G(s_1, \dots, s_n, s) \right] ds_1 \dots ds_n \right) ds < C^{n+1} C(f, V)$$

by means of a special technique for estimating sums of diagrams. To each s_i, $i = 0, \dots, n+1$, where $s_{n+1} = s$ and $s_0 = 0$, we associate the vertex with subscript $n + 1 - i$. We have

$$\begin{aligned} \tau_{s_j}^0(V) = \sum_{i=1}^{d} \sum_{k=1}^{M} & c_k a^*(e^{is_j h} f_{i1}) \dots a^*(e^{is_j h} f_{i,m_i}) \\ & \times a(e^{is_j h} f_{i,m_i+1}) \dots a(e^{is_j h} f_{i,m_i+n_i}). \end{aligned} \tag{28}$$

Expression (27^a) is an $(n+1)$-fold commutator, which we will expand out in $n+1$ steps.

In the first step, we work on the innermost commutator $[\tau_{s_n}^0(V), \tau_{s_n+1}^0(V)]$ and use the canonical anticommutation relations to move the creation and annihilation operators in $\tau_{s_n}^0(V)$, i.e., the factors $a^\#(e^{is_n h} f_{i,j})$ past the operator $\tau_{s_n+1}^0(V)$. In other words, we use the formulas

$$\begin{aligned} a(f)a(g) &= -a(g)a(f), \\ a(f)a^*(g) &= -a^*(g)a(f) + (f, g)I \end{aligned}$$

and their analogs for $a^*(f)$. This shuffling process gives rise to new factors of the form

$$(e^{is_n h} f_{i,j}, e^{is_{n+1} h} f_{i',j'}) \tag{29}$$

or their adjoints. The generation of these factors will be called "pairing", and they will be shuffled one by one: first we move the leftmost creation/annihilation operator $a^\#(f)$ in

each term in $\tau^0_{s_n}(V)$ past the entire operator $\tau^0_{s_n+1}(V)$. In each pairing the operator $a^{\#}(f)$ produces a term

$$(e^{is_nh}f_{i,j}, e^{is_{n+1}h}f_{i',j'})W, \tag{30}$$

where W is a monomial in the creation/annihilation operators $a^{\#}(e^{is_hh}f_{i,j})$ or $a^{\#}(e^{is_{n+1}}hf_{i',j'})$. These terms will henceforth not be further disturbed, and instead we shuffle the next operator $a^{\#}(f)$ in $\tau^0_{s_n}(V)$ past $\tau^0_{s_{n+1}}(V)$, again picking up a set of terms of the form (30). Continuing this process and noting that since V is even, the unpaired terms are reduced to $\tau^0_{s_{n+1}}(V)\tau^0_{s_n}(V)$, we see that the commutator $W_1 = [\tau^0_{s_n}(V), \tau^0_{s_{n+1}}(V)]$ is a sum of terms of the type (30). To each such term we associate the edge (s_n, s_{n+1}), with contribution (29).

Continuing, we can represent the commutator $[\tau^0_{s_{n-1}}(V), W_1]$ by means of an analogous shuffling procedure, in which creation/annihilation operators of the form $a^{\#}(e^{is_{n-1}h}f_{i,j})$ are moved past W_1. At the second step we get some factors of the form

$$(e^{is_{n-1}h}f_{i,j}, e^{is_{n+1}h}f_{i',j'})$$

or

$$(e^{is_{n-1}h}f_{i,j}, e^{is_nh}f_{i',j'}).$$

In other words, the commutator $[\tau^0_{s_{n-1}}(V), W_1]$ is a sum of terms

$$(e^{is_{n-1}h}f^{(2)}_{i,j}, e^{is_kh}f^{(2)}_{i',j'})(e^{is_nh}f^{(1)}_{i,j}, e^{is_{n+1}h}f^{(1)}_{i',j'})\widetilde{W},$$

where $k = n$ or $n+1$, and $\widetilde{W}$ is a monomial involving creation operators of the form $a^{\#}(e^{is_{n-1}h}f)$, $a^{\#}(e^{is_nh}f')$, $a^{\#}(e^{is_{n+1}h}f'')$. We associate the edge (s_{n-1}, s_k) to the factor $(e^{is_{n-1}h}f^{(2)}_{i,j}, e^{ish}f^{(2)}_{i',j'})$. Continuing this procedure, we get at each step v an edge $(s_v, s_{v'})$, where $v' = v'(v) > v$, with a factor of the form

$$r_{v,v'} = (e^{is_vh}f_{i,j}, e^{is_{v'}h}f_{i',j'}).$$

Clearly, for all $1 < i, j, i', j' \le d$, $0 < v, v' < n+1$ we have the estimate

$$|r_{v,v'}| < \frac{C}{(|s_v - s_{v'(v)}| + 1)^{\delta}}, \tag{30^a}$$

where $\delta = \nu/2 - \delta' > 1$, δ' is an arbitrarily small constant, and $C > 0$ does not depend on i, j, i', j', v, v', s_v, $s_{v'}$. In the last step we shuffle the operator $a^{\#}(f)$, so that the $(n+1)$-fold commutator

$$a^{\#}(f)[\tau^0_{s_1}(V)[\dots[\tau^0_{s_n}(V)\tau^0_{s_{k+1}}(V)]]]$$

is now expressed as a sum of terms of the form

$$\prod_v r(v, v')\widetilde{W}, \tag{31}$$

where $\widetilde{W}$ is a monomial in creation/annihilation operators of the form $a^{\#}(e^{is_nh}f)$, and the set $\{(v, v')\}$ forms a diagram G with vertices at the points $0 = s_0 < s_1 < \cdots < s_{n+1}$ lying on the real line and listed in increasing order. From each vertex s_v there is exactly one

right-directed edge $(s_v, s_{v'})$, i.e., one such that $v' > v$, and at most $2m_{\max}$ edges $(s_v, s_{v''})$ going to the left $(v'' < v)$. Such a diagram G will be called *admissible.* Noting that the number of monomials of the form (31) corresponding to the same diagram G is at most C_0^n, and the norm of each monomial is also bounded by C_1^n, where C_1 and C_0 are absolute constants, we obtain finally the estimate (27^a), where the sum on the right in (27^a) is over all the admissible diagrams G described above, and the weight W_G is given by the formula

$$W_G = (C_0 C_1)^n \prod_v \frac{C}{(|s_v - s_{v'(v)}| + 1)^\delta}.$$

It follows from (27^a) that the integral (27) is bounded by

$$\begin{aligned} &\int_{\Delta_{n+1}^\infty} \sum W_G(s_1, \dots, s_n, s_{n+1}) ds_1 \dots ds_n \\ &\qquad = (\overline{C})^n \int_{\Delta_{n+1}^\infty} \sum_{\{v'(v)\}} \prod_v \frac{C}{(|s_v - s_{v'(v)}| + 1)^\delta} ds_1 \dots ds_{n+1}, \end{aligned} \tag{32}$$

where each v corresponds to a unique number $n+1 \geq v'(v) > v$ and the sum $\sum_{\{v'(v)\}}$ is over all sets $\{v'(v),\ v = 0, \dots, n\}$ such that at most $2m_{\max}$ of the numbers in $\{v'(0), \dots, v'(n)\}$ coincide with l, where l takes the values $0, 1, \dots, n+1$.

The next lemma will be used to estimate the total contribution from all the diagrams.

Lemma 6. *Let $g \in L_1(R)$, $g(t) \geq 0$ for all $t \in R^1$. Then for all n we have the bound*

$$\int_{\Delta_{n+1}^\infty} \Big(\sum_{\{v'(v)\}} \prod_v g(t_v - t_{v'(v)}) \Big) dt_1 \dots dt_{n+1} \leq c^n \Big[\int_R g(t)\, dt \Big]^{n+1}, \tag{33}$$

where the sum $\sum$ is over all sets of admissible diagrams, and the constant $c > 0$ is independent of n.

Lemma 5 clearly follows from the estimates (32) and (33).

Proof of Lemma 6. We consider the Riemann sum approximations to both sides of inequality (33) and prove that for them, (33) holds for any d satisfying

$$\begin{aligned} d^n \Big(\sum_{0 < t_1 < \dots < t_n} \sum_{\{v'(v)\}} \prod_v g(t_v - t_{v'(v)}) \Big) &\leq d^{n+1} c^n \Big(\sum_{s \neq 0} g(s) \Big)^{n+1} \\ &= d^n c^n \Big(\sum_{s_1 \neq 0} \cdots \sum_{s_{n+1} \neq 0} \prod_{i=1}^{n+1} g(s_i) \Big), \end{aligned} \tag{34}$$

where d is the step size used in the approximation. The sum in (34) is over all $t_i, s_i \in Z_d$, where Z_d is a one-dimensional lattice with spacing d.

We will use an algorithmic procedure to show that for any set $(s_0, \dots, s_{n+1})$ with $s_0 = 0$, to the corresponding term on the right in (34) there are associated at most C^n admissible diagrams on the left-hand side, with contribution

$$g(s_1) \dots g(s_{n+1}).$$

The algorithm stops after at most $2m_{\max}n$ steps, which we enumerate

$$(1,1),\dots,(1,2m_{\max}),\dots,(n,1),\dots,(n,2m_{\max}).$$

At step $(1,1)$ we take s_1 and construct an edge from the first vertex in s_1; we have thus constructed vertices 1, s_1 and an edge joining them. We now continue inductively. Assume that edges of length $s_1,\dots,s_q$ have been constructed and that we have reached step (i,j); we now proceed according to the following rules:

1. At each step we either construct one edge, or else we do not construct any edge (and correspondingly, one vertex or no vertex at all).

2. If an edge is not constructed at step (i,j), then no edge will be constructed in the subsequent steps (i,j'), $j'>j$.

3. At step $(i,1)$ we choose one of the vertices v_i already constructed, and in the subsequent steps $(1,1),\dots,(1,2m_{\max})$ we can construct edges only from the vertex v_i. We say that v_i is "used" at step $(i,1)$.

4. The choice of the vertex v_i is uniquely determined by the following rule: v_i is the first (has the smallest index) of the vertices that have already been constructed but have not been used in the preceding steps, with the exception of the zero vertex; if all the vertices constructed have been used, then we pick a new (not previously constructed) vertex with smallest possible index (excluding the zero vertex).

5. The algorithm stops either at step $(n,2m_{\max})$, or when there are no unused vertices, or when all n edges have been constructed, i.e., all the $s_1,\dots,s_{n+1}$ have been exhausted.

Evidently, any diagram G can be constructed using this algorithm, and each set $s_1,\dots,s_{n+1}$ will be used at most C^N times, where C depends on $m_{\max}$, since at steps $(i,1),\dots,(i,2m_{\max})$ (for each i, $1\le i\le n$) the algorithm can branch only in the following ways:

a) an edge can go to the right or to the left;

b) during these steps, at most $2m_{\max}$ edges can be constructed from the v_ith vertex;

c) if the edge is directed to the right, then it may terminate in the zero vertex or it may not.

Inequality (33) follows from (34) by letting $d\to 0$. This proves Lemma 6, and hence also Theorem 4 for an interaction V in the class A_l^0.

To generalize the proof of Theorem 4 to interactions of class A_l, it is clearly enough to express $V\in A_l$ in the form (12) in such a way that the estimates (13)–(15) are satisfied, and it was shown in § 2 that this is possible. Theorem 4 is thus completely proved.

4. Unitary equivalence of the Hamiltonians for a free and bounded perturbed Fermi gas in the ground state. In the previous sections we proved the existence of the Møller morphisms as $*$-automorphisms of the CAR algebra for two dynamics: the free Fermi gas, and a Fermi gas with a bounded interaction. In the case of an interaction $V\in A$, i.e., when each monomial in V contains annihilation operators (appearing as always to the left of the creation operators), the vacuum $\Omega\in\mathcal{F}_{as}$ as remains a ground-state vector for both the free and the perturbed dynamics (there is no "polarization" of the vacuum), i.e., the ground states for both dynamics are of the form

$$\langle A\rangle=(A\Omega,\Omega),\qquad A\in\mathfrak{A}(\mathcal{H}).$$

The GNS representation of the CAR algebra constructed from this state is defined in a Fock Fermi space, and the operators H^0_{GNS} and H^V_{GNS} for the free and perturbed dynamics coincide with the Hamiltonians $H_0=d\Gamma(h)$ and H_0+V, respectively. Thus in essence we need to determine when the operators H_0 and H acting on $\mathcal{F}_{as}$ are unitarily equivalent.

We prove the following theorem.

THEOREM 7. *Let $\mathcal{H} = L_2(R^\nu, d^\nu x)$, $H_0 = d\Gamma(h)$, $H_\varepsilon = H_0 + \varepsilon V$, where $h \in \mathbb{H}$, $V \in A$. Then for $\nu \geq 3$ there exists an $\varepsilon_0 > 0$ such that when $|\varepsilon| < \varepsilon_0$ the forward wave operators exist and are invertible, and*

$$W_\pm = s - \lim_{t \to \pm\infty} e^{-it(H_0+\varepsilon V)} e^{itH_0}, \tag{35}$$

$$H_0 + \varepsilon V = W_\pm H_0 W_\pm^{-1}. \tag{36}$$

REMARK. For an interaction $V \in A_l \cap A$, Theorem 7 follows from Theorem 4. Indeed, because there is no polarization of the vacuum in this case, i.e., $V\Omega = 0$, we have

$$e^{it(H_0+V)}\Omega = e^{itH_0}\Omega = \Omega \tag{37}$$

for all t.

Therefore, for

$$A = a^*(f_1) \dots a^*(f_n), \qquad \widetilde{f}_i \in C_0^\infty(R^\nu), \tag{38}$$

the limit

$$\lim_{t \to \pm\infty} e^{-itH_0} e^{it(H_0+\varepsilon V)} A\Omega$$
$$= \lim_{t \to \pm\infty} e^{-itH_0} e^{it(H_0+\varepsilon V)} A e^{-it(H_0+\varepsilon V)} e^{itH_0}\Omega = \widehat{\gamma}_\pm(A)\Omega$$

exists by Theorem 4 for sufficiently small ε. But since linear combinations of vectors of the form $A\Omega$, with A given by (38), are dense in $\mathcal{F}_{as}$, the inverse wave operators exist. The same is evidently also true for the forward wave operators.

PROOF OF THEOREM 7. Again, there is no polarization of the vacuum when $V \in A$; however, we cannot use Theorem 4 here because V may contain odd monomials. But we can use the idea of the proof of Theorem 4.

Let

$$\Phi_N = (\varphi_0, \varphi_1, \dots, \varphi_N, 0, \dots) \in \mathcal{F}_{as,0}, \qquad \widetilde{\varphi}_k \in C_0^\infty(R^{\nu k}).$$

The inverse wave operator $\widehat{W}_+\Phi_N$ is expressible as a perturbation-theoretic series

$$\widehat{W}\Phi_N = \Phi_N + \sum_{n=1}^{\infty} (-i\varepsilon)^n \int_{\Delta_n^{0,t}} dt_1 \dots dt_n V(t_n) \dots V(t_1)\Phi_N. \tag{39}$$
$$\widehat{W}_\pm = \lim_{t \to +\infty} \widehat{W}_+(t),$$

where $\Delta_n^{0,t} = \{(t_1, \dots, t_n),\ 0 < t_1 < t_2 < \dots < t_n < t\}$,

$$V(t) = e^{itH_0} V e^{-itH_0}.$$

The integrand is a product of Wick monomials and can be expressed as a sum

$$V(t_n) \dots V_{t_1} = \sum_G V_G(t_1, \dots, t_n) \tag{40}$$

of Wick moments indexed by *Friedrichs diagrams* (see §2 below). But there are too many of these diagrams, so we will use another expansion, in which we in fact carry out a partial resummation of Friedrichs diagrams. We write the vector Φ_N as a polynomial P in the creation operators, applied to the vacuum vector Ω, and insert this expression for Φ_N into equation (39).

We further associate to the monomial $V(t_v)$ the vertex v of some graph and single out the right-most annihilation operator in $V(t_v)$ of the form $a(f^{(v)}(t_v))$, where

$$f^{(v)}(t_v) = e^{it_v h} f^{(v)},$$

which, by using the anticommutation relations

$$a(f^{(v)}(t_v))a^*(f^{(v')}(t_{v'})) = -a^*(f^{(v')}(t_{v'}))a(f^{(v)}(t_v)) + (f^{(v')}(t_{v'}), f^{(v)}(t_v)) \tag{41}$$

can be moved over to the right so as to act on the vacuum vector Ω.

During this shuffling, one of the two terms on the right in (41) is generated. If it is the first term, we continue the shuffle, while if the second term is generated (i.e., there is a pairing) then we say that a line (v, v') of the diagram has been generated. If no pairing occurs when the annihilation operator is moved over to the right, then the term gives a zero contribution because $a(f)\Omega = 0$ for all $f \in \mathcal{H}$. We thus get precisely n edges $(v, v'(v))$, $v = 1, \dots, n$. If a pairing occurs with a creation operator in the polynomial P then we say that $v'(v) = 0$.

Evidently, we have

$$|(f^{(v')}(t_{v'}), f^{(v)}(t_v))| < \frac{C}{(1 + |t_v - t_{v'}|)^{\nu/2}}. \tag{42}$$

Consider the resulting graph G with vertices $n, n+1, \dots, 1, 0$ and edges $(v, v'(v))$, which by construction is connected. We denote the class of all such graphs by $\mathbb{C}_N^n$.

We note that the graphs in $\mathbb{C}_N^n$ differ from the corresponding admissible graphs described above only in that the zero vertex may belong to several edges (not just one), but in any case this number is at most N.

It is easy to see that in this case we have the estimate

$$\|V(t_n) \dots V(t_1)\Phi_N\| \le C(\Phi_N)c^N \Bigg(\sum_{\{v'(v)\} \in \mathbb{C}_N^n} \prod_v \frac{1}{(1 + |t_v - t_{v'(v)}|)^\delta} \Bigg). \tag{43}$$

LEMMA 8. *Let $g \in L_1(R)$, $g(t) \ge 0$ for all $t \in R^1$. Then for all n we have the estimate*

$$\int_{\Delta_n^\infty} \Bigg(\sum_{\{v'(v)\} \in \mathbb{C}_N^n} \prod_v g(t_v - t_{v'(v)}) \Bigg) dt_1 \dots dt_n \le N! c^n \Bigg(\int_R g(t) dt \Bigg)^n, \tag{44}$$

where the sum $\sum$ is over all sets of admissible diagrams, and the constant c does not depend on n.

The proof of Lemma 8 is completely analogous to that of Lemma 6, and Lemma 8 implies Theorem 7.

5. Unitary equivalence of the Hamiltonians for the free and boundedly perturbed Fermi gas in KMS states. We consider a free Fermi gas with dynamics τ_t^0 defined by a one-particle Hamiltonian $h = -\Delta + \mu$ on the space $L_2(R^\nu)$, and its unique β-KMS state $\langle\cdot\rangle_\beta^0$ on the CAR algebra. Assume further that we are given an interaction $V = V^* \in \mathfrak{A}$ generating the dynamics τ_t^V (see (2^a)). We have the next lemma, which is proved in [49].

LEMMA. *For the dynamics τ_t^V there exists a unique β-KMS state $\langle\cdot\rangle_\beta^V$. It is given by*

$$\langle A\rangle_\beta^V = \frac{\langle F^*AF\rangle_\beta^0}{\langle F^*F\rangle_\beta^0},$$

where the element $F \in \mathfrak{A}$ is defined by

$$\begin{aligned} F &\stackrel{\text{def}}{=} e^{-\beta/2(H_0+V)}e^{\beta/2H_0} \\ &= 1 + \sum_{n=1}^{\infty}\int_0^{\beta/2}\int_0^{s_1}\cdots\int_0^{s_{n-1}} \tau_{is_1}^0(V)\tau_{is_2}^0(V)\ldots\tau_{is_n}^0(V)\,ds_1\ldots ds_n, \end{aligned} \tag{45}$$

and the series (45) *converges in norm.*

DEFINITION. A C^*-dynamical system $(\mathfrak{A}, \tau)$ is said to be $L_1(\mathfrak{A}^0)$ asymptotically abelian if

$$\int_{-\infty}^{\infty} \|[A, \tau_t(B)]\|dt < \infty \tag{46}$$

for all A, B in a norm-dense $*$-subalgebra $\mathfrak{A}^0$ of $\mathfrak{A}$.

The KMS states are related to Møller morphisms in the following simple way.

ASSERTION 10 (see [49]). *Let $(\mathfrak{A}(\mathcal{H}), \tau_t^0)$ be $L_1(\mathfrak{A}^0)$ asymptotically abelian. Then for every $V = V^* \in \mathfrak{A}^0$ the forward Møller morphisms*

$$\gamma_\pm(A) = \lim_{t\to\pm\infty} \tau_{-t}^V(\tau_t^0(A)), \qquad A \in \mathfrak{A}(\mathcal{H})$$

exist, and if $\langle\cdot\rangle_V$ is a (τ_t^V, β) KMS state with $\beta \neq 0$, then $\langle\gamma_\pm(\cdot)\rangle_V$ is a (τ_t^0, β) KMS state.

Note that $(\mathfrak{A}_l(\mathcal{H}), \tau_t^0)$ is $L_1(\mathfrak{A}_l^0(\mathcal{H}))$ asymptotically abelian, where τ_t^0 is generated by the one-particle Hamiltonian h.

THEOREM 11. *Let $\nu \geq 3$, $\beta \neq 0$, and $V = V^* \in A_l$. Then there exists an $\varepsilon_0 = \varepsilon_0(V, \nu) > 0$ such that for $|\varepsilon| < \varepsilon_0$ the operators H_{GNS}^0 and $H_{\text{GNS}}^{\varepsilon V}$ constructed from the corresponding KMS states are unitarily equivalent.*

PROOF. By Theorem 4, there exists an $\varepsilon_0 = \varepsilon_0(V, \nu) > 0$ such that for $|\varepsilon| < \varepsilon_0$ the Møller morphisms $\gamma_\pm$ exist and are invertible. It follows from Assertion 10 that $\langle\gamma_\pm(\cdot)\rangle_{\varepsilon V}$ is a (τ_t^0, β) KMS state on $\mathfrak{A}_l(\mathcal{H})$; but such a state is unique (see [49]). By the gauge invariance of $\langle\cdot\rangle_0$ and $\langle\cdot\rangle_{\varepsilon V}$ we have

$$\langle\gamma_\pm(A)\rangle_{\varepsilon V} = \langle A\rangle_0 \tag{47}$$

for all $A \in \mathfrak{A}(\mathcal{H})$.

We now define the operators $U_\pm\colon \mathcal{H}^0_{\mathrm{GNS}} \to V^{\varepsilon V}_{\mathrm{GNS}}$ by

$$U_\pm(\pi_0(A)\Omega_0) = \pi_{\varepsilon V}(\gamma_\pm(A))\Omega_{\varepsilon V}, \tag{48}$$

where for brevity we have set $\pi_0 = \pi_{\langle\cdot\rangle_0}$, $\pi_{\varepsilon V} = \pi_{\langle\cdot\rangle_{\varepsilon V}}$, $\Omega_0 = \Omega_{\langle\cdot\rangle_0}$, $\Omega_{\varepsilon V} = \Omega_{\langle\cdot\rangle_{\varepsilon V}}$.

The operators $U_\pm$ are unitary. Indeed,

$$\begin{aligned}(U_\pm\pi_0(A)\Omega_0, U_\pm\pi_0(B)\Omega_0) &= (\pi_{\varepsilon V}(\gamma_\pm(A))\pi_{\varepsilon V}(\gamma_\pm(\beta))\Omega_{\varepsilon V}, (\gamma_\pm(B))\Omega_{\varepsilon V})\\ &= \langle(\gamma_\pm(B))^*\gamma_\pm(A)\rangle_{\varepsilon V} = \langle\gamma_\pm(B^*A)\rangle_{\varepsilon V}\\ &= \langle B^*A\rangle_0 = (\pi_0(A)\Omega_0, \pi_0(B)\Omega_0).\end{aligned}$$

Clearly, $\operatorname{Ran} U_\pm = \mathcal{H}^{\varepsilon V}_{\mathrm{GNS}}$ and the invertible operators $U_\pm^{-1}$ are given as follows:

$$U_\pm^{-1}(\pi_{\varepsilon V}(A)\Omega_{\varepsilon V}) = \pi_0(\gamma_\pm^{-1}(A)\Omega_0) \tag{49}$$

for all $A \in \mathfrak{A}(\mathcal{H})$.

By the definition of H^0_{GNS} and $H^{\varepsilon V}_{\mathrm{GNS}}$, we have

$$\begin{aligned}e^{itH^0_{\mathrm{GNS}}}\pi_0(A)\Omega_0 &= \pi_0(\tau_t^0(A))\Omega_0,\\ e^{itH^{\varepsilon V}_{\mathrm{GNS}}}\pi_{\varepsilon V}(A)\Omega_{\varepsilon V} &= \pi_{\varepsilon V}(\tau_t^{\varepsilon V}(A))\Omega_{\varepsilon V}.\end{aligned}$$

Hence since $\tau_t^{\varepsilon V}(A) = \gamma_\pm\tau_t^0\gamma_\pm^{-1}$, we have

$$\begin{aligned}U_\pm^{-1}e^{itH^{\varepsilon V}_{\mathrm{GNS}}}U_\pm\pi_0(A)\Omega_0 &= U_\pm^{-1}e^{itH^{\varepsilon V}_{\mathrm{GNS}}}\pi_{\varepsilon V}(\gamma_\pm(A))\Omega_{\varepsilon V}\\ &= U_\pm^{-1}\pi_{\varepsilon V}(\tau_t^{\varepsilon V}(\gamma_\pm(A)))\Omega_{\varepsilon V} = \pi_0(\gamma_\pm^{-1}\tau_t^{\varepsilon V}\gamma_\pm(A))\Omega_0\\ &= \pi_0(\tau_t^0(A))\Omega_0 = e^{itH^0_{\mathrm{GNS}}}\pi_0(A)\Omega_0.\end{aligned}$$

Thus,

$$e^{itH^0_{\mathrm{GNS}}} = U_\pm^{-1}e^{itH^{\varepsilon V}_{\mathrm{GNS}}}U_\pm, \qquad t \in R, \tag{50}$$

and consequently,

$$U_\pm^{-1}H^{\varepsilon V}_{\mathrm{GNS}}U_\pm = H^0_{\mathrm{GNS}}. \tag{51}$$

6. The operator H_{GNS} on the Hilbert space H_{GNS}. Let $(\mathcal{H}^0_{\mathrm{GNS}}, \pi_0, \Omega_0)$, $(\mathcal{H}^{\varepsilon V}_{\mathrm{GNS}}, \pi_{\varepsilon V}, \Omega_{\varepsilon V})$ be cyclic representations of $\mathfrak{A}(\mathcal{H})$ with respect to the β-KMS states $\langle\cdot\rangle_0$ and $\langle\cdot\rangle_{\varepsilon V}$, respectively. Let H^0_{GNS} and $H^{\varepsilon V}_{\mathrm{GNS}}$ be the generators of the unitary groups acting on $\mathcal{H}^0_{\mathrm{GNS}}$ and $\mathcal{H}^{\varepsilon V}_{\mathrm{GNS}}$ and induced by the dynamics τ_t^0 and $\tau_t^{\varepsilon V}$ on $\mathfrak{A}(\mathcal{H})$.

We define the operator $U\colon \mathcal{H}^{\varepsilon V}_{\mathrm{GNS}} \to \mathcal{H}^0_{\mathrm{GNS}}$ by

$$U\pi_{\varepsilon V}(A)\Omega_{\varepsilon V} = C_{\varepsilon V}\pi_0(AF)\Omega_0, \tag{52}$$

where

$$C_{\varepsilon V} = \langle F^*F\rangle, \tag{53}$$

and F is given by expression (45). We observe that U is unitary. Indeed, for all $A, B \in \mathfrak{A}(\mathcal{H})$ we have

$$\begin{aligned}(U\pi_{\varepsilon V}(A)\Omega_{\varepsilon V}, U\pi_{\varepsilon V}(B)\Omega_{\varepsilon V}) &= C_{\varepsilon V}^2\langle (BF)^* AF\rangle_0 \\ &= C_{\varepsilon V}^2\langle F^* B^* AF\rangle_0 = \langle B^* A\rangle_{\varepsilon V} = (\pi_{\varepsilon V}(A)\Omega_{\varepsilon V}, \pi_{\varepsilon V}(B)\Omega_{\varepsilon V}).\end{aligned}$$

We will assume that the perturbation εV is polynomial in the creation and annihilation operators $a^{\#}(f)$, where F is an analytic vector for the operator h (see [49]). For small ε the element F of the C^*-algebra $\mathfrak{A}(\mathcal{H})$ is invertible, and (see [49])

$$\begin{aligned}F^{-1} &= e^{-\beta/2H_0} e^{\beta/2(H+\varepsilon V)} \\ &= 1 + \sum (-\varepsilon)^n \int_0^{\beta/2} \int_0^{s_n} \cdots \int_0^{s_2} \tau_{is_1}(V)\tau_{is_2}(V)\ldots\tau_{is_n}(V) ds_1 \ldots ds_n.\end{aligned} \tag{54}$$

Therefore, the inverse operator $U^{-1}\colon \mathcal{H}_{\mathrm{GNS}}^0 \to \mathcal{H}_{\mathrm{GNS}}^{\varepsilon V}$ acts by

$$U^{-1}\pi_0(A)\Omega_0 = C_{\varepsilon V}^{-1}\pi_{\varepsilon V}(AF^{-1})\Omega_{\varepsilon V}.$$

Assertion 12. *The operator $H_{\mathrm{GNS}}^{\varepsilon V}$ defined on the space $\mathcal{H}_{\mathrm{GNS}}^{\varepsilon V}$ is unitarily equivalent to the operator H' acting on $\mathcal{H}_{\mathrm{GNS}}^0$ by*

$$H'\pi_0(A)\Omega_0 = H_{\mathrm{GNS}}^0\pi_0(A)\Omega_0 + i\varepsilon(\pi_0(VA)\Omega_0 - \pi_0(AV_{\beta/2})\Omega_0), \tag{55}$$

for all $A \in \mathfrak{A}(\mathcal{H})$ such that $\pi_0(A)\Omega_0 \in D(H_{\mathrm{GNS}}^0)$; here $V_{\beta/2} = \tau_{i\beta/2}^0(V) \in \mathfrak{A}(\mathcal{H})$ and $D(H_{\mathrm{GNS}}^0)$ is the domain of the operator H_{GNS}^0.

Proof. We will prove that the operator in (55) coincides with the operator $UH_{\mathrm{GNS}}^{\varepsilon V}U^{-1}$. We have

$$\begin{aligned}UH_{\mathrm{GNS}}^{\varepsilon V}U^{-1}\pi_0(A)\Omega_0 &= UH_{\mathrm{GNS}}^{\varepsilon V}(C_{\varepsilon V}^{-1}\pi_{\varepsilon V}(AF^{-1})\Omega_{\varepsilon V}) \\ &= C_{\varepsilon V}^{-1}U\pi_{\varepsilon V}(i[H_0+\varepsilon V, AF^{-1}])\Omega_{\varepsilon V}.\end{aligned} \tag{56}$$

Manipulating (56) further, we obtain

$$\begin{aligned}UH_{\mathrm{GNS}}^{\varepsilon V}U^{-1}\pi_0(A)\Omega_0 &= \pi_0(i[H_0+\varepsilon V, AF^{-1}]F)\Omega_0 \\ &= \pi_0(i[(H_0+\varepsilon V)A])\Omega_0 - \pi_0((iAF^{-1}H_0 + V)F)\Omega_0 \\ &= \pi_0(i[(H_0+\varepsilon V)A])\Omega_0 - \pi_0(iA\tau_{i\beta/2}(H_0+\varepsilon V))\Omega_0,\end{aligned} \tag{57}$$

since

$$\begin{aligned}F^{-1}(H_0+\varepsilon V)F &= e^{-\beta/2H_0}e^{\beta/2(H_0+\varepsilon V)}(H_0+\varepsilon V) \\ e^{-\beta/2(H_0+\varepsilon V)}e^{\beta/2H_0} &= e^{-\beta/2H_0}(H_0+\varepsilon V)e^{\beta/2H_0} = \varepsilon e^{-\beta/2H_0}Ve^{\beta/2H_0} + H_0.\end{aligned} \tag{58}$$

Here we have used the following facts: 1) the operators $H_0 + \varepsilon V$ and $e^{\pm\beta(H_0+\varepsilon V)}$ commute; 2) we have

$$e^{\beta/2(H_0+\varepsilon V)}e^{-\beta/2(H_0+\varepsilon V)} = \mathbf{E}_{\mathcal{F}_{as}(\mathcal{H})} \tag{59}$$

on the set of analytic vectors of H_0 in $\mathcal{F}_{as}(\mathcal{H})$; 3) the operators H_0 and $e^{\pm\beta/2H_0}$ commute; and 4) we have

$$e^{-\beta/2H_0}e^{\beta/2H_0} = \mathbf{E}_{\mathcal{F}_{as}(\mathcal{H})} \tag{60}$$

on the set of analytic vectors of H_0. Formula (55) evidently follows from (57) and (58).

§2. Asymptotic completeness for interactions with vacuum polarization (linked cluster theorem)

In subsection 5 of the previous section we considered a perturbation $V \in A$ of the free Hamiltonian H_0 that preserves the vacuum: $V\Omega = 0$, and we established under some general hypotheses that the operators H_0 and H_0+V acting on $\mathcal{F}_a(\mathcal{H})$ are unitarily equivalent. In the present section, we consider a more general perturbation $V \in A_l$ which no longer preserves the vacuum. In this case, strictly speaking, the operators H_0 and $H_0 + \varepsilon V$ are no longer unitarily equivalent, but this is not very important: for small ε there exists a real λ_ε such that the operator $H_0 + \varepsilon V$ is unitarily equivalent to $H_0 + \lambda_\varepsilon V$, i.e., the spectra of these operators differ only by a translation.

On a formal level this fact is well known, e.g., from the famous "linked cluster" theorem, which is a basic computational tool of perturbation theory in many-body quantum theory. The main result of this chapter is the proof of this theorem. This will allow us to rigorously establish all the formal consequences presented in the classical books by K. Friedrichs [42] and K. Hepp [44].

We will consider the following two situations:

1) $\mathcal{F}_a = \mathcal{F}_{as}(L_2(R^\nu))$ is the antisymmetric Fock space over $L_2(R^\nu)$, $\nu \geq 3$, $H = H_0 + \varepsilon V$, where $H_0 = d\Gamma(h)$, or in the k-representation

$$H_0 = \int_{R^\nu} h(k) a^*(k) a(k)\, d^\nu k, \tag{1}$$

and $V \in A_l$, with either

$$h(k) = (k^2 + m^2)^{1/2}, \quad m > 0 \quad \text{(relativistic case)},$$

or

$$h(k) = Lk^2, \quad L > 0 \quad \text{(massless case)}.$$

Since $V = V^* \in \mathfrak{A}(\mathcal{H})$, we have that $H = H_0 + \varepsilon V$ is a selfadjoint operator on $\mathcal{F}_a$ with the same dense domain of definition as H_0.

2) $\mathcal{F}_a = \mathcal{F}_{as}(l_2(Z^\nu))$, $H_0 = d\Gamma(h)$, $h = -\Delta + \mu$, where Δ is the lattice Laplacian, $\mu \geq 0$. We express the operator H_0 in the k-representation. The Fourier transform takes $l_2(Z^\nu)$ into $L_2(T^\nu)$, where $T^\nu = [0, 2\pi]^\nu$ is the ν-dimensional torus, $\mathcal{F}_a = \mathcal{F}_{as}(L_2(T^\nu))$, and

$$H_0 = \int_{T^\nu} h(k) a^*(k) a(k) dk, \tag{2}$$

where

$$h(k) = \sum_{i=1}^{\nu} 2(1 - \cos(k_i)) + \mu, \qquad k = (k_1, \ldots, k_\nu). \tag{3}$$

Since we are assuming that $\mu \geq 0$, the spectrum of H_0 is nonnegative. In the present case we limit ourselves to an interaction $V \in A_l$ of the form

$$V = \sum_{i=1}^{d} c_i a^*(f_{i,1}) \ldots a^*(f_{i,m_i}) a(f_{i,m_i+1}) \ldots a(f_{i,m_i+n_i}), \tag{4}$$

where $m_i + n_i$ is even and $f_{i,j} \in C^\infty(T^\nu)$ for all i, j.

Although all the results proved below are valid in both cases, for ease of exposition we will give the proof only for case 2).

Later, in subsections 5 and 6, we will dispense with the requirement that the one-particle Hamiltonian be nonnegative (i.e., the condition $\mu \geq 0$).

1. Friedrichs diagrams. Algebra of Wick exponentials. The operations $\Gamma_\pm$ and Γ. We consider the vacuum (ground) state

$$\langle A\rangle = (A\Omega, \Omega) \tag{5}$$

on the C^* CAR algebra $\mathfrak{A}(L_2(T^\nu))$; this is a quasifree gauge-invariant state.

The effect of the Wick ordering with respect to this state (see Chapter 2) is simply to reorder the factors so that the annihilation operators appear to the right of the creation operators and the rule of signs is observed. In other words, the Wick ordering has no effect on monomials of the form

$$W = a^*(f_1)\dots a^*(f_m)a(f_{m+1})\dots a(f_{m+n}).$$

The product of several Wick monomials can be expanded as a sum of Wick monomials. The relevant rule is easily formulated in the language of diagrams. To a monomial

$$\begin{aligned} W_i = \int & w_i(k_{i1},\dots,k_{im_i},k_{i,m_i+1},\dots,k_{i,m_i+n_i}) \\ & \times a^*(k_{i1})\dots a^*(k_{im_i})a(k_{im_i+1})\dots a(k_{im_i+n_i}) \\ & \times dk_{i1}\dots dk_{im_i}dk_{im_i+1}\dots dk_{im_i+n_i} \end{aligned} \tag{6}$$

we associate a diagram G_i with m_i labeled left (right) legs, where the jth left leg corresponds to $a^*(k_{ij})$ and the jth right leg to $a(k_{im_i+j})$.

Then

$$W_1W_2\dots W_s = \sum_G W_G, \tag{7}$$

where the sum is over all possible pairings (giving rise to the diagram G) in the disjoint union of the diagrams G_i, and

$$W_G = \int \prod w_i \prod_{i,j} dk_{ij} \prod{}' \delta(k_{ij} - k_{i'j'}) : \prod{}'' a^{\#}(k_{ij}) : (-1)^{\pi(G)}, \tag{8}$$

where the product $\prod'$ is over all lines (paired legs i, j and i', j') of the diagram G, and $\prod''$ is over all the unpaired legs. Formulas (7) and (8) are easily proved by using the anticommutation relations to move each annihilation operator to the right of the creation operators. Then $\pi(G)$ is equal to the total number of such transpositions.

The following example will clarify our notation:

$$\begin{aligned} W_1 &= a^*(f_1)a(f_2), W_2 = a^*(f_3)a^*(f_4)a(f_5), \\ W_1W_2 &= :W_1W_2: + W_1 - \cdot - W_2 \\ &= a^*(f_1)a^*(f_3)a^*(f_4)a(f_2)a(f_5) \\ &\quad + (f_3f_2)a^*(f_1)a^*(f_4)a(f_5) - (f_4,f_2)a^*(f_1)a^*(f_3)a(f_5). \end{aligned}$$

Alternatively, we can express this graphically as

Note that $:W_1W_2:$ is the term in the expansion of W_1W_2 in which no pairings are formed when the annihilation operators are moved.

We introduce the following terminology:

$(W_1,\dots,W_s)_c$ is the sum over all connected diagrams G in (7);

$(W_1,\dots,W_s)_{0,0}$ is the sum over all connected diagrams without external legs in (7);

$(\dots)_L = (\dots)_c(\dots)_{0,0}$ is the sum over all connected diagrams in (7) with at least one external leg;

$(\dots)_{CR}$ is the sum over all diagrams in $(\dots)_L$ whose external legs involve only creation operators.

We will need some algebraic properties satisfied by series involving Wick monomials.

Given a formal power series

$$A = \sum_{m,n} x^m y^n A_{m,n},$$

where the $A_{m,n}$ are Wick monomials, we define

$$:A: := \sum_{m,n} x^m y^n :A_{m,n}: .$$

ASSERTION 1. *Let*

$$A = \sum_{m,n} x^m y^n A_{m,n}, \qquad B = \sum_{m,n} x^m y^n B_{m,n}$$

be formal power series in even Wick monomials $A_{m,n}$, $B_{m,n}$ ($m+n$ is even). Then the identities

$$:A:\exp B :=: (:A:\exp B :)_c \exp B :, \tag{9}$$

$$:\exp B:A:=: (:\exp B:A:)_c \exp B : \tag{10}$$

hold, where as before $(\cdot)_c$ indicates that only connected diagrams are considered.

The proof reduces to simple combinatorial arguments, in which the coefficients of like powers $x^m y^n$ are compared (see [42, 44]).

DEFINITION. The left connected product $W_1 \leftthreetimes :W_2\dots W_s:$ (respectively, right connected product $:W_2\dots W_s: \rtimes W_1$) is the sum of all Wick monomials in $W_1:W_2\dots W_s:$ (respectively, in $:W_2\dots W_s:W_1$) whose graphs are connected. (We recall that each vertex of the graph is labeled by the factor W_i.)

REMARK. The identities (9) and (10) are usually written as

$$:A:\exp B :=: (:A\perp:\exp B :)\exp B :,$$
$$:\exp B:A:=: (:\exp B:\perp A)\exp B : .$$

DEFINITION. We define the *Friedrichs operation* $\Gamma_{\pm\kappa}$ on the monomials

$$U_{mp} = \int u_{mp}(k_1,\dots,k_{m+p})a^*(k_1)\dots a^*(k_m)a(k_{m+1})\dots a(k_{m+p})dk_1\dots dk_{m+p}$$

for $\kappa > 0$ to be given by

$$\begin{aligned}\Gamma_{\pm\kappa}(U_{mp}) &\stackrel{\text{def}}{=} i\int_{\pm\infty}^{0} e^{-\kappa|t|}\tau_t^0(U_{mp})dt \\ &= \int u_{mp}(k_1,\dots,k_{m+p})[E_c - E_A \pm i\kappa]^{-1} \\ &\quad \times a^*(k_1)\dots a^*(k_m)a(k_{m+1})\dots a(k_{m+p})dk_1\dots dk_{m+p},\end{aligned} \tag{11}$$

and $\Gamma_{\pm}(U_{mp})$ is defined to be the strong limit of $\Gamma_{\pm\kappa}(U_{mp})$ as $\kappa \to 0$, where

$$E_c = \sum_{j=1}^{m} h(k_j), \qquad E_A = \sum_{j=m+1}^{m+p} h(k_j).$$

The *Glimm operation* Γ is defined by

$$\begin{aligned}\Gamma(U_{mp}) = \int U_{mp}(k_1,\dots,&k_{m+p})E_c^{-1}a^*(k_1)\dots a^*(k_m) \\ &\times a(k_{m+1})\dots\dots a(k_{m+p})dk_1\dots dk_{m+p}.\end{aligned} \tag{12}$$

We also define the operation

$$[H_0^{\pm\kappa}, U_{mp}] \stackrel{\text{def}}{=} \frac{d}{dt}[e^{-\kappa|t|}\tau_t^0(U_{mp})]\Big|_{t=\pm 0}. \tag{13}$$

It amounts to replacing the kernel U_{mp} by the kernel

$$iU_{mp}(k_1,\dots,k_{m+p})(E_c - E_A \pm i\kappa).$$

Assertion 2. *For $\kappa > 0$ we have the following formulas*

$$[H_0^{\pm\kappa}, \Gamma_{\pm\kappa}(U_{mp})] = U_{mp}, \tag{14}$$

$$[H_0^{\pm\kappa}, :\exp(\Gamma_{\pm\kappa}(U_{mp})):] =: U_{mp}\exp(\Gamma_{\pm\kappa}(U_{mp})):. \tag{15}$$

Proof. The first equality (14) follows from the definition of the operation $\Gamma_{\pm\kappa}$. In turn, it implies that

$$\begin{aligned}&[H_0^{\pm\kappa}, :\exp(\Gamma_{\pm\kappa}(U_{mp})):] \\ &\qquad =: [H_0^{\pm\kappa} - \overset{\circ}{\underset{1}{}} - \Gamma_{\pm\kappa}(U_{mp}) - \Gamma_{\pm\kappa}(U_{mp}) - \overset{\circ}{\underset{1}{}} - H_0^{\pm\kappa}]\exp(\Gamma_{\pm\kappa}(U_{mp})):,\end{aligned}$$

where $\overset{\circ}{\underset{1}{}}$ stands for all the Wick monomials with a single pairing. But the expression in square brackets is equal to U_{mp}.

2. Adiabatic wave operators (linked cluster theorem). We introduce some more notation. For $-\infty < t, s < \infty$ and $\kappa > 0$, we define the evolution operator (with no adiabatic cutoff) to be

$$U(t,s) = e^{itH_0} e^{-i(t-s)H} e^{isH_0} \tag{16}$$

and $U^k(t,s)$ is the corresponding operator with adiabatic cutoff, defined by the equation

$$U^\kappa(t,s) = 1 - i\int_s^t V^\kappa(r) U^\kappa(r,s)\,dr, \tag{17}$$

where $V^\kappa(r) = e^{-i\kappa\tau} e^{irH_0} V e^{-irH_0}$.

It is well known that for finite t, s the formula

$$U^\kappa(t,s) = 1 + \sum_{n=1}^{\infty} (-i\varepsilon)^n \int_{\Delta_n^{s,t}} dt_1 \dots dt_n V^\kappa(t_1)\dots V^\kappa(t_n), \tag{18}$$

is valid for $\kappa \geq 0$ (see [44]), where $\Delta_n^{s,t} = \{(t_1,\dots,t_n)\colon s < t_1 < \dots < t_n < t\}$ and the series converges in norm.

The integrand in (18) is a product of Wick monomials and can be expressed as a sum

$$V^\kappa(t_1)\dots V^\kappa(t_n) = \sum_G W_G(t_1,\dots,t_n) \tag{19}$$

of Wick monomials indexed by Friedrichs diagrams.

Using the equality

$$(\Gamma_{\pm\kappa}(V))(t) = i\int_{\pm\infty}^t V^\kappa(s)\,ds,$$

for $\kappa > 0$, after integrating over $\Delta^{0,\pm\infty}$ in each term of the series (18), we obtain that

$$U^\kappa(0,\pm\infty) = 1 + \sum_{n=1}^{\infty} (-i\varepsilon)^n \Gamma_{\pm n\kappa}(V\dots\Gamma_{\pm 2\kappa}(V\Gamma_{\pm\kappa}(V)), \tag{20}$$

and upon integrating over $\Delta^{\pm\infty,0}$, we get that

$$U^\kappa(\mp\infty,0) = 1 + \sum_{n=1}^{\infty} (-i\varepsilon)^n \Gamma_{\pm n\kappa}(\dots(\Gamma_{\pm 2\kappa}(\Gamma_{\pm\kappa}(V)V)\dots)V). \tag{21}$$

THEOREM 3. *There exists an $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$, the series*

$$\sum_{n=1}^{\infty} (-i\varepsilon)^n \int_{\Delta_n^{s,t}} dt_1 \dots dt_n (V^\kappa(t_1)\dots V^\kappa(t_n))_c \stackrel{\text{def}}{=} U^\kappa(t,s)_c \tag{22}$$

is norm-convergent in each of the two cases

$$-\infty < t, s < \infty \quad \text{and } \kappa \geq 0;$$
$$-\infty \leq t, s \leq \infty \quad \text{and } \kappa > 0.$$

Moreover, ε_0 does not depend on t, s, or κ. The same result is valid if $(\dots)_c$ is replaced by $(\dots)_{0,0}, (\dots)_L$, or $(\dots)_{CR}$.

The next result can be derived from this theorem.

THEOREM 4 (linked cluster theorem). *Under the hypotheses of Theorem* 3 *we have the equalities*

$$U^\kappa(t,s) =: \exp(U^\kappa(t,s)_c) :, \tag{23}$$

$$\frac{U^\kappa(t,s)}{(\Omega, U^\kappa(t,s)\Omega)} =: \exp(U^\kappa(t,s)_L) :, \tag{24}$$

where the exponentials on the right in (23), (24) *are defined by the usual series expansion in powers of* $U^\kappa(t,s)_{c,L}$, *which are norm-convergent.*

Let

$$T^\kappa_{t,s} \stackrel{\text{def}}{=} \frac{U^\kappa(t,s)}{(\Omega, U^\kappa(t,s)\Omega)}. \tag{25}$$

THEOREM 5. *For* $\nu \geq 3$ *there is an* ε_0 *such that for* $|\varepsilon| < \varepsilon_0$ *the following limits exist*:

$$\operatorname*{s-lim}_{\kappa\to 0} T^\kappa_{0,\pm\infty} \stackrel{\text{def}}{=} T^\pm \quad (\textit{toward adiabatic wave operators}), \tag{26}$$

$$\operatorname*{s-lim}_{\kappa\to 0} T^\kappa_{\pm\infty,0} \stackrel{\text{def}}{=} \widehat{T}^\pm \quad (\textit{inverse adiabatic wave operators}). \tag{27}$$

THEOREM 6. *Under the hypotheses of Theorem* 5, *the renormalization constant*

$$Z^{-1} \stackrel{\text{def}}{=} \left\| \exp\left(\sum_n (-\varepsilon)^n (\underbrace{\Gamma(V \dots \Gamma(V)) \dots}_{n \text{ times}})_{CR} \right) \Omega \right\|^2 \tag{28}$$

is finite, where n *is the number of Friedrichs operations. The operator* $\sqrt{Z}T^\pm$ *is unitary and gives a unitary equivalence*

$$HT^\pm = T^\pm(H_0 + \lambda_\varepsilon), \tag{29}$$

where

$$\lambda_\varepsilon = (\Omega, (\varepsilon V \perp T^\pm)\Omega) \tag{30}$$

($V_\perp T^\pm$ *is a left connected product*).

Before turning to the proof of Theorem 3–6, we will give a formal proof of Theorem 6 (see [44]), in which all series will simply be assumed to converge.

Formal proof of Theorem 6. By (20) and (24), we have

$$T^\kappa(0, \pm\infty) =: \exp\left(\sum_{n=1}^\infty (-i\varepsilon)^n \Gamma_{\pm n\kappa}(V \dots \Gamma_{\pm 2\kappa}(V\Gamma_{\pm\kappa}(V)) \dots)_L \right) :, \tag{31}$$

hence for $\kappa \to 0$ we get

$$T^\pm =: \exp(\Gamma_\pm(Q_\pm)) :, \tag{32}$$

where

$$Q_\pm = \sum (-i\varepsilon)^n (V\Gamma_\pm(V \dots \Gamma_\pm(V\Gamma_\pm(V)) \dots))_L.$$

Using Assertions 1 and 2, we have

$$H_0T^{\pm} = T^{\pm}H_0 + :QT^{\pm}: \tag{33}$$

and

$$\begin{aligned}\varepsilon VT^{\pm} &= \varepsilon : (V \prec T^{\pm})T^{\pm} := \varepsilon : (VT^{\pm})_c T^{\pm} : \\ &= \varepsilon : (VT^{\pm})_L T^{\pm} : + \varepsilon : (VT^{\pm})_{0,0} T^{\pm} : \\ &=: \sum_{n=1}^{\infty} (-\varepsilon)^n (V\Gamma_{\pm}(V \dots \Gamma_{\pm}(V\Gamma_{\pm}(V)) \dots))_L T^{\pm} : \\ &\quad + (\Omega, \varepsilon(VT^{\pm})_{0,0}\Omega)T^{\pm},\end{aligned}$$

where the last step follows from the relation (see [42])

$$Q_{\pm} = \varepsilon V \prec : \exp\{-\Gamma_{\pm}(Q_{\pm})\} : -\varepsilon(\Omega, V \prec : \exp(-\Gamma_{\pm}(Q_{\pm})) : \Omega). \tag{34}$$

From (33) and (34) we get Theorem 6 with

$$\lambda_{\varepsilon} = (\Omega, \varepsilon(VT^{\pm})_{0,0}\Omega) = \sum_{1}^{\infty} (-\varepsilon)^n (\Omega, (V\Gamma(V \dots \Gamma(V\Gamma(V)) \dots))_{0,0}\Omega), \tag{35}$$

where n is the number of operations Γ, and we have replaced $\Gamma_{\pm}$ by Γ, since in (35) a contribution can be nonzero only if there are no annihilation operators, since $a(f)\Omega = 0$ for all $f \in \mathcal{H}$.

REMARK. These results can all be proved without recourse to the operations $\Gamma_{\pm}$ by using the identity

$$\begin{gathered}\frac{d}{dt}U^{(\kappa)}(t,s)_c = -i\varepsilon(V^{(\kappa)}(t) : \exp U^{(\kappa)}(t,s)_c :) \\ U^{(\kappa)}(t,t)_c = 1.\end{gathered} \tag{36}$$

DEFINITION. The S-matrix (or *scattering matrix*) is defined by

$$S = ZT_{+}^{*}T_{-} =: \exp\bigl(2\pi i \sum (-\varepsilon)^n \Delta(V\Gamma_{-}(V \dots \Gamma_{-}(V\Gamma_{-}(V))) \dots)_L\bigr), \tag{37}$$

where the operation Δ applied to a Wick monomial U_{mp} means that its kernel U_{mp} is replaced by

$$U_{mp}(k_1, \dots, k_{mp})\delta(E_c - E_A).$$

It follows from Theorem 6 that the S-matrix is unitary.

3. Proof of Theorem 3. Decomposition into clusters and mode expansion. We first consider the case $\kappa = 0$, $-\infty < s, t < \infty$, $\nu \geq 3$. We will be occupied with estimating the expression

$$\int_{\Delta_n^{s,t}} (V(t_1) \dots V(t_n))_c \, dt_1 \dots dt_n \tag{38}$$

whose absolute value will be shown to be bounded by $|t - s|c^n$, where c is a constant independent of s, t, and n. Theorem 3 will be derived from this estimate.

The main obstacle to proving the estimate is the large number of diagrams involved. This number will be reduced by working in "time clusters or sectors".

Partitions. The subscripts $1, \dots, n$ are the vertices of the diagrams. Any subset $\alpha = (\alpha_1, \dots, \alpha_k)$, $\alpha_1 < \alpha_2 < \dots < \alpha_k$ of the set $(1, \dots, n)$ defines a *partition* of $(1, \dots, n)$ into intervals

$$I_1 = [1, \alpha_1) = \{i : 1 \le i \le \alpha_1\},$$
$$\dots$$
$$I_k = [\alpha_{k-1}, \alpha_k), \qquad I_{k+1} = [\alpha_k, n].$$

Sectors. Every partition α defines a subset Δ_α of the region $\Delta_n^{s,t}$, which will be called a *sector.* It is uniquely determined by the following conditions:

a) if i, j lie in the same interval I_j of the partition α, then there exists an M such that

$$t_i, t_j \in [M, M+1) \stackrel{\text{def}}{=} \widehat{I}_M,$$

b) if i, j belong to I_i and I_j, respectively, then t_i and t_j belong to distinct intervals $\widehat{I}_M$ and $\widehat{I}_L$, $M \ne L$.

Clearly, $\bigcup_\alpha \Delta_\alpha = \Delta_n^{s,t}$. In the sequel we will refer to I_j as the jth group of the sector.

Subsectors. A *subsector* $\Delta_\alpha(M_1, \dots, M_{k+1})$ of a sector Δ_α is defined by the partition α and integers $M_1 < M_2 < \dots < M_{k+1}$ to be the set of all $(t_1, \dots, t_n)$ such that if $i \in I_j$ then $t_i \in [M_j, M_{j+1})$.

Modes. We choose an orthonormal basis $\{e_N\}_{N \in Z^\nu}$ in the space $L_2(T^\nu)$, where $e_N = C \exp\{i(N, k)\}$, $k \in T^\nu$. The members of this basis will be called *modes.* Let

$$S \stackrel{\text{def}}{=} \bigcup_{i=1}^{d} \{f_{i,1}, \dots, f_{i,m_i+n_i}\}.$$

Fix a subsector $\Delta_\alpha(M_1, \dots, M_{k+1})$. Since if $t \in [M_1, M_1 + 1)$ we have $t = M_1 + \delta t$, $0 \le \delta t \le 1$, it follows that

$$V(t) = \sum_{i=1}^{d} a^*(e^{iM_2 h} e^{i\delta t h} f_{i,1}) \dots a(e^{iM_1 h} e^{i\delta t h} f_{i,m_i+n_i}). \tag{39}$$

Mode expansion. We expand the vectors $e^{i\delta t h} f$, $f \in S$, in terms of the basis $\{e_N\}$:

$$e^{i\delta t h} f = \sum_{N \in Z^\nu} c_{N,f}(\delta t) e_N. \tag{40}$$

Note that for every $\gamma > 0$, there exists a constant $c(\gamma)$ such that the coefficients $c_{N,f}(\delta t)$ of the series (40) are bounded by

$$|c_{N,f}(\delta t)| \le \frac{c(\gamma)}{|N|^\gamma}, \qquad |N| = \sum_{i=1}^{\nu} |N^{(i)}| \tag{41}$$

uniformly in $f \in S$, $|\delta t| < 1$. The bound (41) is proved by integrating by parts. We now choose $\gamma > \nu + 1$ so that the inequality

$$\sum_{N \in Z^\nu} |c_{N,f}(\delta t)| < C < \infty \tag{42}$$

holds, where the constant C is independent of f and $|\delta t| < 1$.

Given a partition $\alpha = (\alpha_1, \ldots, \alpha_k)$, we write B_α for the following subset of $[0,1]^n$:

$$B_\alpha = \{(\delta t_1, \ldots, \delta t_n) : 0 \le \delta t_1 < \cdots < \delta t_{\alpha_1 - 1}, \\ 0 \le \delta t_{\alpha_1} < \cdots < \delta t_{\alpha_2 - 1}, \ldots, 0 \le \delta t_{\alpha_k} < \cdots < \delta t_n\}.$$

Using the above notation, we can write expression (38) as

$$\sum_{\alpha} \sum_{M_1, \ldots, M_{k+1}} \int_{\mathcal{B}_\alpha} \prod_{i=1}^{n} d(\delta t_i) \sum_{\{N_i, f_j\}} \sum_{G} W_G, \tag{43}$$

where we have (scanning (43) from left to right)

1) sums over all partitions;
2) sums over all subsectors;
3) integrals over each subsector;
4) sums over all modes;
5) sums over all admissible diagrams.

Diagrams. A *diagram* is a graph with vertices $1, \ldots, n$; m_i right legs (corresponding to annihilation operators) and n_i left legs (corresponding to creation operators) are incident on each vertex. A choice of modes amounts to placing each leg in correspondence with an element e_N of the basis $\{e_N\}$. Each leg is labeled by an index (v, p), where v is the index of the vertex and p is the index of the leg at the vertex v. An admissible diagram is a connected graph formed by pairing some of the legs (v_1, p_1) and (v_2, p_2), $v_1 < v_2$, where the first leg is a right leg and the second is a left leg. The paired legs are the edges of the admissible diagram and will be referred to as internal edges (*int*). The unpaired legs are called external edges (*out*).

A weight is associated to each diagram appearing in expression (43).

Weight of a diagram. Assume that a partition $\alpha = (\alpha_1, \ldots, \alpha_k)$, subsector $(M_1, \ldots, M_{k+1})$, vector $(\delta t_1, \ldots, \delta t_n)$, and modes $N_{v,p}$ have been fixed. We define the function $M(v)$, $v = 1, \ldots, n$, to be $M(v) = M_i$ if $v \in I_i$, $i = 1, \ldots, k+1$. The *weight* of a diagram G is defined to be

$$\begin{aligned} W_G =& (-1)^{\pi(G)} \prod_{\text{int}} (e^{ih(M(v) - M(v'))} e_{N_{v,p}}, e_{N_{v',q}}) \\ & \times c_{N_{v,p}}(\delta t_v) c_{N_{v',p}}(\delta t_{v'}) \prod_{\text{out}} a^{\#}(e^{iM(v)h} e_{N_{v,p}}), \end{aligned} \tag{44}$$

where $a^{\#} = a^*$ for a right leg (v, p) and $a^{\#} = a$ for a left leg (v, p).

REMARK. Suppose G is a diagram having an internal edge (v, p, v', q) that lies completely in an interval I_l for some l. Then (44) shows that if $N_{v,p} \neq N_{v',q}$ then $W_G = 0$, because $M(v) = M(v')$ and $(e_{N_{v,p}}, e_{N_{v,q}}) = 0$. This fact will enable us to eliminate many of the diagrams from consideration.

The subset I_l of the vertices of a diagram G will be called the lth group of vertices. Let $A_l(B_l)$ be the set of edges (external and internal) of G that originate from the lth group of vertices, are directed to the right (left), and do not pair any two vertices in the lth group.

Fix a sector $\alpha = (\alpha_1, \ldots, \alpha_k)$, the set $\Delta_\alpha(M_1, \ldots, M_{k+1})$, and the modes $\{N_{v,p}\}$. For each $l = 1, \ldots, k+1$ we denote by $\mathcal{N}_l$ some fixed set of modes $\mathcal{N}_l = \{e_1, \ldots, e_{i_l}\}$; the modes of the legs A_l coincide precisely with $\mathcal{N}_l$.

LEMMA 7. a) *There exists at most one diagram G in $G(N_1, \dots, N_{k+1})$ with a nonzero weight W_G.*

b) *For each such diagram the sets $\mathcal{N}_l$ contain distinct modes.*

PROOF OF LEMMA 7. Both assertions are obvious consequences of the fact that

$$\{a^{\#}(e^{iMh}e_{N_1}), a^{\#}(e^{iMh}e_{N_2})\} = 0, \tag{45}$$

if $N_1 \neq N_2$ (where $\{\cdot, \cdot\}$ denotes an anticommutator) and

$$(a^{\#}(e^{iMh}e_N))^2 = 0. \tag{46}$$

If $I_l = \{i_1, \dots, i_q\}$ then we write $V_{I_l} = V(t_{i_1}) \dots V(t_{i_q})$.

We fix some V_{I_l} and express it as a sum of Wick monomials. According to (45) and (46), there exists just one nonzero diagram with the property that the modes for the unpaired right legs coincide with N_l and are all distinct. Statements a) and b) of Lemma 7 now follow from these observations.

We next turn to estimating the expression (43). Since all the sums there (except for the sums over the modes) are finite, we can move the mode summation and the integration over $\mathcal{B}$ to the left. In view of inequality (41) and the fact that $\mathcal{B}_\alpha \subset [0,1]^N$, to prove Theorem 3 it suffices to establish the following bound:

$$\left\| \sum_{\alpha} \sum_{M_1, \dots, M_{k+1}} \sum_{G} \widetilde{W}_G \right\| \leq c^n |t - s| \tag{47}$$

uniformly in the modes; here c is independent of n, s, t, and the modes, while $\widetilde{W}_G$ is given as in (44), but with all the factors $c_{N_{v,p}}(\delta t_v)$ removed.

Now assume that the set of modes has been fixed. We can choose the sets $N_1, \dots, N_{k+1}$ in at most $(2^{m_{\max}})^n$ ways, where $m_{\max}$ is the maximum number of creation operators in the Wick monomials appearing in V. We may therefore assume also that $N_1, \dots, N_{k+1}$ have been fixed. By Lemma 1, every admissible diagram G can be associated to a connected diagram $\widetilde{G}$ with $k+1$ vertices $M_1, \dots, M_{k+1}$ and total number of edges at most $m_{\max} n$.

The weight $\widehat{W}_{\widetilde{G}}$ of the diagram $\widetilde{G}$ is given by the formula

$$\widehat{W}_{\widetilde{G}} = \prod_{\text{int}} |(e^{ih(M(v) - M(v'))} e_{N_{v,p}}, e_{N_{v',q}})|, \tag{48}$$

where

$$\|\widetilde{W}_G\| < c_1^n \widehat{W}_{\widetilde{G}}, \tag{49}$$

for some constant c_1.

For all $M, N \in Z^\nu$ we have the bound

$$\left| \int_{T^\nu} dk e_N(k) E_M(k) e^{ith(k)} \right| < \frac{c}{(1 + |t|)^{\nu/2}}, \tag{50}$$

with c independent of M, N, and t, whence, recalling (49) and (50), we see that

$$\|\widetilde{W}_G\| \leq (\text{const})^n \prod_{\text{int}} \frac{1}{(|M_{l(v,p)} - M_{l(v',q)}| + 1)^{\nu/2}}, \tag{51}$$

where $l(v,p)$ is the index of the group α to which the vertex v belongs.

Putting together all the above observations, we have

$$\begin{aligned}\Big\|\sum_{\alpha}\sum_{M_1,\dots,M_{k+1}}\sum_{G}\widetilde{W}_G\Big\| \\ \le (\text{const})^n\sum_{M_1,\dots,M_{k+1}}\sum_{\widetilde{G}}\prod\frac{1}{(|M_{l(v,p)}-M_{l(v',q)}|+1)^{\nu/2}}.\end{aligned}\tag{52}$$

Let M_1 be an integer lying in the closed interval $[s,t]$.

LEMMA 8. *We have the bound*

$$\begin{aligned}\Big|\sum_{k=1}^{\infty}\sum_{M_2,\dots,M_{k+1}}\sum_{\widetilde{G}}\prod_{\text{int}}\frac{1}{(|M_{l(v,p)}-M_{l(v',q)}|+1)^{\nu/2}}\Big| \\ \le\Big(\sum_{M=-\infty}^{\infty}\frac{c}{(1+M)^{\nu/2}}\Big)^{nm_{\max}}.\end{aligned}\tag{53}$$

This result follows by applying the standard cluster expansion technique (see [26]) to the left-hand side of (53). The summation over M_1 gives the factor $|t-s|$ in (47). This completes the proof of Theorem 3 when $\kappa=0$, and the case $\kappa>0$ is treated similarly.

4. Asymptotic completeness. We can now give a rigorous (nonformal) proof of Theorems 5 and 6 which will imply the asymptotic completeness of the Hamiltonian $H_0+\varepsilon V$ for small ε, in the case when the interaction V polarizes the vacuum but is even.

PROOF OF THEOREM 5. We show that for $\psi=a^*(f_1)\dots a^*(f_m)\Omega$, where $f_i\in C^\infty(T^\nu)$, the limit

$$\lim_{\kappa\to0}\sum_n(-i\varepsilon)^n\int_{\pm\infty}^0dt\cdots\int_{\pm\infty}^0dt_n(V^{(\kappa)}(t_1)\dots V^{(\kappa)}(t_n))_L\psi\tag{54}$$

exists. Consider the nth term in the series (54). We first prove the existence of the limit for the sum over the diagrams in L, i.e., which have at least one external edge with an annihilation operator. Repeating the proof of Theorem 3, we see that this sum is bounded from above by $C^n\varepsilon^n$, uniformly in κ. In addition, each term in the sum is easily seen to have a limit as $\kappa\to\infty$.

We next analyze the sum over the diagrams having no external edges with annihilation operators. The external edges here are therefore associated with creation operators, which give the contribution

$$\prod a^*(e^{it_vh-\kappa|t_v|}e_{N_{v,p}})=\int\prod_{v,p}a^*(k_{v,p})e^{it_vh-\kappa|t_v|}e_{N_{v,p}}(k_{v,p})dk_{v,p}.\tag{55}$$

We make the change of variables $t_1'=t_1$, $t_2'=t_2-t_1,\dots,t_n'=t_n-t_{n-1}$ and integrate over t_1'. Note that

$$\lim_{\kappa\to0}\int_{\pm\infty}^0dt_1'\exp\Big\{it_1'\sum_{v,p}h(k_{v,p})-\kappa|t_1'|\Big\}=\frac{1}{\sum_{v,p}h(k_{v,p})}\tag{56}$$

belongs to L_2 if $\mu \geq 0$. We may therefore expand $(\sum_{v,p} h(k_{v,p}))^{-1}$ in terms of the modes. The method developed in the proof of Theorem 3 can now be applied.

Remark concerning adiabatic cutoffs. In the preceding we considered wave operators depending on two parameters: the adiabatic cutoff parameter κ and the time t, and it was necessary to investigate the repeated limits $k \to 0$, $t \to \infty$. According to Theorems 3–6, these limits exist in the strong sense, i.e., with respect to the norm in $\mathcal{F}_a$. The question arises of whether the adiabatic cutoff plays any role and whether it might be avoided somehow. (We note that adiabatic cutoffs are a typical feature in stationary scattering theory.)

We observe also that if the limit

$$\operatorname*{s-lim}_{t\to 0} U(0,t)$$

exists with no cutoff, then so does the limit

$$\operatorname*{s-lim}_{\kappa\to 0} U^{(\kappa)}(0,\infty),$$

with an adiabatic cutoff, and the two coincide. This situation occurs, for instance, when $V \in \mathfrak{A}$ and $V\Omega = 0$.

Analysis of the proof of Theorems 3–6 shows that their analogs hold for the corresponding wave operators without an adiabatic cutoff. However, in this case we are obliged to replace all the norm (strong) limits by weak limits. It remains unclear how the existence of the weak limits might be exploited to prove asymptotic completeness.

5. Existence of a perturbed vacuum vector. In this and the next subsection we will show that the dynamics of the fermion system under consideration does not depend strongly on the chemical potential μ. The results in subsections 2–4 show that the perturbed system is unitarily equivalent to a "shifted" free system in $\mu \geq 0$. This condition is used in an essential way in the proof of Theorems 3–6.

Notice that if $\mu \geq 0$, then the point $\lambda_0 = 0$ of the discrete spectrum of $H_0 = d\Gamma(h)$ lies outside or on the boundary of the continuous spectrum of H_0. When the interaction εV is "switched on", λ_0 is shifted to the value λ_ε.

If $\mu < 0$, then the discrete spectrum of H_0 is contained in the interior of the continuous spectrum of H_0. It turns out that even in this case the eigenvalue does not disappear, as might have been expected. For example, such a situation is encountered for the model of an interacting Fermi gas with spin, in which case an eigenvalue imbedded in the continuous spectrum disappears when the interaction is turned on ($\varepsilon \neq 0$).

In the present subsection we prove that the perturbed operator has an eigenvector for arbitrary real μ. We will use the method for estimating diagrams developed in subsections 2–4 and retain all the terminology used there.

THEOREM 9. *Let the one-particle Hamiltonian h be given by* (3), $\mu \in R^1$, *and let the operator V be as in* (4). *Then for ε sufficiently small, the operator $H_\varepsilon = H_0 + \varepsilon V$ has an eigenvector Ω_ε.*

PROOF. One can show as in Theorem 6 that for small enough the quantity

$$Z^{-1} = \left\| \exp\left\{ \sum (-\varepsilon)^n (\Gamma(V,\dots,\Gamma(V))\dots)_{CR} \right\} \Omega \right\|$$

is finite. On the other hand, one can show as we did in subsections 2–4 that

$$Z^{-1} = \lim_{t\to\infty} \frac{1}{|(\Omega, U(0,t)\Omega)|^2} = \lim_{t\to\infty} \frac{1}{|(e^{iH_\varepsilon t}\Omega, \Omega)|^2}. \tag{57}$$

Therefore, the operators

$$\frac{e^{itH_\varepsilon}}{(e^{itH_\varepsilon}\Omega,\Omega)}$$

are bounded uniformly in t. The next lemma can be proved by the same method as Theorem 3.

LEMMA 10. *There is a dense subset D of $\mathcal{F}_a$ for which a finite limit*

$$\langle F\rangle = \lim_{t\to\infty}\frac{(e^{itH_\varepsilon}\Omega,F)}{(e^{itH_\varepsilon}\Omega,\Omega)},\qquad F\in D \tag{58}$$

exists.

The following general lemma asserts that this limit exists and is finite for all $F\in\mathcal{F}_a$.

LEMMA 11. *Let be a separable Hilbert space and $\alpha\colon R\to\mathcal{H}$ be a uniformly bounded function, i.e.,*

$$\|\alpha_t\| < M < \infty. \tag{59}$$

Assume that a finite limit

$$\lim_{t\to\infty}(\alpha_t,F) \tag{60}$$

exists for a set of vectors $\{F\}$ that is dense in $\mathcal{H}$. Then the limit exists and is finite for all $F\in\mathcal{H}$.

Since Hilbert spaces are weakly complete, there exists a vector Ω_ε such that for all $F\in\mathcal{F}_a$

$$\lim_{t\to\infty}\frac{(e^{itH_\varepsilon}\Omega,F)}{(e^{itH_\varepsilon}\Omega,\Omega)} = (\Omega_\varepsilon,F). \tag{61}$$

We will show that Ω_ε is an eigenvector of the perturbed operator H_ε. First of all, setting $F=\Omega$ in (61) we see that $(\Omega_\varepsilon,\Omega)=1$, i.e., $\Omega_\varepsilon\neq 0$. Then for every $s\in R$ we have

$$\begin{aligned}(\Omega_\varepsilon,F) &= \lim_{t\to\infty}\frac{(e^{i(t+s)H_\varepsilon}\Omega,F)}{(e^{i(t+s)H_\varepsilon}\Omega,\Omega)} = \lim_{t\to\infty}\frac{(e^{itH_\varepsilon}\Omega,e^{-isH_\varepsilon}F)}{(e^{itH_\varepsilon}\Omega,e^{-isH_\varepsilon}\Omega)}\\ &= \lim_{t\to\infty}\frac{(e^{itH_\varepsilon}\Omega,e^{-isH_\varepsilon}F)}{(e^{itH_\varepsilon}\Omega,\Omega)}\frac{(e^{itH_\varepsilon}\Omega,\Omega)}{(e^{itH_\varepsilon}\Omega,e^{-isH_\varepsilon}\Omega)}\\ &= \frac{(\Omega_\varepsilon,e^{-isH_\varepsilon}F)}{(\Omega_\varepsilon,e^{-isH_\varepsilon}\Omega)} = \frac{(e^{isH_\varepsilon}\Omega_\varepsilon,F)}{(e^{isH_\varepsilon}\Omega_\varepsilon,\Omega)}.\end{aligned} \tag{62}$$

That is, for all $s\in R$ and $F\in\mathcal{F}_a$ we have the identity

$$(\Omega_\varepsilon,F) = \frac{(e^{isH_\varepsilon}\Omega_\varepsilon,F)}{(e^{isH_\varepsilon}\Omega_\varepsilon,\Omega)}$$

or

$$(\Omega_\varepsilon,F)(e^{itH_\varepsilon}\Omega_\varepsilon,\Omega) = (e^{isH_\varepsilon}\Omega_\varepsilon,F). \tag{63}$$

Differentiating both sides of (63) with respect to s and setting $s=0$, we get

$$(\Omega_\varepsilon,F)(H_\varepsilon\Omega_\varepsilon,\Omega) = (H_\varepsilon\Omega_\varepsilon,F). \tag{64}$$

Since (64) holds for all $F\in\mathcal{F}_a$, we have $H_\varepsilon\Omega_\varepsilon = (H_\varepsilon\Omega_\varepsilon,\Omega)\Omega_\varepsilon$, i.e., Ω_ε is an eigenvector of H_ε. This concludes the proof of Theorem 9.

6. Unitary equivalence, general case. Here we consider a further generalization of the results in subsection 1 to the case of a chemical potential μ of arbitrary sign. We will prove the next result.

THEOREM 12. *Let the one-particle Hamiltonian h be given by* (3), *let $\mu \in R^1$, and let V be given by* (4). *Then there exists an $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$ there is an λ_ε such that the operators H_ε and $H_0 + \lambda_\varepsilon E$ are unitarily equivalent.*

REMARK. The proof of Theorem 12 is based on the results of the previous subsection on the existence of a perturbed eigenvector.

PROOF. Consider the following two dynamics in the C^*-algebra $\mathfrak{A}(\mathcal{H})$:

$$\tau_t^0 = e^{itH_0} A e^{-itH_0},$$
$$\tau_t^{\varepsilon V} = e^{it(H_0+\varepsilon V)} A e^{-it(H_0+\varepsilon V)}.$$

By Theorem 4.1, for small enough ε the Møller morphisms

$$\gamma_\pm(A) = \operatorname*{s-lim}_{t\to\pm\infty} \tau_{-\tau}^{\varepsilon V}(\tau_t^0(A)), \qquad A \in \mathfrak{A}(\mathcal{H}) \tag{64^a}$$

exist and are invertible in the C^*-algebra $\mathfrak{A}(\mathcal{H})$. Let Ω_ε be a perturbed vacuum vector for the operator $H_0 + \varepsilon V$, whose existence is guaranteed by Theorem 9; we may assume that it is normalized, i.e., $\|\Omega_\varepsilon\| = 1$. We set $\gamma \equiv \gamma_\pm$.

LEMMA 13. *For every $f \in \mathcal{H}$ we have*

$$(\gamma a(f))\Omega_\varepsilon = 0. \tag{65}$$

PROOF. Writing $\widetilde{a}(f) = \gamma a(f)$, we have

$$\lim_{t\to\infty} \tau_{-\tau}^{\varepsilon V}(\tau_t^0(a(f)))\Omega_\varepsilon = \widetilde{a}(f)\Omega_\varepsilon. \tag{66}$$

On the other hand, using the fact that Ω_ε is an eigenvector of $H_\varepsilon = H_0 + \varepsilon V$, we have

$$\|\tau_{-\tau}^{\varepsilon V}(\tau_t^0(a(f)))\Omega_\varepsilon\| = \|e^{-itH_\varepsilon} a(e^{ith} f) e^{itH_\varepsilon}\Omega_\varepsilon\| = \|a(e^{ith} f) e^{itH_\varepsilon}\Omega_\varepsilon\| \to 0 \tag{67}$$

as $t \to +\infty$. Indeed, suppose that $\Omega_\varepsilon^{(n)} \to \Omega_\varepsilon$ as $n \to \infty$, where $\Omega_\varepsilon^{(n)}$ is a finite linear combination of vectors of the form

$$a^*(f_1)a^*(f_2)\dots a^*(f_m)\Omega, \qquad f_i \in C_0^\infty(R^\nu).$$

Let $\|\Omega_\varepsilon^{(n)} - \Omega_\varepsilon\| < \delta$ for $n > N_0$. Then

$$\|a(e^{-ith} f)\Omega_\varepsilon\| \le \|a(e^{-ith} f)\Omega_\varepsilon^{(n)}\| + \delta\|f\|. \tag{68}$$

Using the anticommutation relations, we can move the operator $a(e^{ith} f)$ from left to right past the operators $a^*(f_1)a^*(f_2)\dots a^*(f_m)$; $a(e^{ith} f)\Omega_\varepsilon^{(n)}$ will contain a finite number of terms, each involving a factor of the form

$$(e^{ith} f, f_j). \tag{69}$$

It follows from spectral theory and Lebesgue's theorem (if we use the absolute continuity of the spectrum of h) that (69) tends to zero as $t \to \infty$. Since δ is arbitrary, this implies the assertion of Lemma 13.

The next lemma follows easily from Lemma 13.

LEMMA 14. *For every $A \in \mathfrak{A}(\mathcal{H})$ we have*

$$(A\Omega, \Omega) = (\gamma(A)\Omega_\varepsilon, \Omega_\varepsilon). \tag{70}$$

Define an operator $U\colon \mathcal{F}_a \to \mathcal{F}_a$ by

$$U(A\Omega) = \gamma(A)\Omega_\varepsilon. \tag{71}$$

Then U is norm-preserving, since

$$\|A\Omega\|^2 = (A^*A\Omega, \Omega) = (\gamma(A^*A)\Omega_\varepsilon, \Omega_\varepsilon) = \|\gamma(A)\Omega_\varepsilon\|^2.$$

It is therefore well defined and isometric. Since $\mathfrak{A}(\mathcal{H})$ is irreducible on $\mathcal{F}_a$, we have $\mathfrak{A}\Omega_\varepsilon = \mathcal{F}_a$, and hence the image of U is all of $\mathcal{F}_a$, i.e., U is unitary.

For every $A \in \mathfrak{A}(\mathcal{H})$ we have

$$\begin{aligned} e^{it(H_0+\varepsilon V)}UA\Omega &= e^{it(H_0+\varepsilon V)}\gamma(A)\Omega_\varepsilon \\ &= \tau_t^{\varepsilon V}\gamma(A)e^{it(H_0+\varepsilon V)}\Omega_\varepsilon = e^{it\lambda_\varepsilon}\gamma(\tau_t^0(A))\Omega_\varepsilon \\ &= e^{it\lambda_\varepsilon}U\tau_t^0(A)\Omega = e^{it\lambda_\varepsilon}Ue^{itH_0}A\Omega. \end{aligned} \tag{72}$$

Here we have used the intertwining property of the Møller morphisms: $\tau_t^{\varepsilon V} = \gamma\tau_t^0$, and the fact that Ω_ε is an eigenvector of $H_0 + \varepsilon V$ with eigenvalue λ_ε.

It follows from expression (72) that

$$e^{it(H_0+\varepsilon V)} = e^{it\lambda_\varepsilon}Ue^{itH_0}U^*$$

or

$$H_0 + \varepsilon V = U(H_0 + \lambda_\varepsilon E)U^*.$$

This proves Theorem 12.

We also have the next

THEOREM 15. *The Møller morphisms $\gamma_\pm$ defined by* (64^a) *are unitarily representable* (*i.e., they are given by an inner automorphism*).

PROOF. For every $B \in \mathfrak{A}$ and operator U given by (71), we have

$$\gamma(A)B\Omega_\varepsilon = \gamma(A\gamma^{-1}(B))\Omega_\varepsilon = UA\gamma^{-1}(B)\Omega = UAU^*B\Omega_\varepsilon.$$

Since B is arbitrary, it follows that

$$\gamma(A) = UAU^*. \tag{73}$$

§3. Fermi gas interacting weakly with a particle

In the previous sections we studied a small perturbation of the free dynamics for an ideal Fermi gas and showed that under mild hypotheses the Hamiltonian for the perturbed dynamics is unitarily equivalent either to the original Hamiltonian or to a "shift" of it by some constant.

Here we will study a small perturbation of a free system consisting of an ideal Fermi gas and an independently moving particle and show that the previous results remain valid in this case.

1. Statement of the main theorem. We consider the Hilbert space

$$K = \mathcal{F}_a(L_2(R^\nu)) \otimes L_2(R^\nu),$$

where $\mathcal{F}_a = \mathcal{F}_a(L_2(R^\nu))$ is the antisymmetric Fock space over $L_2(R^\nu)$, i.e., K is the space of infinite sequences

$$\{F_0(y), F_1(x_1, y), F_2(x_1, x_2, y), \dots, F_n(x_1, \dots, x_n, y), \dots\}, \tag{1}$$

where $x_j, y \in R^\nu$ and the functions F_n are antisymmetric in the x_j.

Let selfadjoint operators h_1 and h_2 acting on $L_2(R^\nu)$ be given. We take the free Hamiltonian to be the selfadjoint operator

$$H_0 = d\Gamma(h_1) \otimes \mathbf{1} + \mathbf{1} \otimes (h_2).$$

In the sequel we will consider the case when h_1 and h_2 are both equal to $-\Delta$.

Let the total Hamiltonian have the form

$$H = H_0 + \varepsilon V, \qquad \varepsilon \in R^1, \tag{2}$$

where V belongs to one of the operator classes A or B.

Let $f = a^*(\psi_r) \dots a^*(\psi_1)\Omega \otimes \psi$, where ψ and $\psi_j \in S(R^\nu)$; then $V \in A$ means that

$$\begin{aligned} Vf = &\sum_{j=1}^{M} \int_{R^\nu} dz \int_{R^{\nu(m_j+n_j+1)}} dx_1 \dots dx_{m_j+n_j}\, dy_1 \\ &\times K_j(x_1 - z, \dots, x_{m_j} - z, x_{m_j+1} - z, \dots, x_{m_j+n_j} - z, y_1 - z, y_2 - z) \\ &\times a^*(x_1) \dots a^*(x_m) a(x_{m_j+1}) \dots a(x_{m_j+n_j}) a^*(\psi_r) \dots a^*(\psi_1)\Omega \otimes \psi(y_1), \end{aligned} \tag{3}$$

i.e., in terms of the variables describing the independently moving particle, V is a translation-averaged integral operator; if on the other hand $V \in B$, then

$$\begin{aligned} Vf = &\sum_{j=1}^{M} \int_{R^\nu} dz \int_{R^{\nu(m_j+n_j+1)}} dx_1 \dots dx_{m_j+n_j} \\ &\times K(x_1 - z, \dots, x_{m_j} - z, x_{m_j+1} - z, \dots, x_{m_j+n_j} - z, y - z) \\ &\times a^*(x_1) \dots a^*(x_m) a(x_{m_j+1}) \dots a(x_{m_j+n_j}) a^*(\psi_r) \dots a^*(\psi_1)\Omega \otimes \psi(y), \end{aligned} \tag{4}$$

i.e., in terms of the variables of the distinguished particle, V is a translation-averaged multiplication operator; here $M < \infty$, and $K_j \in S(R^{\nu(m_j+n_j+2)})$ or $K_j \in S(R^{\nu(m_j+n_j+1)})$ (in cases A and B, respectively); $x_p, y_s \in R^\nu$, $p = 1, \dots, m_j + n_j$, $s = 1, 2$.

The operator V is bounded in K and will be taken to be selfadjoint. Then if the conditions

$$m_j \geq 1, \quad n_j \geq 1 \quad \text{and } m = \max_j m_j = \max_j n_j < \infty \tag{5}$$

are satisfied, H is a symmetric operator on the subset D of vectors of the form

$$a^*(\psi_r) \dots a^*(\psi_1)\Omega \otimes \psi, \qquad \psi, \psi_j \in S(R^\nu), \tag{5^a}$$

which is dense in K. In fact, H is known to be essentially selfadjoint on D, and this will also follow from the convergence of the expansions derived below.

We define the inverse and forward wave operators for finite t by

$$W_t = \exp\{-itH_0\} \exp\{itH\},$$
$$\widehat{W}_t = \exp\{-itH\} \exp\{itH_0\}.$$

THEOREM 1. *If $\nu \geq 3$, condition* (5) *holds, and V is in class A or B, then there exists an $\varepsilon_0 = \varepsilon_0(\nu, m, V)$ such that for $|\varepsilon| < \varepsilon_0$ the strong limits*

$$\widehat{W}_\pm = \operatorname*{s-lim}_{t\to\pm\infty} \widehat{W}_t$$

and

$$W_\pm = \operatorname*{s-lim}_{t\to\pm\infty} W_t$$

exist for the forward and inverse wave operators, respectively.

REMARK 1. The hypotheses in Theorem 1 are in some sense best possible. In dimension $\nu = 1, 2$, even for particles in an external field, bound states may be present for arbitrarily small ε. When $m_{\min} = 0$, where $m_{\min} = \min m_j$, then as we have seen, the operator H is "shifted". The requirements concerning the smoothness of the kernels K_j can be weakened, but only slightly.

A standard consequence of Theorem 1 is the following:

COROLLARY. *The operators $W_+ = \widehat{W}_+^*$ and $W_- = \widehat{W}_-^*$ are unitary and give a unitary equivalence of H and H_0, i.e., for instance,*

$$H = W_+^* H_0 W_+.$$

2. Perturbation-theoretic series. We consider the following series for $0 \leq t < \infty$, $F \in D$:

$$F + \sum_{n=1}^{\infty} (i\varepsilon)^n \int_{\Delta_t^n} V_{t_n} \dots V_{t_1} dt_1 \dots dt_n F, \tag{6}$$

where

$$V_t = \exp\{-itH_0\} V \exp\{itH_0\},$$

and we recall that Δ_t^n is the region

$$\Delta_t^n = \{0 \leq t_1 < t_2 < \dots < t_n < t\} \subset R^n.$$

The main result to be proved reads as follows:

THEOREM 2. *Under the hypotheses of Theorem* 1, *there exists an $\varepsilon_0 > 0$ such that when $|\varepsilon| < \varepsilon_0$,*

1) *the norm of the nth term of the series* (6) *is less than $(\varepsilon c_1)^n c_2(F)$, where c_1 does not depend on n, t, F, and $c_2(F)$ is independent of n and t.*

2) *As $t \to \infty$, the nth term of the series* (6) *has a limit in the topology of K defined by the norm.*

The first assertion in Theorem 2 implies in the standard way that the series (6) is equal to $W_t F$, while the second assertion implies the existence of the inverse wave operators. The corresponding arguments for the forward operators $\widehat{W}_+$ as well as for W_- and $\widehat{W}_-$ are all similar.

REMARK 2. When $\nu = 1$ or 2 one cannot prove analyticity in ε, even for the forward wave operators; however, their existence (for arbitrary dimension $\nu = 1, 2, 3, \dots$) is easily proved using Cook's method without the assumption that ε is small.

We will prove Theorem 2 for interactions V in the class A; the proof for $V \in B$ is similar, except that in equation (11) below the function $G_v(\overline{K}_v)$ has to be somewhat modified. We will first prove Theorem 2 for a dense subset of A and then extend it to include all interactions in A.

We consider interactions which in the k-representation have the special form

$$V = \sum_{j=1}^{M} \int_{R^\nu} V_j^{(1)}(z) \otimes V_j^{(2)}(z)\, dz, \tag{7}$$

where

$$V_j^{(1)}(z) = \int \prod_p e^{\pm i(z,k,j,p)} f_{j,p}(k_{j,p}) a^{\#}(k_{j,p})\, dk_{j,p} \tag{8}$$

for $f_{j,p} \in S(R^\nu)$, and the plus or minus sign in $e^{\pm i(z,k,j,p)}$ is chosen according as $a^{\#}(k_{j,p})$ is a creation or an annihilation operator; $\# = *$ for $p = 1, \dots, m_j$, the order of the creation and annihilation operators in the product is the same as in (4), and

$$V_j^{(2)}\psi = (\psi, g_j) f_j, \qquad \psi \in L_2(R^\nu) \tag{9}$$

for suitable f_j and $g_j \in S(R^\nu)$.

Let us write out in more detail the nth term of the series (6) for vectors of the form (5^a):

$$\begin{aligned} \sum_\pi \int_{\Delta_t^n} dt_1 \dots dt_n \int_{R^{\nu n}} dz_1 \dots dz_n \int \prod_{v,p} & f_{\pi(v),p}(k_{v,p}) a^{\#}(k_{v,p}) e^{\pm(it_v h_1(k_{v,p}))} \\ & \times e^{\pm(iz_v, k_{v,p})} dk_{v,p} a^*(\psi_r) \dots a^*(\psi_1)\Omega \\ \otimes \int \prod_{v=1}^{n} & e^{i(t_{v-1}-t_v)h_2(k_v) + i((z_{v-1}-z_v),k_v)} f_{\pi(v-1)}(k_v) \overline{g}_{\pi(v)} dk_v \\ & \times e^{-it_n(h_2(k_{n+1}) - i(z_n, k_{n+1}))} f_{\pi(n)}(k_{n+1}), \end{aligned} \tag{10}$$

where π is an arbitrary function $(1, \dots, n) \to (1, \dots, M)$, $t_0 = z_0 = 0$, and $f_{\pi(0)} = \psi$. The sign $\pm$ is chosen depending on whether the variable $k_{v,p}$ corresponds to a creation or an annihilation operator, respectively. Since the subsequent estimates will be uniform in π, we will take $\pi(v) = v$ for brevity.

We integrate over the space variables $z_1, \dots, z_n$ and then eliminate the resulting δ-functions by integrating over the variables $k_1, \dots, k_n$ describing the particle. Straightforward

computations transform (10) into

$$\begin{aligned}
&\int_{\Delta_t^n} dt_1 \dots dt_n \int \prod_{v,p} f_{v,p}(k_{v,p}) a^{\#}(k_{v,p}) \\
&\quad \times e^{\pm(ith_1(k_{v,p}))} dk_{v,p} \int \prod_{v=1}^{n} e^{i(t_{v-1}-t_v)h_2(k_v)} \\
&\quad \times G_v(k_v) \delta\Big(\sum_p (\pm k_{v,p}) + k_v - k_{v+1}\Big) dk_v \\
&\quad \times a^*(\psi_r) \dots a^*(\psi_1)\Omega \otimes (e^{-ith_2(k_{n+1})} f_n(k_{n+1})) \\
&= \int_{\Delta_t^n} dt_1 \dots dt_n \prod_{v,p} \int f_{v,p}(k_{v,p}) a^{\#}(k_{v,p}) \qquad (11) \\
&\quad \times e^{\pm(it_v(k_{v,p})} dk_{v,p} \prod_{v=1}^{n} e^{i(t_{v-1}-t_v)h_2(\overline{k}_v)G_v(k_v)} \\
&\quad \times a^*(\psi_r) \dots a^*(\psi_1)\Omega \otimes (e^{-it_n h_2(k_{n+1})} f_n(k_{n+1})),
\end{aligned}$$

where we have written $G_v = f_{\pi(v-1)}\overline{g}_{\pi(v)}$,

$$\overline{k}_v = -\sum_{v'=v}^{n} \Big(\sum_p (\pm k_{v',p}) + k_{n+1} \Big)$$

and the choice of sign $\pm$ is as in (10).

3. Expansion in resummed diagrams. Although the norm of the integrand on the right in (11) is finite, we still need to perform an integration over an infinite time interval. In order to get the required time decay, we note that the last product $\prod_{v,p}$ on the right in (11) is a product of Wick monomials indexed by Friedrichs diagrams. It is well known that each diagram decays rapidly in the variable t_v; however, the number of diagrams increases as $n!$. It is therefore necessary to show that cancellations occur. We will employ another expansion which is, of course, equivalent to a partial summation of the diagrams.

To this end, for each vertex v we pick the leftmost creation operator $a(k_{v1})$ and use the anticommutation relations

$$a(k_{v1})a^*(k_{v',j}) = -a^*(k_{v',j})a(k_{v1}) + \delta(k_{v1} - k_{v',j}) \qquad (12)$$

to move it over to the right toward the vacuum vector Ω.

Each such transposition produces one of the two terms on the right in (12). If the first term is generated, we continue the shuffle, while if a δ-function appears (i.e., there is a pairing) then we say that a line $((v,1),(v',j))$ has been generated in the diagram. This procedure thus generates exactly n lines $((v-1),(v'(v),j(v)))$, $v = 1,\dots,n$. Moreover, if the pairing involves a creation operator appearing in the product

$$a^*(\psi_r) \dots a^*(\psi_1)\Omega = \int \Big(\prod a^*(k_{v,j})\psi(k_{v,j}) dk_{v,j} \Big)\Omega,$$

then we say that $v'(v) = 0$. Here, of course, all pairs $(v'(v), j(v))$ are distinct.

The result of these operations is to eliminate from (10) the corresponding creation/annihilation operators $a(k_{v1})$, $a^*_{k(v'(v),j(v))}$ and to produce a new term

$$\sum_{\{v'(v),j(v)\}} \prod \delta(k_{v,1} - k_{v'(v),j(v)})$$

in the integrand, where the sum $\sum$ is over all sets of pairwise distinct pairs such that $v'(v) < v$. We take this sum outside the integral and integrate over all variables $k_{v'(v),j(v)}$, $v = 1, \dots, n$. All δ-functions then disappear, and if we set

$$F = \sum_{v=1}^{n} f_{v,1}(k_{v,1}) f_{v'(v),j(v)}(k_{v,1}) G_v(\overline{k}_v),$$

the integrand becomes

$$\begin{aligned} & F \prod_{v,p} [f_{v,p}(k_{v,p}) e^{\pm(ith_1(k_{v,p})} a^{\#}(k_{v,p})] \\ & \qquad \times \exp\bigl(-i \sum (t_v - t_{v-1}) h_2(\overline{k}_v) + (t_v - t_{v'(v)}) h_1(k_{v,1})\bigr) \\ & \qquad \times \Omega \otimes (e^{-ith_2(k_{n+1})} f_n(k_{n+1})), \end{aligned} \tag{13}$$

where in $\overline{k}_v$ all the $k_{v'(v),j(v)}$ have been replaced by $k_{v,1}$, and no factors corresponding to v, 1 and $(v'(v), j(v))$ appear in the product $\prod$; $f_{0,p} = \psi_p$.

Consider the graph G with vertices $n, n-1, \dots, 1, 0$ and edges (lines) $(v, v'(v))$. We note that this graph is connected by construction.

4. Stationary-phase estimates and summation of diagrams. Let us consider in (13) the integral

$$\begin{aligned} & \int F \exp\Bigl(- i \sum_{v=1}^{n} [(t_v - t_{v-1})(\overline{k}_v)^2 + (t_v - t_{v'(v)})(k_{v,1})^2] \Bigr) \prod_{\nu=1}^{n} dk_{v,1} \\ & \qquad = \int F_1 \exp\Bigl[- \frac{1}{2}(B\underline{k}, \underline{k}) + i(\underline{a}, \underline{k}) \Bigr] d\underline{k}, \end{aligned} \tag{14}$$

where we recall that $h_1(k) = h_2(k) = (k)^2$; here $\underline{k} = (k_{1,1}, \dots, k_{n,1})$, and the components of the vector $\underline{a}$ are linear combinations of the $k_{v,j}$, $j \neq 1$, obtained by expanding out the square $(\overline{k}_v)^2$.

We have set

$$\begin{aligned} -\frac{1}{2}(B\underline{k}, \underline{k}) &= -\Bigl[\sum_{v=1}^{n} (t_v - t_{v-1})(\overline{k}_v^{(1)})^2 \Bigr] - \Bigl[\sum_{v=1}^{n} (1 + t_v - t_{v'(v)})(k_{v,1})^2 \Bigr] \\ &= -\frac{1}{2}(B_1\underline{k}, \underline{k}) - \frac{1}{2}(B_2\underline{k}, \underline{k}), \end{aligned} \tag{15}$$

$$F_1 = F \exp\Bigl\{ i \sum_{v=1}^{n} \Bigl[(t_v - t_{v-1})(\overline{k}_v^{(2)})^2 \Bigr] + \sum_{v=1}^{n} k_{v,1}^2 \Bigr\},$$

and $\overline{k}_v^{(1)} = \overline{k}_v$, where all variables other than $k_{1,1}, \dots, k_{n,1}$ are set equal to zero; $\overline{k}^{(2)} = \overline{k}_v - \overline{k}^{(1)}$.

Taking the Fourier transform in the variables $(k_{1,1},\dots,k_{n,1})$, we can rewrite (14) as

$$\frac{1}{(2\pi i)^{n\nu/2}(\operatorname{Det} B)^{1/2}}\int \exp\left[i\frac{1}{2}(B^{-1}\underline{\kappa},\underline{\kappa})\right]\widetilde{F}_1(\kappa-a)d\underline{\kappa}, \tag{16}$$

where $\kappa=(\kappa_{11},\dots,\kappa_{n1})$.

Observe that (16) is a function of the parameters $k_{v,p}$, where $p\neq 1$ and $(v,p)=(v'(v),j(v))$.

The next two results will be needed to estimate (16).

ASSERTION 3. *Since the matrices B_1 and B_2 are both positive, we have*

$$(\operatorname{Det} B)^{-1} < (\det B_2)^2 = \prod_{v=1}^{n}\frac{1}{(|t_v-t_{v'(v)}|+1)^{\nu/2}}. \tag{17}$$

ASSERTION 4. *The integral in* (16), *which is a function of the variables $k_{v,p}$ such that $j(v)\neq 1$ and $(v,p)\neq(v'(v),j(v))$, belongs to $S(R^{\nu N})\cap L_2(R^{\nu N})$, where N is the number of variables. Moreover, its norm $\|\cdot\|_{L_2}$ is bounded by $c_1(\psi)c_2^n$ uniformly in $\{t_v\}$, where $c_2>0$ is a constant independent of n.*

PROOF. We now set

$$f_1=\prod_{v=1}^{n} f_{v1}(k_{v,1})\exp\{ik_{v,1}^2\},\quad f_2=\prod_{v}{}' f_{v'(v),j(v)}(k_{v,1}),$$
$$f_3=\prod_{v}{}'' \psi_{j(v)}(k_{v,1}),\qquad f_4=\prod_{v=1}^{n} G_v(k_v),$$

where the product $\prod''$ (respectively, $\prod'$) is over all vertices v paired (respectively, not paired) with the zero vertex. Then

$$F_1=f_1f_2f_3f_4\exp\left\{i\sum_{v=1}^{n}[(t_{v-1}-t_v)(\overline{k}_v^{(2)})^2]\right\},$$
$$\left|\int \exp\left[i\frac{1}{2}(B^{-1}\underline{\kappa},\underline{\kappa})\right]\widetilde{F}_1(\kappa-a)d\underline{\kappa}\right| \le \|\widetilde{F}_1\|_1$$
$$\le \|f_1\widetilde{f_2f_3}f_4\|_1=\|\widetilde{f}_1*(\widetilde{f_2f_3})*\widetilde{f}_4\|_1\le\|\widetilde{f}_1\|_1\|\widetilde{f_2f_3}\|_1\|\widetilde{f}_4\|_1.$$

Here we have written the Fourier transform of a product of functions as a convolution and used Young's theorem. The norm of each factor can be estimated uniformly in $\{k_v^{(2)}, v=1,\dots,n)$ by

$$\|\widetilde{f}_1\|_1=\prod_{v=1}^{n}\|f_{v,1}(k_{v,1})\exp\{ik_{v,1}^2\}\|_1,$$
$$\|\widetilde{f_2f_3}\|_1=\prod_{v}{}'\|\widetilde{f}_{v'(v),j(v)}\|_1\prod_{v}{}''\|\psi_{j(v)}\|_1,$$
$$\|\widetilde{f}_4\|_1=\frac{1}{(2\pi)^{n\nu/2}}\int\left|\int\exp\left\{-i\sum_{v=1}^{n}(\kappa_v,k_{v,1})\right\}G_v(\overline{k}_v)\prod_{v=1}^{n}\right|\prod_{v=1}^{n}d\kappa_v \tag{18}$$
$$=\frac{1}{(2\pi)^{n\nu/2}}\int|\widetilde{f}_4((D^{-1})^*\kappa)\det(D^{-1})|\prod_{v=1}^{n}d\kappa_v=\left(\prod_{v=1}^{n}\int|\widetilde{G}_v(\overline{\kappa}_v)|d\kappa_v\right),$$
$$(\det(D^{-1}))(\det(D)^*)=\prod_{v=1}^{n}\|\widetilde{G}_v\|_1\le\prod_{v=1}^{n}\|\widetilde{f}_v\|_1\|\widetilde{g}\|_1,$$

and the constant-coefficient matrix D satisfies $k = D\overline{k}$, where

$$\overline{k} = (\overline{k}_1, \dots, \overline{k}_n); \qquad k = (k_{1,1}, \dots, k_{n,1}),$$
$$\kappa = (\kappa_1, \dots, \kappa_n); \qquad \overline{\kappa} = (\overline{\kappa}_1, \dots, \overline{\kappa}_n),$$
$$\overline{\kappa} = (D^{-1})^* \kappa.$$

Since D is triangular with identity diagonal entries, (18) yields the bound

$$\|\widetilde{f_1 f_2 f_3} f_4\|_1 < (C_f)^{4n} C_\psi^r,$$

where

$$C_f = \max_{v,j}\{\|f_{v,j}\|_2, \|\widetilde{f}_{v,j}\|_1, \|f_{v1} \exp\{-ik_{v,1}\}\|_1\} > 1,$$
$$C_\psi = \max_{j}\{\|\psi_j\|_2, \|\widetilde{\psi}_j\|_1) > 1.$$

Using the estimate

$$\left\| \int Q(y_1, \dots, y_N) a^\#(y_1) \dots a^\#(y_N)\, dy_1 \dots dy_N \right\| \le \|Q\|_2$$

valid for any $Q \in L_2$, we obtain Assertion 4 with $C_2 = (C_f)^{2m+2}$.

Assertions 3 and 4 imply that the nth term in the series (6) is bounded by

$$C^n \Big(\sum_{\{v(v), j(v)\}} \prod_v \frac{1}{(|t_v - t_{v'(v)}| + 1)^{\nu/2}} \Big). \tag{19}$$

Lemma (6.1) can then be used to estimate the total contribution from all the diagrams.

5. Two-particle interaction. Here we consider an operator H that preserves particle number and whose restriction to $\mathcal{F}_a^{(N)}(L_2(R^\nu)) \otimes L_2(R^\nu)$ is given by

$$H_N = -\sum_{j=1}^N \Delta_{x_j} - \Delta_y + \varepsilon \sum_{j=1}^N V(x_j - y) = H_0 + \varepsilon \sum_{j=1}^N V(x_j - y), \tag{20}$$

where $V \in S(R^\nu)$ and $x_j, y \in R^\nu$.

THEOREM 3. *For $\nu \ge 3$ and $V \in S(R^\nu)$ there exists an $\varepsilon_0 = \varepsilon_0(\nu, V) > 0$, independent of N, such that for all N and $|\varepsilon| < \varepsilon_0$ the system* (20) *is asymptotically complete and H_N is unitarily equivalent to the free Hamiltonian H_0.*

This result differs from the previous ones in that the interaction operator in the "second quantization representation" is given by

$$\varepsilon \int_{R^\nu} V(x - y) a^*(x) a(x)\, dx,$$

i.e., an extra δ-function is present.

Since the proof of this theorem is a verbatim repetition of the proof of Theorems 1 and 2 apart from the difference just noted, we will merely indicate the changes required in the proof.

Formally, the difference consists in the fact that in expression (7) $M = 1$, $m_j = n_j = 1$, and there is an additional δ-function:

$$V = \int_{R^\nu} V(x_1 - y)\delta(x_1 - x_2)a^*(x_1)a^*(x_2)\,dx_1\,dx_2 \otimes \delta_y\,dy,$$

where δ_y is the δ-function at the point y.

Let $\widetilde{V}(k)$ be the Fourier transform of $V(x)$, so that in the k-representation the nth term of the series (6) for vectors of the form (5^a) in the present case takes the form

$$\begin{aligned}\int_{\Delta_t^n} dt_1 \dots dt_n \prod_v \Big\{ \int \widetilde{V}(k_{v,2} - k_{v,1}) \exp\{it_v[h(k_{v,1}) - h(k_{v,2})]\} \\ \times a^*(k_{v,1})a(k_{v,2})dk_{v,1}dk_{v,2}\Big\} a^*(\psi_r)\dots a^*(\psi_1)\Omega \\ \otimes \Big[\prod_{v=1}^n \int e^{i(t_{v-1}-t_v)h(k_v)}\delta(-k_{v,1} + k_{v,2} - k_v + k_{v+1})dk_v\Big]\, dk_{n+1}\psi(k_1),\end{aligned} \tag{21}$$

where $h(k) = k^2$.

We note that this differs from (11) in that all the functions $f_v \equiv 1$, $g_v \equiv 1$ for $v = 1, \dots, n$; $f_0 = \psi$.

Eliminating the δ-function in (21), we obtain

$$\begin{aligned}\int_{\Delta_t^n} dt_1 \dots dt_n \prod_v \Big\{ \int \widetilde{V}(k_{v,2} - k_{v,1}) \exp\{it_v[h(k_{v,1}) - h(k_{v,2})]\} \\ \times a^*(k_{v,1})a(k_{v,2})dk_{v,1}dk_{v,2}\Big\} \\ \times a^*(\psi_{r_1})\dots a^*(\psi_1)\Omega \otimes \Big[\prod_{v=1}^n \int e^{i(t_{v-1}-t_v)h(\overline{k}_v)}\, dk_{n+1}\psi(\overline{k}_1)\Big],\end{aligned} \tag{22}$$

where

$$\overline{k}_v = k_{v,1} - k_{v,2} + k_{n+1}.$$

As in (11), we can represent (22) as sums over Friedrichs diagrams. Let $v'(v)$ be the number of vertices paired with vertex v. Setting $\psi_{j(v)} \equiv 1$ if $v'(v) \neq 0$, we see that each term of the sum is of the form

$$\int F\Big\{\prod_{v\in L_2} a^{\#}(k_{v,2})\Big\}\Big\{\prod_{j\in I} a^*(\psi_j)\Big\}\Omega \otimes e^{-it_n h(k_{n+1})}\psi(\overline{k}_1)\prod_{v=1}^n dk_{v,1}\prod_{v\in I_2} dk_{v,2}, \tag{23}$$

where

$$\begin{aligned}F = &\Big\{\prod_{v\in I_1} \widetilde{V}(k_{v'(v),1} - k_{v,1})e^{-i(t_v - t_{v'(v)})h(k_{v,2})}\psi_{j(v)}(k_{v,1})\Big\} \\ &\times \Big\{\prod_{v\in I_2} \widetilde{V}(k_{v,2} - k_{v,1})e^{-i(t_v - t_{v'(v)})h(k_{v,1})\psi_{j(v)}(k_{v,1})}\Big\} \\ &\times \exp\Big\{i\sum_{v=1}^n[(t_{v-1} - t_v)h(\overline{k}_v))\Big]\Big\},\end{aligned} \tag{24}$$

and I_1 is the set of vertices with which there is a pairing, I_2 is the set of vertices with which there is no pairing, and I is the set of unpaired legs of the zero vertex.

Consider the function F_1 of the variables $\{k_{v,1},\, v=1,\dots,n\}$ and $\{k_{v,2},\, v\in I_2\}$, where

$$F_1=\Big\{\prod_{v\in I_1}\widetilde{V}(k_{v'(v),1}-k_{v,1})\psi_{j(v)}(k_{v,1})\Big\}\Big\{\prod_{v\in I_2}\widetilde{V}(k_{v,2}-k_{v,1})\psi_{j(v)}(k_{v,1})\Big\}.$$

As in (15), we take the Fourier transform in (23). It remains only to prove the analog of Assertion 4 for the function

$$\begin{aligned}&F_2(k_{v,2},v\in I_2,k_{n+1})\\ &\quad=\frac{1}{(2\pi)^{n\nu/2}}\int\Big|\int\exp\Big(-i\sum_{v=1}^n(x_v,k_{v,1})\Big)F_1(k)\prod_{v=1}^n dk_{v,1}\Big|\prod_{v=1}^n dx_v.\end{aligned}\tag{25}$$

Assertion 6. *The function $F_2(k_{v,2},\, v\in I_2,k_{n+1})$ is in $L_2(R^{\nu N})$, where N is the number of variables $\{k_{v,2},\, v\in I_2\}$, k_{n+1}; moreover, its L_2 norm is bounded by*

$$\|F_2\|<C^nC(\psi),$$

where $C>0$ is a constant independent of n, $\psi,\psi_1,\dots,\psi_r$, and $C(\psi)$ depends only on $\psi,\psi_1,\dots,\psi_r$.

Proof. Set $k=(k_{1,1},\dots,k_{n,1})$, $x=(x_1,\dots,x_n)$. For each component of the variable $k_{v,1}=(k_{v,1}^{(1)},\dots,k_{v,1}^{(\nu)})$ we split the inner integral into the parts over $|x_v^{(i)}|\ge 1$ and $|x_v^{(i)}|<1$. Integrating the first one twice by parts with respect to $k_{v,1}^{(1)}$, we obtain the estimate

$$\begin{aligned}&F_2(k_{v,2},v\in I_2,k_{n+1})\le C^n\int\int\prod_{v=1}^n\frac{1}{(|x_v|+1)^{2\nu}}\Big|\frac{\partial^{2\nu n}F_1(k)}{\partial^2k_{11}^{(1)}\dots\partial^2k_{n,1}^{(v)}}\Big|,\\ &\prod_{v=1}^n dk_{v,1}\prod_{v=1}^n dx_v\le C^n\int\Big|\frac{\partial^{2\nu n}F_1(k)}{\partial^2(k^{(1)})\dots\partial^2(k^{(v)})}\Big|\prod_{v=1}^n dk_{v,1},\end{aligned}\tag{26}$$

where $|x_v|=|x_v^{(1)}|+\dots+|x_v^{(\nu)}|$.

Note that $\psi(\overline{k}_1)$ depends on at most $2r+1$ variables, since

$$\overline{k}_1=-\sum_{v\in I}k_{v,2}+k_{n+1}+\sum_{v\in I_3}k_{v,1},$$

where I_3 is the set of vertices joined by edges to the zero vertex.

Expanding the product in (26), we therefore get at most $C^n(2r)^{2r}$ terms, which involve derivatives of order at most 4ν in the functions V, ψ_1, and ψ_r, and of order at most $2r\nu$ in the function ψ.

Consequently,

$$|F_2(k_{v,2},v\in I_2,k_{n+1})|<C^n\sum_j\int|F_1^j(k)|\prod_{v=1}^n dk_{v,1},\tag{27}$$

where $F_1^{(j)}$ has the same form as F_1, except that V, ψ_j, and ψ are replaced by derivatives of order $\le 4\nu$ or $\le 2r\nu$, as described above. Hence

$$\|F_2\| < C^n \sum_j \sum_i \int \Big(\int |F_1^{(j)}(k)| \prod_{v=1}^{n} dk_{v,1} \times \int |F_1^{(i)}(k')| \prod_{v=1}^{n} dk'_{v,1} \Big) \prod_{v\in I_2} dk_{v,2}\, dk_{n+1}. \tag{28}$$

Each integral in (28) is easily estimated by applying Fubini's theorem and then using the estimate $|V^{(i)}(k_{v,2} - k'_{v,1})| < \overline{C}$ for $v \in I_2$, and $|\psi(\overline{k}'_1)| < C(\psi)$. Making the obvious change of variables, we get the desired estimate.

REMARK. The result of this subsection remains valid for symmetric Fock space. Indeed, consider a particle-number-conserving operator H whose restriction to $\mathcal{F}_s^N(L_2(R^\nu)) \otimes L_2(R^\nu)$ is given by

$$H_N = -\sum_{i=1}^{N} \Delta x_i - \Delta y + \varepsilon \sum_{i=1}^{N} V(x_i - y), \tag{29}$$

where $\mathcal{F}_s^N(L_2(R^\nu)) \subset \mathcal{F}_s(L_2(R^\nu))$ is the N-particle subspace of symmetric Fock space, $V \in S(R^\nu)$, and $x_i, y \in R^\nu$.

THEOREM 4. *For $\nu \ge 3$ and $V \in S(R^\nu)$ there exists an $\varepsilon_0 = \varepsilon_0(v, V) > 0$, independent of N, such that for all N and $|\varepsilon| < \varepsilon_0$, system* (29) *is asymptotically complete and the operator H_N is unitarily equivalent to the free Hamiltonian H_N^0.*

The proof of Theorem 4 is the same as for Theorem 3, except that the boson creation and annihilation operators $a^{\#}(f)$, which are unbounded when considered on all of $\mathcal{F}_s(L_2(R^\nu))$, are bounded on each subspace $\mathcal{F}_s^r$ by $(r+1)^{1/2}$ times the L_2 norm of f. In our case we get exactly r unpaired creation operators, which for each r enables us to estimate the norm of their product independently of N.

6. One-particle operators h_1 and h_2 of general form. Here we mention a result related to asymptotic completeness for operators

$$H = H_0 + \varepsilon V,$$

where as before H_0 is of the form

$$H_0 = d\Gamma(h_1) \otimes \mathbf{1} + 1 \otimes h_2$$

and the operators h_1 and h_2 need not coincide with the Laplace operator, which was assumed in the previous theorems. Specifically, we assume that after taking the Fourier transform, h_1 and h_2 are given by multiplication by smooth functions $\widetilde{h}_1$ and $\widetilde{h}_2$ which have finitely many critical points, which are all of Morse type (see [3]). However, the price to be paid for working in this generality is that some conditions must be imposed on the interaction operator

$$V = \sum_{i=1}^{M} \int dx_1 \dots dx_{m_i+n_i} K_i(x_1, \dots, x_{m_i+n_i}, y) \times a^*(x_1) \dots a^*(x_{m_i}) a(x_{m_i+1}) \dots a(x_{m_i+n_i}),$$

where the K_i are smooth kernels invariant under translations of their arguments.

In the theorem stated below, it is essential that the dimension ν be strictly greater than 4; in addition, the nonpolarization of the vacuum

$$m = \min m_i = \min n_i > 0$$

by the operator V is required to hold in the stronger sense, that is

$$m > \frac{\nu+2}{\nu-4}.$$

We now have the next theorem.

THEOREM 5. *Under the above hypotheses, for sufficiently small ε the forward and inverse Møller operators $W_\pm$ and $\widehat{W}_\pm$ for the operators H_0 and $H_0+\varepsilon U$ exist.*

The proof of this theorem again makes full use of the complicated machinery employed in the previous proofs (partitions, sectors, diagrams). For simplicity we consider the case when

$$V = \int_{R^\nu} V_2^{(1)} \otimes V_z^{(2)}\, dz = \int V_z\, dz,$$

where

$$\begin{aligned} V_z^{(1)} &= a^*(f_1(\cdot - z)) \dots a^*(f_m(\cdot - z)) a(f_m(\cdot - z)) a(f_1(\cdot - z)), \\ V_z^{(2)} g &= (g, f(\cdot - z)) f(\cdot - z), \qquad \widetilde{f}, \widetilde{f}_i \in C_0^\infty(R^\nu). \end{aligned} \tag{30}$$

To estimate the nth term

$$A_n(t,\psi) = \int_{\Delta_n^t} \int_{(R^\nu)^n} V_{z_1}(t_1) \dots V_{z_n}(t_n) \psi^{(1)} \otimes \psi^{(2)} dz_1 \dots dz_n dt_1 \dots dt_n, \tag{31}$$

in the series (6), where $\psi = \psi^{(1)} \otimes \psi^{(2)}$, we appeal to the following inequality for $u \in C_0^\infty(R^\nu)$, $\operatorname{supp} u \subset [-R, R]^\nu$:

$$\left| \int_{R^\nu} e^{i(x,k) + ith(k)} u(k)\, dk \right| < C(u) I(t,x), \tag{31^a}$$

where

$$I(t,x) = \begin{cases} \frac{1}{(1+|t|)^{\nu/2}}, & |x| \le M|t|, \\ \frac{c(d)}{(1+|t|+|x|)^d}, & |x| > M|t|, \end{cases} \qquad M = \max_{x \in \operatorname{supp} u} |\nabla h| \tag{32}$$

for every $d \ge 1$, and $c(d)$ depends on d. Using (31) and (32), one proves without difficulty that

$$\|V_{z_1}^{(1)}(t_1) \dots V_{z_n}^{(1)}(t_n) \psi^{(1)}\| < c(V^{(1)}, d)^n c(V^{(1)}, d, \psi) \prod_{i=1}^n I(t_i - t_{i-1}, z_i - z_{i-1}), \tag{33}$$

where $t_0 = 0$, $z_0 = 0$. Then (30) and (33) imply the bound

$$\begin{aligned} \|A_n(t,\psi)\| < c^n c(\psi) &\int_{\Delta_n^t} A_n^{(1)}(t_1, \dots, t_n, \psi^{(1)}) \\ &\times \prod \frac{1}{(1+|t_i - t_{i-1}|)^{\nu/2}}\, dt_1 \dots dt_n, \end{aligned} \tag{34}$$

where $d = 2\nu$ and

$$A_n^{(1)}(t_1, \dots, t_n, \psi^{(1)}) = \operatorname{supp}_{z_1,\dots,z_n} \|V_{z_1}^{(1)}(t_1) \dots V^{(1)} z_n(t_n) \psi^{(1)}\|, \tag{35}$$

and the constant c depends on ν, $V^{(1)}$, $R = R(V^{(2)})$, where R is such that $\operatorname{supp} \widetilde{f} \subseteq [-R, R]^\nu$.

We then expand (35) as a sum of Friedrichs diagrams and carry out a partial summation of the latter which differs from the one employed above.

Namely, we introduce partitions and sectors as was done in subsection 3 of §2. Since there are at most 2^n distinct sectors, we can fix some sector $\alpha = (\alpha_1, \dots, \alpha_k)$ and the right endpoints of its intervals: $\Gamma_1, \dots, \Gamma_k$, where $\Gamma_1 = 1$, $\Gamma_i = \alpha_{i-1} + 1$, $i = 2, \dots, k$. Using the canonical anticommutation relations, we move all the annihilation operators at the vertices Γ_i, $i = 1, \dots, k$, to the right until they act on the vacuum vector. This shuffle gives rise to lines in the diagrams, each corresponding to a "propagator"

$$(e^{it_j h_1} f_q^{(j)}(x_j), e^{it_i h_1} f_p^{(i)}(x_i)), \tag{36}$$

which as in (30^a) can be estimated by

$$\frac{C(R^{(1)})}{(1 + |t_i - t_{i-1}|)^{\nu/2}}.$$

Here $R^{(1)} = R^{(1)}(V^{(1)})$ is such that $\operatorname{supp} \widetilde{f}_q \subset [-R^{(1)}, R^{(1)}]^\nu$ for $q = 1, \dots, n$.

We thus obtain graphs with $n + 1$ vertices $n, \dots, 1, 0$. A graph will be called admissible if:

1) each vertex $i \in I$ is the left-hand endpoint of at least m lines (these result from shuffling the annihilation operators from the vertex $i \in I$ to the vacuum vector);

2) each vertex i is the right-hand endpoint of at most m lines (they result from pairing with the creation operators at the vertex i).

Since $\|a^\#(f)\| = \|f\|$ for every f, we can estimate expression (34) by the quantity

$$\begin{aligned} c^n c(\psi) \sum_\alpha \sum_G \int \prod_{(i,j)\in G} \frac{1}{(1 + |t_i - t_{i-1}|)^{\nu/2}} \prod_{v=j}^{n} |t_v - t_{v-1}|^{\nu/2} dt_1 \dots dt_n \\ < c^n c(\psi) \sum_\alpha \sum_G \int \prod_{(i,j)\in G} \frac{1}{(1 + |t_i - t_{i-1}|)^{\nu/2(1-\frac{1}{m})}} dt_1 \dots dt_n, \end{aligned} \tag{37}$$

where the sum $\sum_G$ is over all admissible graphs for the sector α.

LEMMA 9. *For $\beta = (\nu/2)(1 - 1/m) > 2$ we have the bound*

$$\sum_\alpha \sum_G \int \prod_{(i,j)\in G} \frac{1}{(1 + |t_i - t_{i-1}|)^\beta} dt_1 \dots dt_n < C^n,$$

where C is independent of m.

We refer to [48] for the proof. Theorem 5 now follows from this lemma and estimate (37).

§4. Weak-interaction limit for a quantum Schrödinger particle interacting with Fermi gas

In the interaction of a particle with Fermi gas, the latter can be regarded as a "heat reservoir" vis-à-vis the particle, and the motion of the latter can be analyzed under the influence of the reservoir. More precisely, this means the following. The interaction of the particle with the gas generates a perturbed dynamics τ_t^ε (where ε is the interaction parameter) in the algebra $\mathfrak{A} = \mathfrak{A}_s \otimes \mathfrak{A}_R$, where the C^*-algebra $\mathfrak{A}_s$ describes the particle and the C^*-algebra (CAR algebra) $\mathfrak{A}_R$ describes the reservoir. This dynamics and an equilibrium state ω for the reservoir can be used to define a family of maps γ_t^ε of $\mathfrak{A}_s$ into itself by

$$\gamma_t^\varepsilon(B) = \omega(\tau_t^\varepsilon(\tau_{-t}^0(B) \otimes 1)),$$

where τ_t^0 is the free dynamics on $\mathfrak{A}_s$, and $A \in \mathfrak{A}_R$. In general, the family of maps $\gamma_t^\varepsilon \colon \mathfrak{A}_s \to \mathfrak{A}_s$ (sometimes called the *reduced dynamics* of the particle) does not form a group, nor even a semigroup of transformations. However, in the *weak-interaction limit*

$$\lim_{\substack{\varepsilon \to 0 \\ \varepsilon^2 t \to s}} \gamma_t^\varepsilon(B) = T_s(B) \tag{1}$$

the maps $T_s \colon \mathfrak{A}_s \to \mathfrak{A}_s$ can be taken to define a semigroup. In this section we will study the existence of the limit (1) in some particular cases.

1. Reduced dynamics and the weak-interaction limit. Let $\mathcal{H}_s = L_2(R^\nu)$, $\nu \geq 3$, and let $\mathcal{H}_R = \mathcal{F}_a(\mathcal{H}_s)$ be the antisymmetric Fock space over $\mathcal{H}_s$, where $\mathrm{Com}(\mathcal{H}_s)$ is the algebra of compact operators on $\mathcal{H}_s$, and $\mathfrak{A}_s$ is the C^*-algebra of compact operators on $\mathcal{H}_s$ with the identity element adjoined. Let $\mathfrak{A}_R$ be the C^*-algebra generated by the creation/annihilation operators $\{a(f), a^*(f), f \in \mathcal{H}_s\}$ on $\mathcal{H}_R$. We have the selfadjoint operator

$$H_0 = H_s \otimes 1 + 1 \otimes H_R, \tag{2}$$

where $H_s = -\Delta$ is the Laplace operator on $\mathcal{H}_s$ and $H_R = d\Gamma(H_s)$ is the second-quantized Laplace operator on $\mathcal{H}_R$; then H_0 defines a free dynamics

$$\tau_t^0(A) = \exp\{itH_0\} A \exp\{-itH_0\}, \qquad A \in \mathfrak{A}, \qquad t \in R$$

on the C^*-algebra $\mathfrak{A}_s \otimes \mathfrak{A}_R$.

Let ω_β be a KMS state on the algebra $\mathfrak{A}_R$ corresponding to the inverse temperature β relative to the dynamics τ_t^0; ω_β is a quasifree state (see Chapter 2). Consider the interaction $\mathbb{V}$

$$\mathbb{V} = \sum_{j=1}^{\nu} p_j \otimes V_j, \tag{3}$$

where $p = (p_1, \ldots, p_\nu) = i\nabla$ is the momentum operator on $\mathcal{H}_s$, and

$$V_j = \sum_{k=1}^{M_j} : a^*(f_{j,1}^{(k)}) \ldots a^*(f_{j,m_j}^{(k)}) a(f_{j,m_j+1}^{(k)}) \ldots a(f_{j,m_j+n_j}^{(k)}) :,$$

$$\widetilde{f}_{j,i}^{(k)} \in C_0^\infty(R^\nu), \qquad 1 \leq j \leq \nu, \tag{4}$$

$$1 \leq i \leq m_j + n_j, \qquad 1 \leq k \leq M_j, \qquad M_j < \infty,$$

where $: \cdots :$ are the Wick dots relative to the quasifree gauge-invariant state ω_β (see Chapter 2). Let

$$I = \{f_{j,i}^{(k)} \in C_0^\infty(R^\nu), 1 \le j \le \nu, 1 \le i \le m_j + n_j, 1 \le k \le M_j\}$$

be the set of all functions by means of which $\mathbb{V}$ is defined.

REMARK. The choice of $\mathbb{V}$ in the special form (4) is not accidental but corresponds to a renormalization of the interaction V_j. If the Wick dots are omitted in (4), then the weak-interaction limit may fail to exist.

We set $m_{\max} = \max_j(m_j + n_j)$, $m_{\min} = \min_j(m_j + n_j)$. Let D_0 be the linear span of the functions $f \in \mathcal{H}_s$ such that $\widetilde{f} \in C_0^\infty(R^\nu)$, and let $\mathcal{F}_0$ be the subspace of $\mathcal{H}_R$ spanned by the vectors

$$a^*(f_1) \dots a^*(f_k)\Omega, \qquad f_j \in D_0, \quad j = 1, \dots, k,$$

where Ω is the vacuum vector.

The operator $H_\varepsilon = H_0 + \varepsilon\mathbb{V}$, $\varepsilon \in R$, is essentially selfadjoint on $D_0 \otimes \mathcal{F}_0$ and defines a perturbed dynamics

$$\tau_t^{\varepsilon\mathbb{V}}(A) = \exp\{itH_\varepsilon\} A \exp\{-itH_\varepsilon\}, \qquad A \in \mathfrak{A}, \qquad t \in R \tag{5}$$

on $\mathfrak{A}$. Let $A \otimes B \in \mathfrak{A}$ and set $\omega(A \otimes B) = A\omega_\beta(B)$. Extending ω by linearity and continuity to all of $\mathfrak{A}$, we can define $\gamma_t^\varepsilon : \mathfrak{A}_s \to \mathfrak{A}_s$ as follows:

$$\gamma_t^\varepsilon(A) = \omega(\tau_t^{\varepsilon\mathbb{V}} \tau_{-t}^0(A \otimes 1)), \qquad A \in \mathfrak{A}_s.$$

In this section we will be interested in proving the existence of the weak-interaction limit

$$\lim_{\substack{\varepsilon \to 0, t \to \infty \\ \varepsilon^2 t \to s}} \gamma_t^\varepsilon(A) = T_s(A), \qquad A \in \mathfrak{A}_s, \quad s \ge 0, \tag{6}$$

where T_s is a quantum dynamical semigroup on $\mathfrak{A}_s$. For this it suffices to prove the existence of the limit on the left in (6) for all $A \in \mathfrak{A}_s^0$ for $0 < s < s_A$, where $\mathfrak{A}_s^0 \subseteq \mathfrak{A}_s$ is a dense subalgebra and $s_A > 0$.

2. Existence of the weak-interaction limit. Let $\mathfrak{A}_{s,r}^0$ be the subalgebra of $\mathfrak{A}_s$ generated by the identity operator and the projections $(g, \cdot)f$, where the Fourier transforms $\widetilde{f}$ and $\widetilde{g}$ are in C_0^∞ and have support contained in S_r, the ball of radius r in R^ν. We set

$$\mathfrak{A}_s^0 = \bigcup_{r>0} \mathfrak{A}_{s,r}^0.$$

The correlation functions $\{g_{ij},\ 1 \le i, j \le \nu\}$ are defined by

$$g_{ij}(s) = \omega_\beta(V_{i,s} V_j), \tag{7}$$

$$V_{k,s} = \exp\{isH_R\} V_k \exp\{-isH_R\}. \tag{8}$$

The definition of the operator V_k and the properties of the KMS state ω_β imply that:

a)

$$|g_{ij}(s)| < \text{const}(1 + |s|)^{-\nu m_{\min}}; \tag{9}$$

b) if $\alpha_{ij} = \int_0^\infty g_{ij}(s)\,ds$, then

$$\operatorname{Im}\alpha_{ij} = \operatorname{Im}\alpha_{ji}, \qquad 1 \le i,j \le \nu; \tag{10}$$

c) the matrix $\{\operatorname{Re} a_{ij},\ 1 \le i,j \le \nu\}$ is positive definite on $\mathbb{C}^\nu$.

Let $Q = \{Q_{ij},\ 1 \le i,j \le \nu\}$ be the positive square root of the matrix $\{\operatorname{Re}(a_{ij} - a_{ji})\}$. For $u = (u_1, \dots, u_\nu) \in R^\nu$ and $p = (p_1, \dots, p_\nu)$, we define

$$P = \sum_{i,j=1}^{\nu} \operatorname{Im}\alpha_{ij} p_i p_j, \quad (u, Qp) = \sum_{i,j=1}^{\nu} Q_{ij} u_i p_j, \quad u^2 = \sum_{i=1}^{\nu} u_i^2. \tag{11}$$

THEOREM 1. *There exists a quantum dynamical semigroup $\{T_s,\ s \ge 0\}$ on $\mathfrak{A}_s$ such that for every $A \in \mathfrak{A}_s^0$ there exists a number s_A such that the limit*

$$\lim_{\substack{\varepsilon \to 0, t \to \infty \\ \varepsilon^2 t = s}} \gamma_t^\varepsilon(A) = T_s(A)$$

exists uniformly in $s \in [0, s_A]$. If $A \in \mathfrak{A}_{s,r}^0$ then s_A depends only on τ. The generator L of the semigroup T_s is given by

$$L(A) = -\sum_{k,l=1}^{\nu} \left\{ \frac{1}{2} \operatorname{Re}(\alpha_{k,l} + \alpha_{l,k})[p_k, [p_k, A]] + i \operatorname{Im}\alpha_{kl}[p_k p_l, A] \right\}, \tag{12}$$

for $A \in \mathfrak{A}_s^0$, and the semigroup itself acts on $\mathfrak{A}_s$ as follows:

$$T_s(A) = (2Pt)^{-\nu/2} \int e^{-(u^2/(2t)^\nu)} \times \exp\{itp + i(u, Qp)\} A \exp\{-itp - i(u, Qp)\}\, du. \tag{13}$$

PROOF. We will only sketch the main steps in the proof of this theorem. For $A \in \mathfrak{A}_{s,r}$ we have a convergent power series for $\gamma_t^\varepsilon(A)$:

$$\gamma_t^\varepsilon(A) = A + \sum_{n=1}^{\infty} (i\varepsilon)^n \int_{\Delta_n^t} \omega_\beta(\mathbb{V}_{t_n}, [\mathbb{V}_{t_{n-1}}, \dots [\mathbb{V}_{t_1}, A \otimes \mathbf{1}] \dots] dt_1 \dots dt_n, \tag{14}$$

where

$$\mathbb{V}_s = \exp\{isH_0\} \mathbb{V} \exp\{-isH_0\} = \sum_{j=1}^{\nu} p_j \otimes V_{j,s}.$$

The proof of Theorem 1 is based on the following estimate for the nth term $I_n^t(A)$ of the series (14):

$$\|I_n^t(A)\| \le c^n |\varepsilon|^n t^{[n/2]} C(A), \tag{15}$$

where $[\varepsilon]$ denotes the largest integer $\le \varepsilon$, the constant $c > 0$ is independent of n, and $C(A)$ depends on A.

The main difficulty in deriving (15) is that after the commutators are expanded out, one needs to estimate the integrals

$$\int_{\Delta_n^t} \omega_\beta(V_{i_1,t_{i_1}} \dots V_{i_n,t_{i_n}}) dt_1 \dots dt_n, \tag{16}$$

where $\pi\colon (1,\dots,n) \to (i_1,\dots,i_n)$ is a permutation. Associating to each V_{i,t_i} the vertex with index i, we can express the integrand as a sum of diagrams which we will call *admissible*. Each edge in an admissible diagram contributes the multiplicative factor

$$r_{ij}(t_i - t_j) = \omega_\beta(a^\#(f_{t_i})a^\#(g_{t_j})),$$

to the weight of the diagram, where $f, g \in S(R^\nu)$. We note that two vertices may be joined by more than one edge. Plainly, $r_{ij} \in L_1(R)$ for all i, j. The admissible diagrams do not contain any loops, as follows directly from the properties of the Wick dots. Among the admissible diagrams there are many that are not connected.

The proof of estimate (15) is simple in the case when $m_{\max} = m_{\min} = 2$: we need only apply Lemma 6.1 after adding some more edges so as to make each admissible diagram connected.

The case $m_{\max} > 2$ is much more difficult and requires a combination of the methods used to derive the estimates in §§2, 3.

The algebra $\mathfrak{A}_s$ admits a state ρ_0 that is invariant under every quantum dynamical semigroup, namely

$$\rho_0(a\mathbf{1} + B) = a, \qquad a \in \mathbb{C}, \quad B \in \mathrm{Com}(\mathcal{H}_s).$$

THEOREM 2. *If the operator $\mathbb{V} \neq 0$, then ρ_0 is the only invariant state for the semigroup T_s (see* [15]*).*

REMARK. Theorem 1 is valid for interactions $\mathbb{V}$ of the type

$$\mathbb{V} = \sum_{j=1}^{\nu} f_j(p_j) \otimes V_j,$$

where V_j is given by (4) and the $f_j \in C^\infty(R^\nu)$ are bounded in absolute value by a polynomial.

CHAPTER V

THE METHOD OF BETHE-SALPETER KERNELS (DYSON'S EQUATION)

In Chapter 3 we discussed a general technique (sometimes called the "Moscow method") for studying the highest branches of the spectrum of the transfer matrix for a Gibbs field on the lattice Z^ν, where the field takes values in a compact set S. Here we will give another method for analyzing the one-particle spectrum of the transfer matrix, which can also be applied when S is not compact. This method, which is somewhat arbitrarily called the Bethe-Salpeter kernel method, can also be used to investigate bound states in two-, three-, and higher-particle subspaces, but we will touch on this only briefly. We note that although the Bethe-Salpeter kernel method applies to a larger class of models than does the Moscow method, it is less informative: although it can provide information on, say, the one-particle spectrum of the transfer matrix, this sheds little light on the properties of the one-particle subspace itself.

In order to illustrate the essence of the method, and also to relate it to the constructions in Chapter 3, we will first describe it for the example of the transfer matrix for the Ising field on the lattice $Z^{\nu+1}$ (this field was introduced in §2, Chapter 2, and its transfer matrix in subsection 1 of §4, Chapter 3). We then apply the method to the study of the one-particle spectral branches of the transfer matrix for a Gibbs field on $Z^{\nu+1}$ taking arbitrary real values (this case has so far not been amenable to the Moscow approach).

§1. The Dyson equation and the Bethe-Salpeter kernel method for the Ising field

1. Dyson's equation and the one-particle spectrum. Let $\{\sigma(x),\ x = (\overline{x}, x^{(0)}) \in Z^\nu \times Z^1,\ \sigma(x) = \pm 1\}$ be a Gibbs Ising field defined on the $(\nu+1)$-dimensional lattice $Z^{\nu+1}$ for small values of the parameter β (see Chapter 2, §5). Consider the covariance function

$$\begin{aligned} s_2(x,y) &= s_2(x-y) = \langle \sigma(x), \sigma(y)\rangle \\ &\equiv \langle \sigma(x)\sigma(y)\rangle - \langle \sigma(x)\rangle\langle\sigma(y)\rangle = \langle \sigma(x)\sigma(y)\rangle \end{aligned} \tag{1}$$

of this field (here we have used the fact that $\langle\sigma(x)\rangle = 0$ for all $x \in Z^{\nu+1}$ for the Ising field). The average is over the distribution of the field in the space $\Omega = \{-1,1\}^{Z^{\nu+1}}$. Evidently, for $x = (\overline{x}, x^{(0)})$ we have

$$s_2(x) = (U_{\overline{x}} \mathcal{J}^{|x^{(0)}|}\sigma(0), \sigma(0))_{\mathcal{H}}, \tag{2}$$

where $(\cdot,\cdot)_{\mathcal{H}}$ is the inner product in the physical Hilbert space $\mathcal{H}$ (which as we recall consists of functionals that depend only on the values of the field $\{\sigma(\overline{x}, 0)\}$ on the zero

slice: $\{x = (\overline{x}, x^{(0)}),\ x^{(0)} = 0\}$); $\{U_s, s \in Z^\nu\}$ is a unitary representation of the group of translations acting on $\mathcal{H}$, and $\mathcal{J}$ denotes the transfer matrix of the field. In what follows we will use the decomposition

$$\sigma(0) = h_1 + h_2, \tag{3}$$

where $h_1 \in \mathcal{H}_1$ is the projection of the functional $\sigma(0) \in \mathcal{H}$ onto the one-particle invariant subspace of the transfer matrix $\mathcal{J}$, and $h_2 \in \widetilde{\mathcal{H}}_1$ is the projection of $\sigma(0)$ onto the orthogonal complement of $\mathcal{H}_1$ (see Chapter 3, §4). The constructions in Chapter 3 imply that h_1 and h_2 are functionals with rapidly (exponentially) decaying dependence on the values of the configuration $\sigma = \{\sigma(\overline{x}),\ x \in Z^\nu\}$ at points far from zero:

$$|h_1(\sigma_1) - h_2(\sigma_2)| < \lambda^d, \tag{4}$$

where σ_1 and σ_2 are two configurations coinciding in a d-neighborhood of 0: $\sigma_1(\overline{x}) = \sigma_2(\overline{x})$ if $|\overline{x}| \le d$ and $0 < \lambda < 1$ (with a similar estimate for h_2). We showed in Chapter 3 that the spectrum of the transfer matrix $\mathcal{J}$ restricted to the invariant subspace $\widetilde{\mathcal{H}}_1$ is contained in the interval

$$\mathrm{Sp}(\mathcal{J}/\widetilde{\mathcal{H}}_1) \subset [-C\beta^2, C\beta^2], \tag{5}$$

where C is a constant. From this and (4) it follows that

$$|(U_{\overline{x}}\mathcal{J}^{|x^{(0)}|}h_2, h_2)| < B\lambda^{|\overline{x}|}(C\beta^2)^{|x^{(0)}|}, \tag{6}$$

where B is a constant. Furthermore, as was shown in Chapter 3, §7, the space $\mathcal{H}_1$ can be mapped unitarily onto the space $L_2(T^\nu, dp)$, where T^ν is the ν-dimensional torus, in such a way that the operators $\{U_s|_{\mathcal{H}_1},\ s \in Z^\nu\}$ go into multiplication by the functions $\{\exp\{i(s;p),\ p \in T^\nu\}$, while $\mathcal{J}|_{\mathcal{H}_1}$ becomes multiplication by some smooth (analytic) function $\varepsilon(p)$ satisfying

$$c_2\beta < \varepsilon(p) < c_1\beta \tag{7}$$

for suitable constants $c_1 > c_2 > 0$. Under this isomorphism $\mathcal{H}_1 \overset{\approx}{\to} L_2(T^\nu, dp)$ the functional goes into a smooth (analytic) function $\varphi_0(p)$ defined on T^ν. Thus,

$$(U_{\overline{x}}, \mathcal{J}^{|x^{(0)}|}h_1, h_1)_{\mathcal{H}} = \int_{T^\nu} |\varphi_0(p)|^2 \varepsilon^{|x^{(0)}|}(p) e^{i(p,\overline{x})}\, dp. \tag{8}$$

We now consider the Fourier transform of the function $s_2(x) = s_2(\overline{x}, x^{(0)})$, the so-called Green function:

$$\begin{aligned} \widetilde{s}_2(p,k) &= \sum_{(\overline{x},x^{(0)}) \in Z^{\nu+1}} s_2(\overline{x}, x^{(0)}) e^{-i(p,\overline{x})} e^{-ikx^{(0)}} \\ &= \widetilde{s}_2'(p,k) + \widetilde{s}_2''(p,k), \qquad p \in T^\nu, \qquad k \in T^1, \end{aligned} \tag{9}$$

where $\widetilde{s}_2'(p,k)$ and $\widetilde{s}_2''(p,k)$ are the Fourier transforms of the functions $(U_{\overline{x}}, \mathcal{J}_{h_1}^{|x^{(0)}|}, h_1)$ and $(U_{\overline{x}}, \mathcal{J}_{h_2}^{|x^{(0)}|}, h_2)$, respectively. From estimate (6) we see that for each fixed $p \in T^\nu$, the function $\widetilde{s}_2''(p,k)$ is analytic in k on the strip

$$|\operatorname{Im} k| < |\ln C\beta^2| = 2|\ln \beta| + \text{const}. \tag{10}$$

Equality (8) and estimate (7) also show that for fixed p, $s_2'(p,k)$ is meromorphic in the strip (10) and has two first-order poles at the points

$$k_\pm = \pm i \ln \varepsilon(p). \tag{11}$$

The determination and analysis of $\varepsilon(p)$ thus reduces to studying the poles of the Green function $\widetilde{s}_2(p,k)$ (relative to the variable k) which have the smallest imaginary part.

In this method the poles (11) of $\widetilde{s}_2(p,k)$ are found using a special equation for the function $s_2(x)$, called the *Dyson equation*:

$$s_2(x-y) = I(x-y) + \sinh\beta \sum_{\substack{(x',y')\in Z^{\nu+1}\\ |x'-y'|=1}} I(x-y')s_2(x'-y), \qquad x,y\in Z^{\nu+1}. \tag{12}$$

In the next sections we will discuss the derivation of Dyson's equation for the correlation functions s_2 of Gibbs fields of a rather general type. We will also discuss in detail the function $I(x)$ appearing in (12). Here we merely briefly mention that the derivation of (12) is based on a general cluster expansion of s_2 discussed in detail in [26]. That is, $s_2(x-y)$ is represented as a sum of contributions from a collection of connected diagrams (graphs) containing the points x and y. The sum is over the "one-particle irreducible" diagrams, i.e., the ones for which there exist at least two nonintersecting paths going from x to y, and the sum gives rise to the function $I(x-y)$ (see below for more details). From the definition of $I(x)$ it follows that

$$I(x) < \overline{c}(C\beta^2)^{|x|} \tag{13}$$

and consequently, for any fixed $p\in T^\nu$, the Fourier transform

$$\widetilde{I}(p,k) = \sum_{(x,x^0)\in Z^{\nu+1}} I(\overline{x},x^{(0)})e^{-i(p,\overline{x})}e^{ik|x^{(0)}|}, \qquad (p,k)\in T^\nu\times T^\nu$$

can be continued analytically to k in the strip (10). Taking the Fourier transform of equation (12), we obtain that

$$\widetilde{s}_2(p,k) = \widetilde{I}(p,k) + \sinh\,\beta\widetilde{I}(p,k)\biggl(\sum_{j=1}^{\nu}\cos p^{(j)} + \cos k\biggr)\widetilde{s}_2(p,k),$$

where $p=(p^{(1)},\dots,p^{(\nu)})$. Hence the Green function is

$$\widetilde{s}_2(p,k) = \widetilde{I}(p,k)\biggl[1-\sinh\beta\widetilde{I}(p,k)\biggl(\sum_{j=1}^{\nu}\cos p^{(j)} + \cos k\biggr)\biggr]^{-1}. \tag{14}$$

Thus for fixed $p\in T^\nu$, the function $\widetilde{s}_2(p,k)$ has poles in the strip (10) that coincide with the roots of the equation

$$1-\sinh\beta\widetilde{I}(p,k)\biggl(\sum_{j=1}^{\nu}\cos p^{(j)} + \cos k\biggr) = 0. \tag{15}$$

A detailed analysis of this equation in the general case will be given in the following sections; it confirms that (as we already know) $\widetilde{s}_2(p,k)$ has two poles in the strip (10) which lie at the points $k_\pm = i\varepsilon(p)$ on the imaginary axis.

2. The Bethe-Salpeter kernel and bound two-particle states of the transfer matrix. The approach described above can be suitably generalized to find two-particle bound states of the transfer matrix for the Ising model. We recall that these are defined as follows (see §4 and §7 in Chapter 3). We decompose the physical Hilbert space $\mathcal{H}$ as a direct integral

$$\mathcal{H} = \int_{T^\nu} \mathcal{H}(P)\, dP \tag{16}$$

of spaces $\{\mathcal{H}(P),\, P \in T^\nu\}$ consisting of eigenvectors of the translation group $\{U_s,\, s \in Z^\nu\}$ with eigenvalues $\{\exp\{i(s, P),\, P \in T^\nu\}$. The transfer matrix $\mathcal{J}$ commutes with the group $\{U_s,\, s \in Z^\nu\}$ and is also expressible as a direct integral

$$\mathcal{J} = \int_{T^\nu} \mathcal{J}(P)\, dP \tag{17}$$

of operators $\{\mathcal{J}(P),\, P \in T^\nu\}$ acting on the respective spaces $\mathcal{H}(P)$. We next recall that for small β, in addition to the one-particle subspace $\mathcal{H}_1$ the space $\mathcal{H}$ also contains a two-particle subspace $\mathcal{H}_2$ which is invariant under $\mathcal{J}$ and U_s, so that we have the decomposition

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2 + \widetilde{\mathcal{H}}_2, \tag{18}$$

where $\mathcal{H}_0 = \{\text{const}\}$, $\mathcal{H}_1$ is defined above, and $\widetilde{\mathcal{H}}_2 = (\mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2)^\perp$ is also invariant under $\mathcal{J}$. The spectrum also satisfies

$$\mathrm{Sp}(\mathcal{J}|_{\mathcal{H}_2} \subset [-C\beta^3, C\beta^3], \tag{19}$$

where C is a constant.

The space $\mathcal{H}_2$ and the operator $\mathcal{J}_2 = \mathcal{J}|_{\mathcal{H}_2}$ acting on it also admit decompositions

$$\mathcal{H}_2 = \int_{T^\nu} \mathcal{H}_2(P)\, dP, \qquad \mathcal{J}_2 = \int_{T^\nu} \mathcal{J}_2(P)\, dP \tag{20}$$

analogous to (16) and (17);

$$\mathcal{H}_2(P) \subset \mathcal{H}(P), \qquad \mathcal{J}_2(P) = \mathcal{J}(P)/\mathcal{H}_2(P).$$

As was shown in Chapter 2, §6, the spectrum of the operator $\mathcal{H}_2(P)$ consists of a continuous (Lebesgue) part which contains the range of the function

$$\varepsilon_P(p) = \varepsilon(p)\varepsilon(P - p), \qquad p \in T^\nu, \tag{20^a}$$

(where $\varepsilon(p)$ was defined above) plus, possibly, a finite number of eigenvalues $\widetilde{\varepsilon}_1(P)$, $\ldots, \widetilde{\varepsilon}_s(P)$ with eigenvectors $\varphi_P^{(1)}, \varphi_P \in \mathcal{H}_2(P)$, which are called coupled states (with total quasimomentum P). In general, it is not known if for small β bound states can exist for the $Z^{\nu+1}$ Ising model for some range of values of the total quasimomentum P. It is known (from explicit calculations of Onsager and Kaufman, see [47a]) that for $\nu = 1$ there are no bound states, and moreover it can be seen that (for small β) no bound states are present when $\nu > 2$ (see Chapter 3, §7). The existence of bound states

when $\nu = 2$ has not been ruled out. However, since the subsequent constructions apply to more or less arbitrary models, the Ising model having been chosen merely as a very simple illustration, we will postulate without further justification that there is a region $G \subseteq T^\nu$ of values of P for which the space $\mathcal{H}(P)$ contains a unique bound state $\overline{\varphi}_P$ with eigenvalue $\overline{\varepsilon}(P)$ lying to the right of the continuous spectrum:

$$\overline{\varepsilon}(P) > \max_{p \in T^\nu} \varepsilon_P(p). \tag{21}$$

This eigenvalue $\overline{\varepsilon}(P)$ can be found and analyzed by exploiting the analytic properties of the correlation function

$$s_4(T, T') = \langle \widetilde{\sigma}_T, \widetilde{\sigma}_{T'} \rangle = \langle \sigma_T \sigma_{T'} \rangle - \langle \sigma_T \rangle \langle \sigma_{T'} \rangle,$$

where $T = (x_1, x_2)$, $T = (x_1', x_2') \subset Z^{\nu+1}$ for the two-point subsets of the lattice $Z^{\nu+1}$, and

$$\sigma_T = \sigma(x_1)\sigma(x_2), \qquad \widetilde{\sigma}_T = \sigma_T - \langle \sigma_T \rangle.$$

Expanding the monomial $\langle \sigma_T \sigma_{T'} \rangle$ (see [26]), we obtain that

$$s_4(T, T') = s_2(x_1 - x_1')s_2(x_1 - x_2') + s_2(x_1 - x_2')s_2(x_2 - x_1') + s_4^c(T, T'), \tag{22}$$

where $s_2(x - y)$ is the correlation function defined above, and the "connected part" $s_4^c(T, T')$ is the semi-invariant

$$s_4^c(T, T') = \langle \sigma_{x_1}, \sigma_{x_2}, \sigma_{x_1'}, \sigma_{x_2'} \rangle \tag{23}$$

of the values of the fields at the points x_1, x_2, x_1', x_2'. Using the standard cluster bounds for the semi-invariants of Gibbs fields (see [26]), we find that

$$|s_4^c(T, T')| < (C\beta)^{d_{T \nu T'}}, \tag{24}$$

where d_B, $B \subset Z^{\nu+1}$ is the length of the smallest connected graph constructed on the points in the set B (see Chapter 3), and C is a constant. Let $\overline{C}^{(2)}_{Z^{\nu+1}} \subset C^{(2)}_{Z^{\nu+1}}$ be the set of "simultaneous" two-point sets $T = ((\overline{x}_1, x_1^{(0)}), (\overline{x}_2, x_2^{(0)}))$, i.e., such that $x_1^{(0)} = x_2^{(0)}$. Let $l_2(\overline{C}^{(2)}_{Z^{\nu+1}}) = l_2$ be the Hilbert space of functions $f = (f(T), T \in \overline{C}^{(2)}_{Z^{\nu+1}})$, and let the operator R be defined by

$$(Rf)(T) = \sum_{T' \in \overline{C}^{(2)}_{Z^{\nu+1}}} s_4(T, T')f(T'), \qquad f \in l_2. \tag{25}$$

Note that in view of (22), the estimates (24) and $|s_2(x - y)| < (C\beta)^{|x-y|}$ follow from the representation (2), and R is a cluster operator (with cluster parameter $C\beta$).

We now take the Fourier transform of functions in l_2, i.e., we consider the functions

$$f(p_1, p_2, k) = \sum_{T = \{(\overline{x}_1, x^{(0)}), (\overline{x}_2, x^{(0)})\}} f(T) e^{-i(\overline{x}_1, p_1) - i(\overline{x}_2, p_2) - ikx^{(0)}}. \tag{26}$$

It is easy to check that f is symmetric in the variables p_1, p_2 and satisfies the condition

$$\int\limits_{p_1+p_2=P} f(p_1,p_2,k)\,dp_1\,dp_2 = 0 \tag{27}$$

for all $P \in T^\nu$ and $k \in T^1$. We denote the space of such functions by $\widetilde{L}_2$. The Fourier transform takes R into the operator $\widetilde{R}$ on L_2 with kernel

$$\begin{aligned}\widetilde{R}(p_1,p_2,k;p_1',p_2',k') =& \widetilde{s}_2(p_1,k)\widetilde{s}_2(p_2,k)\delta(p_1-p_1')\delta(p_2-p_2')\delta(k-k')\\ &+\delta(p_1+p_2-p_1'-p_2')\delta(k-p')\widetilde{S}_4^c(p_1,p_2;p_1',p_2',k),\end{aligned} \tag{28}$$

where $\widetilde{S}_4^c(p_1,p_2;p_1',p_2',k)$ is a smooth function defined on the manifold $p_1+p_2=p_1'+p_2'$ and $\widetilde{s}_2(p,k)$ is the Fourier transform of $s_2(x)$ (the Green function). This formula shows that when $\widetilde{L}_2$ is decomposed as a direct integral

$$\widetilde{L}_2 = \int_{T^\nu\times T^1} \widetilde{L}_2(P,k)\,dP\,dk \tag{29}$$

of spaces $\widetilde{L}_2(P,k)$, consisting of functions $f(p)$, $p \in T^\nu$ such that $f(p) = f(P-p)$ and $\int f(p)\,dp = 0$, $\widetilde{R}$ can be expressed as the direct integral

$$\widetilde{R} = \int_{T^\nu\times T^1} \widetilde{R}(P,k)\,dP\,dk, \tag{30}$$

where the $\widetilde{R}(P,k)$ act on $\widetilde{L}_2(P,k)$ by the formula

$$(\widetilde{R}(P,k)f)(p) = \widetilde{s}_{P,k}(p)f(p) + \int \widetilde{s}^c_{P,k}(p,p')f(p')\,dp'. \tag{31}$$

Here

$$\widetilde{s}_{P,k}(p) = \widetilde{s}_2(P,k)\widetilde{s}(P-p,k)$$

and

$$\widetilde{s}^c_{P,k}(p,p') = \widetilde{s}_4(p,P-p;p',P-p';k).$$

The next lemma relates the operators $\widetilde{R}(P,k)$ to the eigenvalues $\{\overline{\varepsilon}(P),\ P\in G\}$ in (21).

LEMMA. *For any fixed $P \in G$, the operator-valued function $\widetilde{R}(P,k)$ of the variable k is meromorphic in the strip*

$$|\operatorname{Im} k| < \min_{p_1+p_2=P}\{|\ln\varepsilon(p_1)| + |\ln\varepsilon(p_2)|\} \tag{32}$$

and has there two poles at the points

$$\overline{k}_\pm = \pm|\ln\overline{\varepsilon}(P)| \tag{33}$$

(*we recall that the function $\varepsilon(p)$ defines the one-particle branch of the spectrum of $\mathcal{J}$*).

PROOF. We denote by $T_{\overline{y}} \subset \overline{C}^{(2)}_{Z^{\nu+1}}$ the set

$$T_{\overline{y}} = ((\overline{0}, 0), (\overline{y}, 0)), \qquad \overline{y} \in Z^{\nu},$$

and use the decompositions

$$\widetilde{\sigma}_{T\overline{y}} = h^{(1)}_{T\overline{y}} + h^{(2)}_{T\overline{y}}, \qquad \widetilde{\sigma}_{T\overline{y}'} = h^{(1)}_{T\overline{y}'} + h^{(2)}_{T\overline{y}'}, \tag{34}$$

where $h^{(1)}_{T\overline{y}}$, $h^{(1)}_{T\overline{y}'} \in \mathcal{H}_2$, $h^{(2)}_{T\overline{y}}, h^{(2)}_{T\overline{y}'} \in \overline{\mathcal{H}}_2$ (the projections of the vectors $\widetilde{\sigma}_{T\overline{y}}$ and $\widetilde{\sigma}_{T\overline{y}'}$ on the subspaces $\mathcal{H}_0$ and $\mathcal{H}_1$ are equal to zero).

Thus,

$$\begin{aligned}
\langle \widetilde{\sigma}_{T\overline{y}}, \widetilde{\sigma}_{T\overline{y}'+(\overline{\xi},\xi^0)} \rangle =& (U_{\overline{\xi}} \mathcal{J}^{|\xi^{(0)}|} \widetilde{\sigma}_{T\overline{y}}, \widetilde{\sigma}_{T\overline{y}'})_{\mathcal{H}} \\
=& (U_{\overline{\xi}} \mathcal{J}^{|\xi^{(0)}|} h^{(1)}_{T\overline{y}}, h^{(1)}_{T\overline{y}'})_{\mathcal{H}} + (U_{\overline{\xi}} \mathcal{J}^{|\xi^{(0)}|} h^{(2)}_{T\overline{y}}, h^{(2)}_{T\overline{y}'})_{\mathcal{H}} \\
=& \int_{T^{\nu}} e^{i(P,\overline{\xi})} (\mathcal{J}_2^{|\xi^{(0)}|}(P) h^{(1)}_{T\overline{y}}(P), h^{(1)}_{T\overline{y}'}(P))_{\mathcal{H}(P)}\, dP \\
&+ \int_{T^{\nu}} e^{i(P,\overline{\xi})} (\mathcal{J}_2^{|\xi^{(0)}|}(P) h^{(2)}_{T\overline{y}}(P), h^{(2)}_{T\overline{y}'}(P))_{\mathcal{H}(P)}\, dP.
\end{aligned}$$

Here $h^{(i)}_{T\overline{y}}(P), h^{(i)}_{T\overline{y}'}(P)$, $i = 1, 2$, are the components of the vectors $h^{(i)}_{T\overline{y}}, h^{(i)}_{T\overline{y}'}$, $i = 1, 2$ in the direct-integral decomposition (20). One computes readily that the kernel of the operator $\widetilde{R}(P, k)$ is

$$\begin{aligned}
\widetilde{R}_{P,k}(p, k') =& \sum_{\xi, \xi^0, \overline{\xi}, \overline{\xi}'} \langle \widetilde{\sigma}_{T\overline{y}}, \widetilde{\sigma}_{T\overline{y}'+(\overline{\xi},\xi^0)} \rangle e^{-i(\overline{\xi},P)-ik\xi^0} e^{i(\overline{y},p)-i(\overline{y}',p')} \\
=& \sum_{\overline{y}, \overline{y}', \xi^{(0)}} (\mathcal{J}_2^{|\xi^{(0)}|}(P) h^{(1)}_{T\overline{y}}(P), h^{(1)}_{T\overline{y}'}(P))_{\mathcal{H}_2(P)} e^{i(\overline{y},p)-i(\overline{y}',p')} e^{-ik\xi^0} \\
&+ \sum_{\overline{y}, \overline{y}', \xi^{(0)}} (\mathcal{J}_2^{|\xi^{(0)}|}(P) h^{(2)}_{T\overline{y}}(P), h^{(2)}_{T\overline{y}'}(P))_{\mathcal{H}(P)} e^{i(\overline{y},p)-i(\overline{y}',p')} e^{-ik\xi^0}.
\end{aligned}$$

According to estimate (19), the second term (for fixed P) can be analytically continued as a function of k into the strip $|\operatorname{Im} k| < 3|\ln \beta| + \text{const}$. Next, we expand the vector $h^{(1)}_{T\overline{y}}(P)$ as

$$h^{(1)}_{T\overline{y}}(P) = C_{\overline{y}}(P) \overline{\varphi}_P + \widehat{h}^{(1)}_{T\overline{y}}(P),$$

where $\widetilde{\varphi}_P$ is an eigenvector of $\mathcal{J}_2(P)$ with eigenvalue $\overline{\varepsilon}(P)$, and $\widehat{h}^{(1)}_{T\overline{y}}(P) \perp \overline{\varphi}_P$. Using also the analogous decomposition for $h^{(1)}(P)$, we find that

$$\begin{aligned}
&\sum_{\xi^{(0)}} (\mathcal{J}_2^{|\xi^{(0)}|}(P) h^{(1)}_{T\overline{y}}(P), h^{(1)}_{T\overline{y}'}(P))_{\mathcal{H}_2(P)} e^{-ik\xi^0} \\
&\qquad = \sum_{\xi^{(0)}} C_{\overline{y}}(P) \overline{C}_{\overline{y}'}(P) (\overline{\varepsilon}_1(P))^{|\xi^{(0)}|}(P) e^{-ik\xi^0} \\
&\qquad\quad + \sum_{\xi^{(0)}} (\mathcal{J}_2^{|\xi^{(0)}|}(P) \widehat{h}^{(1)}_{T\overline{y}}(P), \widehat{h}^{(1)}_{T\overline{y}'}(P))_{\mathcal{H}_2(P)} e^{-ik\xi^0}.
\end{aligned}$$

The second term is analytic in k in the strip (32), while the first has poles at the points (33). This proves the lemma.

The following method can be used to find the poles of (33) directly. Consider the operator $\widetilde{R}^0$ on l_2 with kernel

$$s_4^0(T,T') = s_2(x_1 - x_1')s_2(x_2 - x_2') + s_2(x_1 - x_2')s_2(x_2 - x_1').$$

Again taking Fourier transforms, we can decompose $\widetilde{R}^0$ as a direct integral of operators $\widetilde{R}^0(P,k)$ given by the kernels

$$\widetilde{R}^0_{P,k}(p,p') = \widetilde{s}_{P,k}(p)\delta(p-p') + s^{0,c}_{P,k}(p,p').$$

Using the constructions in Chapter 3, §7, one shows without difficulty that for fixed P, $\widetilde{R}^0(P,k)$ has an analytic continuation for k in the strip (32). In addition, it can be shown that $\widetilde{R}(P,k)$ and $\widetilde{R}^0(P,k)$ are invertible and that their difference

$$(\widetilde{R}^0(P,k))^{-1} - (\widetilde{R}(P,k))^{-1} = K(P,k) \tag{35}$$

is an integral operator having a smooth kernel $K_{P,k}(p,p')$ which is analytic in k for k in the strip (32). The definition (35) implies that

$$\widetilde{R}(P,k) = \widetilde{R}^0(P,k) + \widetilde{R}(P,k)K(P,k)\widetilde{R}^0(P,k)$$

and consequently,

$$\widetilde{R}(P,k) = (E + K(P,k)\widetilde{R}^0(P,k))^{-1}\widetilde{R}^0(P,k).$$

This formula shows that the poles (33) of $\widetilde{R}(p,k)$ coincide with the zeros of the Fredholm determinant: $\mathrm{Det}(E + K(P,k)\widetilde{R}^0(P,k)) = 0$. The kernel $K_{P,k}(p,p')$ is called the Bethe-Salpeter kernel and can be expressed as a series summed over a set of diagrams in much the same way as the function $\widetilde{I}$ appearing in Dyson's equation.

To find bound states in three-, four-, ... , k-particle invariant subspaces of the transfer matrix, one must analyze the higher-order covariance functions of the field; this proceeds along the same lines as in the case of two-particle bound states. A higher-order Bethe-Salpeter kernel arises naturally in this context (see [12]).

§2. One-particle spectrum of the transfer matrix for a Gibbs field with unbounded spin. Description of the model and results

In the previous section we saw that for an Ising field, the Fourier transform $\widetilde{s}_2(p,k)$ of its covariance function extends as a meromorphic function of k to a strip in the complex plane, in which the poles of $\widetilde{s}_2(p,k)$ are related in a very simple way to the one-particle spectrum of the transfer matrix. The situation turns out to be much the same for a larger class of Gibbs lattice fields, and this will be discussed in the present section.

Consider the Gibbs modification of the free (independent) measure

$$\mu_0 = \lambda_0^{Z^{\nu+1}}, \tag{1}$$

where λ_0 is a probability measure on the number line R^1. We assume that the modified measure μ_1 is given by means of an interaction

$$U_\Lambda(\xi) = \beta \sum_{x,x' \in \Lambda} \Phi(\xi(x), \xi(x')), \qquad \xi = \{\xi(x), x \in \Lambda\}, \quad \xi(x) \in R^1 \tag{2}$$

where $\Lambda \subset Z^{\nu+1}$ is a bounded set and the sum in (2) is over all pairs of adjacent points $x, x' \in \Lambda$. Here $\Phi(\xi_1, \xi_2)$ is a nonnegative real-valued symmetric function of the real variable ξ_1, ξ_2 with finite moments

$$\int_{R^1} \int_{R^1} \Phi^n(\xi_1, \xi_2)\, d\lambda_0(\xi_1)\, d\lambda_0(\xi_2) < n, \qquad n = 1, 2, \ldots. \tag{3}$$

According to the results in [26] (see Chapter 4, §2), the limit Gibbs modification

$$\mu = \lim_{\Lambda \uparrow Z^{\nu+1}} \mu_\Lambda \tag{4}$$

exists for sufficiently small β. Here the random field $\{\xi(x),\ x \in Z^{\nu+1}\}$, with values in R^1 on the lattice $Z^{\nu+1}$ and with probability distribution μ, is Markov.

Let φ_1 and φ_2 be two arbitrary real-valued bounded functions defined on R^1. Consider the correlation function

$$\begin{aligned} s_{\varphi_1,\varphi_2}(x) =& \langle \varphi_1(\xi(0)), \varphi_2(\xi(x)) \rangle \\ =& \langle \varphi_1(\xi(0)) \varphi_2(\xi(x)) \rangle - \langle \varphi_1(\xi(0)) \rangle \langle \varphi_2(\xi(x)) \rangle, \end{aligned} \tag{5}$$

where the mean $\langle \cdot \rangle = \langle \cdot \rangle_\mu$ is taken with respect to the distribution μ. Let further

$$\widetilde{s}_{\varphi_1,\varphi_2}(p,k) = \sum_{x=(\overline{x},x^{(0)}) \in Z^{\nu+1}} s_{\varphi_1,\varphi_2}(x) e^{i(p,\overline{x}) + ikx^{(0)}}$$

be the Fourier transform of $s_{\varphi_1,\varphi_2}(x)$, i.e., the Green function.

Consider now the integral operator $\widehat{\Phi}$ on $L_2(R^1, d\lambda_0)$ with kernel $\Phi(u, v)$,

$$(\widehat{\Phi} f)u = \int_{R^1} \Phi(u, v) f(v)\, d\lambda_0(v). \tag{6}$$

This operator is symmetric and compact on $L_2(R^1, d\lambda_0)$, because of condition (3). We introduce the operator $M = (E - P)\widehat{\Phi}(E - P)$, where P projects $L_2(R^1, d\lambda_0)$ onto the subspace of constant functions. Let $\psi_1, \psi_2, \ldots, \psi_N, \ldots$ be a system of orthonormal eigenvectors of M with nonzero eigenvalues $\mu_1, \ldots, \mu_N$ such that $|\mu_1| > |\mu_2| > \cdots > |\mu_N| > \ldots$, and we assume that all the μ_i are simple. We consider the two cases

1) there are finitely many eigenvalues μ_i, $i = 1, \ldots, N$;

2) there are infinitely many: $i = 1, \ldots, N, \ldots$.

We have the following two theorems.

THEOREM 1. *Assume that case* 1) *holds. Then for sufficiently small* β *and any two bounded functions* φ_1, φ_2, *the Fourier transform* $\widetilde{s}_{\varphi_1,\varphi_2}(p,k)$, *regarded as a function of the variable* k *for any fixed* $p \in T^\nu$, *can be continued to a meromorphic function in the strip*

$$|\operatorname{Im} k| < 2|\ln c_1\beta|, \tag{7}$$

where the constant $c_1 = c_1(\Phi, \lambda_0, \nu)$ *does not depend on* β. *Furthermore, the Green function* $\widetilde{s}_{\varphi_1,\varphi_2}(p,k)$ *has* $2N$ *poles in the strip* (7); *they lie on the imaginary axis,*

$$k_j^0 = \pm i \ln|\varepsilon_j(p)|, \qquad j = 1, \dots, N, \tag{8}$$

where the $\varepsilon_j(p)$ *are functions of* $p \in T^\nu$ *and satisfy*

$$|\varepsilon_j(p) - \mu_j\beta| < c_2\beta^2 \tag{9}$$

with a constant $c_2 = c_2(\Phi, \lambda_0, \nu)$ *independent of* β.

THEOREM 2. *Assume that case* 2 *holds. Then for any* N *there exists a* $\beta_0 = \beta_0(N) > 0$ *such that for all* $0 < \beta < \beta_0(N)$ *the function* $\widetilde{s}_{\varphi_1,\varphi_2}(p,k)$ *admits a continuation in the variable* k *to the strip*

$$|\operatorname{Im} k| < \ln \frac{2}{(|\mu_N| + |\mu_{N+1}|)p} \tag{10}$$

and has poles at the points (8); *estimate* (9) *is again valid.*

We now relate the poles of $\widetilde{s}_{\varphi_1,\varphi_2}(p,k)$ described above to the spectrum of the transfer matrix $\mathcal{J}$ of our field. Let Φ_φ be the functional

$$\Phi_\varphi(\xi) = \varphi(\xi(0)) - \langle\varphi(\xi(0))\rangle, \tag{11}$$

where is an arbitrary bounded function defined on R^1. Clearly, we have

$$s_{\varphi_1,\varphi_2}(\overline{x}, x^{(0)}) = (U_{\overline{x}}\mathcal{J}^{|x^{(0)}|}\Phi_{\varphi_1}, \Phi_{\varphi_2})_{\mathcal{H}}, \tag{12}$$

where $\{U_{\overline{x}},\ \overline{x} \in Z^\nu\}$ is the unitary group of spatial translations acting on $\mathcal{H}_{\text{phys}} = \mathcal{H}$. As before, we have direct-integral decompositions

$$\mathcal{H} = \int \mathcal{H}(p)\,dp, \qquad \mathcal{J} = \int \mathcal{J}(p)\,dp, \tag{13}$$

where the $\mathcal{H}(p)$ are the eigenspaces for the group $\{U_{\overline{x}},\ \overline{x} \in Z^\nu\}$, with eigenvalues $\exp\{i(p,\overline{x})\}$. The right hand side of (12) then becomes

$$\begin{aligned}(U_{\overline{x}}\mathcal{J}^{|x^{(0)}|}\Phi_{\varphi_1}, \Phi_{\varphi_2})_{\mathcal{H}} &= \int_{T^\nu} e^{i(p,\overline{x})}(\mathcal{J}^{|x^{(0)}|}(p)\Phi_{\varphi_1}(p), \Phi_{\varphi_2}(p))_{\mathcal{H}_p}\,dp \\ &= \int_{T^\nu} e^{i(p,\overline{x})} \int_{-1}^{1} u^{|x^{(0)}|}\,d\rho^{(p)}_{\varphi_1,\varphi_2}(\mu),\end{aligned} \tag{14}$$

where $\Phi_{\varphi_i}(p) \in \mathcal{H}(p)$, $i = 1, 2$, is the component of the vector Φ_{φ_i} in the decomposition (13), and

$$\rho^{(p)}_{\varphi_1,\varphi_2}(\mu) = (E_p(\mu)\Phi_{\varphi_1}(p), \Phi_{\varphi_2}(p))_{\mathcal{H}(p)} \tag{15}$$

is the spectral measure corresponding to the elements $\Phi_{\varphi_1}(p)$ and $\Phi_{\varphi_2}(p)$ and generated by the spectral family of projections $\{E_p(\mu)\}$ for the selfadjoint operator $\mathcal{J}(p)$ acting on the space $\mathcal{H}(p)$. ¿From (12), (14) and (15) we get

$$\begin{aligned}\widetilde{s}(p,k) &= \int_{-1}^{1} \left(\sum_{x^0 \in Z^1} u^{|x^{(0)}|} e^{-ikx^{(0)}} \right) d\rho^{(p)}_{\varphi_1,\varphi_2}(\mu) \\ &= \int_{-1}^{1} \left(1 + \frac{\mu}{e^{ikx^{(0)}} - \mu} + \frac{\mu}{e^{-ikx^{(0)}} - \mu} \right) d\rho^{(p)}_{\varphi_1,\varphi_2}(\mu).\end{aligned}$$

Together with Theorems 1 and 2, this formula implies that for any φ_1 and φ_2 and any fixed p, the support of the measure $\rho^{(p)}(\mu)$ is concentrated at the points $\varepsilon_1(p), \dots,$ $\varepsilon_N(p)$, and some set that belongs to the segments

$$|\mu| < C_1\beta^2 \tag{16}$$

and

$$|\mu| < \frac{|\mu_{N+1}| + |\mu_N|}{2}\beta \tag{17}$$

in cases 1 and 2, respectively.

Writing $\widehat{\mathcal{H}}(p) \subset \mathcal{H}(p)$ for the smallest closed subspace invariant under $\mathcal{J}(p)$ and containing the functional $\Phi_\varphi(p)$, we see from the above result that the space $\widehat{\mathcal{H}}(p)$ contains N eigenvectors $\psi_1(p), \dots, \psi_N(p)$ of the transfer matrix with eigenvalues $\varepsilon_1(p), \dots, \varepsilon_N(p)$, and on the orthogonal complement of these vectors the spectrum of $\mathcal{J}(p)$ lies in the intervals (16) or (17). If $\{C\psi_j(p)\}$ denotes the one-dimensional subspace spanned by $\psi_j(p)$, then it is clear that the spaces

$$\mathcal{H}_j = \int \{C\psi_j(p)\}\, dp \subset \mathcal{H}, \qquad j = 1, \dots, N,$$

are one-particle invariant subspaces for the operator $\mathcal{J}$ and group $\{U_{\overline{x}}, \overline{x} \in Z^\nu\}$.

Remark. It is plausible that on the entire orthogonal complement of the sum of the subspaces $\mathcal{H}_j$, the spectrum of the transfer matrix $\mathcal{J}$ should also be of order $O(\beta^2)$ (in case 1). In other words, this means that the smallest closed subspace invariant under $\mathcal{J}$ and containing all the functionals $\{U_{\overline{x}}\Phi_\varphi\}$ should contain the spectral subspace $E([-1,1] \setminus [-C\beta^2, C\beta^2])\mathcal{H}$, where $\{E(\Delta), \Delta \subset [-1,1]$, a closed interval$\}$ is the spectral family of projections for $\mathcal{J}$. More generally, we may conjecture the following. For every $\varepsilon > 0$ there exists an n such that the smallest invariant subspace for $\mathcal{J}$ containing the monomials of the form

$$U_{\overline{x}_1}\Phi_{\varphi_1} U_{\overline{x}_2}\Phi_{\varphi_2} \dots U_{\overline{x}_n}\Phi_{\varphi_n}, \tag{18}$$

contains $E([-1,1] \setminus [-\varepsilon, \varepsilon])\mathcal{H}$ (i.e., the polynomial states are complete). This question, as well as the problem of finding the two-, three-, ... , k-particle subspaces for the transfer matrix, requires an analysis of the correlation functions for monomials of the form (18), i.e., of the higher-order correlation functions of the field, as we explained in the previous section. Here it is natural to seek Bethe-Salpeter kernels that might be used to find and analyze bound states in the k-particle subspaces. Some parts of this program have been carried out for the $P(\varphi)_2$ Euclidean continuous quantum field models (see [12]).

§3. Cluster expansion of the covariance operators

The rest of this chapter is devoted to the proof of Theorems 1 and 2. In this section we will establish a cluster expansion for the correlation function $s_{\varphi_1,\varphi_2}(x)$ that will be needed below, while in the next section we will derive a Dyson equation similar to equation (12.1) in §1; finally, in the last section we use this equation to prove both Theorems 1 and 2.

Thus we begin by considering the cluster expansion of the correlation function $s_{\varphi_1,\varphi_2}(x)$ for small β. To this end, consider the graph $\mathcal{L}(Z^{\nu+1})$ whose vertices are the points of the lattice $Z^{\nu+1}$ and whose edges are all possible pairs of adjacent points (x, x') of $Z^{\nu+1}$. To each connected finite subgraph $\Gamma \subset \mathcal{L}(Z^{\nu+1})$ we define

$$K_\Gamma = \left\langle \prod_{(y,y')\in\Gamma} (\exp\{-\beta\Phi(\xi(y),\xi(y'))\} - 1) \right\rangle_0,$$

$$K_\Gamma(x;\varphi) = \left\langle \varphi(\xi(x)) \prod_{(y,y')\in\Gamma} (\exp\{-\beta\Phi(\xi(y),\xi(y'))\} - 1) \right\rangle_0$$

(here $\langle\cdot\rangle_0 = \langle\cdot\rangle_{\mu_0}$) and

$$K_\Gamma(x_1,x_2;\varphi_1,\varphi_2) = \left\langle \varphi_1(\xi(x_1))\varphi_2(\xi(x_2)) \prod_{(y,y')\in\Gamma} (\exp\{-\beta\Phi(\xi(y),\xi(y'))\} - 1) \right\rangle_0,$$

where $x, x_1, x_2 \in \widetilde{\Gamma}$ and φ, φ_1, φ_2 are bounded functions. Henceforth, $\widetilde{\Gamma}$ will always denote the set of vertices of the graph Γ.

Then for sufficiently small $\beta > 0$ (see [26]) we have

(1)

$$\begin{aligned}\langle\varphi_1(\xi(0)),&\varphi_2(\xi(x))\rangle_\mu \\ &= \sum_{n=0}^{\infty} \Bigg\{ {\sum_{\Gamma_0,\dots,\Gamma_n}}' \frac{1}{n!} \varphi_{\mathrm{con}}(\Gamma_0,\dots,\Gamma_n) K_{\Gamma_0}(0,x;\varphi_1,\varphi_2) K_{\Gamma_1}\dots K_{\Gamma_n} \\ &\qquad + {\sum_{\Gamma_0,\dots,\Gamma_{n+1}}}'' \frac{1}{n!} \varphi_{\mathrm{con}}(\Gamma_0,\dots,\Gamma_{n+1}) K_{\Gamma_0}(0,\varphi_1) K_{\Gamma_1}\dots K_{\Gamma_n} K_{\Gamma_{n+1}}(x,\varphi_2) \Bigg\},\end{aligned}$$

where the sum $\sum'$ is over all ordered sets of finite connected graphs $\Gamma_0,\dots,\Gamma_n \subset \mathcal{L}(Z^{\nu+1})$ such that $0, x \in \widetilde{\Gamma}_0$, and the graph $\Gamma_0 \cup \dots \cup \Gamma_n$ is connected; the sum $\sum''$ is over all ordered sets of finite connected graphs $\Gamma_0,\dots,\Gamma_{n+1} \subset \mathcal{L}(Z^{\nu+1})$ such that $0 \in \widetilde{\Gamma}_0$, $x \in \widetilde{\Gamma}_{n+1}$, the graph $\Gamma_0 \cup \dots \cup \Gamma_{n+1}$ is connected, and for all $\Gamma_0,\dots,\Gamma_n$

$$\varphi_{\mathrm{con}}(\Gamma_0,\dots,\Gamma_n) = \sum_G^{(\mathrm{con})} (-1)^{|G|},$$

the sum being over all connected graphs G with vertex set $\{0,\dots,n\}$ and edges (i,j), $0 \le i < j \le n$, where $(i,j) \in G$ only if $\widetilde{\Gamma}_i \cap \widetilde{\Gamma}_j \neq \emptyset$; $|G|$ is the number of edges of G.

The proof that the series (1) converges for small enough follows straightforwardly from the general theory of cluster expansions for correlation functions (see [26, Chapter 3]) by use of the cluster estimates derived in [26, Chapter 4, §2], together with estimates that follow easily from the positivity of Φ:

$$|K_\Gamma(x;\varphi)| < \|\varphi\|_{L_2} |K_{\Gamma'}|, \tag{2}$$

where $\|\cdot\|_{L_2}$ is the norm on $L_2(R^1, d\lambda_0)$, and the graph $\Gamma' \subset \Gamma$ contains all the edges of Γ, except those that leave from the vertex $x \in \widetilde{\Gamma}$, and

$$|K_\Gamma(x_1, x_2; \varphi_1, \varphi_2)| \leq \|\varphi_1\|_{L_2}\|\varphi_2\|_{L_2}|K_{\Gamma''}|, \tag{3}$$

where the graph $\Gamma'' \subset \Gamma$ contains all the edges of Γ except for the ones leaving the vertices $x_1 \in \widetilde{\Gamma}$ and $x_2 \in \widetilde{\Gamma}$.

We fix $\beta_0 > 0$ so that series (1) converges absolutely for all $\beta \in (0, \beta_0)$. It will be assumed henceforth that $0 < \beta < \beta_0$.

We will need the following estimate for the function $\varphi_{\text{con}}(\cdot)$:

$$|\varphi_{\text{con}}(\Gamma_0, \dots, \Gamma_n)| \leq |\mathcal{J}(\Gamma_0, \dots, \Gamma_n)| \tag{4}$$

for all $\Gamma_0, \dots, \Gamma_n \subset \mathcal{L}(Z^{\nu+1})$, where $\mathcal{J}(\Gamma_0, \dots, \Gamma_n)$ is the set of all trees with vertices $\{0, 1, \dots, n\}$ and edges (i, j), where $0 \leq i < j \leq n$ and $\widetilde{\Gamma}_i \cap \widetilde{\Gamma}_j \neq \emptyset$. In [26, Chapter 4, §2] the estimate

$$|K_\Gamma| < (c\beta)^{\gamma|\Gamma|} \tag{5}$$

was derived for the quantities K_Γ, where $\Gamma \subset \mathcal{L}(Z^{\nu+1})$, $|\Gamma| < \infty$; here $\gamma = 1/(\nu+1)$ and $c > 0$ is a constant.

Inequalities (2)–(5) imply in particular that for small enough $\beta > 0$, $\beta < \beta_0$,

$$|\langle \varphi_1(\xi(0)), \varphi_2(\xi(x))\rangle| \leq \|\varphi_1\|_{L_2}\|\varphi_2\|_{L_2} c_2 (c_1\beta)^{\gamma|x|}, \tag{6}$$

for all φ_1 and φ_2, where $c_1, c_2 > 0$ are constants (see [26], Chapter 3, §3).

We consider the system of covariance operators $\widehat{S}_x$, $x \in Z^{\nu+1}$ on $L_2(R^1, d\lambda_0)$, where for every $x \in Z^{\nu+1}$, $\widehat{S}_x$ is defined by

$$\langle \varphi_1, S_x \varphi_2\rangle = \langle \varphi_1(\xi(0)), \overline{\varphi_2(\xi(x))}\rangle, \qquad \varphi_1, \varphi_2 \in L_2(R^1, d\lambda_0)$$

and $(\cdot, \cdot)$ is the inner product on $L_2(R^1, d\lambda_0)$.

It is easily seen that for each $x \in Z^{\nu+1}$ the operator $\widehat{S}_x$ is selfadjoint, and by (6) we have.

$$\|\widehat{S}_x\| < \text{const}(c_1\beta)^{\gamma|x|}.$$

We next write out the cluster expansion for the operators $\widehat{S}_x$, $x \in Z^{\nu+1}$. For this purpose, if $\Gamma \subset \mathcal{L}(Z^{\nu+1})$ is any finite connected graph, we consider the conditional expectations with respect to the free measure μ_0:

$$k_\Gamma(x; u) = \left\langle \prod_{(y,y')\in\Gamma} (\exp\{-\beta\Phi(\xi(y), \xi(y'))\} - 1) | \xi(x) = u \right\rangle_0$$

and

$$k_\Gamma(x, x'; u, u') = \left\langle \prod_{(y,y')\in\Gamma} (\exp\{-\beta\Phi(\xi(y), \xi(y'))\} - 1) | \xi(x) = u, \xi(x') = u' \right\rangle_0.$$

Since $\Phi(u,v)>0$, we see easily that

$$k_\Gamma(x;\cdot)\in L_2(R^1,d\lambda_0),$$
$$k_\Gamma(x,x';\cdot,\cdot)\in L_2(R^1,d\lambda_0\times d\lambda_0)\quad \text{for } x\neq x'.$$

We also set

$$k_\Gamma(x,x';u,u')=\delta_{uu'}k_\Gamma(x;u),$$

where $\delta_{uu'}$ is the δ-function in the space $L_2(R^1,d\lambda_0)$:

$$\int_{R^1}\delta_{u,u'}\psi(u')\,d\lambda_0(u')=\int_{R^1}\delta_{u',u}\psi(u')\,d\lambda_0(u')=\psi(u).$$

Let $\widehat{k}_\Gamma(x,x')$ be the integral operator on $L_2(R^1,d\lambda_0)$ with kernel

$$k_\Gamma(x,x';u,u'),u,u'\in R^1,$$

and let $\widehat{k}_{\Gamma_1,\Gamma_2}(x,x')$ be the integral operator on $L_2(R^1,d\lambda_0)$ with kernel $k_{\Gamma_1}(x;u)k_{\Gamma_2}(x';u')$. Then for small enough $\beta>0$, $\beta<\beta_0$ and any $x\in Z^{\nu+1}$, we obtain from (1) that

$$\widehat{S}_x=\sum_{n=0}^{\infty}\frac{1}{n!}\Big\{\sum_{\Gamma_0,\dots,\Gamma_n}{}'\varphi_{\mathrm{con}}(\Gamma_0,\dots,\Gamma_n)\widehat{k}_{\Gamma_0}(0,x)K_{\Gamma_1}\dots K_{\Gamma_n}+\sum_{\Gamma_0,\dots,\Gamma_{n+1}}{}''\varphi_{\mathrm{con}}(\Gamma_0,\dots,\Gamma_{n+1})K_{\Gamma_1}\dots K_{\Gamma_n}\widehat{k}_{\Gamma_0,\Gamma_{n+1}}(0,x)\Big\},\tag{7}$$

or, for the kernel $S_x(u,u')$, $u,u'\in R^1$ of the integral operator $\widehat{S}_x$, $x\in Z^{\nu+1}$,

$$S_x(u,u')=\sum_{n=0}^{\infty}\frac{1}{n!}\Big\{\sum_{\Gamma_0,\dots,\Gamma_n}{}'\varphi_{\mathrm{con}}(\Gamma_0,\dots,\Gamma_n)k_{\Gamma_0}(0,x;u,u')K_{\Gamma_1}\dots K_{\Gamma_n}+\sum_{\Gamma_0,\dots,\Gamma_{n+1}}{}''\varphi_{\mathrm{con}}(\Gamma_0,\dots,\Gamma_{n+1})k_{\Gamma_0}(0,u)K_{\Gamma_1}\dots K_{\Gamma_n}k_{\Gamma_{n+1}}(x,u')\Big\},\tag{8}$$

where the sums are the same as in (1).

We will need the following estimates for the functions

$$k_\Gamma(x,x';\cdot,\cdot),k_\Gamma(x;\cdot),\Gamma\subset\mathcal{L}(Z^{\nu+1}),\qquad x,x'\in\widetilde{\Gamma}.$$

Assertion 1. *There exists $b_1>0$ such that*
1) *for every finite graph $\Gamma\subset\mathcal{L}(Z^{n+1})$ the inequalities*

$$\|k_\Gamma(x,x';\cdot,\cdot)\|_{L_2(R^2,d\lambda_0\times d\lambda_0)}<(b_1\beta)^{|\Gamma|}$$

hold for all $x,x'\in\widetilde{\Gamma}$, $x\neq x'$,

$$\|k_\Gamma(x,\cdot)\|_{L_2(R^2,d\lambda_0)}<(b_1\beta)^{|\Gamma|}$$

for every $x \in \widetilde{\Gamma}$, *and*

$$\sup_{u \in R^1} |k_\Gamma(x,u)| < (b_1\beta)^{|\Gamma|-2(\nu+1)}$$

for every $x \in \widetilde{\Gamma}$;

2) *for any finite graphs* $\Gamma_1, \Gamma_2 \subset \mathcal{L}(Z^{\nu+1})$ *and any* $x_1 \in \widetilde{\Gamma}_1$ *and* $x_2 \in \widetilde{\Gamma}_2$, $|x_1 - x_2| = 1$, *we have*

$$\|k_{\Gamma_1}(x_1;\cdot) \otimes k_{\Gamma_2}(x_2;\cdot)(\exp\{\cdot\beta\Phi(\cdot,\cdot)\} - 1)\|_{L_2(R^2, d\lambda_0 \times d\lambda_0)} \le (b_1\beta)^{|\Gamma_1|+|\Gamma_2|+1},$$

where

$$(k_{\Gamma_1}(x_1;\cdot) \otimes k_{\Gamma_2}(x_2;\cdot))(u_1,u_2) = k_{\Gamma_1}(x_1;u_1)k_{\Gamma_2}(x_2;u_2), \qquad u_1, u_2 \in R^1.$$

Proof. For the proof, we note that

$$\begin{aligned}
\|k_\Gamma(x,\cdot)\|_{L_2(R^2,d\lambda_0)} &\le \|k_\Gamma(x,x';\cdot,\cdot)\|_{L_2(R^2,d\lambda_0\times d\lambda_0)} \\
&\le \left(\left\langle \prod_{(y,y')\in\Gamma} (\exp\{-\beta\Phi(\xi(y),\xi(y'))\} - 1)^4 \right\rangle_0\right)^{1/4}, \\
\sup_{u\in R^1} |k_\Gamma(x,u)| &\le \text{const}\left(\left\langle \prod_{(y,y')\in\Gamma}{}' (\exp\{-\beta\Phi(\xi(y),\xi(y'))\} - 1)^2 \right\rangle_0\right)^{1/2},
\end{aligned}$$

where the product $\prod'$ is over all the edges $(y,y') \in \Gamma$ that do not enter the point $x \in \widetilde{\Gamma}$, and

$$\begin{aligned}
\|(k_{\Gamma_1}(x_1;\cdot) \otimes k_{\Gamma_2}(x_2;\cdot))(\exp\{-\beta\Phi(\cdot,\cdot)\} - 1)\|_{L_2(R^2,d\lambda_0\times d\lambda_0)} \\
\le &\left(\left\langle \prod_{(y,y')\in\Gamma_1} (\exp\{-\beta\Phi(\xi(y),\xi(y'))\} - 1)^4 \right\rangle_0\right)^{1/4} \\
&\times (\langle(\exp\{-\beta\Phi(\xi(x_1),\xi(x_2))\} - 1)^4\rangle_0)^{1/4} \\
&\times \left(\left\langle \prod_{(y,y')\in\Gamma} (\exp\{-\beta\Phi(\xi(y),\xi(y'))\} - 1)^4 \right\rangle_0\right)^{1/4}.
\end{aligned}$$

Thus, to prove Assertion 1 it is enough to show that for every finite connected graph $\Gamma \subset \mathcal{L}(Z^{\nu+1})$ we have

$$\left(\left\langle \prod (\exp\{-\beta\Phi(\xi(y),\xi(y'))\} - 1)^4 \right\rangle_0\right)^{1/4} \le (b_1\beta)^{|\Gamma|}, \tag{9}$$

where $b_1 = \text{const} > 0$.

In proving (9) it is helpful to introduce the notion of the dimension of a graph $\Gamma \subset \mathcal{L}(Z^{\nu+1})$.

Definition 1. A graph $\Gamma \subset \mathcal{L}(Z^{\nu+1})$ is said to have *dimension* n, $0 \le n \le \nu+1$, if n is the smallest number m, $0 \le m \le \nu+1$, such that Γ is homeomorphic to some graph $\Gamma' \subset \mathcal{L}(Z^m)$.

For the proof of (9) we consider the next lemma.

LEMMA 1. *For any finite graph $\Gamma \subset \mathcal{L}(Z^{\nu+1})$ we have the inequality*

$$\Big(\Big\langle \prod_{(y,y')\in\Gamma} (\exp\{-\beta\Phi(\xi(y),\xi(y'))\}-1)^{2^m}\Big\rangle_0\Big)^{2^{-m}} \le \prod_{(y,y')\in\Gamma} (\langle(\exp\{-\beta\Phi(\xi(y),\xi(y'))\}-1)^{2^{m+n}}\rangle_0)^{2^{-m-n}}.$$

PROOF OF LEMMA 1. The proof is by induction on $n = 1,\dots,\nu+1$. The result is obvious for a graph $\Gamma \subset \mathcal{L}(Z^{\nu+1})$, $|\Gamma| < \infty$, of dimension 1. Indeed, we can then label the vertices of Γ in the order in which they are joined by edges. Let $\widetilde{\Gamma} = (y_1,\dots,y_{|\widetilde{\Gamma}|})$, $\Gamma = \{(y_1,y_2),\dots,(y_{|\widetilde{\Gamma}|-1},y_{|\widetilde{\Gamma}|})\}$. Here the points $y_1,\dots,y_{|\widetilde{\Gamma}|}$ are all distinct, and

$$|y_{j+1}-y_j| = 1, \qquad j = 1,\dots,|\widetilde{\Gamma}|-1.$$

It is easy to show that

$$\begin{aligned}\Big(\Big\langle \prod_{j=1}^{|\widetilde{\Gamma}|-1} (\exp\{-\beta\Phi(\xi(y_j),\xi(y_{j+1}))\}-1)^{2^m}\Big\rangle_0\Big)^{2^{-m}} &\le \Big(\Big\langle \prod_j{}^{(1)} (\exp\{-\beta\Phi(\xi(y_j),\xi(y_{j+1}))\}-1)^{2^{m+1}}\Big\rangle_0\Big)^{2^{-m-1}} \\ &\quad\times \Big(\Big\langle \prod_j{}^{(2)} (\exp\{-\beta\Phi(\xi(y_j),\xi(y_{j+1}))\}-1)^{2^{m+1}}\Big\rangle_0\Big)^{2^{-m-1}} \\ &\le \prod_{j=1}^{|\widetilde{\Gamma}|-1} (\langle(\exp\{-\beta\Phi(\xi(y_j),\xi(y_{j+1}))\}-1)^{2^{m+1}}\rangle_0)^{2^{-m-1}},\end{aligned}$$

where the product $\prod^{(1)}$ is over all odd j with $1 \le j \le |\widetilde{\Gamma}|-1$, and $\prod^{(2)}$ is over all even j with $1 \le j \le |\widetilde{\Gamma}|-1$. Lemma 1 is thus proved in the case when $n = 1$.

Now let $n > 1$ and take a graph $\Gamma \subset \mathcal{L}(Z^{\nu+1})$ of dimension n. We divide it into subgraphs $\Gamma_1,\dots,\Gamma_k$ in such a way that no two subgraphs $\Gamma_1,\dots,\Gamma_k$ have a common edge, no two of $\Gamma_1,\dots,\Gamma_k$ with even indices have a common vertex (with the analogous statement for the graphs with odd indices), and all the graphs $\Gamma_1,\dots,\Gamma_k$ have dimension less than n.

If $n = 2$, this can be achieved, for instance, by splitting each vertex of the lattice Z^2 into two parts, one joining an upward and rightward directed edge, the other a downward and leftward directed edge. Then the graph $\Gamma' \subset \mathcal{L}(Z^2)$ homeomorphic to Γ splits into several subgraphs $\Gamma_1,\dots,\Gamma_k$ of dimension 1, where for each $j = 1,\dots,k$ the graph Γ_j can have vertices in common only with Γ_{j-1} for $j > 1$, and with Γ_{j+1} for $j < k$.

For $n > 2$ we proceed similarly and divide each point of Z^ν joining the edges $(y-e_j,y)$, $(y,y+e_j)$, $j = 1,\dots,n$, where $e_j = (e_j^{(1)},\dots,e_j^{(n)})$, $e_j^i = \delta_{ij}$, $j = 1,\dots,n$, into two nodes, one joining the edges $(y-e_j,y)$, $j = 1,\dots,n$, and the other the edges $(y,y+e_j)$, $j = 1,\dots,n$. The graph $\Gamma' \subset \mathcal{L}(Z^n)$ homeomorphic to Γ then decomposes into finitely

many subgraphs of dimension $< n$, say $\Gamma'_1, \dots, \Gamma'_N$, where for each j, $1 \le j \le N$, the graph Γ'_j can have vertices in common only with Γ'_{j-1} if $j > 1$, and with Γ'_{j+1} if $j < N$.

The decomposition of Γ into subgraphs $\Gamma_1, \dots, \Gamma_N$ is given by the homeomorphism $\rho\colon \Gamma \to \Gamma'$ such that $\rho\colon \Gamma_j \to \Gamma'_j$, $j = 1, \dots, N$. It is easy to see that for any $m \in Z_+$, $m \ge 1$,

$$
\begin{aligned}
&\Big(\Big\langle \prod_{(y,y')\in\Gamma} (\exp\{-\beta\Phi(\xi(y),\xi(y'))\} - 1)^{2^m} \Big\rangle_0\Big)^{2^{-m}} \\
&\quad \le \Big(\Big\langle \prod_{j}^{(1)} \prod_{(y,y')\in\Gamma_j} (\exp\{-\beta\Phi(\xi(y),\xi(y'))\} - 1)^{2^{m+1}} \Big\rangle_0\Big)^{2^{-m-1}} \\
&\qquad \times \Big(\Big\langle \prod_{j}^{(2)} \prod_{(y,y')\in\Gamma_j} (\exp\{-\beta\Phi(\xi(y),\xi(y'))\} - 1)^{2^{m+1}} \Big\rangle_0\Big)^{2^{-m-1}} \\
&\quad = \prod_{j=1}^{N} \Big(\Big\langle \prod_{(y,y')\in\Gamma_j} (\exp\{-\beta\Phi(\xi(y),\xi(y'))\} - 1)^{2^{m+1}} \Big\rangle_0\Big)^{2^{-m-1}},
\end{aligned} \tag{10}
$$

where the products $\prod^{(1)}$ and $\prod^{(2)}$ are over all the graphs $\Gamma_1, \dots, \Gamma_N$ with odd and even indices, respectively. By induction on n, (10) gives the assertion of Lemma 1 for all $n \in Z_+$, $n \ge 1$. This completes the proof of Lemma 1.

The proof of Assertion 1 is concluded by using Lemma 1 and the inequality

$$
(\langle(\exp\{-\beta\Phi(\xi(y),\xi(y'))\} - 1)^{2^{n+2}}\rangle_0)^{2^{-n-2}} \le b_1\beta, \qquad 1 \le n \le \nu + 1,
$$

where

$$
b_1 = \max_{k=1,\dots,\nu+3} (\langle(\Phi(\xi(y),\xi(y')))^{2^k}\rangle_0)^{2^{-k}},
$$

which holds by virtue of (3.2) and the positivity $\Phi(u, v) \ge 0$. This proves Assertion 1.

The next result is a corollary of Assertion 1.

ASSERTION 2. *For $\beta > 0$ sufficiently small, we have $\|\widehat{S}_x\| < c_2(b_2\beta)^{|x|}$ for every $x \in Z^{\nu+1}$, where*

$$
|x| = \sum_{j=1}^{\nu+1} |x^{(j)}|, \qquad x = (x^{(1)}, \dots, x^{(\nu+1)})
$$

and b_2 and c_2 are positive constants.

PROOF. We remark that for any finite graph $\Gamma \subset \mathcal{L}(Z^{\nu+1})$ we have

$$
K_\Gamma = \langle k_\Gamma(x; \xi(x))\rangle_0, x \in \widetilde{\Gamma}
$$

and hence by Assertion 1,

$$
|K_\Gamma| \le (b_1\beta)^{|\Gamma|}. \tag{11}
$$

The rest of the proof is now routine, using the estimates (4), (11), and Assertion 1.

§4. Dyson's equation for the covariance operators

ASSERTION 1. *For sufficiently small β, $0 < \beta < \beta_1$, the operators $\widehat{S}_x$ satisfy the system of equations*

$$\widehat{S}_x = I_x + \sum_{\substack{y,y' \in Z^{\nu+1} \\ |y-y'|=1}} I_y \widehat{F} \widehat{S}_{x-y'}, \tag{1}$$

where $\widehat{F}$ is an integral operator on $L_2(R, d\lambda_0)$ with the kernel

$$F(u,v) = \exp\{-\beta\Phi(u,v)\} - 1,$$

where $u, v \in R^1$, and I_y, $y \in Z^{\nu+1}$, are linear bounded selfadjoint operators on $L_2(R, d\lambda_0)$ satisfying the following conditions:

a) *for all $y \in Z^{\nu+1}$, $y \neq 0$, I_y is an integral operator on $L_2(R, d\lambda_0)$ with kernel $I_y(u,v)$ satisfying*

$$\begin{gathered} I_y(\cdot,\cdot) \in L_2(R, d\lambda_0 \times d\lambda_0), \\ \|I_y(\cdot,\cdot)\|_{L_2(R,d\lambda_0\times d\lambda_0)} \leq c_3 (b_3\beta)^{2|y|}, \end{gathered} \tag{2}$$

for suitable positive constants c_3 and b_3;

b) *for $y = 0$, we have $I_0 = I_0' + I_0'' + P_0$, where I_0' is multiplication by the bounded function $I'(u)$, $I'(\cdot) : R \to R$, and*

$$\|I_0' F\| < (b_3\beta)^2, \tag{3}$$

whereas I_0'' is an integral operator on $L_2(R, d\lambda_0)$ with kernel $I''(u,v) \in L_2(R, d\lambda_0 \times d\lambda_0)$ and

$$\|I_0''\| \leq b_3\beta. \tag{4}$$

Here P_0 is the orthogonal projection of $L_2(R, d\lambda_0)$ onto the subspace orthogonal to the subspace of constant functions in $L_2(R, d\lambda_0)$:

$$P_0 f = f - \langle f \rangle_0, \qquad f \in L_2(R, d\lambda_0),$$

c) *for every $y \in Z^{\nu+1}$ we have $I_y \mathbf{1} = 0$, where $\mathbf{1} \in L_2(R, d\lambda_0)$ is the constant function on R taking the value 1.*

PROOF. Consider all graphs with edges in $\mathcal{L}(Z^{\nu+1})$, where multiple edges are allowed. Let $\mathfrak{A}$ be the set of all such finite connected graphs.

DEFINITION 1. A finite connected graph $D \in \mathfrak{A}$ with two distinguished vertices y and $y' \in \widetilde{D}$ (where $\widetilde{D}$ is the set of vertices of the graph D) is called a *diagram* and denoted by $D(y, y')$. We have $D(y, y') \neq D(y', y)$, and y and y' are called the first and last vertices of $D(y, y')$.

Given a diagram $D(y,y')$, we define an integral operator $R_{D(y,y')}$ on the space $L_2(R,d\lambda_0)$ with kernel

$$R_{D(y,y')}(u,v)=\sum_n\Bigg\{\sum_{\Gamma_0,\dots,\Gamma_n}{}'\frac{1}{n!}\varphi_{\mathrm{con}}(\Gamma_0,\dots,\Gamma_n)k_{\Gamma_0}(y,y';u,v)K_{\Gamma_1}\dots K_{\Gamma_n}$$
$$+\sum_{\Gamma_0,\dots,\Gamma_{n+1}}{}''\frac{1}{n!}\varphi_{\mathrm{con}}(\Gamma_0,\dots,\Gamma_{n+1})\tag{5}$$
$$\times k_{\Gamma_0}(y;u)K_{\Gamma_1}\dots K_{\Gamma_n}k_{\Gamma_{n+1}}(y';v)\Bigg\},$$

where the sum $\sum'$ is over all ordered sets of connected graphs $\Gamma_0,\dots,\Gamma_n$ with edges of multiplicity one; here $y\in\widetilde{\Gamma}_0$, $y'\in\widetilde{\Gamma}_0$, and the union $\Gamma_0\cup\dots\cup\Gamma_n$, with allowance for the multiplicity of the edges, is D; $|\Gamma_j|\geq 1$ for $1\leq j\leq n$; Γ_0 may reduce to a single vertex if $y=y'$. Similarly, the sum $\sum''$ is over all ordered sets of graphs $\Gamma_0,\dots,\Gamma_{n+1}$ with edges of multiplicity one, where $y\in\widetilde{\Gamma}_0$, $y'\in\widetilde{\Gamma}_{n+1}$, the union $\Gamma_0\cup\dots\cup\Gamma_{n+1}$ (allowing for edge multiplicity) is D, $|\Gamma_j|\geq 1$ for $j=1,\dots,n$, and Γ_0 and Γ_{n+1} may reduce to a single vertex y and y', respectively.

LEMMA 1. 1) *For $y\neq y'$, $y,y'\in Z^{\nu+1}$, we have $R_{D(y,y')}(\cdot,\cdot)\in L_2(R,d\lambda_0\times d\lambda_0)$ for any diagram $D(y,y')$, and for every $N\in Z_+$ we have*

$$\sum_{D(y,y')}^{(N)}\|R_{D(y,y')}\|\leq c_4(b_4\beta)^N,\tag{6}$$

where the sum $\sum^{(N)}$ is over all diagrams $D(y,y')$ with fixed vertices y and y', and the number of edges of $D(y,y')$ (counting multiplicity) is equal to

$$|D|=N,$$

where c_4 and b_4 are positive constants.

2) *If $y=y'$, $y\in Z^{\nu+1}$, then for every diagram $D(y,y)$ we have*

$$R_{D(y,y)}=R'_{D(y,y)}+R''_{D(y,y)},$$

where the operator $R'_{D(y,y)}$ on $L_2(R,d\lambda_0)$ is given by multiplication by a bounded function $r_{D(y,y)}(\cdot)\colon R^1\to R^1$, $R''_{D(y,y)}$ is an integral operator on $L_2(R,d\lambda_0)$ with kernel $R''_{D(y,y)}(u,v)\in L_2(R,d\lambda_0\times d\lambda_0)$, and

$$\sum_{D(y,y)}^{(N)}\sup_{u\in R^1}|r_{D(y,y)}(u)|<c_4(b_4\beta)^{N-2(\nu+1)},\tag{7}$$

$$\sum_{D(y,y)}^{(N)}\|R''_{D(y,y)}\|\leq c_4(b_4\beta)^N,\tag{8}$$

for all $N\in Z_+$.

3) *For all* $y, y' \in Z^{\nu+1}$ *with* $|y - y'| = 1$, *and* $N_1, N_2 \in Z_+$, *we have*

$$\sum_{D_1(y,y)}^{(N_1)} \sum_{D_2(y',y')}^{(N_2)} \|R'_{D_1(y,y)} \widehat{F} \widehat{R}'_{D_2(y',y')}\| \le c_4 (b_4 \beta)^{N_1+N_2+1}, \tag{9}$$

where $\widehat{F}$ *is the integral operator on* $L_2(R, d\lambda_0)$ *with kernel* $\exp\{\beta\Psi(u,v)\} - 1$.

4) *For any diagram* $D(y, y')$, $y, y' \in Z^{\nu+1}$, *we have*

$$R_{D(y,y')}\mathbf{1} = 0.$$

Proof of Lemma 1. Let $R'_{D(y,y)}$ be given on $L_2(R^1, d\lambda_0)$ by multiplication by the function

$$r_{D(y,y)}(u) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\Gamma_0,\dots,\Gamma_n}{}' \varphi_{\mathrm{con}}(\Gamma_0, \dots, \Gamma_n) k_{\Gamma_0}(y; u) K_{\Gamma_1} \dots K_{\Gamma_n}$$

and let $R''_{D(y,y)}$ be the integral operator on $L_2(R^1, d\lambda_0)$ with kernel

$$R''_{D(y,y)}(u,v) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\Gamma_0,\dots,\Gamma_{n+1}}{}'' \varphi_{\mathrm{con}}(\Gamma_0, \dots, \Gamma_{n+1}) \times k_{\Gamma_0}(y; u) K_{\Gamma_1} \dots K_{\Gamma_n} k_{\Gamma_{n+1}}(y'; v),$$

the sums $\sum'$ and $\sum''$ being as in (5). All the sums here are finite, and the function $k_\Gamma(y; v)$ is bounded for every finite $\Gamma \subset \mathcal{L}(Z^{\nu+1})$ and $y \in \widetilde{\Gamma}$ (see Assertion 1.3); consequently, the functions $r_{D(y,y)}(\cdot)$ and $R''_{D(y,y)}(\cdot,\cdot)$ are bounded, and the operators $R'_{D(y,y)}$ and $R''_{D(y,y)}$ are defined on $L_2(R^1, d\lambda_0)$. By (5) we have $R_{D(y,y)} = R'_{D(y,y)} + R''_{D(y,y)}$.

To prove the estimates (6)–(9) we use Assertion 1.3 and the bounds (4.3), (11.3). Assume that $y \ne y'$; then

$$\sum_{D(y,y')}^{(N)} \|R_{D(y,y')}\| \le \sum_{n=0}^{\infty} \frac{1}{n!} \Big\{ \sum_{\Gamma_0,\dots,\Gamma_n^{(N)}}{}' |\mathcal{J}(\Gamma_0, \dots, \Gamma_n)| (b_1\beta)^N + \sum_{\Gamma_0,\dots,\Gamma_{n+1}^{(N)}}{}'' |\mathcal{J}(\Gamma_0, \dots, \Gamma_{n+1})| (b_1\beta)^N \Big\}, \tag{10}$$

where b_1 is the constant in Assertion 1.3, $\mathcal{J}(\Gamma_0, \dots, \Gamma_n)$ is the set of all trees with vertices $\{0, \dots, n\}$ and edges (i, j), $0 \le i < j \le n$, where $\widetilde{\Gamma}_i \cap \widetilde{\Gamma}_j \ne \emptyset$. The sum $\sum'_{(N)}$ is over all ordered sets of finite connected graphs $\Gamma_0, \dots, \Gamma_n \subset \mathcal{L}(Z^{\nu+1})$ (possibly coinciding) such that $y, y' \in \widetilde{\Gamma}_0$, the graph $\Gamma_0 \cup \dots \cup \Gamma_n$ is connected, and $|\Gamma_j| \ge 1$, $j = 1, \dots, n$,

$$\sum_{j=1}^{n} |\Gamma_j| = N.$$

The sum $\sum''_{(N)}$ is over all ordered sets of finite connected graphs $\Gamma_0, \dots, \Gamma_{n+1} \subset \mathcal{L}(Z^{\nu+1})$ such that $y \in \widetilde{\Gamma}_0$, $y' \in \widetilde{\Gamma}_{n+1}$, the graph $\Gamma_0 \cup \dots \cup \Gamma_{n+1}$ is connected, $|\Gamma_j| \ge 1$, $j = 1, \dots, n$;

Γ_0 and Γ_{n+1} may reduce to the single vertex y or y', respectively, and $|\Gamma_0|+\cdots+|\Gamma_{n+1} = N$. Similarly,

$$\begin{aligned}(11)\qquad &\sum_{D(y,y)}^{(N)} \sup_{u\in R} |r_{D(y,y)}(u)| \\ &\qquad \le \sum_{n\ge 0} \frac{1}{n!} \sum_{\Gamma_0,\dots,\Gamma_n^{(N)}}{}' |\mathcal{J}(\Gamma_0,\dots,\Gamma_n)|(b_1\beta)^{\sum_{j=1}^n |\Gamma_j|}(b_1\beta)^{|\Gamma_0|-2(\nu+1)},\end{aligned}$$

where the sum $\sum'_{(N)}$ is as in (10):

$$(12)\qquad \sum_{D(y,y)^{(N)}}{}'' \|R''_{D(y,y)}\| \le \sum_{n\ge 0} \frac{1}{n!} \sum_{\Gamma_0,\dots,\Gamma_{n+1}^{(N)}}{}'' |\mathcal{J}(\Gamma_0,\dots,\Gamma_{n+1})|(b_1\beta)^N,$$

where again the sum $\sum''_{(N)}$ is as in (10), and for $y, y' \in Z^{\nu+1}$ with $|y-y'|=1$ we have

$$\begin{aligned}(13)\qquad &\sum_{D(y,y)}^{(N_1)} \sum_{D(y',y')}^{(N_2)} \|R'_{D_1(y,y)} \widehat{F} R'_{D_2(y',y')}\| \\ &\qquad \le \sum_{n_1\ge 0} \frac{1}{n_1!} \sum_{n_2\ge 0} \frac{1}{n_2!} \sum_{\Gamma_0^1,\dots,\Gamma_{n_1}^{1\,(N_1)}}{}' |\mathcal{J}(\Gamma_0^1,\dots,\Gamma_{n_1}^1)| \\ &\qquad\quad \times \sum_{\Gamma_0^2,\dots,\Gamma_{n_2}^{2\,(N_2)}}{}'' |\mathcal{J}(\Gamma_0^2,\dots,\Gamma_{n_2}^2)|(b_1\beta)^{N_1+N_2+1}.\end{aligned}$$

Here the sums $\sum'_{N_1}$ and $\sum'_{N_2}$ over $\Gamma_0^1,\dots,\Gamma_n^1 \subset \mathcal{L}(Z^{\nu+1})$ and over $\Gamma_0^2,\dots,\Gamma_n^2 \subset \mathcal{L}(Z^{\nu+1})$, respectively, are as in (10).

Given (10)–(13), the rest of the proof of the estimates (6)–(9) is now routine (see [26]).

The first three statements of Lemma 1 have thus been proved. To establish the fourth, we observe that it follows directly from (5); indeed, for every $\psi \in L_2(R^1, d\lambda_0)$ we obtain from (5) that

$$\begin{aligned}(\psi, R_{D(y,y')}\mathbf{1}) &= \sum_{n\ge 0} \frac{1}{n!} \Bigg\{ \sum_{\Gamma_0,\dots,\Gamma_n}{}' \varphi_{\mathrm{con}}(\Gamma_0,\dots,\Gamma_n) K_{\Gamma_1}\dots K_{\Gamma_n} k_{\Gamma_0}(y,y';\psi,\mathbf{1}) \\ &\qquad + \sum_{\Gamma_0,\dots,\Gamma_{n+1}}{}'' \varphi_{\mathrm{con}}(\Gamma_0,\dots,\Gamma_{n+1}) K_{\Gamma_1}\dots K_{\Gamma_{n+1}} k_{\Gamma_0}(y;\psi) \Bigg\} \\ &= \sum_{n\ge 0} \frac{1}{n!} \Bigg\{ \sum_{\Gamma_0,\dots,\Gamma_n}{}' \varphi_{\mathrm{con}}(\Gamma_0,\dots,\Gamma_n) k_{\Gamma_0}(y;\psi) K_{\Gamma_1}\dots K_{\Gamma_n} \\ &\qquad + \sum_{\Gamma_0,\dots,\Gamma_{n+1}}{}'' \varphi_{\mathrm{con}}(\Gamma_0,\dots,\Gamma_{n+1}) k_{\Gamma_0}(y;\psi) K_{\Gamma_1}\dots K_{\Gamma_{n+1}} \Bigg\}\end{aligned}$$

since $k_{\Gamma_0}(y,y';\psi,\mathbf{1}) = k_{\Gamma_0}(y;\psi)$ for every finite graph $\Gamma \subset \mathcal{L}(Z^{\nu+1})$ and $\psi \in L_2(R^1, d\lambda_0)$. Recall that as in (5), all the sums here are finite. We break the sum $\sum''$ into two parts:

$$\sum_{\Gamma_0,\dots,\Gamma_{n+1}}{}'' = \sum_{\Gamma_0,\dots,\Gamma_{n+1}:|\widetilde{\Gamma}_{n+1}|=1}{}'' + \sum_{\Gamma_0,\dots,\Gamma_{n+1}:|\widetilde{\Gamma}_{n+1}|>1}{}''$$

(the sum $\sum''$ here being defined in the same way as in (5)). We will show that

$$\sum_{n\geq 0}\frac{1}{n!}\Big\{\sum_{\Gamma_0,\dots,\Gamma_n}{}' \varphi_{\mathrm{con}}(\Gamma_0,\dots,\Gamma_n)k_{\Gamma_0}(y;\psi)K_{\Gamma_1}\dots K_{\Gamma_n} + \sum_{\Gamma_0,\dots,\Gamma_{n+1}:|\widetilde{\Gamma}_{n+1}|>1}{}'' \varphi_{\mathrm{con}}(\Gamma_0,\dots,\Gamma_{n+1})k_{\Gamma_0}(y;\psi)K_{\Gamma_1}\dots K_{\Gamma_n}\Big\} = -\sum_{n\geq 0}^{\infty}\frac{1}{n!}\sum_{\Gamma_0,\dots,\Gamma_n,\Gamma_{n+1}=\{y'\}}\varphi_{\mathrm{con}}(\Gamma_0,\dots,\Gamma_{n+1})k_{\Gamma_0}(y;\psi)K_{\Gamma_1}\dots K_{\Gamma_n}. \tag{14}$$

For this we note that if the graph Γ_{n+1} in $\sum''$ consists of a single vertex $\widetilde{\Gamma}_{n+1}=\{y'\}$, then $y'\in\widetilde{\Gamma}_0\cup\dots\cup\widetilde{\Gamma}_n$ and the graph $\Gamma_0\cup\dots\cup\Gamma_n$ is connected. Moreover, if exactly m of the graphs $\Gamma_0,\dots,\Gamma_n$ contains the vertex y' $(0\leq m\leq n)$, then

$$\varphi_{\mathrm{con}}(\Gamma_0,\dots,\Gamma_{n+1}) = -m\varphi_{\mathrm{con}}(\Gamma_0,\dots,\Gamma_n). \tag{15}$$

To prove (15) it suffices to consider

$$\varphi_{\mathrm{con}}(\Gamma_0,\dots,\Gamma_{n+1}) = \sum_G{}^{(1)}(-1)^{|G|} + \sum_G{}^{(2)}(-1)^{|G|},$$

where the sum $\sum^{(1)}$ is over all connected graphs G with vertices $\{0,\dots,n+1\}$ and edges (i,j), $0\leq i<j\leq n$, where $\widetilde{\Gamma}_i\cap\widetilde{\Gamma}_j\neq\emptyset$, $0\leq i<j\leq n+1$. It is easy to see that if $|\Gamma_{n+1}|=1$ and exactly m of the graphs $\Gamma_0,\dots,\Gamma_n$ contain the vertex y', then we have

$$\sum_G{}^{(1)}(-1)^{|G|} = -m\sum_G{}^{\mathrm{con}}(-1)^{|G|} = -m\varphi_{\mathrm{con}}(\Gamma_0,\dots,\Gamma_n)$$

and

$$\sum_G{}^{(2)}(-1)^{|G|} = 0.$$

Thus the right-hand side of (14) is equal to

$$\sum_{n\geq 0}\frac{1}{n!}\sum_{m=1}^{n+1}\sum_{\Gamma_0,\dots,\Gamma_n}{}^{(m)}\varphi_{\mathrm{con}}(\Gamma_0,\dots,\Gamma_n)k_{\Gamma}(\psi;y)K_{\Gamma_1}\dots K_{\Gamma_n}, \tag{16}$$

where the sum $\sum^{(m)}$, $m=1,\dots,n+1$, is over all ordered sets of finite connected graphs $\Gamma_0,\dots,\Gamma_n\subset\mathcal{L}(Z^{\nu+1})$ (possibly coinciding) such that $y\in\widetilde{\Gamma}_0$, $y'\in\widetilde{\Gamma}_0\cup\dots\cup\widetilde{\Gamma}_n$, and the union $\Gamma_0\cup\dots\cup\Gamma_n$ (with allowance for edge multiplicities) is D (the diagram $D(y,y')$ is fixed); here $|\Gamma_j|\geq 1$ for $j=1,\dots,n$, Γ_0 may reduce to the single vertex y, and exactly m of the graphs $\Gamma_0,\dots,\Gamma_n$ contain the vertex y'. We now observe that

$$\sum_{\Gamma_0,\dots,\Gamma_n}{}' + n\sum_{\Gamma_0,\dots,\Gamma_n:|\Gamma_n|\geq 1}{}'' = \sum_{m=1}^{n+1} m\sum_{\Gamma_0,\dots,\Gamma_n}{}^{(m)},$$

where the sums $\sum'$ and $\sum''$ are the same as in (5), except that we always have $|\Gamma_n|\geq 1$. By (16), this implies (15), proving Lemma 1.

DEFINITION 2. We say that $D(y, y')$ is a *one-particle irreducible* diagram if either $y = y'$ or the graph D contains at least two paths joining the vertices y and y' which are either disjoint or satisfy the cndition that the multiplicity of each common edge of these two graphs is greater than 1.

Let us not than if the diagram $D(y, y')$ is not one-particle irreducible, then D contains an edge $(y_1, y_1') \in D$ of multiplicity one, such that when the edge is removed we get two nonintersecting graphs D_1 and D_2:

$$D = D_1 \cup \{(y_1, y_1')\} \cup D_2,$$

where $y, y_1 \in \widetilde{D}_1$ and $y', y_1' \in \widetilde{D}_2$. In this case, we say that the edge (y_1, y_1') splits $D(y, y')$ into the two disjoint diagrams $D_1(y, y_1)$ and $D(y_1', y')$.

LEMMA 2. *Assume that the edge $(y_1, y_1') \in D$, $D \in \mathfrak{A}$, splits the diagram $D(y, y')$, $y \neq y'$, into two disjoint diagrams $D_1(y, y_1)$and$D_2(y_1', y')$. Then*

$$R_{D(y,y')} = R_{D_1(y,y_1)} \widehat{F} R_{D_2(y_1',y')}.$$

PROOF. Indeed, let $\Gamma_0, \ldots, \Gamma_n \subset \mathcal{L}(Z^{\nu+1})$ be such that the union $\Gamma_0 \cup \cdots \cup \Gamma_n$ is D (counting edge multiplicity), and suppose that $(y_1, y_1') \in \Gamma_j$ for some j, $0 \leq j \leq n$. Then $\Gamma_j = \Gamma_j' \cup \Gamma_j''\{(y, y')\}$, where the graphs $\Gamma_j', \Gamma_j'' \subset \mathcal{L}(Z^{\nu+1})$ have no vertices in common, each is connected, and $y_1 \in \Gamma_j' \subset D_1$, $y_1' \in \Gamma_j'' \subset D_2$. We note that if $j = 0$ and $y, y' \in \widetilde{\Gamma}_0$, then

$$\widehat{k}_{\Gamma_0}(y, y') = \widehat{k}_{\Gamma_0'}(y, y_1) \widehat{F} \widehat{k}_{\Gamma_0''}(y_1', y')$$

and similarly,

$$k_{\Gamma_j}(y; u) = \int_{R^2} k_{\Gamma_j'}(y, y_1; u, u_1) F(u_1, u_1') k_{\Gamma_j''}(y_1'; u_1')\, d\lambda_0(u_1)\, d\lambda(u_1'),$$

if $y \in \widetilde{\Gamma}_j'$,

$$k_{\Gamma_j}(y'; u') = \int_{R^2} k_{\Gamma_j'}(y_1; u_1) F(u_1, u_1') k_{\Gamma_j''}(y_1', y'; u_1', u')\, d\lambda_0(u_1)\, d\lambda(u_1'),$$

if $y \in \widetilde{\Gamma}_j''$, and

$$K_{\Gamma_j} = \int_{R^2} k_{\Gamma_j'}(y_1; u_1) F(u_1, u_1') k_{\Gamma_j''}(y_1'; u_1')\, d\lambda_0(u_1)\, d\lambda(u_1').$$

Thus to prove Lemma 2, it remains to show that if $\Gamma_0, \ldots, \Gamma_n \subset \mathcal{L}(Z^{\nu+1})$ are such that the union $\Gamma_0 \cup \cdots \cup \Gamma_n$ (counting edge multiplicity) is D, and if $(y_1, y_1') \in \Gamma_j$ for some j, $0 \leq j \leq n$, then

$$\begin{aligned}\varphi_{\mathrm{con}}(\Gamma_0, \ldots, \Gamma_n) = {} & \varphi_{\mathrm{con}}(\Gamma_j', \Gamma_i \subset D_1,\ i \neq j,\ 0 \leq i \leq n) \\ & \times \varphi_{\mathrm{con}}(\Gamma_j'', \Gamma_i \subset D_1,\ i \neq j,\ 0 \leq i \leq n).\end{aligned}$$

For this it suffices to note that if the union $\Gamma_0 \cup \cdots \cup \Gamma_n$ (with edge multiplicity) is D and $(y_1, y_1') \in \Gamma$ for some j, $0 \leq j \leq n$, then every graph G with vertex set $\{0, 1, \ldots, n\}$ and

edges (i,j), $0 \le i < j \le n$, where $\widetilde{\Gamma}_i \cap \widetilde{\Gamma}_j \neq \emptyset$, is expressible as a union of two connected graphs G_1 and G_2 which have a single vertex j in common, and which have the property that $i \in G_l$ if and only if $\Gamma_i \in D_l$, $i = 0, \dots, n$, $i \neq j$, $l = 1, 2$. This concludes the proof of Lemma 2.

We now observe that in the statement of Lemma 2, the edge $(y_1, y') \in D$ can be chosen so that the diagram $D_1(y, y_1)$ is one-particle irreducible. We can now repeat the preceding arguments for the diagram $D_2(y_1', y')$, and so on. Thus we remove edges $(y_1, y_1'), \dots, (y_k, y_k') \in D$, which divide the diagram $D(y, y')$ into one-particle irreducible parts

$$D_1(y, y_1), D_2(y_1', y_2), \dots, D_{k+1}(y_k', y').$$

Successive application of Lemma 2 then leads to the following expression for $R_{D(y,y')}$:

$$R_{D(y,y')} = R_{D_1(y,y_1)} \widehat{F} R_{D_2(y_1',y_2)} \widehat{F} \dots \widehat{F} R_{D_{k+1}(y_k',y')}. \tag{17}$$

Now observe that by (8.3) and (15), for every $x \in Z^{\nu+1}$ we have

$$\widehat{S}_x = \sum_{D(0,x)} R_{D(0,x)},$$

where the sum is over all diagrams $D(0, x)$. Substituting expression (17) for $R_{D(0,x)}$, we find that

$$\begin{aligned} \widehat{S}_x = \sum_{D(0,x)}^{(1)} R_{D(0,x)} + \sum_{k=1}^{\infty} \sum_{(y_1,y_1'),\dots,(y_k,y_k')}^{(2)} \sum_{D_1(0,y_1),\dots,D_{k+1}(y_k',x)}^{(3)} R_{D_1(0,y_1)} \widehat{F} R_{D_2(y_1',y_2)} \\ \dots \widehat{F} R_{D_{k+1}(y_k',x)} \prod_{1 \le i < j \le k+1} (1 + \chi(D_i, D_j)), \end{aligned} \tag{18}$$

where the sum $\sum^{(1)}$ is over all one-particle irreducible diagrams $D(0, x)$, and $\sum^{(2)}$ is over all ordered sets of pairs of points $(y_1, y_1'), \dots, (y_k, y_k')$, $y_j, y_j' \in Z^{\nu+1}$ such that $|y_j - y_j'| = 1$, $j = 1, \dots, k$ (some of these pairs may coincide). The sum $\sum^{(3)}$ is over all one-particle irreducible diagrams $D_1(0, y_1)$, $D_2(y_1', y_2), \dots, D_{k+1}(y_k', x)$, and

$$\begin{aligned} \chi(D_i, D_j) &= -1, \quad \text{if } \widetilde{D}_i \cap \widetilde{D}_j \neq \emptyset, \\ \chi(D_i, D_j) &= 0, \quad \text{if } \widetilde{D}_i \cap \widetilde{D}_j = \emptyset. \end{aligned}$$

We now use the diagram expansion (18) to derive Dyson's equation. Let $g(D_1, \dots, D_{k+1})$, $D_1, \dots, D_{k+1} \in \mathfrak{A}$, be the graph with vertices $\{1, \dots, k+1\}$ and edges (i, j), $1 \le i < j \le k+1$, where $(i, j) \in g(D_1, \dots, D_{k+1})$ if and only if $\widetilde{D}_i \cap \widetilde{D}_j \neq \emptyset$, $1 \le i < j \le k+1$. Then

$$\prod_{1 \le i < j \le k+1} (1 + \chi(D_i, D_j)) = \sum_{G \le g(D_1,\dots,D_{k+1})} (-1)^{|G|},$$

where the sum is over all subgraphs $G \le g(D_1, \dots, D_{k+1})$ with vertex set $\{1, \dots, k+1\}$; $|G|$ is the number of edges in G.

DEFINITION 3. A graph G with vertex set $\{1,\dots,n\}$ is *decomposable* if it consists of two disjoint parts G_1 and G_2, where for some j, $1 \le j \le n$

$$\widetilde{G}_1 = \{1,\dots,j\}, \qquad \widetilde{G}_2 = \{j+1,\dots,n\}.$$

If this is not the case, G is said to be *indecomposable* (a graph G on a single vertex, $n=1$, is considered to be indecomposable).

If the graph $g(D_1,\dots,D_{k+1})$ is indecomposable, we set

$$\varphi_H(D_1,\dots,D_{k+1}) = \sum_G (H)(-1)^{|G|},$$

where the sum is over all indecomposable subgraphs $G \subseteq g(D_1,\dots,D_{k+1})$ with vertex set $\{1,\dots,k+1\}$; otherwise (if $g(D_1,\dots,D_{k+1})$ is decomposable), we define $\varphi_H(D_1,\dots,D_{k+1}) = 0$. It is easy to see that

$$\begin{aligned}\sum_{G\subseteq g(D_1,\dots,D_{k+1})} (-1)^{|G|} &= \varphi_H(D_1,\dots,D_{k+1}) \\ &\quad + \sum_{n=1}^{k} \varphi_H(D_1,\dots,D_n) \sum_{G\subseteq g(D_{n+1},\dots,D_{k+1})} (-1)^{|G|}.\end{aligned} \tag{19}$$

Define the operators I_y, $y \in Z^{\nu+1}$ for $y \ne 0$ by

$$\begin{aligned}I_y = \sum_{D(0,y)}^{(1)} R_{D(0,y)} + \sum_{k=1}^{\infty} \sum_{(y_1,y_1'),\dots,(y_k,y_k')}^{(2)} \sum_{D_1(0,y_1),\dots,D_{k+1}(y_k',y)}^{(3)} \varphi_H(D_1,\dots,D_{k+1}) \\ \times R_{D_1(0,y_1)} \widehat{F} R_{D_2(y_1',y_2)} \dots \widehat{F} R_{D_{k+1}(y_k',y)},\end{aligned} \tag{20}$$

and for $y = 0$ set

$$I_0' = \sum_{D(0,0)}^{(1)} R'_{D(0,0)},$$

$$\begin{aligned}I_0'' = \sum_{D(0,0)}^{(1)} R''_{D(0,0)} + \sum_{k=1}^{\infty} \sum_{(y_1,y_1'),\dots,(y_k,y_k')}^{(2)} \sum_{D_1(0,y_1),\dots,D_{k+1}(y_k',0)}^{(3)} \varphi_H(D_1,\dots,D_{k+1}) \\ \times R_{D_1(0,y_1)} \widehat{F} R_{D_2(y_1',y_2)} \dots \widehat{F} R_{D_{k+1}(y_k',0)}.\end{aligned}$$

The sums $\sum^{(2)}$ and $\sum^{(3)}$ are the same as in (18), and $\sum^{(1)}$ is over all diagrams $D(0,0)$ with more than one vertex, i.e., $|\widetilde{D}| > 1$.

Lemma 1 and the next result readily imply that each of these series is norm-convergent in the space of bounded linear operators on $L_2(R^1, d\lambda_0)$.

LEMMA 3. *For any $D_1,\dots,D_k \in \mathfrak{A}$ we have*

$$|\varphi_H(D_1,\dots,D_k)| < q_1^k,$$

for some positive constant q_1.

PROOF. We will need the concept of a minimal indecomposable graph on a set of vertices.

DEFINITION 4. An indecomposable graph G with vertex set $\{1,\dots,k\}$ is *minimal* if a decomposable graph results whenever an edge is removed from G.

To every indecomposable graph G with vertex set $\{1,\dots,k\}$ we associate the sequence of all its edges:

$$\{(i_1,j_1),\dots,(i_{|G|},j_{|G|})\}=G,$$

where $i_s<j_s$, $s=1,\dots,|G|$, and $i_s\le i_{s+1}$, $s=1,\dots,|G|-1$, $i_1=1$. Note that for a minimal graph, this sequence is uniquely determined and satisfies the condition

$$i_{s+1}<j_s, s=1,\dots,|G|-1$$

and

$$j_{s-1}<i_{s+1}, s=2,\dots,|G|-1.$$

We order the set of minimal graphs with vertices $\{1,\dots,k\}$ as follows. Let $\{(i_1,j_1),\dots,(i_{|G|},j_{|G|})\}$ be the sequence of edges of the minimal graph G and $\{(i'_1,j'_1),\dots,(i'_{|G|},j'_{|G|})\}$ the corresponding sequence for another minimal graph G'. Then define $G<G'$ if for some s, $1\le s\le\min\{|G|,|G'|\}$ we have $(i_m,j_m)=(i'_m,j'_m)$ for $1\le m<s$ (provided $s>1$), and either $j_s>j'_s$, or else $j_s=j'_s$ but $i_s<i'_s$. It is easy to see that this gives a total ordering on the set of all minimal graphs.

We use the above ordering to enumerate all the minimal graphs with vertex set $\{1,\dots,k\}$ contained in $g(D_1,\dots,D_k)$, $G_1,\dots,G_N$, $G_i<G_j$, $1\le i<j\le N$.

Let G_j, $j=1,\dots,N$, be the set of all indecomposable graphs $G\subseteq G(D_1,\dots,D_k)$ with vertex set $\{1,\dots,k\}$, each of which contains the jth minimal indecomposable graph G_j but not any of the preceding minimal indecomposable graphs G_i, $1\le i<j$. Defining

$$\sigma_j(D_1,\dots,D_k)=\sum_{G\in G_j}(-1)^{|G|}$$

we have

$$\varphi_H(D_1,\dots,D_k)=\sum_{j=1}^{N}\sigma_j(D_1,\dots,D_k).$$

We next derive an explicit form for the quantities

$$\sigma_j(D_1,\dots,D_k),\qquad j=1,\dots,N.$$

Let $\{(i_1,j_1),\dots,(i_{|G_2|},j_{|G_2|})\}$ be the sequence of edges of the minimal graph G_l, $1\le l\le N$. We remove all the edges $(i,j)\in g(G_1,\dots,G_k)\setminus G_l$ such that for some m, $1\le m\le l$, $i\le j_{m-1}$ and either $j>j_m$, or else $j=j_m$ but $i<i_m$. The set of edges removed in this way will be denoted by $O(G_l)$. Note that if $G<G_l$, $1\le l\le N$, then G contains no edges in $O(G_l)$, because otherwise we would have $G\supset G_m$ for some $1\le m<l$. And conversely, if $G\cap O(G_l)=\emptyset$ and $G_l\in G$, then $G<G_l$. Thus,

$$\sigma_j(D_1,\dots,D_k)=(-1)^{|G_j|}\sum_{G}(-1)^{G},\qquad j=1,\dots,N,$$

where the sum is over all graphs $G\subset g(D_1,\dots,D_k)\setminus(O(G_j)\cup G_j)$, and consequently $\sigma_j(D_1,\dots,D_k)=0$ if $g(D_1,\dots,D_k)\setminus(O(G_j)\cup G_j)\ne\emptyset$ and $\sigma_j(D_1,\dots,D_k)=(-1)^{|G_j|}$

if $g(D_1,\dots,D_k) = (O(G_j)\cup G_j)$. Thus, $|\varphi_H(D_1,\dots,D_k)|$ is less than the number of minimal indecomposable subgraphs $g(D_1,\dots,D_k)$ with vertices $\{1,\dots,k\}$. It is easy to see that the number of minimal indecomposable graphs with vertex set $\{1,\dots,k\}$ is less than 4^k. Lemma 3 is proved.

We can now conclude the proof of Assertion 1. Let us show that for any $y\in Z^{\nu+1}$ the operator I_y is selfadjoint. Indeed, $\widehat{F}^* = \widehat{F}$ by the symmetry of the function Φ: $\Phi(u,v) = \Phi(v,u)$, and it follows easily from (5) that $R^*_{D(y,y')} = R_{D(y,y')}$ for every diagram $D(y,y')$; thus, $I_y^* = I_{-y} = I_y$. We further have by definition that $I_0 = I_0' + I_0'' + P_0$, where P_0 is the orthogonal projection in $L_2(R^1, d\lambda_0)$ given by

$$P_0 f = f - \langle f\rangle_0, \qquad f\in L_2(R^1, d\lambda_0),$$

and I_0' on $L_2(R^1,d\lambda_0)$ is multiplication by the function

$$I_0' = \sum_{D(0,0)}^{(1)} r'_{D(0,0)}(u),$$

where the sum is over all diagrams $D(0,0)$ with more than one vertex: $|\widetilde{D}|>1$. The boundedness of this function follows easily from (7). For $y\neq 0$, $y\in Z^{\nu+1}$, I_0'' and I_y are Hilbert-Schmidt operators on $L_2(R^1,d\lambda_0)$ (i.e., they have a kernel in $L_2(R1, d\lambda_0\times d\lambda_0)$).

Estimate (3) follows easily from (9) and Lemma 3. To prove (2) and (4), we notice that for any $y_j, y_j'\in Z^{\nu+1}$ such that $|y_j - y_j'| = 1$, $j=1,\dots,k$, and all $N_1,\dots,N_{k+1}\in Z_+$, we have by (6), (8), and (9) that

$$\sum^{(N_1,\dots,N_{k+1})} \|R_{D_1(0,y_1)}\widehat{F}R_{D_2(y_1',y_2)}\dots\widehat{F}R_{D_{k+1}(y_k',y)}\| \le (b'\beta)^{k+N_1+\dots,N_{k+1}}, \tag{21}$$

where b' is a positive constant and the sum is over all one-particle irreducible diagrams $D_1(0,y_1), D_2(y_1',y_2),\dots,D_{k+1}(y_k',y)$ such that $|D_j| = N_j$, $j=1,\dots,k+1$. Inequality (4) now follows easily from (21), (8), and Lemma 3.

To prove (2) we observe that if

$$D_1(0,y_1), D_2(y_1',y_2),\dots,D_{k+1}(y_k',y),$$

where $|y_j - y_j'| = 1$, $j=1,\dots,k$, are one-particle irreducible diagrams and the graph $g(D_1,\dots,D_{k+1})$ is indecomposable, then $\sum_{j=1}^{k+1}|D_j| + k \ge 2|y|$, whence (21) implies (2) easily.

Part c) of Assertion 1 follows automatically from the fourth statement in Lemma 1.

The Dyson equations themselves (1) now follow readily from the diagram expansions (18) and (20) upon recalling (19).

This completes the proof of Assertion 1.

§5. Analytic structure of the Green functions

Here we will use Assertion 1.4 above to prove Theorems 1 and 2, which were stated in §2.

Let β be sufficiently small, $0 < \beta < \beta_1$, where β_1 is the constant appearing in Assertion 1.4. Consider the Fourier transform of the operators $\widehat{S}_x$:

$$\underline{S}(p,k) = \sum_{x=(\overline{x},x^{(0)})\in Z^{\nu+1}} e^{-(p,\overline{x})-ikx^{(0)}}\,\widehat{S}_x, \qquad (p,k)\in T^\nu\times T^1,$$

$$(p,\overline{x}) = \sum_{j=1}^{\nu} p^{(j)}x^{(j)}, \qquad p=(p^{(1)},\dots,p^{(\nu)}), \quad \overline{x}=(x^{(1)},\dots,x^{(\nu)}).$$

By Assertion 2.3, for any fixed $p\in T^\nu$ and any k in the strip $|\operatorname{Im} k| < \ln(b_2\beta)^{-1}$ (where b_2 is the constant in Assertion 2.3), we have that $\underline{S}(p,k)$ is a bounded linear operator on $L_2(R^1,d\lambda_0)$ such that

$$\underline{S}(p,k)\mathbf{1} = 0, \tag{1}$$

since by definition $\widehat{S}_x\mathbf{1} = 0$ for every $x\in Z^\nu$, and $\underline{S}^*(p,k) = \underline{S}(p,\overline{k})$ in view of the obvious symmetry $S_x^* = S_{-x} = S_x$, $x\in Z^{\nu+1}$. Similarly, for the Fourier transform of I_x, where $x=(\overline{x},x^{(0)})\in Z^{\nu+1}$, Assertion 4.1 implies that, for any fixed $p\in T^\nu$ and any k in the strip $|\operatorname{Im} k| < \ln(b_3\beta)^{-2}$ (where the constant b_3 is as in Assertion 4.1),

$$\underline{I}(p,k) = \sum_{x\in Z^{\nu+1}} e^{-(p,\overline{x})-ikx^{(0)}} I_x$$

is a bounded linear operator on $L_2(R^1,d\lambda_0)$ such that

$$\underline{I}(p,k)\mathbf{1} = 0, \qquad \underline{I}^*(p,k) = \underline{I}(p,\overline{k}), \tag{2}$$

and

$$\underline{I}(p,k) = P_0 + I_0' + \underline{I}''(p,k). \tag{3}$$

Here $P_0 f = f - \langle f\rangle_0$, $f\in L_2(R^1,d\lambda_0)$, I_0' is multiplication by the bounded function

$$I_0'\colon R^1\to R^1$$

and $I''(p,k)$ is the Hilbert-Schmidt operator on $L_2(R^1,d\lambda_0)$ with kernel

$$\underline{I}''(u,u';p,k) = I_0''(u,u') + \sum_{\substack{x\in Z^{\nu+1}\\ x\neq 0}} e^{-(p,\overline{x})-ikx^{(0)}} I_x(u,u').$$

Let $\mathcal{H}\subset L_2(R^1,d\lambda_0)$ be the subspace spanned by the functions $f-\langle f\rangle_0$, $f\in L_2(R^1,d\lambda_0)$. By (1) and (2), $\mathcal{H}$ is invariant under $\underline{S}(p,k)$ and $\underline{I}(p,k)$, where

$$\underline{S}(p,k) = P_0\underline{S}(p,k)P_0,$$
$$\underline{I}(p,k) = P_0\underline{I}(p,k)P_0,$$

and P_0 is the orthogonal projection of $L_2(R^1,d\lambda_0)$ on $\mathcal{H}$. We will henceforth regard $\underline{S}(p,k)$ and $\underline{I}(p,k)$ as operators on $\mathcal{H}$.

We can then rewrite equation (3) as

$$\underline{I}(p,k) = E + P_0I_0' + P_0\underline{I}''(p,k),$$

where E is the identity operator on $\mathcal{H}$.

LEMMA 1. *For any $h > 0$ there exist a $\beta_2 > 0$, $\beta_2 h < 1$, and a $\kappa_1 > 0$ such that for any β with $0 < \beta < \beta_2$, any k in the strip $|\operatorname{Im} k| < \ln(h\beta)^{-1}$, and any $p \in T^\nu$ the inequalities*

$$\|\underline{I}(p,k)\widehat{F}\| < \kappa_1\beta, \qquad \|(\underline{I}(p,k) - E)\widehat{F}\| < \kappa_1\beta^2 \tag{4}$$

hold, where $\|\cdot\|$ is the norm on the space of bounded linear operators on $\mathcal{H}$.

PROOF. The first inequality follows automatically from the second, since

$$\|\widehat{F}\| \le \|(e^{-\beta\Phi} - 1)\|_{L_2(R^2, d\lambda_0 \times d\lambda_0)} \le \beta\|\Phi\|_{L_2(R^2, d\lambda_0 \times d\lambda_0)}.$$

To prove the second inequality, we appeal to the estimates in Assertion 1.4. Consider

$$\|(\underline{I}(p,k) - E)\widehat{F}\| \le \|I_0'\widehat{F}\| + \|I_0''\widehat{F}\| + \sum_{\substack{x \in Z^{\nu+1} \\ x \ne 0}} |e^{-ikx^{(0)}}|\,\|I_x\widehat{F}\|.$$

In view of (2.4), (3.4), and (4.4), for small enough $\beta > 0$ this gives

$$\begin{aligned}\|(\underline{I}(p,k) - E)\widehat{F}\| \le{}& (b_3\beta)^2 + b_3\beta^2\|\Phi\|_{L_2(R^2, d\lambda_0 \times d\lambda_0)} \\ &+ \sum_{\substack{x \in Z^{\nu+1} \\ x \ne 0}} \beta(b_3\beta)^{2|x|} e^{x^{(0)} \operatorname{Im} k}\|\Phi\|_{L_2(R^2, d\lambda_0 \times d\lambda_0)},\end{aligned}$$

and for $|\operatorname{Im} k| < \ln(h\beta)^{-1}$, $h > 0$, we have

$$\begin{aligned}\|(\underline{I}(p,k) - E)\widehat{F}\| \le{}& (b_3\beta)^2 + b_3\beta^2\|\Phi\|_{L_2(R^2, d\lambda_0 \times d\lambda_0)} \\ &+ \sum_{\substack{x \in Z^{\nu+1} \\ x \ne 0}} \beta\left(\frac{b_3\beta}{h}\right)^{|x_0|} (b_3\beta)^{2|x|}\|\Phi\|_{L_2(R^2, d\lambda_0 \times d\lambda_0)},\end{aligned}$$

where b_3 is the constant in Assertion 1.4. This last inequality easily implies the second inequality in (4) for $0 < \beta < hb_3^{-2}$. This proves Lemma 1.

We now take Fourier transforms in (1.4):

$$\underline{S}(p,k) = \underline{I}(p,k) + 2\left(\sum_{j=1}^{\nu} \cos p^{(j)} + \cos k\right)\underline{I}(p,k)\widehat{F}\underline{S}(p,k), \tag{5}$$

or

$$\left(E - 2\left(\sum_{j=1}^{\nu} \cos p^{(j)} + \cos k\right)\underline{I}(p,k)\widehat{F}\right)\underline{S}(p,k) = \underline{I}(p,k). \tag{5^a}$$

For any fixed $p \in T^\nu$, $\underline{S}(p,k)$ and $\underline{I}(p,k)$ are functions of k with values in the space of bounded linear operators on $L_2(R^1, d\lambda_0)$, where the first is defined and analytic on the strip $|\operatorname{Im} k| < \ln(b_2\beta)^{-1}$, and the second on the strip

$$|\operatorname{Im} k| < \ln(b_3\beta)^{-2} \tag{6}$$

(where b_2 is the constant in Assertion 2.3 and b_3 the constant in Assertion 1.4). In what follows, $p \in T^\nu$ will be fixed. Consider the operator

$$\underline{B}(p,k) = E - 2\Big(\sum_{j=1}^{\nu} \cos p^{(j)} + \cos k\Big)\underline{I}(p,k)\widehat{F}.$$

The analyticity of $\underline{I}(p,k)$ on the strip (6) implies that if the operator $\underline{B}(p,k)\colon \mathcal{H} \to \mathcal{H}$ is invertible for $k = k_0$, $|\operatorname{Im} k_0| < \ln(b_3\beta)^{-2}$, then by (5^a) the operator-valued function $\underline{S}(p,k)$ is analytic at the point $k = k_0$, which implies that the Green function

$$\widetilde{s}_{\varphi_1,\varphi_2}(p,k) = \sum_{x \in Z^{\nu+1}} e^{i(p,\overline{x})+ikx^{(0)}}(\varphi_1, S_x\varphi_2) = (\varphi_1, \underline{S}(p,k)\varphi_2)$$

is analytic at $k = k_0$, where $\varphi_1, \varphi_2 \in L_2(R^1, d\lambda_0)$ and $(\cdot,\cdot)$ is the inner product on $L_2(R^1, d\lambda_0)$. For any k in the strip (6) the spectrum of $\underline{B}(p,k)$ is discrete, as follows from the compactness of $\widehat{F}$. Consequently, $\underline{B}(p,k)$ is invertible if and only if its set of eigenvalues stays away from zero.

Let $0 \le \operatorname{Im} k < \ln(b_3\beta)^{-2}$ (all the arguments are analogous for $\operatorname{Im} k < 0$). We consider the operator

$$\underline{B}_0(k) = E - \beta e^{-ik} P_0 \widehat{\Phi} P_0$$

on $\mathcal{H}$.

LEMMA 2. *For any $h > 0$ there exist a β_3, $\beta_3 h < 1$, and a $\kappa_2 > 1$ such that for any β, $0 < \beta < \beta_3$, and any k in the strip $0 \le |\operatorname{Im} k| < \ln(h\beta)^{-1}$ we have*

$$\|\underline{B}(p,k) - \underline{B}_0(k)\| < \kappa_2\beta.$$

The proof follows easily from Lemma 1. Indeed,

$$\begin{aligned}\|\underline{B}(p,k) - \underline{B}_0(k)\| <{}& (2\nu + |e^{ik}|)\|\underline{I}(p,k)\widehat{F}\| + |e^{ik}|\,\|(\underline{I}(p,k) - E)\widehat{F}\| \\ &+ |e^{ik}|\,\|e^{-\beta\Phi} - 1 + \beta\Phi\|_{L_2(R^2, d\lambda_0 \times d\lambda_0)}.\end{aligned}$$

By (4), the first term here is less than $(2\nu+1)\kappa_1\beta$ and the second is less than $\kappa_1\beta/h$, while the third is bounded by

$$\frac{1}{h\beta}\left\|\Phi^2 \int_0^\beta d\beta' \int_0^{\beta'} d\beta'' e^{-\beta''\Phi}\right\|_{L_2(R^2, d\lambda_0 \times d\lambda_0)} \le \frac{\beta}{2h}\|\Phi^2\|_{L_2(R^2, d\lambda_0 \times d\lambda_0)},$$

by virtue of the condition $\Phi \ge 0$. This proves Lemma 2.

We now consider the eigenvalues

$$\mu_1, \dots, \mu_N, \dots, \mu_j \in R^1, \qquad j = 1, 2, \dots$$

of the operator $P_0\widehat{\Phi}P_0$, where $|\mu_1| \ge |\mu_2| \ge \cdots \ge |\mu_N| \ge 0$. By hypothesis $\mu_i \ne \mu_j$ for $i \ne j$, $i, j = 1, 2, \dots$. We set $h = (|\mu_N| + |\mu_{N+1}|)/2$.

LEMMA 3. *There exist $\beta_4 > 0$ and $\kappa_3 > 0$ such that for every β, $0 < \beta < \beta_4$, and any k in the strip $0 < \operatorname{Im} k < \ln(1/(h\beta))$, satisfying the condition*

$$\left| e^{-ik} - \frac{1}{\mu_i \beta} \right| > \frac{\kappa_3}{|\mu_N|}, \qquad j = 1, \dots, N, \tag{7}$$

the operator $\underline{B}(p,k)$ is invertible.

PROOF. The operator $\underline{B}_0(k)$ is invertible for every k in the strip $0 < \operatorname{Im} k < \ln(1/(h\beta))$ with the exception of the points k for which $|e^{ik} - \frac{1}{\mu_j \beta}| = 0$, $j = 1, \dots, N$. We have $\|\underline{B}_0(k))^{-1}\| = \max_{j=1,\dots,N} |1 - \beta e^{-ik}\mu_j|^{-1}$. Let $0 < \operatorname{Im}\kappa < \operatorname{Im}(1/(h\beta))$ and $1 - \beta e^{-ik}\mu_j \neq 0$, $j = 1, \dots, N$, and consider the representation

$$\underline{B}(p,k) = \underline{B}_0(k)[E + \underline{B}_0^{-1}(k)(\underline{B}(p,k) - \underline{B}_0(k))].$$

To prove Lemma 3 it is enough to show that

$$\|\underline{B}_0^{-1}(k)(\underline{B}(p,k) - \underline{B}_0(k))\| < 1, \tag{8}$$

if $\beta > 0$ is sufficiently small and

$$\left| e^{-ik} - \frac{1}{\mu_i \beta} \right| > \frac{\kappa_3}{|\mu_N|}, \qquad j = 1, \dots, N,$$

where κ_3 is some positive constant.

In view of Lemma 2, there exist $\beta_3 > 0$ and $\kappa_2 > 0$ such that for any $\beta > 0$, $\beta < \beta_3$ and any k in the strip $0 < \operatorname{Im} k < \ln(1/(h\beta))$ we have

$$\|\underline{B}(p,k) - \underline{B}_0(k)\| < \kappa_2 \beta. \tag{9}$$

Setting $\kappa_3 = 2\kappa_2$, we see that for every k in the strip $0 < \operatorname{Im} k < \ln(1/(h\beta))$ satisfying (7), we have

$$\|\underline{B}_0(k)\| < \left(\frac{\kappa_3}{|\mu_N|} \right)^{-1} \max_{j=1,\dots,N} (\beta|\mu_j|)^{-1} = \frac{1}{2\kappa_2 \beta}. \tag{10}$$

Estimate (8) now follows from (9) and (10), completing the proof of Lemma 3.

Let the constant $\kappa_3 > 0$ of Lemma 3 be fixed, and take $\kappa_2 = \kappa_3/2$. We now examine the case when

$$\left| e^{-ik} - \frac{1}{\mu_i \beta} \right| < \frac{\kappa_3}{|\mu_N|}$$

for some j, $1 \leq j \leq N$. Let $\psi_1, \dots, \psi_N$ be the eigenvectors of the operator $P_0 \widehat{F} P_0$ on $\mathcal{H}$ corresponding to the eigenvalues $\mu_1, \dots, \mu_N$. Let $\mathcal{H}_j^1$ be the subspace of $\mathcal{H}$ spanned by the vector ψ_j and $\mathcal{H}_j^2$ be the orthogonal complement of $\mathcal{H}_j^1$ in $\mathcal{H}$. Then on $\mathcal{H} = \mathcal{H}_j^1 \oplus \mathcal{H}_j^2$ we can write the operators $\underline{B}(p,k)$ and $\underline{B}_0(k)$ in block form

$$\begin{pmatrix} \underline{B}_j^{11}(p,k) & \underline{B}_j^{12}(p,k) \\ \underline{B}_j^{21}(p,k) & \underline{B}_j^{22}(p,k) \end{pmatrix}$$

and

$$\begin{pmatrix} \underline{B}_{0j}^{11}(k) & \underline{B}_{0j}^{12}(k) \\ \underline{B}_{0j}^{21}(k) & \underline{B}_{0j}^{22}(k) \end{pmatrix}$$

respectively, where

$$\underline{B}_j^{ml}(p,k)\colon \mathcal{H}_j^l \to \mathcal{H}j^m,$$
$$\underline{B}_{0j}^{ml}(k)\colon \mathcal{H}_j^l \to \mathcal{H}j^m, m,l \in \{1,2\}.$$

It is easily seen that $\underline{B}_{0j}^{12}(k) = 0$, $\underline{B}_{0j}^{21}(k) = 0$, and $\underline{B}_{0j}^{11}(k) = (1 - \mu_j\beta e^{-ik})E$. We show that for small enough $\beta > 0$, the operator $\underline{B}_j^{22}(p,k)$ is invertible. Indeed, for sufficiently small β we have

$$\|(\underline{B}_{0j}^{22}(p,k))^{-1}\| < \frac{1}{\kappa_3\beta}, \tag{11}$$

because for small $\beta > 0$ we have

$$\left|e^{-ik} - \frac{1}{\mu_l\beta}\right| > \frac{\kappa_3}{|\mu_N|}$$

for all $l \neq j$, $1 \leq l \leq N$. Moreover,

$$\|\underline{B}_j^{22}(p,k) - \underline{B}_{0j}^{22}(k)\| < \|\underline{B}(p,k) - \underline{B}_0(k)\| < \kappa_2\beta, \tag{12}$$

where $\kappa_2 = \kappa_3/2$ is the constant in Lemma 2. The invertibility of $\underline{B}_j^{22}(p,k)$ follows readily from (11) and (12), and the norm of the inverse satisfies

$$\|(\underline{B}_j^{22}(p,k))^{-1}\| < \frac{2}{\kappa_3\beta} \tag{13}$$

for any k such that $|e^{-ik} - 1/(\mu_j\beta)| > \kappa_3/|\mu_N|$. From the invertibility of $\underline{B}_j^{22}(p,k)$ we see easily that $\underline{B}(p,k)$ is invertible if and only if

$$\underline{B}_j^{11}(p,k) - \underline{B}_j^{12}(p,k)(\underline{B}_j^{22}(p,k))^{-1}\underline{B}_j^{21}(p,k) \neq 0.$$

LEMMA 4. *For sufficiently small $\beta > 0$, $0 < \beta < \beta_4$, where β_4 is a suitable positive constant, the equation*

$$\underline{B}_j^{11}(p,k) - \underline{B}_j^{12}(p,k)(\underline{B}_j^{22}(p,k))^{-1}\underline{B}_j^{21}(p,k) = 0 \tag{14}$$

has for every $j = 1,\dots,N$ and any fixed $p \in T^\nu$ a unique solution in the region

$$\left\{k \in \mathbb{C} : \left|e^{-ik} - \frac{1}{\mu_j\beta}\right| < \frac{\kappa_3}{|\mu_N|}, \quad \operatorname{Im} k \in \left(0, \ln\frac{1}{h\beta}\right), \quad \operatorname{Re} k \in [0, 2\pi]\right\}. \tag{15}$$

PROOF. Indeed, by Lemma 2

$$\|\underline{B}_j^{12}(p,k)\|, \|\underline{B}_j^{21}(p,k)\|, |\underline{B}_j^{11}(p,k) - \underline{B}_{0j}^{11}(k)| < \frac{1}{2}\kappa_3\beta$$

and hence by (13) we have

$$|\underline{B}_j^{11}(p,k) - \underline{B}_j^{12}(p,k)(\underline{B}_j^{22}(p,k))^{-1}\underline{B}_j^{21}(p,k) - \underline{B}_{0j}^{11}(k)| < \kappa_3\beta$$

for every k in the region (15) for $\beta > 0$ sufficiently small. Thus, for small $\beta > 0$ and all k such that $|e^{-ik} - 1/(\mu_j\beta)| = \kappa_3/|\mu_N|$, we have

$$|\underline{B}_{0j}^{11}(k)| > |\underline{B}_j^{11}(p,k) - \underline{B}_{0j}^{11}(k) - \underline{B}_j^{12}(p,k)(\underline{B}_j^{22}(p,k))^{-1}\underline{B}_j^{21}(p,k)|.$$

An application of Rouché's theorem [31] now gives Lemma 4.

Theorem 2.2 now follows easily from Lemmas 3 and 4, because the solution of equation (14) is a pole of the Green function

$$S_{\psi_j,\psi_j}(p,k) = \sum_{x=(\overline{x},x^{(0)})\in Z^{\nu+1}} e^{-(p,\overline{x})-ikx^{(0)}}\langle\psi_j(\xi(0)), \overline{\psi_j(\xi(x))}\rangle.$$

The proof of Theorem 1.2 will use the next lemma.

LEMMA 5. *Suppose that $\mu_k = 0$ for all $k > N$, $k \in Z_+$, and*

$$|\mu_1| \geq |\mu_2| \geq \cdots \geq |\mu_N| > 0.$$

Then there exist $\beta_5 > 0$ and q, $0 < q < 1$, such that for every $\beta > 0$, $\beta < \beta_5$ and any k satisfying

$$\ln\frac{2}{|\mu_N|\beta} < |\operatorname{Im} k| < \ln\left(\frac{q}{\beta b_3}\right)^2,$$

the operator $\underline{B}(p,k)$ is invertible (here b_3 is the constant in Assertion 1.4).

PROOF. We consider $\operatorname{Im} k > 0$, the case $\operatorname{Im} k < 0$ being analogous. It is enough to show that if $\beta > 0$ is small enough, then for any k satisfying

$$\ln\frac{2}{|\mu_N|\beta} < \operatorname{Im} k < \ln\frac{q^2}{(\beta b_3)^2},$$

where q is a positive constant such that $2b_3\beta/|\mu_N| \leq q^2 < 1$, we have the inequality

$$\|\beta e^{-ik}P_0\widehat{\Phi}P_0\| = |\mu_N\beta e^{-ik}| > 1 + \|B(p,k) - E + \beta e^{-ik}P_0\widehat{\Phi}P_0\|. \tag{16}$$

Indeed, suppose that $|e^{-ik}| = \left(\frac{q}{\beta b_3}\right)^2 > \frac{2}{|\mu_N|\beta}$, $0 < q < 1$. Then

$$\|(I(p,k) - E)\widehat{F}\| < \|(I_0' + I'')F\| + \sum_{\substack{x\in Z^{\nu+1}\\ x\neq 0}} \left(\frac{q}{\beta b_3}\right)^{2x^{(0)}} \|I_x\widehat{F}\|$$

and by Assertion 4.1,

$$\begin{aligned}
&(b_3\beta)^2 + b_3\beta^2 + \beta\sum_{\substack{x\in Z^{\nu+1}\\ x\neq 0}} \left(\frac{q}{\beta b_3}\right)^{2x^{(0)}} (b_3\beta)^{2|x|}\\
&\qquad \leq (b_3\beta)^2 + b_3\beta^2 + \frac{(1+(b_3\beta)^2)^\nu}{(1-(b_3\beta)^2)^\nu}\beta\sum_{x^0}\left(\frac{q}{\beta b_3}\right)^{2x^{(0)}}(b_3\beta)^{2|x^0|}\\
&\qquad\quad + \frac{2\nu(b_3\beta)^2}{1-(b_3\beta)^2}\frac{(1+(b_3\beta)^2)^{\nu-1}}{(1-(b_3\beta)^2)^{\nu-1}}\beta\sum_{x^0\in Z^1} q^{2x^{(0)}}(b_3\beta)^{2|x^0|-2x^0}.
\end{aligned}$$

We note that for small enough β,

$$\sum_{x^0} q^{2x^{(0)}} (b_3\beta)^{2(|x^0|-x^0)} < \frac{q^2}{1-q^2} + \frac{b_3^2\beta^3|\mu_N|}{1-b_3^2\beta^3|\mu_N|},$$

since $q^2 > b_3\beta/|\mu_N|$, and for $\beta > 0$ sufficiently small, we have

$$b_3^2\beta^3|\mu_N| < 1.$$

Thus when $|e^{-ik}| = \left(\frac{q}{\beta b_3}\right)^2 > \frac{2}{|\mu_N|\beta}$, $0 < q < 1$, we have

$$\|(I(p,k) - E)\widehat{F}\| < c_1\beta^2 + c_2q^2\beta, \tag{17}$$

where c_1 and c_2 are positive constants. Consequently, we have

$$\|\underline{I}(p,k)\widehat{F}\| < c_1\beta^2 + c_2q^2\beta + \beta\|\widehat{\Phi}\|. \tag{18}$$

For $|e^{-ik}| = \left(\frac{q}{\beta b_3}\right)^2 > \frac{2}{|\mu_N|\beta}$, $0 < q < 1$, we now estimate

$$\|B(p,k) - E - \beta e^{-ik}P_0\widehat{\Phi}P_0\| < (2\nu+1)\|\underline{I}(p,k)\widehat{F}\| + \frac{q^2}{\beta b_3^2}(\|(I(p,k)-E)\widehat{F}\| + \|e^{-\beta\Phi} - 1 + \beta\Phi\|_{L_2(R^2, d\lambda_0\times d\lambda_0)}).$$

For small enough $\beta > 0$, (18) shows that the first term here is bounded by const $\cdot\beta$; by (17), the second term is bounded by const $\cdot q^4\beta^{-1}$; and since $\Phi \geq 0$, the third term is $\leq$ const $\cdot\beta^2$. Thus, if $|e^{-ik}| = \left(\frac{q}{\beta b_3}\right)^2 > \frac{2}{|\mu_N|\beta}$, $0 < q < 1$ and $\beta > 0$ is small enough, we have

$$\|B(p,k) - E - \beta e^{-ik}P_0\widehat{\Phi}P_0\| < \text{const}\, q^4\beta^{-1}.$$

On the other hand,

$$\|\beta e^{-ik}P_0\widehat{\Phi}P_0\| = |\mu_N|\frac{q^2}{b_3^2\beta}$$

and consequently we can find a $q_0 > 0$, $\frac{2b_3^2}{|\mu_N|} \leq q_0 < 1$, such that for every q, $\frac{2b_3^2}{|\mu_N|} \leq q < q_0$, inequality (16) holds for small enough $\beta > 0$ when $|e^{ik}| = \left(\frac{q}{\beta b_3}\right)^2$.

It follows easily from this that the operator $\underline{B}(p,k)$ is invertible for every k in the strip

$$\ln\frac{2}{|\mu_N|\beta} \leq \operatorname{Im} k \leq \ln\left(\frac{q_0}{\beta b_3}\right)^2.$$

The case

$$-\ln\left(\frac{q_0}{\beta b_3}\right)^2 \leq \operatorname{Im} k \leq -\ln\frac{2}{|\mu_N|\beta}$$

is handled similarly. Lemma 5 is proved.

Theorem 1.2 now follows easily from Lemmas 3, 4, and 5.

GUIDE TO THE LITERATURE

The introductory Chapter 0 discusses well-known material, parts of which can be found in the books by Ruelle [37], Glimm and Jaffe [12], Bratteli and Robinson [7, 49], Reed and Simon [36], as well as in the review article by Dobrushin, Sinai, and Sukhov [14].

Part of Chapter 1 is purely expository (§§2 and 3, see the related material in the books by Dixmier [13] and Berezin [5]). Other parts are illustrative in character: §1, where we give a brief exposition of work by Sinai [39] and Dobrushin and Fritz [50] on the dynamics of a one-dimensional gas; §4, in which we construct a dynamics for a nonideal continuous Fermi gas by a method similar to that of Robinson [37]; and finally, §5, which is devoted to a simple example of a "linear dynamics" on the CAR and Weyl algebras (see [5]). Essentially new results appear in §§6 and 7. In §6 we construct a stochastic dynamics and its equilibrium states for a continuous-time Markov field. These constructions are inspired by ideas in "stochastic quantization" (see the review article by Migdal [29]); our treatment follows the work of Ignatyuk, Malyshev, and Sidoravicius [19]. Section 7, where we study in detail the dynamics for a special class of Markov fields with local interaction, follows work of Malyshev, Petrova, and Scacciatelli [56].

Chapter 2 is primarily expository and contains, with few exceptions, known material, suitably reworked for our purposes. However, the material in §3 is new (see the article by Botvich [47]), as is the material in §4 and also Part A of subsection 4 of §5 (see Kashapov and Malyshev [52]).

The foundations of a powerful method for studying the lower spectral branches for the Hamiltonians of lattice quantum field theory models (called the "Moscow method") are laid in Chapter 3. This method goes back to the work of Minlos and Sinai [31], which introduced the construction of multiplicative bases and, in embryonic form, the idea of a cluster operator. No proofs were given in [31]. A central idea in Chapter 3 is the concept of cluster operator introduced by Abdulla Zade, Minlos, and Pogosyan [1], which was subsequently investigated in detail by Malyshev and Minlos [27, 54, 55], Malyshev [25], Kashapov and Malyshev [52], and Zoląadek [17]. Many applications of this method to various specific models are presented in the review by Minlos [57], where a large bibliography is given (see also additional reference list). There exist interesting applications of this method to Markov chain with local interaction (Boldregini, Minlos, and Pellegrinotti, [47^b, 4^*], and to the computation of correlations of Gibbs fields (Minlos and Zhizhina [18, 31^*]).

Finally, Chapter 4 sets forth recent results regarding asymptotic completeness for a weakly perturbed Fermi gas (perturbed either by a weak self-interaction, or by an interaction with some foreign particle). These results should be viewed as an extension and rigorous justification of the work of Friedrichs [42] and Hepp [44] on this subject. The presented results are contained in a series of papers by Botvich, Malyshev, and Domnenkov. Our treatment follows works of Aizenshtadt, Botvich, and Malyshev [2],

Botvich, Malyshev [6], and also Botvich, Domnenkov, and Malyshev [48]. An extensive bibliography of work in this area is given in [2].

Chapter 5 is an introduction to an alternative technique for studying the spectrum of the transfer matrix, somewhat arbitrarily referred to as the Bethe-Salpeter method. This method is used primarily by Western scientists. In §1 we give a brief exposition of the method for the case of the Ising model and compare it with the results discussed in Chapter 3. The bulk of this chapter, in which the Dyson equation is used to analyze the one-particle spectrum of the transfer matrix for Gibbs fields with unbounded spin, is taken from the dissertation of Ignatyuk [19a]. In spite of its frequent use in journal articles, the Bethe-Salpeter method has been discussed hardly at all in the monograph literature. A brief review of the method can be found in [12], where a large bibliography is given.

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