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# PHASE TRANSITION IN A LOSS LOAD SHARING MODEL

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Février 1992

**Transition de phase dans un modèle de répartition de charge avec perte**

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**Résumé.**

Dans ce papier nous analysons le réseau avec perte suivant : quand un client arrive à un des nœuds du réseau, il est servi par ce nœud si celui ci est libre. Sinon le client est déplacé vers un nœud libre du réseau où il est servi avec un autre taux de service. Parmi les systèmes les plus simples de ce type, nous analysons les réseaux avec un grand nombre de nœuds et un partage global sur l'ensemble du réseau. Nous montrons un phénomène de transition de phase de deuxième espèce pour ce modèle et donnons les caractéristiques explicites de celui ci. Dans le cadre d'un partage local, nous analysons le cas d'un réseau infini et présentons des résultats de convergence. Nous terminons par des développements asymptotiques dans les cas des faibles et grandes charges du réseau.

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**Abstract.**

In this paper we analyze the following loss network: when a customer arrives at a node of the network, it is served by this node if this one is not occupied, otherwise it is transmitted to some empty node where it will be served with another rate. For simplest systems of such type with very large number of nodes with global sharing we show the existence of second order phase transitions and present explicit formulas for probability characteristics. For local sharing, we study the case of the infinite network and present some convergence results, formulas for small and large loads are obtained.

# Phase Transition in a Loss Load Sharing Model

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**Abstract**

In this paper we analyze the following loss network: when a customer arrives at a node of the network, it is served by this node if this one is not occupied, otherwise it is transmitted to some empty node where it will be served with another rate. For simplest systems of such type with very large number of nodes with global sharing we show the existence of second order phase transitions and present explicit formulas for probability characteristics. For local sharing, we study the case of the infinite network and present some convergence results, formulas for small and large loads are obtained.

## 1 Introduction

Consider some finite set $\Lambda$ of nodes of a (large) network such that for any $x \in \Lambda$ there is a Poisson arrival stream called $x$-stream. These streams are independent and have a constant rate $\lambda$. A customer of the $x$-stream arriving at time $t$ is not necessarily served on the $x$-station but can be assigned to an empty server (say $y$-server) chosen

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via some policy which is fixed once and for all. Let us denote by $\mu(x, y)$ a nonnegative real function defined on $\Lambda \times \Lambda$. The function $\mu(x, y)$ together with some rule $\mathcal{R}$ to assign the customers completely define the policy via the following rule:
A customer of the $x$-stream arriving at time $t$ is served on the $x$-station if it is empty at time $t$, if not then it can be put to some empty $y$-station or discarded depending of the rule $\mathcal{R}$ (we do not fix it now but assume that this rule $\mathcal{R}$ can depend only on the current state of the system). If all servers are busy then the customer is lost. A customer from the $x$-stream served on the $y$-station is called $\mu(x, y)$-type customer and it is served with the rate $\mu(x, y)$. Denote $q_\Lambda = q_\Lambda(\mu(x, y), \mathcal{R})$, the loss probability of an arriving customer at a given moment in the stationary regime and for the policy defined by $\mu(x, y)$ and by $\mathcal{R}$. So we choose $x$ at random and consider the probability that all $y$-stations with $\mu(x, y) \neq 0$ are occupied.

**Independent policy.**
A customer who finds its arrival queue busy is discarded and $\mu(x, x) = \mu$. This means that we have independent queues of $M|M|1|1$ type. Loss probability here is

$$q_\Lambda = \frac{\lambda}{\mu + \lambda}, \tag{1}$$

the simplest case of Erlang's formula. The obvious problem of practical importance is when it is profitable to use another policy.
**Local sharing - Processes with local interaction**.
If all $\Lambda$'s belong to some countable metric space $\mathcal{L}$ with the distance $\rho(x, y)$, local sharing means that a customer arriving at $x$, is lost if $\rho(x, y) > d$ for all empty site $y$. Then we have a process with local interaction in the standard sense. A customer can be served only in the vicinity of its arrival node.
**Global sharing.**
If all nodes are available for any stream then we say that sharing is global. In particular a customer is lost only if all the queues are busy. If $\mu(x, x) = \mu_1, \mu(x, y) = \mu_2 \leq \mu_1$ for $x \neq y, \mu_2 \neq 0$, we shall see that some important results here do not depend on the rule $\mathcal{R}$.

Let $\xi_t(x), x \in \Lambda, 0 \leq t < +\infty$, be the corresponding continuous time homogeneous Markov process with the state space $\mathcal{A}^\Lambda$ where $\mathcal{A}$ consists of 0 and of the set of values of $\mu(x, y)$. So $\xi_t(y) = 0$ if the node $y$ is empty at time $t$ and $\xi_t(y) = \mu(x, y)$ if at time $t$ there is still a $\mu(x,y)$-customer in the node $y$. We put

$$p_t^\Lambda(s_\Lambda) = P^\Lambda(\xi_t(x) = s_x, x \in \Lambda), \text{ with } s_\Lambda = (s_x)_{x\in\Lambda},$$

$s_x$ is equal to 0 or to one of $\mu(.,x)$, given some initial distribution $p_0^\Lambda(s_\Lambda)$. Let $\pi^\Lambda(s_\Lambda) = p_{+\infty}^\Lambda(s_\Lambda)$ be the corresponding stationary probabilities. For any $A \subset \Lambda$ we define the correlation functions $p_t^\Lambda(s_A) = p_t^\Lambda(A; s_A) = P^\Lambda(\xi_t(x) = s_x, x \in A)$, subscript $\Lambda$ means that we consider the process "in finite volume $\Lambda$".

# 2 Global sharing, $\mathcal{R}$-independent results

**Theorem 1** *Under global sharing and arbitrary rule $\mathcal{R}$, if $\mu_1 = \mu_2$, the following limit exists ($\rho = \frac{\lambda}{\mu_1}$)*

$$q(\rho) = \lim_{\Lambda\to+\infty} q_\Lambda(\rho) = \begin{cases} 1 - \frac{1}{\rho}, & \text{if } \mu_1 < \lambda \\ 0, & \text{if } \mu_1 \geq \lambda \end{cases} \tag{2}$$

Let $\eta^\Lambda$ be a (random) number of customers in the system at a given moment in the stationary regime. Then in probability

$$\frac{1}{|\Lambda|}\eta^\Lambda \to p(\frac{\lambda}{\mu_1}) \equiv \begin{cases} 1, & \text{if } \mu_1 < \lambda \\ \frac{\lambda}{\mu_1}, & \text{if } \mu_1 \geq \lambda \end{cases} \tag{3}$$

So $p(\rho)$ is the probability for a "randomly chosen" server to be busy. This means that the function $q(\rho)$ and $p(\rho)$, $\rho = \frac{\lambda}{\mu_1}$, are continuous but not differentiable in the critical point $\rho_{cr} = 1$. In other words we have a phase transition of the second kind. We get the similar picture for the case $\mu_1 \neq \mu_2$.

**Theorem 2** *Under global sharing and arbitrary rule $\mathcal{R}$,if $\mu_1 \neq \mu_2$,*

$$q(\mu_1) = \lim_{\Lambda\to+\infty} q_\Lambda(\mu_1) = \begin{cases} 0, & \mu_2 \geq \lambda, \\ 1 - \frac{\mu_2}{\lambda}, & \mu_2 \leq \lambda \end{cases} \tag{4}$$

Let $\eta_t$(resp. $\zeta_t$) be the number of $\mu_1$-customers (resp. $\mu_2$-customers) at time $t$ in $\Lambda$, $\eta^\Lambda$(resp. $\zeta^\Lambda$) are the corresponding random variables in the stationary state. Then in probability,

$$\frac{1}{|\Lambda|}\eta^\Lambda \to \begin{cases} \frac{\lambda}{\mu_1+\lambda+\frac{\lambda^2}{\mu_2-\lambda}}, & \text{if } \mu_2 > \lambda \\ 0, & \text{if } \mu_2 \leq \lambda \end{cases} \tag{5}$$

$$\frac{1}{|\Lambda|}\zeta^\Lambda \to \begin{cases} \frac{\lambda^2}{(\mu_2-\lambda)(\mu_1+\lambda)+\lambda^2}, & \text{if } \mu_2 > \lambda \\ 1, & \text{if } \mu_2 \leq \lambda \end{cases} \tag{6}$$

Optimisation rule: from (2) it follows that for $\mu_1 = \mu_2$ global sharing is always better than the independent policy. But for $\mu_2 < \mu_1$ it is the case iff

$$\mu_1 - \mu_2 - \frac{\mu_1\mu_2}{\lambda} < 0. \tag{7}$$

As a corollary we get the following result about all correlation functions in the case when the rule $\mathcal{R}$ is chosen in the specific way: if for an arriving customer its own station is occupied then it is put to any empty station with equal probability.

**Theorem 3** *Let at time* 0 *the distribution be invariant with respect to all permutations of* $\Lambda$. *Then for any fixed stations* $A = \{1, \ldots, s\}$ *the probability*

$$\lim_{|\Lambda|\to+\infty} p^\Lambda(s_A) \to p(s_A) = \prod_{x\in A} p(s_x),$$

**Proof**
Let us consider for example the case when $|A| = 2$ and $s_A = 0_A$. But any pair of empty stations is simultanously empty with equal probability and so the number of (ordered) pairs of empty stations divided by $|\Lambda|(|\Lambda|-1)$ tends to the square of the limit of the number of empty stations divided by $|\Lambda|$. ■

**Proof of Theorem 1**
Let $\eta_t^\Lambda$ be a (random) number of occupied buffers at time $t$. It is easy to see that $\eta_t^\Lambda$ is a Markov chain and it does not depend on the

rule $\mathcal{R}$. In fact it is the continuous time random walk on the set $\{1, ..., N\}, N = |\Lambda|$, with the transition rates

$$\lambda_{i,i+1} = N\lambda, \qquad i = 0, ..., N-1,$$
$$\lambda_{j,j+1} = j\mu_1, \qquad j = 1, ..., N.$$

Let $\pi_i = \pi_i^\Lambda$ be its stationary probabilities. These stationary probabilities are well known

$$\pi_j = \frac{\theta_j}{1 + \theta_1 + ... + \theta_N}, \theta_j = (\frac{\lambda}{\mu_1})^j \frac{N^j}{j!}. \tag{8}$$

It is straightforward to get (3) and also (2) (as $p(\mu_1) = \pi_N$ from (8)). ■

**Proof of Theorem 2**

It is surprisingly enough that Lyapounov functions approach can be used here for a problem which is completely different from the conventional type of stability problems with infinite buffers (see also [2] §4). It is easy to verify that the pair $(\eta_t, \zeta_t)$ is a Markov chain with the state space which is the set $\mathcal{A}_N$ of integer points of $\mathcal{S}_N = \{(k,l)/0 \leq k + l \leq N\} \subset \mathbb{R}^2$ and with transition rates $\lambda_{\alpha\beta}$ which do not depend on the rule $\mathcal{R}$. They are given by

$$\lambda_{\alpha\beta} = \begin{cases} \lambda(N-k-l), & \text{if } \beta = (k+1,l) \\ \lambda(k+l), & \text{if } \beta = (k,l+1) \\ \mu_1 k, & \text{if } \beta = (k-1,l) \\ \mu_2 l, & \text{if } \beta = (k,l-1). \end{cases},$$

if $\alpha = (k,l) \in \mathcal{S}_N$ and $\beta \in \mathcal{A}$. Let $\pi_N(k,l)$ be stationary probabilities of this Markov chain and define the mean jump vector

$$M = M(k,\lambda) = (\lambda(N-k-l) - \mu_1 k, \lambda(k+l) - \mu_2 l)$$

Let us consider the point $(k,l) \in \mathcal{S}_N$ where both components of $M$ change the sign, i.e.

$$\begin{aligned} \lambda(N-k-l) - \mu_1 k = 0 \\ \lambda(k+l) - \mu_2 l = 0. \end{aligned} \tag{9}$$

The system (9) has a unique solution which is given by

$$\begin{aligned} k_0 &= \frac{\lambda}{\mu_1+\lambda+\frac{\lambda^2}{\mu_2-\lambda}}N, \quad l_0 = \frac{\lambda}{\mu_2-\lambda}k_0, \quad \text{for } \mu_2 > \lambda, \\ k_0 &= 0, \qquad\qquad\qquad l_0 = N, \qquad\qquad \text{for } \mu_2 = \lambda. \end{aligned}$$

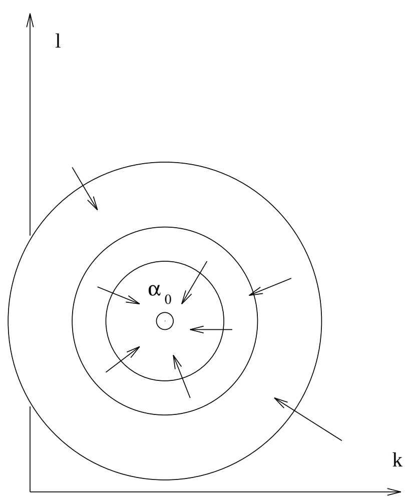
Notice that $(k,l)$ lies inside $\mathcal{S}_N$ iff $\mu_2 > \lambda$.
We want to prove that the stationary measure is concentrated in the vicinity of the point $\alpha_0 = (k_0, l_0)$, i.e. that for any $\varepsilon > 0$ and $N \to +\infty$,

$$\pi_N\left(|\frac{k_0}{N} - \frac{\eta^\Lambda}{N}| < \varepsilon, |\frac{l_0}{N} - \frac{\zeta^\Lambda}{N}| < \varepsilon\right) \to 1. \tag{10}$$

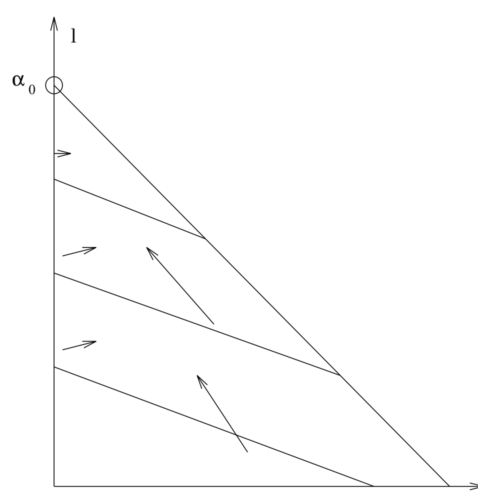
To prove this we shall use the results of [5] which are written for discrete time Markov chains. So we consider the embedded chain $y_n, n = 0, 1, \ldots$, with the same state space $\mathcal{A}_N$ and transition probabilities

$$\begin{aligned} p_{\alpha\beta} &= \lambda_{\alpha\beta}\tfrac{c}{N}, \alpha \neq \beta, \\ p_{\alpha\alpha} &= 1 - \textstyle\sum_{\beta\neq\alpha} p_{\alpha\beta}, \end{aligned}$$

for some $c > 0$ sufficiently small. So one step mean jump vector is equal to $M' = \frac{c}{N}M$ (the vector field of $M'(k,l)$ is shown on Fig. 1 ) and moreover the stationary probabilities for continuous and discrete time Markov chains are the same.




1-a




1-b

for the case $\mu_2 > \lambda$ we choose a "Lyapounov function"

$$f(\alpha) = \rho(\alpha, \alpha_0) \equiv \sqrt{(k-k_0)^2 + (l-l_0)^2}, \quad \alpha = (k,l), \alpha_0 = (k_0, l_0)$$

Define a sequence of random variables $S_n = f(y_n)$. It is easy to check that for any sufficiently small $\varepsilon > 0$ there exist constants $j = j(\varepsilon), \delta = \delta(\varepsilon) > 0$ such that for all sufficiently large $N$,

$$\sum_\beta p_{\alpha\beta}^{(j)} f(\beta) - f(\alpha) < -\delta \text{ for all } \alpha \text{ with } \rho(\alpha, \alpha_0) < \varepsilon N \qquad (11)$$

The proof of this statement proceeds in the following way. For the vector field $M'(k,l) = (M'_k(k,l), M'_l(k,l))$

$$\begin{aligned} \mathrm{sgn} M'_k(k,l) &= -\mathrm{sgn}(k-k_0), \\ \mathrm{sgn} M'_l(k,l) &= -\mathrm{sgn}(l-l_0), \end{aligned}$$

for any $(k,l) \in \mathcal{S}_N$.

For $N$ sufficiently large and $|\alpha - \alpha_0| > \varepsilon N$, let us take the tangent plane $S(N, \alpha, \varepsilon)$ to the sphere with the center $\alpha_0$ passing through $a$. Then the mean jump vector $M'(\alpha)$ looks inside this sphere and its projection onto the perpendicular to the sphere in a is bounded away from zero uniformly in $a$. Then equation (11) follows easily (see the almost linearity principle in [4]) and moreover we can take $j = 1$.
Put for the stationary measure $\pi = \pi_N$ on $\mathcal{A}_N$

$$\pi(A) = \sum_{i \in A} \pi_i.$$

Now we want to prove that there exist $C > 0$ and $\gamma > 0$ such that for any $\beta > 0$,

$$\pi(\{(k,l)/f(k,l) > \beta N\}) < Ce^{-\beta\gamma N}.$$

We need a simple lemma about martingales (see [5]). Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_n, n \geq 0\}$ an increasing family of $\sigma$-algebras: $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_\infty$. Let $\{S_i, i \geq 0\}$ be a sequence of real random variables, such that $S_i$ is $\mathcal{F}_i$-measurable for all $i$, $S_0 \geq 0$ is a constant.

**Lemma 4** *If there exist positive $\varepsilon, d$ such that for all $i$*

$$\begin{aligned} E(S_{i+1}|\mathcal{F}_i) &\leq S_i - \varepsilon \quad a.s., \\ |S_i - S_{i-1}| &\leq d \quad a.s., \end{aligned}$$

*then for any* $\delta_1 < \varepsilon$ *and for any* $n \geq 0$,

$$P(S_n > -\delta_1 n) < Ce^{-\delta_2 n},$$

*where for any sufficiently small* $h > 0$ *we have*

$$C = e^{-hS_0}, \;\; \delta_2 = h(\varepsilon - \delta_1) - \frac{3}{2}h^2 d^2.$$

We stress that $h$ and $\delta_2$ depend only on $\varepsilon, d, \delta_1$ but not from the other characteristics of the sequence $S_n$. We take sufficiently small $\beta > 0$ and put

$$\begin{array}{ll} B & = \{\alpha = (k,l)/\beta N \leq f(\alpha) \leq \beta N + 2\}, \\ C & = \{\alpha = (k,l)/2\beta N + 2 \leq f(\alpha)\}, \end{array}$$

then by lemma 4

$$P(\xi_n \in B, f(\xi_{n-1}), \ldots, f(\xi_1) > 2\beta N + 2 | \xi_0 \in B) < Ce^{-\gamma N}$$

for some constants $C, \gamma > 0$ which do not depend on $N$. By ergodic theorem,

$$\begin{aligned} \pi(C) \; & = \lim_{n \to +\infty} \tfrac{1}{n} \textstyle\sum_{k=0}^{n} 1_{\{\xi_k \in C\}} \\ & \leq \lim_{n \to +\infty} \tfrac{1}{n} \textstyle\sum_{k=0}^{n} \;\; m 1_{\{\xi_{m+k} \in B, f(\xi_{m-1+k}), \ldots, f(\xi_{k+1}) > \beta N + 2, \xi_k \in B, m+k \leq n, m \geq \beta N\}} \\ & \qquad\qquad + \textstyle\sum_{k=0}^{n} \;\; m 1_{\{f(\xi_{m+k}), f(\xi_{m-1+k}), \ldots, f(\xi_{k+1}) > \beta N + 2, \xi_k \in B, m+k = n, m \geq \beta N\}} \\ & \leq \pi(B) \textstyle\sum_{n=\varepsilon N}^{+\infty} \;\; n P(\xi_n \in B, f(\xi_{n-1}), \ldots, f(\xi_1) > 2\beta N + 2 | \xi_0 \in B) \\ & \leq \pi(B) \textstyle\sum_{n=\varepsilon N}^{+\infty} nCe^{-\gamma n} \leq e^{-\gamma' \beta N} \end{aligned} \tag{12}$$

The proof of (10) for the case $\mu_2 > \lambda$ is completed. For the case $\mu_1 > \mu_2 = \lambda$ or $\lambda > \mu_2 > 0$ we take as a Lyapounov function

$$f(k,l) = l + \rho k \tag{13}$$

and want to prove that

$$\pi_N(|\frac{\eta^\Lambda}{N}| < \varepsilon, |1 - \frac{\zeta^\Lambda}{N}| < \varepsilon) \to 1. \tag{14}$$

Up to a constant factor,

$$M'(k,l) = (\lambda(1 - \frac{k+l}{N}) - \mu_1 \frac{k}{N}, \lambda \frac{k}{N})$$

For $j=1$ and $\rho>0$ sufficiently small (such that $\lambda>\rho(\lambda+\mu_1)$) it is easy to show that equation (11) holds for all $(k,l)$ with $l<(1-\frac{\delta}{\lambda_\rho})N$. We have in fact,

$$\begin{aligned}\rho M_k'(k,l)+M_l'(k,l) &= \rho(\lambda(1-\tfrac{k+l}{N})-\mu_1\tfrac{k}{N})+\lambda\tfrac{k}{N} \\ &= \tfrac{k}{N}(\lambda-\rho(\lambda+\mu_1))+\lambda\rho\tfrac{N-l}{N}\end{aligned}$$

which gives the result. If $\lambda>\mu_2>0$ then

$$\begin{aligned}M_l' &= (\lambda-\mu_2)\tfrac{l}{N}+\lambda\tfrac{k}{N}>0,\\ M_k' &= \lambda(1-\tfrac{k}{N}-\tfrac{l}{N})-\mu_1\tfrac{k}{N}\end{aligned}$$

in $\mathcal{S}_N$. We take the same Lyapounov function (13) and (14) follows in a similar fashion. So the assertion (5) of theorem 2 is proved. Zero case of (4) easily follows. It is more complicated to prove (4) for the case $\mu_2\leq\lambda$. ■

**Remark 1**

It is important to understand that this could not be obtained even if we had complete information about the equilibrium measure after the thermodynamic limit. For the system with infinite $\Lambda$ it follows from (5) that in case $\mu_2\leq\lambda$ customer is lost with probability 1 but the stationary probability of loss is converging to $1-\frac{\mu_2}{\lambda}$. We give a heuristic proof of this fact without entering the details.
Choose $\varepsilon$ sufficiently small and put for our Markov chain

$$\mu_t=N-\eta_t^\Lambda-\zeta_t^\Lambda,$$

We can find $M(\varepsilon)$ so that $M(\varepsilon)\to+\infty$ if $\varepsilon\to0$ and

$$\begin{aligned}\left|P(\mu_{t+1}=\mu_t+1/\mu_t,\ldots,\mu_0;\mu_t<M(\varepsilon),|\tfrac{\eta^\Lambda}{N}|<\varepsilon,|1-\tfrac{\zeta^\Lambda}{N}|<\varepsilon)-\mu_2\right|<K\varepsilon,\\ \left|P(\mu_{t+1}=\mu_t-1/\mu_t,\ldots,\mu_0;\mu_t<M(\varepsilon),|\tfrac{\eta^\Lambda}{N}|<\varepsilon,|1-\tfrac{\zeta^\Lambda}{N}|<\varepsilon)-\lambda\right|<K\varepsilon\end{aligned}\quad(15)$$

Most of the time is spent in the domain

$$|\frac{\eta^\Lambda}{N}|<\varepsilon,|1-\frac{\zeta^\Lambda}{N}|<\varepsilon.$$

Moreover for $\varepsilon\to0$, (15) becomes a simple birth-death process on $\mathbb{Z}_+$ with transition probabilities $p_{\mu_t,\mu_t+1}=\mu_2$, $p_{\mu_t,\mu_t-1}=\lambda$. This gives the answer.

# 3 Load sharing on an infinite network

In this section we consider the one-dimensional network $\mathbb{Z}$ with the following rule to assign the customers: If the arrival site of a customer is occupied, then it is moved to the first empty place on its right. If the customer has not been moved [resp. has been moved], it is served with rate $\mu_1$ [resp. $\mu_2$] (we set $\mu_0 = 0$ for convenience) and the state of the site is 1 [resp. 2]. Because of the non local interaction, an additional problem in this case is the construction of the associated Markov process with generator $\Omega$:

$$\begin{aligned}\Omega(f)(\underline{x}) = \textstyle\sum_{i\in\mathbb{Z}} \quad & \mu_{x_i}(f(\underline{x}) - f(\underline{x} - x_i\delta_i)) \\ & +\lambda(f(\underline{x} + \delta_i) - f(\underline{x}))1\{x_i = 0\} \\ & +\lambda(f(\underline{x} + 2\delta_{\nu_i(\underline{x})}) - f(\underline{x}))1\{x_i \neq 0\}, \quad \underline{x} \in \{0,1,2\}^{\mathbb{Z}}.\end{aligned}$$

where $f$ is some function belonging to the domain of $\Omega$ (see Liggett [3]), $\delta_i$, the Dirac function at $i$ and $\nu_i(\underline{x}) = \inf\{j > i/x_j = 0\}$. Let us denote by $\mathcal{E}$, the set of translation invariant and ergodic measures $\phi$ on $\{0,1,2\}^{\mathbb{Z}}$ such that $\phi\{\underline{x}/x_0 = 0\} > 0$. In the following, we will consider Markov processes $\underline{X}(t)$ with generator $\Omega$ and initial distribution in $\mathcal{E}$.

**Proposition 5** *For any $\phi \in \mathcal{E}$, there exists some $h > 0$ and a stochastic process $(\underline{X}(t))_{0\leq t\leq h}$ such that*

- *The distribution of $\underline{X}(0)$ is $\phi$.*
- *$\underline{X}(t)$ is a Markov process on $[0,h]$ with generator $\Omega$.*

The method to construct our process is quite simple. We will show that the real line can be cut into non interacting (random) intervals between 0 and some $h > 0$. For $i \in \mathbb{Z}$, we denote by $N_i$ the Poisson process $(\lambda)$ of the arrivals at site $i$ and

$$\nu(k,h) = \inf\{i \geq 0 / \sum_{j=k}^{i+k} N_j[0,h] - 1_{\{x_j\neq 0\}} \leq 0\},$$

where $\underline{X}_0 = (x_i)_{i\in\mathbb{Z}}$ is some random variable with distribution $\phi$ and $N_j[0,h]$ denotes the number of points of $N_j$ between 0 and $h$.

**Lemma 6**

*a) If $\lambda h < \phi\{\underline{x}/x_0 = 0\} > 0$, then $\nu(k,h)$ is finite a.s. for $k \in \mathbb{Z}$.*

*b)* $H(i) = \sum_{-\infty}^{j=i-1} 1_{\{j+\nu(j,h)\geq i\}}$ *is almost surely finite and* $P(H(i) = 0) > 0$.

**Proof**
According to the ergodic theorem,

$$\frac{1}{i}\sum_{j=k}^{i+k} N_j[0,h] - 1_{\{x_j\neq 0\}} \overset{i\to+\infty}{\longrightarrow} \lambda h - \phi\{\underline{x}/x_0 = 0\} < 0 \text{ a.s.},$$

thus $\nu(k,h)$ is finite a.s. Using that $\{\sum_k^i N_j[0,h] - 1_{\{x_j\neq 0\}} \leq 0\} \subset \{k+\nu(k,h) \leq i\}$, we get that $H(i)$ is also a.s. finite. The maximal lemma of ergodic theory (see [1] appendix 3) gives

$$P\left(\sup_{k<0}\left\{\sum_{j=k}^{-1} N_j[0,h] - 1_{\{x_j\neq 0\}}\right\} \leq 0\right) > 0,$$

hence $P(H(i) = 0) = P(H(0) = 0) > 0$. The lemma is proved. ∎

The statement b) of the above lemma has the following consequence: Even if the customers never leave the network, there is a positive probability that none of the customers arriving during $[0,h]$ on the sites with negative index is assigned to a site with a positive index. This implies that on this event during $[0,h]$, one can discard all the sites on the left of 0. But if we discard the sites $i < 0$, the process is trivial to construct the sample paths of $(x_0(t),\ldots,x_n(t))_t,\ldots$ on $[0,h]$. Using that, almost surely, there is a non-decreasing sequence $(t_i)_{i\in\mathbb{Z}}$ such that $H(t_i) = 0$, we can thus construct our process on the interval $[0,h]$. Theorem 5 is proved. ∎

**Remark 2**
Because of our construction, it is easy to prove that for $0 \leq t \leq h$ the distribution of $\underline{X}(t)$ is in $\mathcal{E}$ (by verifying that the ergodic theorem holds, for example).

So, our process can be constructed as long as $P(x_0(t) = 0) > 0$. We will denote by † the (cemetery) state of $\underline{X}(t)$ whenever this condition is not satisfied and $\tau$ the (deterministic) hitting time of †. We now turn to the asymptotic behavior of our process.

**Theorem 7**

*a) If $\mu_2 < \lambda$, the process almost surely dies, i.e. for any $\phi \in \mathcal{E}$ there exists some $t_0$ such that $\underline{X}(t_0) = \dagger$.*

*b) If $\lambda < \min\{\mu_1, \mu_2\}$, then the Markov process $\underline{X}(t)_{t\geq 0}$ is almost surely defined.*

*c) If $\mu_1 < \lambda < \mu_2$, then if $\phi\{\underline{x}/x(1) \neq 0\}$ is sufficiently small, $\underline{X}(t)_{t\geq 0}$ is almost surely defined, otherwise it dies.*

This result is a consequence of the following lemma which is related in some sense to the fact that $(\eta_t, \xi_t)_t$ is a Markov process (section 2).

**Lemma 8** *If $P_i(t) = P(x_0(t) = i)$ for $i \in \{0,1,2\}$ and $t < \tau$, then*

$$\begin{aligned} \frac{\partial P_1(t)}{\partial t} &= \lambda \quad -\lambda P_2(t) - (\lambda + \mu_1) P_1(t) \\ \frac{\partial P_2(t)}{\partial t} &= \qquad (\lambda - \mu_2) P_2(t) + \lambda P_1(t) \end{aligned}$$

**Proof**

Using the definition of the generator $\Omega$, we have

$$P_2(t+dt) = (1-\mu_2 dt)P_2(t) + \lambda dt \sum_{i\leq -1} P(x_i(t) \neq 0, \ldots, x_{-1}(t) \neq 0, x_0(t) = 0) + o(dt),$$

the invariance by translation gives

$$\frac{\partial P_2(t)}{\partial t} = -\mu_2 P_2(t) + \lambda \sum_{i\geq 1} P(x_0(t) \neq 0, \ldots, x_{i-1}(t) \neq 0, x_i(t) = 0).$$

The sequence $(x_i(t))_{i\geq 1}$ hits 0 with probability 1, hence,

$$\frac{\partial P_2(t)}{\partial t} = -\mu_2 P_2(t) + \lambda P(x_0(t) \neq 0).$$

Replacing $P(x_0(t) \neq 0)$ by $P_1(t) + P_2(t)$ in the above equation gives the second equation of our lemma. The other equation is proved in the same way. ∎

**Proof of Theorem 7**
The limiting values of the solutions of the differential equation of Lemma 8 are given by

$$P_1 = \frac{\lambda}{\mu_1 + \lambda + \frac{\lambda^2}{\mu_2 - \lambda}}, \qquad P_2 = \frac{\lambda^2}{(\mu_2 - \lambda)(\mu_1 + \lambda) + \lambda^2} \tag{16}$$

a) If $\mu_2 < \lambda$, then it is easily checked that $P_1 + P_2 > 1$, hence there exists some $t$ for which $P_0(t) = 0$, $\tau$ cannot be infinite in this case.

b) Using Lemma 8, we get that

$$\frac{\partial (P_1 + P_2)(t)}{\partial t} = \lambda - \mu_1 P_1(t) - \mu_2 P_2(t).$$

A finite extremum of $(P_1 + P_2)(t)$ (if any) satisfies $\lambda - \mu_1 P_1(t_0) - \mu_2 P_2(t_0) = 0$, hence

$$(P_1 + P_2)(t_0) \leq \frac{\lambda}{\inf\{\mu_1, \mu_2\}} < 1.$$

At infinity, it is easy to check that $P_1 + P_2 < 1$. Our process is thus always alive.

c): Trite calculations with the explicit solution of the equation of Lemma 8. ■

In the case where $\tau = +\infty$, there exists at least one invariant measure. Our next proposition shows that there is a unique one:

**Proposition 9** *If $\mu_1 = \mu_2 = \mu$, then for any $\phi \in \mathcal{E}$, $(\underline{X}(t))_t$ is converging in distribution to a unique invariant measure.*

**Proof**
In this case the state space can be reduced to $\{0,1\}^{\mathbb{Z}}$. If $(\underline{X}^0(t))_t$ [resp. $(\underline{X}(t))_t$] denotes the Markov process with the empty network [resp. $\phi$] as starting point. Using our construction of Theorem 5, it is easy to couple the two processes so that $\underline{X}^0(t) \ll \underline{X}(t)$, i.e. $x_i^0(t) \leq x_i(t)$ for all $i \in \mathbb{Z}$. In particular $(\underline{X}^0(t))_t$ is stochastically non-decreasing, hence is converging in distribution to some measure $\pi$. Any limiting distribution $\psi$ of $(\underline{X}(t))$ satisfies $\pi \ll_{st} \psi$ (where $\ll_{st}$ is the stochastic order on measures associated with $\ll$), but according to Lemma 7,

$\pi(x_0 = i) = \psi(x_0 = i)$ for $i = 0, 1$ which implies $\pi = \psi$ (see [3] § 2, for example). ■

**Remark 3**
In the constant case $\mu_1 = \mu_2 = \mu$ when $\lambda = \mu_1$, it is easy to show that $\tau = +\infty$. The process $(\underline{X}(t))_t$ is completely defined but its limiting distribution is the Dirac mass at †.

# 4 Local sharing

We consider now the model on the one-dimensional lattice with $\Lambda = [-L, L] \subset \mathbb{Z}^1$. We put $\mu(x, y) = \mu(y - x)$. We assume that a customer is lost if it cannot find an empty site within a distance $d$ of its arrival point. For simplicity of notations we assume that

$$\mu(0) = \mu_1, \mu(1) = \ldots = \mu(d) = \mu_2 \text{ for some } d > 0 \text{ and } \mu(i) = 0 \text{ otherwise.} \tag{17}$$

From (7) it is seen that important parameters for understanding the qualitative behavior of the loss probability are $\frac{\mu_2}{\mu_1}, \lambda$, and we add the parameter $d$. The following theorem is standard, it gives a satisfactory description of the process in the thermodynamic limit.

**Theorem 10** *Under the assumptions (17) there exists $\lambda_0 > 0$ such that for any $\lambda < \lambda_0$ the following limits exist*

$$p(s_A) = \lim_{L \to +\infty} \lim_{t \to +\infty} p_t^\Lambda(s_A) = \lim_{t \to +\infty} \lim_{L \to +\infty} p_t^\Lambda(s_A) \tag{18}$$

*$p(s_A)$ are analytic in $\lambda$ (including $\lambda = 0$) and define a translation invariant measure on $\mathcal{A}^{\mathbb{Z}^1}$ with exponential decay of correlations, e.g.*

$$|p(s_x, s_y) - p(s_x)p(s_y)| \leq Ce^{-\alpha|x-y|}$$

*for some $C, \alpha > 0$.*

This is known from the cluster expansion theory for the processes with local interaction (see [6]) and references therein). Now we proceed to the calculation of the loss probability. Contrary to Remark 1, here the loss probability is defined by the thermodynamical limit of the finite volume distributions. Moreover it is equal to

$$q_\Lambda(x) = p^\Lambda(\xi(y) \neq \emptyset \text{ for all } y \in [x, x + d]) \tag{19}$$

To provide explicit calculations we use here the methods of mathematical physics: the correlation equations for correlation functions. The reader is not supposed to be acquainted with this techniques so we give a detailed exposition. Let $h(s_\Lambda, s'_\Lambda), s_\Lambda \neq s'_\Lambda$, be transition rates from $s'_\Lambda$ to $s_\Lambda$ for our Markov chain in $\Lambda$ so that (we omit for a while the upper index $\Lambda$ having in mind that we consider Markov chain $\mathcal{A}^\Lambda$)

$$h(s_\Lambda, s'_\Lambda)dt = P(x_{t+dt}(x) = s_x, x \in \Lambda | \xi_t(x) = s'_x, x \in \Lambda) \qquad (20)$$

Then the Kolmogorov's equations are

$$p_{t+dt}(s_\Lambda) = p_t(s_\Lambda) + \sum_{s'_\Lambda} h(s_\Lambda, s'_\Lambda) p_t(s'_\Lambda) dt$$

where

$$h(s_\Lambda, s_\Lambda) = -\sum_{s'_\Lambda, s'_\Lambda \neq s_\Lambda} h(s_\Lambda, s'_\Lambda)$$

For our model only one node can change at a time, i.e. for $s_\Lambda \neq s'_\Lambda$ the rates $h(s_\Lambda, s'_\Lambda)$ can be different from 0 only if $s_x \neq s'_x$ exactly at one point $x \in \Lambda$. One can put so

$$h(s_\Lambda, s'_\Lambda) = h^s_x(s'_\Lambda)$$

when $s_\Lambda$ is different from $s'_\Lambda$ only in the point $x$ and $s_x = s$. From (20) one gets using Kronecker symbols

$$P(\xi_{t+dt}(x) = s | \vec{\xi}_t) = (1 - \delta_s(\xi_t(x)) dt h^s_x(\vec{\xi}_t) + \delta_s(\xi_t(x))(1 - dt \sum_{s' : s' \neq s} h^s_x(\vec{\xi}_t)))$$

Taking expectations we get

$$p_{t+dt}(x; s) = \sum_{s_\Lambda : s_x \neq s} h^s_x(s_\Lambda) p_t(\Lambda; s_\Lambda) dt + p_t(x; s) - dt \sum_{s_\Lambda : s_x = s} \sum_{s' \neq s} h^{s'}_x(s_\Lambda) p_t(\Lambda; s_\Lambda)$$

or for the stationary measure

$$0 = \sum_{s_\Lambda : s_x \neq s} h^s_x(s_\Lambda) p(\Lambda; s_\Lambda) - \sum_{s_\Lambda : s_x = s} \sum_{s' \neq s} h^{s'}_x(s_\Lambda) p(\Lambda; s_\Lambda) \qquad (21)$$

Now we write $\xi(x) = 0$ instead of $\xi(x) = \mu_1, \xi(x) = 1$ instead of $\xi(x) = \mu_2, \xi(x) = \emptyset$ if $x$-station is empty, $\xi(x) = *$ if it is not empty. Put also

$$p(A) = p(\xi(x) = *, x \in A)$$

Then (21) becomes (if we put $s = 0$ and $s = 1$)

$$0 = \lambda p(x; \emptyset) - \mu_1 p(x; 0)$$

$$0 = \lambda \sum_{k=0}^{d-1} p([x-k-1, x]; \underbrace{* \ldots *}_{k} \emptyset) - \mu_2 p(x; 1) =$$

$$\lambda \sum_{k=0}^{d-1} p([x-k-1, x-1]; \underbrace{* \ldots *}_{k}) - \lambda p([x-k-1, x]; \underbrace{* \ldots *}_{k}) - \mu_2 p(x; 1). \tag{22}$$

**Remark 4**
If in (22) one puts $\Lambda = \mathbb{Z}^1, d = +\infty$, and considers translation invariant measure then a closed equations for one-point correlation functions result. Solving them ne can get the same formulas as in the theorem 1.
To get equations for higher order correlation functions we consider

$$P(\xi_t + dt(x) = s_x, x \in A/\vec{\xi_t}) =$$

$$\sum_{x \in A} \left(1 - \delta_{s_x}(\xi_t(x))\right) \left(\prod_{y: y \neq x, y \in A} \delta_{s_y}(\xi_t(y))\right) h_x^{s_x}(\vec{\xi_t}) +$$

$$\left(\prod_{x \in A} \delta_{s_x}(\xi_t(x))\right) \left(1 - dt \sum_{x \in A} \sum_{s'_x \neq s_x} h_x^{s'_x}(\vec{\xi_t})\right).$$

or for $A \cap B = \emptyset$

$$\frac{\partial p_t(1_A 0_B)}{\partial t} = \lambda \sum_{x \in B} p_t(1_A 0_{B-\{x\}} \emptyset_x) +$$

$$\lambda \sum_{x \in A} \sum_{k=0}^{+\infty} p_t \left(1_{A-\{x\}} 0_B \emptyset_x *_{[x-k-1, x-1]}\right) - (\mu_2 |A| + \mu_1 |B|) p_t(1_A O_B)$$

or for the stationary measure

$$p(1_A 0_B)(\mu_2 |A| + \mu_1 |B|) =$$

$$\lambda \sum_{x \in B} (p(1_A 0_{B-\{x\}}) - p(1_A 0_{B-\{x\}} *_x)) +$$

$$\lambda \sum_{x \in A} \sum_{k=0}^{+\infty} p(1_{A-\{x\}} 0_B *_{[x-k-1, x-1]}) - p(1_{A-\{x\}} 0_B *_{[x-k-1, x]}) \tag{23}$$

Coming to optimization problem we note first that for $\mu_1 = \mu_2$ the independent policy is always the worse: if there is an empty available node a customer should be put into it. For $\mu_1 > \mu_2$ it is not always the case.

**Theorem 11** *If $d = 1, \lambda$ is small enough and $\mu_1, \mu_2$ fixed then the loss probability (in the thermodynamic limit) is*

$$q = (\frac{\lambda}{\lambda + \mu_1})^2 + O(\lambda^3)$$

*Now let us assume*

$$\mu_1 = o(\lambda), \mu_2 = o(\mu_1) \tag{24}$$

*Then*

$$q = 1 - 2\frac{\mu_2}{\lambda} + O(\frac{\mu_2}{\lambda}\frac{\mu_2}{\mu_1})$$

**Proof**

We use analyticity result of theorem 10 and calculate the first terms of the expansion in $\lambda$ for $d = 1$ and small $\lambda$ with $\mu_1, \mu_2 > 0$ fixed. Using translation invariance of the limiting measure we get from the second formula (22)

$$p(1) = p(1_x) = O(\lambda)$$

and from the first one

$$p(0) = \frac{\lambda}{\lambda + \mu_1}(1 - p(1)) = \frac{\lambda}{\mu_1} + O(\lambda^2)$$

From (23) we get

$$p(00)2\mu_1 = \lambda(p(0\emptyset) + p(\emptyset 0)) = \lambda(2p(0) - p(0*) - p(*0)^{\circ}) = \lambda(2p(0) - 2p(00) - p(01) - p(10))$$

or

$$p(00) = \frac{\lambda}{\lambda + \mu_1}p(0) - \frac{\lambda}{2(\lambda + \mu_1)}(p(01) + p(10)) = (\frac{\lambda}{\lambda + \mu_1})^2 + O(\lambda^3)$$

Similarly one can show that $p(11), p(01), p(10)$ are all $O(\lambda^3)$ and so

$$q = p(**) = (\frac{\lambda}{\lambda + \mu_1})^2 + O(\lambda^3)$$

Now we pass the case (24). Let us note that equations (24) have been written for correlation functions $p(1_A 0_B)$ as these are small quantities, more exactly $p(1_A 0_B) = O(\lambda|A \cup B|)$. Now the correlation equations should be written for the correlation functions $p(\emptyset_A 0_B)$. ■

**Theorem 12** *Under the assumptions (17) there exists* $\varepsilon_0 > 0$ *such that for* $\frac{\mu_2}{\mu_1}, \frac{\mu_1}{\lambda} < \varepsilon_0$ *the following limits exist*

$$p(s_A) = \lim_{L\to+\infty} \lim_{t\to+\infty} p_t^{\Lambda}(s_A) = \lim_{t\to+\infty} \lim_{L\to+\infty} p_t^{\Lambda}(s_A) \qquad (25)$$

$p(s_A)$ *are analytic in* $\frac{\mu_2}{\mu_1}, \frac{\mu_1}{\lambda}$ *and define a translation invariant measure on* $\mathcal{A}^{\mathbb{Z}^1}$ *with exponential decay of correlations. Moreover* $p(\emptyset_A 0_B) = O(\varepsilon^{|A\cup B|})$, $\varepsilon = \max(\frac{\mu_1}{\lambda}, \frac{\mu_2}{\mu_1})$.

We leave rewriting the equations and the proof to the reader. To end the proof of the theorem 11 for the case (24) we can write

$$q = p(**) = 1 - 2p(\emptyset) + p(\emptyset\emptyset)$$

From (22) we get

$$\begin{aligned} \lambda p(\emptyset) &= \mu_1 p(0) \\ \mu_2(1 - p(\emptyset) - \frac{\lambda}{\mu_1} p(\emptyset)) &= \mu_2(1 - p(\emptyset) - p(0)) = \mu_2 p(1) = \lambda p(*\emptyset) = \lambda(p(\emptyset) - p(\emptyset\emptyset)) \end{aligned}$$

so

$$p(\emptyset) = \frac{\frac{\mu_2}{\lambda} + p(\emptyset\emptyset)}{\frac{\mu_2}{\lambda} + 1 + \frac{\mu_2}{\mu_1}} = \frac{\mu_2}{\lambda} + O(\frac{\mu_2}{\lambda}\frac{\mu_2}{\mu_1})$$

We have proved that local sharing policy is better than the independent one for $\lambda$ small but for the case (24) local sharing is better only if

$$\mu_2 > \frac{\mu_1}{2}$$

# 5 Some open problems

Load sharing models promises to be rich in different phenomena which could be common for more complicated models and we want to indicate some further problems :

**Problem 1**. In the model of the section 4 take $d \to +\infty$ or even equal to infinity, take even $\mu_1 = \mu_2$. The resulting process is not a process with local interaction. Does the series defining the solution of correlation equations converge for $\frac{\lambda}{\mu_1}$ small ? Is the solution analytic in $\frac{\lambda}{\mu_1}$ ?

**Problem 2.** In the model on the one-dimensional lattice with $\Lambda = [-L, L] \subset \mathbb{Z}^1, \mu(x,y) = \mu_1$ for $|x-y| \leq d, \mu(x,y) = \mu_2$ for $|x-y| > d$. Is the invariant measure unique for all values of the parameters $\mu_1, \mu_2, \lambda, d$ ? Find the phase diagram for the loss probability.
**Problem 3.** Let, on $\mathbb{Z}^1$ $\mu(x,y) \sim \frac{\mu_1}{1+|x-y|^\alpha}$. For which $\alpha$ it is true that this policy is better than the independent one with the parameter $\mu_1$ ?
**Problem 4.** What about higher order correlation functions in the cases of section 2 for a one-dimensional lattice when the rule $\mathcal{R}$ is just choosing the closest node.
**Problem 5.** Try similar problems for waiting time and infinite buffers.

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