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Examples of Asymptotic Completeness in Translation Invariant Systems with an Unbounded Number of Particles

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Abstract. A tagged particle interacting with a finite but unbounded number of particles is considered. Some examples of asymptotic completeness are given which are uniform in the number of particles.

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0. Introduction

This report concerns the construction of some missing examples in asymptotically complete scattering theory for a class of self-adjoint operators in Hilbert space

$$\mathscr{H} = F_a(H_1) \otimes L_2(R^\nu),$$

where F_a is the antisymmetric Fock space over the one-particle space H_1, for which we choose either (a) $H_1 = L_2(R^\nu)$, or (b) $H_1 = L_2(R^\nu) \oplus L_2(R^\nu)$.

The Hilbert space F_a corresponds to a system consisting of an unbounded number of free fermions, $L_2(R^\nu)$ is the set space of the tagged particle.

Case (b) corresponds to the presence of two types of particles: 'particles' with spectrum $[0, \infty)$ and 'antiparticles' with spectrum $(-\infty, 0]$.

In quantum mechanics, statistical physics, and quantum field theory, the question of asymptotic completeness (including the existence problem of singular spectra) is the final structural question of the theory. This question has been studied sufficiently well for one or two particle systems. L. D. Faddeev's famous work [1] and its subsequent extensions, settled the question for three particle systems in its main aspects. Many technical problems of n-particle systems nevertheless are not well understood, despite some beautiful results [3].

The essential new difficulty (uniform bounds on the number of particles) was the reason that, until now, there has been no example (except in the trivial noninteracting case) of any proof of the asymptotic completeness for systems with an unbounded number of particles.

The first such example [4], and its generalizations [6–8, 13], gave the solutions of some classical problems of Friedrichs [11] and Hepp [10] for the case when interaction takes place only in a bounded region of space outside of which the system is free. The first translation invariant example was announced in [5]; the proof is given below in Theorem 2.

Our method involves proving the convergence of perturbation series for wave operators. This series can converge only if the following conditions are satisfied:

(1) ε is small;

(2) dimension ν satisfies $\nu \geqslant 3$, since for $\nu = 1, 2$ bound states can appear;

(3) each Wick monomial contains at least one annihilation operator, since in the contrary case, mass renormalization for the tagged particle must be performed (there is no vacuum in the sector under consideration).

Assuming these conditions, we prove asymptotic completeness in the case

(a_1) $H_1 = L_2(R^\nu)$, $\quad h_1 = -\Delta + \mu$, $h_2 = -\Delta$.

However, in the case

(b) $H_1 = L_2(R^\nu) \oplus L_2(R^\nu)$, $\quad h_1 = (-\Delta) \oplus \Delta$, $\quad h_2 = -\Delta$, we suppose additionally the presence of creation and annihilation operators both in 'particles' and 'antiparticles' in any interaction term (see below).

Let us note that in case (b), the one-particle spectrum is not bounded from below (such operators appear in temperature states [4]).

In the case

(a_2) $H_1 = L_2(R^\nu)$, and h_i are of a general nature (see below), our results are not complete (they correspond to [5]) and involve the same difficulties that appear when trying to generalize Iorio–O'Carroll's theorem to one-particle Hamiltonians different from the Laplacian.

We denote all absolute constants by C. As a rule, they depend on ν and on the Hamiltonian in question.

1. Formulation of the Main Results

Let us consider the Hilbert space

$$\mathscr{H} = F_a(H_1) \otimes L_2(R^\nu),$$

where $F_a = F_a(H_1)$ is the antisymmetric Fock space over H_1, i.e. $\mathscr{H}$ is the space of infinite sequences: if $H_1 = L_2(R^\nu)$, then

$$(F_0(y), F_1(x_1, y), \ldots, F_n(x_1, \ldots, x_n, y), \ldots), \tag{1.1a}$$

where $x_j, y \in R^\nu$ and the functions F_n are antisymmetric in x_j; if $H_1 = L_2(R^\nu) \oplus L_2(R^\nu)$, then $\mathscr{H}$ is the space of two index arrays

$$\{F_{n,m}(x_1, \ldots, x_n, z_1, \ldots, z_m, y), \ldots), m, n \geqslant 0\}, \tag{1.1b}$$

where x_j, z_j, $y \in R^\nu$ and the functions $F_{n,m}$ are antisymmetric in x_j and in z_j, simultaneously.

In the case (a), we define creation-annihilation operators $a^*(f)$ and $a(g)$, $f, g \in L_2(R^\nu)$ which satisfy the canonical anticommutation relations (CAR)

$$a^*(f)a^*(g) + a^*(g)a^*(f) = 0,$$
$$a(f)a(g) + a(g)a(f) = 0,$$
$$a^*(f)a(g) + a(g)a^*(f) = (f, g)1,$$

where (f, g) is the scalar product in $L_2(R^\nu)$, which is assumed to be antilinear in the second argument.

In the case (b), we define in addition creation-annihilation operators $b^*(f)$, $b(g)$, $f, g \in L_2(R^\nu)$, which also satisfy the CAR and, moreover

$$a^\#(f)b^\#(g) + b^\#(g)a^\#(f) = 0,$$

where '#' means either '*' or the absence of the asterisk.

Let self-adjoint operators h_1 in H_1, h_2 in $L_2(R^\nu)$ be given. Let us consider the self-adjoint operator in $\mathscr{H}$

$$H_0 = \mathrm{d}\Gamma(h_1) \otimes 1 + 1 \otimes (h_2),$$

where $\mathrm{d}\Gamma(h)$ is the second quantization functor ([3]).

We shall consider Hamiltonians of the type

$$H = H_0 + \varepsilon U, \quad U = \int_{R^\nu} V_z \, \mathrm{d}z, \ \varepsilon \in R, \tag{1.2}$$

where the operator V belongs to one of the classes of operators A or B. These classes of operators are defined differently for cases (a) and (b).

Case (a)

Let $f = a^*(\psi_r) \cdots a^*(\psi_1)\Omega \otimes \psi$, ψ_j, $\psi \in S(R^\nu)$; then $V \in A$ means that

$$\begin{aligned} Vf = \sum_{j=1}^{M} \int & \mathrm{d}x_{m_j+l_j} \cdots \mathrm{d}x_1 \, \mathrm{d}y_1 K_j(x_{m_j+l_j}, \ldots, x_1, y_1, y_2) \times \\ & \times a^*(x_{m_j+l_j}) \cdots a^*(x_{l_j+1}) a(x_{l_j}) \cdots a(x_1) a^*(\psi_r) \cdots a^*(\psi_1)\Omega \otimes \psi(y_1), \end{aligned} \tag{1.3a}$$

$K_j \in S(R^{\nu(k_j+m_j+2)})$,

i.e. V is an integral operator in the tagged particle; if $V \in B$, then

$$\begin{aligned} Vf = \sum_{j=1}^{M} \int & \mathrm{d}x_{m_j+l_j} \cdots \mathrm{d}x_1 K_j(x_{m_j+l_j}, \ldots, x_1, y) \times \\ & \times a^*(x_{m_j+l_j}) \cdots a^*(x_{l_j+1}) a(x_{l_j}) \cdots a(x_1) a^*(\psi_r) \cdots a^*(\psi_1)\Omega \otimes \psi(y), \end{aligned} \tag{1.4a}$$

$K_j \in S(R^{\nu(k_j+m_j+1)})$,

i.e. V is a multiplication operator in the tagged particle.

Case (b)

Let $f = a_\nu^*(\psi_r)\cdots a^*(\psi_1)\Omega_a \otimes (b^*(\varphi_s)\cdots)b^*(\varphi_1)\Omega_b \otimes \psi$, ψ_j, φ_j, $\psi \in S(R^\nu)$, then $V \in A$ means that

$$Vf = \sum_{j=1}^{M} \int dx_{m_j+l_j}\cdots dz_1\, dy_1 K_j(x_{m_j+l_j},\ldots,x_1, z_{m_j+l_j},\ldots,z_1, y_1, y_2) \times$$
$$\times\, a^*(x_{m_j+l_j})\cdots a^*(x_{l_j+1})a(x_{l_j})\cdots a(x_1)a^*(\psi_r)\cdots a^*(\psi_1)\Omega_a \otimes$$
$$\otimes\, b^*(z_{p_j+q_j})\cdots b^*(z_{p_j+1})b(z_{p_j})\cdots b(z_1)b^*(\varphi_s)\cdots b^*(\varphi_1)\Omega_b \otimes \psi(y_1),$$
$$K_j \in S(R^{\nu(k_j+m_j+p_j+q_j+2)}); \tag{1.3b}$$

if $V \in B$, then

$$Vf = \sum_{j=1}^{M} \int dx_{m_j+l_j}\cdots dz_1\, dy_1 K_j(x_{m_j+l_j},\ldots,x_1, z_{m_j+l_j},\ldots,z_1, y) \times$$
$$\times\, a^*(x_{m_j+l_j})\cdots a^*(x_{l_j+1})a(x_{l_j})\cdots a(x_1)a^*(\psi_r)\cdots a^*(\psi_1)\Omega_a \otimes$$
$$\otimes\, b^*(z_{p_j+q_j})\cdots b^*(z_{p_j+1})b(z_{p_j})\cdots b(z_1)b^*(\varphi_s)\cdots b^*(\varphi_1)\Omega_b \otimes \psi(y),$$
$$K_j \in S(R^{\nu(k_j+m_j+p_j+q_j+1)}). \tag{1.4b}$$

Here we use operator-valued distributions defined by

$$a^*(\tilde{f}) = \int a^*(k)\tilde{f}(k)\,dk, \qquad a^*(f) = \int a^*(x)f(x)\,dx,$$
$$a(\tilde{f}) = \int a(k)\bar{\tilde{f}}(k)\,dk, \qquad a(f) = \int a(x)\bar{f}(x)\,dx, \tag{1.5}$$

$$a^*(k)a(k') + a(k')a^*(k) = \delta(k-k'), \qquad \tilde{f}(k) = \frac{1}{(2\pi)}\nu/2 \int e^{-i(k,x)}f(x)\,dx,$$

and similarly for $b(k)$ and $b^*(k)$.

If T_z is the representation of the group of space translations in $\mathscr{H}$, i.e.

$$T_z(F_0(y), F_1(x_1, y),\ldots) = (F_0(y-z), F_1(x_1-z, y-z),\ldots),$$

then V_z is as in (1.3a) (for A) and as in (1.4a) (for B) with a kernel (e.g. for case A) of the form

$$K_j(x_{m_j+l_j}-z,\ldots,x_1-z, y_1-z, y_2-z)$$

and similarly for case B.

The operator V is bounded in $\mathscr{H}$ and we assume it is self-adjoint. Then, if both of the following conditions are fulfilled

$$m_{\max} = \max_j m_j = \max_j l_j < \infty, \quad m_j \geqslant 1,\ l_j \geqslant 1,$$
$$p_{\max} = \operatorname{map}_j p_j = \max_j q_j < \infty, \quad p_j \geqslant 1,\ q_j \geqslant 1, \tag{1.6}$$

then H is symmetric in the domain D (dense in $\mathscr{H}$) of vectors in case (a)

$$a^*(\psi_r)\cdots a^*(\psi_1)\Omega \otimes \psi, \tag{1.7a}$$

where Ω is the vacuum vector in F_a, ψ_j, $\psi \in S(R^\nu)$;

in case (b)

$$a^*(\psi_r) \cdots a^*(\psi_1)\Omega_a \otimes b^*(\varphi_s) \cdots b^*(\varphi_1)\Omega_b \otimes \psi, \tag{1.7b}$$

where $r, s \geqslant 0$, ψ, ψ_j, $\varphi_j \in S(R^\nu)$.

In fact, it is known that H is essentially self-adjoint on D. This follows from theorems of Kato or Nelson, and also from the convergence of the expansions obtained below.

Let us define the inverse and direct wave operators as follows

$$W_t = \exp(-itH_0)\exp(itH), \qquad \hat{W}_t = \exp(-itH)\exp(itH_0).$$

The main results are the following.

THEOREM 1. *Let $\nu \geqslant 3$, and either $h_1 = h_2 = -\Delta$ for case* (a) *or $h_1 = (-\Delta) + \Delta$, $h_2 = -\Delta$ for case* (b). *If* (1.6) *holds and V belongs to* A *or* B, *then there is an $\varepsilon_0 = \varepsilon_0(\nu, m, V) > 0$ such that for $|\varepsilon| < \varepsilon_0$ the following strong limits of direct*

$$\hat{W}_\pm = s - \lim_{t \to +\infty} \hat{W}_t,$$

and inverse

$$W_\pm = s - \lim_{t \to +\infty} W_t,$$

wave operators exist.

Remark 1. In a certain sense, Theorem 1 cannot be improved. In dimension $\nu = 1, 2$ even for a particle in an external field, bound states can appear for arbitrary small ε. For $m_{\min} = 0$, where

$$m_{\min} = \min_j m_j = \min_j l_j,$$

under conventional ideas mass renormalization is required. Smoothness conditions for kernels K_j can be substantially weakened, but this is not required under the present state of affairs. It should be noted that for the existence of direct wave operators, one does not need ε to be small. This can be easily proved by Cook's method.

Let us define the class G of functions $h(k)$, $k \in R^\nu$, such that the matrix of second-order derivatives has a nonzero determinant for $k \in R^\nu$.

Let us note that the functions $h(k) = k^2 \equiv (k, k)$ and $h(k) = k^2 + m^2$, $m > 0$ (one-particle relativistic Hamiltonian) belong to G.

THEOREM 2. *Let h_1 and h_2 belong to G, $\nu > 4$, $m_{\min} > (\nu+2)/(\nu-4)$. If we have* (1.6) *and assume that V belongs to* A, *then there exists an $\varepsilon_0 = \varepsilon_0(\nu, m, V) > 0$ such that for $|\varepsilon| < \varepsilon_0$ strong limits for direct and inverse wave operators exist.*

Remark 2. Limitations on dimension $\nu > 4$ and on the degree of interaction

$m_{\min} > (\nu+2)/(\nu-4)$ in Theorem 2 may perhaps be weakened up to $\nu \geqslant 3$, $m_{\min} \geqslant 1$, as in Theorem 1.

A standard consequence of Theorems 1 and 2 is the following corollary.

COROLLARY 1. *Under the conditions of Theorem* 1 *or* 2, $W_\pm, \hat{W}_\pm$ *are unitary and yield unitary equivalence of* H *and* H_0, i.e. *for example*, $H_0 W_+ = W_+ H$.

We also consider the operator H which preserves the number of particles and which is defined either on $F_a^{(N)}(L_2(R^\nu))_\nu \otimes L_2(R^\nu)$ (antisymmetric case) or on $F_s^{(N)}(L_2(R^\nu))_\nu \otimes L_2(R^\nu)$ (symmetric case) in the following way

$$H_\varepsilon^N = -\sum_{j=1}^N \Delta_{x_j} - \Delta_y + \varepsilon \sum_{j=1}^N V(x_j - y) = H_0^N + \varepsilon \sum_{j=1}^N V(x_j - y), \tag{1.8}$$

where $V(x) \in S(R^\nu)$, x_j, $y \in R^\nu$.

THEOREM 3. *If* $\nu \geqslant 3$, $V \in S(R^\nu)$, *then there exists an* $\varepsilon_0 = \varepsilon_0(\nu, V) > 0$ *which does not depend on* N *and is such that for all* N *and* $|\varepsilon| < \varepsilon_0$ *the system* (1.8) *is asymptotically complete and* H_ε^N *is unitarily equivalent to the free Hamiltonian* H_0^N.

2. Proof of Theorem 1. Perturbation Series

Let us consider the following series for $0 \leqslant t < \infty$, $\psi \in D$

$$\psi + \sum_{n=1}^{\infty} (i\varepsilon)^n \int_{Q_t^n} U_{t_n} \cdots U_{t_1} \, dt_1 \cdots dt_n \psi, \tag{2.1}$$

where

$$U_t = \exp(-itH_0) U \exp(itH_0),$$

$$Q_t^n = \{(t_1, \ldots, t_n) : t > t_n > \cdots > t_1 > 0\} \in R^n.$$

We will now prove that the main result is the following.

LEMMA 1. *Under the conditions of Theorem* 1, *there exists an* $\varepsilon_0 > 0$ *such that for* $|\varepsilon| < \varepsilon_0$:

(1) *the norm of* nth *term of the series* (2.1) *is bounded by* $(\varepsilon C)^n C(\psi)$, *where* C *does not depend on* n, ψ, t *and* $C(\psi)$ *does not depend on* n, t;

(2) *the* nth *term of the series* (2.1) *tends to a limit in the norm topology as* $t \to \infty$.

From the first assertion of Lemma 1, it follows in the standard way that (2.1) is equal to $W_t \psi$ and, from the second, that there exist inverse wave operators $W_\pm$. For $\hat{W}_t$ everything is the same and that is why Theorem 1 follows from Lemma 1.

Remark 3. For $\nu = 1, 2$, analyticity in ε cannot be proved, even for direct wave operators; however, their existence can be easily proved by Cook's method for arbitrary ε.

Let us consider only case (a). Case (b) is similar.

The operators h_1 and h_2 in the k-representation are the multiplication operators by the functions $h_1(k) = h_2(k) = k^2 \equiv (k, k)$.

We shall prove Lemma 1 for interactions V of class A. For class B, the proof is similar.

First, we shall prove Lemma 1 for a dense subset in A and then we shall extend this proof to all interactions in A.

Let the interaction V in the k-representation be of the following special type

$$V = \sum_{j=1}^{M} V_j^{(1)} \otimes V_j^{(2)},$$
$$V_j^{(1)} = \int \prod_{p=1}^{m_j + l_j} f_{jp}(k_{jp}) a^{\#}(k_{jp})\, \mathrm{d}k_{jp}, \tag{2.2}$$

where $\# = *$ for $p = m_j + l_j, \ldots, l_j + 1$, the order of creation-annihilation operators in the product being the same as in (1.4)

$$V_j^{(2)}\psi = (\psi, \bar{g}_j) f_j, \quad \psi \in L_2(R^\nu), \tag{2.3}$$

for some $f_{jp}, f_j, g_j \in S(R^\nu)$.

Let us rewrite more thoroughly the nth term of the series (2.1) for vectors of type (1.7)

$$\begin{aligned}
&\sum_{\pi} \int_{Q_t^n} \mathrm{d}t_n \cdots \mathrm{d}t_1 \int_{R^{\nu n}} \mathrm{d}z_n \cdots \mathrm{d}z_1 \times \\
&\quad \times \int \prod_{v,p} f_{\pi(v),p}(k_{vp}) a^{\#}(k_{vp})\, \mathrm{e}^{\pm(it_v h_1(k_{vp}) \pm i(z_v, k_{vp}))}\, \mathrm{d}k_{vp} \times \\
&\quad \times a^*(\psi_r) \cdots a^*(\psi_1)\Omega \otimes \\
&\quad \otimes \int \prod_{v=1}^{n} \mathrm{e}^{i(t_{v-1} - t_v)h_2(k_v) + i(z_{v-1} - z_v, k_v)} f_{\pi(v-1)}(k_v) g_{\pi(v)}(k_v)\, \mathrm{d}k_v \times \\
&\quad \times \mathrm{e}^{-it_n h_2(k_{n+1}) - i(z_n, k_{n+1})} f_{\pi(n)}(k_{n+1}),
\end{aligned} \tag{2.4}$$

where π is an arbitrary permutation $(1, \ldots, n) \to (1, \ldots, M)$, $t_0 = z_0 = 0$, $f_{\pi(0)} = \psi$. The sign $\pm$ depends on whether the variable k_{vp} corresponds to creation or annihilation operators (in the first case it is '$+$', in the second it is '$-$').

Our estimates will be uniform in π, so for conciseness, we put $\pi(v) \equiv v$.

Let us integrate w.r.t. the space variables $z_1, \ldots, z_n$. We get rid of the arising δ-functions by integrating w.r.t. the variables $k_1, \ldots, k_n$, which are connected with 'the particle'. After simple calculations, we reduce (2.4) to (we denote $G_v = f_{\pi(v-1)}g_{\pi(v)}$)

$$\begin{aligned}
&\int_{Q_t^n} \mathrm{d}t_n \cdots \mathrm{d}t_1 \int \prod_{v,p} f_{v,p}(k_{vp}) a^{\#}(k_{vp}) \mathrm{e}^{\pm i t_v h_1(k_{vp})} \mathrm{d}k_{vp} \times \\
&\quad \times \int \prod_{v=1}^{n} \mathrm{e}^{i(t_{v-1} - t_v) h_2(k_v)} G_v(k_v) \delta\left(\sum_p (\pm k_{vp}) + k_v - k_{v+1} \right) \mathrm{d}k_v \times \\
&\quad \times a^*(\psi_r) \cdots a^*(\psi_1)\Omega \otimes (e^{-i t_n h_2(k_{n+1})} f_n(k_{n+1}) \\
&= \int_{Q_t^n} \mathrm{d}t_n \cdots \mathrm{d}t_1 \prod_{v,p} \int f_{v,p}(k_{vp}) a^{\#}(k_{vp}) \mathrm{e}^{\pm i t_v h_1(k_{vp})} \mathrm{d}k_{vp} \times \\
&\quad \times \prod_{v=1}^{n} \int \mathrm{e}^{i(t_{v-1} - t_v) h_2(\bar{k}_v)} G_v(\bar{k}_v) \times \\
&\quad \times a^*(\psi_r) \cdots a^*(\psi_1)\Omega \otimes (e^{-i t_n h_2(k_{n+1})} f_n(k_{n+1}),
\end{aligned} \tag{2.5}$$

where

$$\bar{k}_v = -\sum_{v'=v}^{n} \sum_p (\pm k_{v'p}) + k_{n+1}$$

and the sign $\pm$ is defined as in (2.4).

3. Expansion in Resummed Diagrams

The integrand in the right-hand side of (2.5) has finite norm, however we must also perform integration in the time variables over an infinite interval. To obtain the necessary decay in time, let us note that the last product $\Pi_{v,p}$ in the right-hand side of (2.5) is the product of Wieck monomials and can be expanded into a sum of Wieck monomials which are enumerated by Friedrich diagrams [11]. It is well known that an arbitrary diagram does have the necessary decay in variables t_j, but the number of diagrams has factorial growth in n. That is why it is necessary to perform a mutual contraction of diagrams. So we shall use another expansion, which is, nevertheless, equivalent to some resummation of diagrams.

We single out in any vertex v the extreme right creation operator $a(k_{v1})$ and, using anticommutation relations,

$$a(k_{v1}) a^*(k_{v'j}) = -a^*(k_{v'j}) a(k_{v1}) + \delta(k_{v1} - k_{v'j}), \quad v' < v, \tag{3.1}$$

we carry this to the right of the vacuum vector Ω.

After the applications of (3.1), one of two terms in its right-hand side appears. If the first term appears, we continue carrying to the right and if a δ-function, i.e. a coupling occurs, then we say that the line $(v1, v'j)$ of the diagram appears. So we shall have exactly n lines $(v1, v'(v)j(v))$, $v = 1, \ldots, n$. Moreover, if coupling occurs with the creation operator from the vector

$$a^*(\psi_r) \cdots a^*(\psi_1)\Omega = \int \prod a^*(k_{vj}) \psi_j(k_{vj}) \, \mathrm{d}k_{vj} \, \Omega,$$

we say that $v'(v) = 0$. All pairs $(v'(v), j(v))$ are pairwise different.

After all these operations in (2.4), instead of the operators $a(k_{v1})$, $a^*(k_{v'(v),j(v)})$ under the integral, we get

$$\sum_{\{(v'(v),j(v))\}} \prod_{v=1}^{n} \delta(k_{v1} - k_{v'(v)j(v)}),$$

where the sum Σ is taken over all arrays of pairwise different pairs such that $v'(v) < v$. We take this out from under the integral sign and integrate over all variables $k_{v'(v),j(v)}$; $v = 1, \ldots, n$. Then all δ-functions disappear. We put

$$F = \prod_{v=1}^{n} f_{v1}(k_{v1}) f_{v'(v)j(v)}(k_{v1}) G_v(\bar{k}_v),$$

then the integrand reduces to

$$\begin{aligned} &F \prod_{v,p}' [f_{vp}(k_{vp}) e^{\pm i t_v h_1(k_{vp})} a^{\#}(k_{vp})] \times \\ &\times \exp\left(-i \sum_{v=1}^{n} [(-t_{v-1} + t_v) h_2(\bar{k}_v) + (-t_{v'(v)} + t_v) h_1(k_{v1})]\right) \times \qquad (3.2) \\ &\times \Omega \otimes (e^{-i t_n h_2(k_{n+1})} f_n(k_{n+1})), \end{aligned}$$

where in $\bar{k}_v$ all the $k_{v'(v),j(v)}$ are replaced by k_{v1}, Π' means that the factors corresponding to $(v1)$ and $(v'(v), j(v))$ are absent and $f_{0p} = \psi_p$.

Let us consider the graph G with the vertices $n, n-1, \ldots, 1, 0$ and lines $(v, v'(v))$. This graph is connected by construction.

4. Stationary Phase Bounds and Summation of Diagrams

Let us single out the integral in (3.2)

$$\begin{aligned} &\int F \exp\left(-i \sum_{v=1}^{n} [-(t_{v-1} - t_v)(\bar{k}_v)^2 - (t_{v'(v)} - t_v)(k_{v1})^2]\right) \prod_{v=1}^{n} dk_{v1} \qquad (4.1) \\ &\quad = \int F_1 \exp[-\tfrac{1}{2}(B\mathbf{k}, \mathbf{k}) + i(\mathbf{a}, \mathbf{k})] \, d\mathbf{k}, \end{aligned}$$

where $\mathbf{k} = (k_{11}, \ldots, k_{n1})$, $\mathbf{a}$ is the vector linearly depending on the k_{vj}, $j \neq 1$, which appear when one unfolds the brackets in $(\bar{k}_v)^2$. Let us put

$$\begin{aligned} -\tfrac{1}{2}(B\mathbf{k}, \mathbf{k}) &= -\left[\sum_{v=1}^{n} (t_v - t_{v-1})(\bar{k}_v^{(1)})^2\right] - \left[\sum_{v=1}^{n} (1 + t_v - t_{v'(v)}) k_{v1}^2\right] \\ &= -\tfrac{1}{2}(B_1 \mathbf{k}, \mathbf{k}) - \tfrac{1}{2}(B_2 \mathbf{k}, \mathbf{k}), \qquad (4.2) \\ F_1 = F \exp &\left\{ i \sum_{v=1}^{n} [(t_{v-1} - t_v)(\bar{k}_v^{(2)})^2 + \sum_{v=1}^{n} k_{v1}^2 \right\}, \end{aligned}$$

where $k_v^{(1)} = \bar{k}_v$ and all variables different from $k_{11}, \ldots, k_{n1}$ are set equal to zero, $\bar{k}_v^{(2)} = \bar{k}_v - \bar{k}_v^{(1)}$.

Using the Fourier transform w.r.t. the variables $(k_{11}, \ldots, k_{n1})$, (4.1) can be rewritten as

$$\frac{1}{(2\pi i)^{n\nu/2}\sqrt{\det B}} \int \exp[i\tfrac{1}{2}(B^{-1}\mathbf{x}, \mathbf{x})]\tilde{F}_1(\mathbf{x} - \mathbf{a})\,d\mathbf{x}, \tag{4.3}$$

where $\mathbf{x} = (x_{11}, \ldots, x_{n1})$.

Let us note that (4.3) is a function of the parameters k_{vp}, where $p \neq 1$, and $(v, p) \neq (v'(v), j(v))$.

To estimate (4.3), we use two propositions.

PROPOSITION 1. *Since* $B_1 \geqslant 0$, $B_2 \geqslant 0$, *we have*

$$(\det B)^{-1} \leqslant (\det B_2)^{-1} = \prod_{v=1}^{n} \frac{1}{(|t_v - t_{v'(v)}| + 1)^{\nu/2}}.$$

PROPOSITION 2. *The integral in* (4.3) (*which is a function of the parameters* k_{vp} *such that* $j \neq 1$ *and* $(v, p) \neq (v'(v), j(v))$) *belongs simultaneously to the spaces* $S(R)^{\nu N}$ *and* $L_2(R^{\nu N})$, *where* N *is the number of variables; moreover, its norm* $\| \ \|_2$ *is bounded* $C_\psi C^n$ *uniformity in* $\{t_v\}$, *where* $C > 0$ *is some constant not depending on* n.

Proof. Let us put

$$f_1 = \prod_{v=1}^{n} f_{v1}(k_{v1}) \exp(+ik_{v1}^2), \qquad f_4 = \prod_{v=1}^{n} G_v(\bar{k}_v),$$

$$f_2 = \prod_v{}' f_{v'(v)j(v)}(k_{v1}), \qquad f_3 = \prod_v{}'' \psi_{j(v)}(k_{v1}),$$

where Π' is the product over all v that are not coupled with the zero vertex and Π'' is the product over all vertices that are coupled with the zero vertex.

Then

$$F_1 = f_1 f_2 f_3 f_4 \exp\left\{i \sum_{v=1}^{n} [(t_{v-1} - t_v)(\bar{k}_v^{(2)})^2]\right\}, \tag{4.4}$$

$$\begin{aligned}
&\left|\int \exp[i\tfrac{1}{2}(B^{-1}\mathbf{x}, \mathbf{x})]\tilde{F}_1(\mathbf{x} - \mathbf{a})\,d\mathbf{x}\right| \\
&\quad \leqslant \|\tilde{F}_1\|_1 \leqslant \|\widetilde{(f_1 f_2 f_3 f_4)}\|_1 \\
&\quad =_{(1)} \|\tilde{f}_1 * \widetilde{(f_2 f_3)} * \tilde{f}_4\|_1 \leqslant_{(2)} \|\tilde{f}_1\|_1 \, \|\widetilde{f_2 f_3}\|_1 \, \|\tilde{f}_4\|_1,
\end{aligned}$$

where in (1) we use the fact that the Fourier transform of multiplication is convolution and in (2) we use Young's inequality.

Let us estimate uniformly in $\{\bar{k}_v^{(2)}, v = 1, \ldots, n\}$ the norm of the factors

$$\|\tilde{f}_1\|_1 = \prod_{v=1}^{n} \|f_{v1}(k_{v1})\widetilde{\exp}(+ik_{v1}^2)\|_1,$$

$$\|\widetilde{f_2 f_3}\|_1 = \prod_v{}' \|\tilde{f}_{v'(v)j(v)}\|_1 \prod_v{}'' \|\tilde{\psi}_{j(v)}\|_1, \tag{4.5a}$$

$$\|f_4\|_1 = \frac{1}{(2\pi)^{nv/2}} \int \left| \int \exp\left(-i \sum_{v=1}^{n} (x_v, k_{v1})\right) G_v(\bar{k}_v) \prod_{v=1}^{n} \mathrm{d}k_{v1} \right| \prod_{v=1}^{n} \mathrm{d}x_v$$

$$= \frac{1}{(2\pi)^{nv/2}} \int \left| \int \exp(-i(x, D^{-1}\bar{k})) f_4(\bar{k}) \prod_{v=1}^{n} \mathrm{d}\bar{k}_{v1} \{\det(D^{-1})\} \right| \prod_{v=1}^{n} \mathrm{d}x_v \tag{4.5b}$$

$$= \frac{1}{(2\pi)^{v/2}} \int \left| \tilde{f}_4((D^{-1})^* x)\{\det(D^{-1})\} \right| \prod_{v=1}^{n} \mathrm{d}x_{v1}$$

$$= \left\{ \prod_{v=1}^{n} \int |\tilde{G}_v(\bar{x}_v)|\, \mathrm{d}\bar{x}_v \right\} \{\det(D^{-1})\}\{\det(D)^*)\}$$

$$= \prod_{v=1}^{n} \|\tilde{G}_v\|_1 \leqslant \prod_{v=1}^{n} \|\tilde{f}_v\|_1 \, \|\tilde{g}_v\|_1,$$

where D is the matrix with constant coefficients such that $\bar{k} = Dk$, where

$$\bar{k} = (\bar{k}_1, \ldots, \bar{k}_n); \qquad k = (k_{11}, \ldots, k_{n1});$$
$$x = (x_1, \ldots, x_n); \qquad \bar{x} = (\bar{x}_1, \ldots, \bar{x}_n); \qquad \bar{x} = (D^{-1})^* x.$$

Since D is triangular with units on the diagonal, it follows from (4.5) that we have the bound

$$\|(\widetilde{f_1 f_2 f_3 f_4})\|_1 \leqslant (C_f)^{4n} C_\psi^r,$$

where

$$C_f = \max_{v,j} \{1, \|f_{vj}\|_2, \|\tilde{f}_{vj}\|_1, \|(f_{v1}\widetilde{\exp}(-ik_{v1}^2))\|_1\} \geqslant 1,$$

$$C_\psi = \max_j (1, \|\psi_j\|_2, \|\tilde{\psi}_j\|_1) \geqslant 1.$$

Using the following bound (valid for any function $Q \in L_2$)

$$\left\| \int Q(y_1, \ldots, y_N) a^{\#}(y_1) \cdots a^{\#}(y_N)\, \mathrm{d}y_1 \cdots \mathrm{d}y_N \right\|_2 \leqslant \|Q\|_2,$$

we get Proposition 2 for $C = (C_f)^{2m+2}$.

From Propositions 1 and 2, it follows that the nth term of the series is bounded by

$$C^n \left(\sum_{\{v'(v), j(v)\}} \prod_v \frac{1}{(|t_v - t_{v'(v)}| + 1)^{v/2}} \right). \tag{4.6}$$

To estimate the sum of all diagrams, we use the following lemma [4].

LEMMA 2. *For $\nu \geqslant 3$ the following bound holds*

$$\int_{Q_T^n}\left(\sum_{\{v'(v),j(v)\}}\prod_{v=1}^{n}\frac{1}{(|t_v - t_{v'(v)}|+1)^{\nu/2}}\right) \leqslant (C(m_{\max}))^n r!\left[\int_R \frac{dt}{(1+|t|)^{\nu/2}}\right]^n, \tag{4.7}$$

where the sum Σ is taken over all arrays $v'(v) < v$ such that among the numbers $v'(1), \ldots, v'(n)$ can be not more than $m_{\max} = \max_i m_i$ repetitions of a nonzero value.

From (4.6) and (4.7), Lemma 1 now follows.

Proof of Lemma 2. The proof of the bound (4.6) is similar to the proof of (4.1) in [4], where it was proved for the case $m_{\min} = m = 2$, $r = 1$.

Remark 4. Let us point out some special features of the proof of Theorem 1 in case (b). Expanding in the diagrams, we carry over to the right-hand side only the terms $a(k_{v1})$, $v = 1, \ldots, n$. Then, as in case (a), we also obtain a sum over connected diagrams. The main observation here is that Proposition 1 also holds. This was the reason we used only an $a(k)$-operator but no $b(k)$-operator. So again we have the inequality

$$B = B_1 + B_2 \leqslant B_2,$$

and, hence, Proposition 1. The remaining part of the proof of Theorem 1 for case (b) is quite similar to that in case (a).

5. General Type of Interaction

To prove Theorem 1 for interactions of a general type, it is sufficient to prove Proposition 2. For this, one needs to establish estimate (4.4).

Let the Fourier transform of the kernels K_i (case A) be

$$\hat{K}_i \in C_0^\infty(R^{\nu(l_i+m_i+2)});$$

then one can choose N so that

$$\operatorname{supp}\hat{K}_i \in [-N, N]^{\nu(l_i+m_i+2)} \quad \text{for any } i.$$

Let

$$e_n(k) = \chi(k)\exp(2\pi i n k/A), \quad k \in R,$$

where the function $\chi(k)$ is from $C_0^\infty(R)$, $0 \leqslant \chi(k) \leqslant 1$, $\chi(k) = 1$ for $|k| \leqslant B$ and $\chi(k) = 0$ for all $|k| \geqslant B + 1$. Choose A and B so that $N < B < 2N < A$; then

$$\hat{K}_i = \sum_{\bar{n}} c_{\bar{n}} \prod_i e_{n_i}(k_i), \tag{5.1}$$

where $n = (n_1, \ldots, n_M)$, $M = \nu(l_i + m_i + 1)$. It is evident that for any q, there exists a $c(q)$, such that

$$|c_{\bar{n}}| \leqslant \frac{c(q)}{|\bar{n}|^q}, \qquad |\bar{n}| = \sum_i n_i. \tag{5.2}$$

So we have represented the operator V by the series

$$\hat{V} = \sum_{\bar{n}} c_{\bar{n}} V_{\bar{n}}^{(1)} \otimes V_{\bar{n}}^{(2)}, \tag{5.3}$$

where the $V_n^{(1)}$ are Wieck monomials and $V_n^{(2)}$ can be written as

$$V_{\bar{n}}^{(2)}\psi = \sum_{\bar{n}} (V_{\bar{n}}^{(2)}\psi, e_{\bar{m}})e_{\bar{m}} = \sum_{\bar{m}} (\psi, V_{\bar{n}}^{(2)*}e_{\bar{m}})e_{\bar{m}} = \sum_{\bar{m}} b_{\bar{n}\cdot\bar{m}}(\psi)e_{\bar{m}}, \tag{5.4}$$

where the $b_{\bar{n}}(\psi)$ satisfy a bound similar to (5.2)

$$|b_{\bar{n}}(\psi)| \leqslant \frac{c(q)}{|\bar{n}|} q\|\psi\|, \quad |\bar{n}| = \sum_i n_i. \tag{5.5}$$

Choose $q > 2\nu + 1$ to be sure that the series of coefficients converges.

Using the bounds (5.2) and (5.5) and the fact that $\Sigma_\pi \leqslant M^n$, one can repeat the proof of Theorem 1 (and Lemma 1) for interactions V of type (5.3) and (5.4).

If $\hat{K}_i \in S(R^{\nu(l_i+m_i+2)})$, we shall use the partition of unity

$$\sum \chi_i(k) = 1, \tag{5.6}$$

where diam supp $\chi_{\bar{n}} \leqslant$ const uniformly in $\bar{n}$.

Then we represent the kernel $\hat{K}$ by the sum $\hat{K}_i\chi_{\bar{l}}$ of C_0^∞-kernels and, using an expansion similar to (5.1) for $\hat{K}_i\chi_{\bar{l}}$ (with correspondingly shifted functions $e_{\bar{n}}$), we repeat the proof.

6. Proof of Theorem 2

We prove Theorem 2 for the inverse Moller morphisms W_+. For W_-, the proof is the same.

We shall prove Theorem 2 first for a simpler form of the interaction V and then use this simpler form to approximate the general case (1.3a).

We assume further on that the Fourier transforms $f_{i,j}$, f_i, g_i belong to $C_0^\infty(R^\nu)$ and that (2.2) is formally symmetric.

Let us put

$$f_x(k) = f(k)\,\mathrm{e}^{i(k,x)}, \quad x \in R^\nu, \tag{6.1}$$

and suppose that V_x, $V_{i,x}^{(1)}$, $V_{i,x}^{(2)}$ are the operators defined above into which $f_{i,j,x}$, $f_{i,x}$, $g_{i,x}$ are substituted in place of $f_{i,j}$, f_i, g_i. Theorem 2 then follows from the following Lemma.

LEMMA 3. *The* nth *term of* (2.1) *converges in the norm topology when* $t \to \infty$ *and the norm of the* nth *term is bounded by* $(\varepsilon C)^n C(\psi)$, *uniformly in* t, *where* C *does not depend on* n, ψ, *and* $C(\psi)$ *does not depend on* n.

First we shall prove Lemma 3 for interactions of type (6.1).

The nth term of (2.1) can be rewritten as

$$\sum_\pi \int_{Q_t^n} \int_{(R^\nu)} V_n(\pi(n), x_n, t_n) \cdots V(\pi(1), x_1, t_1)\, \mathrm{d}t_1 \cdots \mathrm{d}t_n\, \mathrm{d}x_1 \cdots \mathrm{d}x_n\, \psi, \tag{6.2}$$

where

$$V(i, x, t) = \exp(-itH_0)(V^{(1)}_{i,x} \otimes V^{(2)}_{i,x})\exp(itH_0)$$

and the sum Σ_π is taken over all maps $\pi: \{1, \dots, n\} \to \{1, \dots, M\}$.

So if $\psi = \psi \otimes t^{(2)} \in D$, the norm of (6.2) is bounded by

$$\sum_\pi \| V^{(1)}(\pi(n), x_n, t_n) \cdots V^{(1)}(\pi(1), x_1, t_1)\psi^{(1)}\| \times \\ \times \| V^{(2)}(\pi(n), x_n, t_n) \cdots V^{(2)}(\pi(1), x_1, t_1)\psi^{(2)}\|. \tag{6.3}$$

First, we shall obtain an estimate for the second factor in (6.3). For $f \in L_2(R^\nu)$, we get

$$V^{(2)}(j, x, t)f = (f, e^{-ith(k) - i(k,x)}g_j)e^{-ith(k) - i(k,x)}f_j. \tag{6.4}$$

For any $u(k)$, let us put

$$D(u) = \{\text{grad } h(k) : k \in \text{supp } u\},$$

$$R = \inf\{r : D(u) \in \text{sphere of radius } r \text{ with center at the origin}\},$$

where inf is over all $u = \psi g_i, f_i g_i, \psi = \psi^{(2)}$. We shall use the following estimate (see Theorems XI.14 and XI.15 in [3])

$$\left|\int \exp(i(x, k) + ith_j(k))u(k)\,dk\right| \\ \leqslant C\|u\|_{d,\alpha}\,\dot{\mathrm{I}}_{R,d}(t, x), \quad j = 1, 2, \tag{6.5}$$

where

$$\|u\|_{d,\alpha} = \sum_{\alpha_1 + \cdots + \alpha_\nu \leqslant d} \left\| \frac{\partial^{\alpha_1 + \cdots + \alpha_\nu}}{\partial^{\alpha_1} \cdots \partial^{\alpha_\nu}} \right\|_\alpha$$

$$\dot{\mathrm{I}}(t, x) = \dot{\mathrm{I}}_{R,d}(t, x) = \begin{cases} 1/(1+t)^{\nu/2}, & \|x\| \leqslant R|t| \\ C(d)/(1+t+\|x\|)^d, & \|x\| \geqslant R|t|. \end{cases} \tag{6.6}$$

Further, we put $d = \nu + 2$. Using (6.4), we get

$$\| V^{(2)}(\pi(n), x_n, t_n) \cdots V^{(2)}(\pi(1), x_1, t_1)\psi^{(2)}\| \\ = \|\psi_{\pi(n)}\| \, |(\psi, e^{-itk^2 - i(x_1,k)}g_{\pi(1)})| \prod_{j=1}^n C^n \|f_{\pi(n)}\|_2 \|\psi g_{\pi(1)}\|_{\nu+2,\alpha} \times \\ \times \prod_{j=2}^n \|f_{\pi(j-1)}g_{\pi(j)}\|_{j+2,\alpha} \prod_{j=1}^n \dot{\mathrm{I}}(t_j - t_{j-1}, x_i - x_{j-1}), \tag{6.7}$$

where $t_0 = 0$, $x_0 = 0$.

For fixed $\psi^{(1)}$ let us denote,

$$A(t_1, \dots, t_n) \\ = \sup_\pi \sup_{x_1, \dots, x_n} \times \\ \times \| V^{(1)}(\pi(n), x_n, t_n) \cdots V^{(1)}(\pi(1), x_1, t_1)\psi^{(1)}\|$$

Now using $\Sigma_\pi \leqslant M^n$ and performing x-integrations in (6.2),

$$dx_1 \cdots dx_n = dx_1\, d(x_2 - x_1) \cdots d(x_n - x_{n-1})$$

using (6.6) and (6.7) and the bound

$$\int_{|x_i - x_{i-1}| < R|t_i - x_{i-1}|} d(x_i - x_{i-1}) \leqslant R^\nu |t_i - t_{i-1}|^\nu,$$

we get the following estimate for the norm of (6.2)

$$C^n C(\psi^{(2)}) \int_{Q_t^n} A(t_1, \ldots, t_n) \prod_{i=1}^{n} |t_i - t_{i-1}|^{\nu/2}\, dt_1 \cdots dt_n. \tag{6.8}$$

We shall prove here that (6.8) is bounded by $C^n C(\psi)$ uniformly in t. Convergence as $t \to \infty$ also follows from our proof. To do this, we must compensate for extra powers of $|t_i - t_{i-1}|$ in (6.8). To simplify notations, we take $A(t_1, \ldots, t_n)$ to be the norm of the individual term

$$V(t_1) \cdots V(t_n) a^*(\psi_1) \cdots a^*(\psi_r)\Omega, \tag{6.9}$$

where $V(t_i) = V^{(1)}(\pi(i), x_i, t_i)$ for fixed π and $x_1, \ldots, x_n$.

We shall use an expansion of (6.9) into the Friedrich diagrams. Complete expansion gives too many diagrams, and we use a partial expansion which is more involved than the expansion in Section 4.

First of all, we subdivide Q_t^n into sectors the Q_α

$$Q_t^n = U_\alpha Q_\alpha, \qquad Q_{\alpha_1} Q_{\alpha_2} = 0, \quad \alpha_1 \equiv \alpha_2$$

DEFINITION. The index α denotes any partition of $(n, \ldots, 0)$ into intervals

$$\Gamma_k, \ldots, \Gamma_1, \quad k = k(\alpha),$$
$$G_k = (n, n - q_k + 1), \ldots, G_1 = (q_1, 0), \quad q_i = n, \quad |\Gamma_i| = q_i;$$

given the n-tuple

$$(t_n, \ldots, t_1), t_n > \cdots > t_1 > 0 = t_0,$$

we say that t_i and t_{i-1} belong to the same interval if $|t_i - t_{i-1}| < 1$. This defines a map $P: Q_t^n \to \{\alpha\}$. We say that $(t_n, \ldots, t_1)$ and $(\hat{t}_n, \ldots, \hat{t}_1)$ belong to the same sector if $P(t_n, \ldots, t_1) = P(\hat{t}_n, \ldots, \hat{t}_1)$.

The number of sectors does not exceed 2^n, so we must establish the uniform bound C^n for any sector. So let us fix some sector α and the right end points of its intervals $\Gamma_k, \ldots, \Gamma_2$. The set of these points is

$$\dot{1} = (q_1 + \cdots + q_{k-1} + 1, \ldots, q_1 + q_2 + 1, q_1 + 1).$$

For any annihilation operator which appears in $V(t_i)$, $i \in \dot{1} = \dot{1}(\alpha)$, one can pull it through to the right in (6.9), i.e. to vacuum, by using the anticommutation relations. In the process, either we get zero or a line of the Friedrichs diagram with 'propagator'

$$(\exp(it_j h_1(k) + i(x_j, k)) f_{jm}, \exp(it_s h_1(k) + i(x_s, k)) f_{sp}) \tag{6.10}$$

for some $j \in \dot{\mathrm{I}}$ and $s < j$, or

$$(\exp(it_jk^2 + i(x_j, k))f_{jm}, \psi_s) \tag{6.11}$$

for some s, $1 \leqslant s \leqslant r$.

We process all annihilation operators in all $V(t_i)$, $i \in \dot{\mathrm{I}}$, in this way in some order (e.g. from right to left).

Then we use the bounds $\|a^*(f)\| \leqslant \|f\|$, $\|a(f)\| \leqslant \|f\|$ for all those a^* and a which remain.

For (6.10) and (6.11), we use the bound (6.6) and get

$$\frac{\text{const}}{(1 + |t_j - t_s|)}^{\nu/2}, \tag{6.12}$$

so we obtain graphs with $n + 1$ vertices $n, \ldots, 1, 0$. The graph is called admissible iff

(1) any vertex $i \in \dot{\mathrm{I}}$ has not less than $m = \min(m_i, k_i)$ lines to the right (i.e. which appear from coupling of the annihilation operator at the vertex i),

(2) any vertex $i \in \dot{\mathrm{I}}$ has not more than $m = \min(m_i, k_i)$ lines to the left (i.e. from the contraction of a creation operator from the vertex i).

Since $\|a^*(f)\| \leqslant \|f\|$, $\|a(f)\| \leqslant \|f\|$, the expression (6.8) is bounded by

$$\begin{aligned} &C^n C(\psi) \sum_\alpha \sum_g \prod_{\text{lines}} \frac{1}{(1 + |t_j - t_s|)}^{\nu/2} \prod_{i \in \mathrm{I}} |t_i - t_j|^{\nu/2} \\ &\leqslant C^n C(\psi) \sum_\alpha \sum_g \prod_{\text{lines}} \frac{1}{(1 + |t_j - t_s|)}^{\nu/2 - \nu/2m}, \end{aligned} \tag{6.13}$$

where the sum Σ_G is taken over all admissible graphs for a fixed sector α.

Let us consider the quotient graph $G = G/\{\Gamma_1, \ldots, \Gamma_k\}$ with k vertices $1, \ldots, k$. It is clear from the construction that G_α is connected and has no loops. To any vertex $j \in G_\alpha$ we assign 'time' $s_j = t_i$, where $i = i(\Gamma_j)$ is the right end point of Γ_j. The set L_j of lines G_α incident to the vertex j consists of the 'left' lines $L_{j,l}$ connecting j with $i < j$. Then

$$L_j = L_{j,l} \cup L_{j,r}, \quad m_{\min} \leqslant |L_{j,r}| \leqslant m_{\max}. \tag{6.14}$$

Instead of t-interactions, we obtain s-integrations with the key property that

$$|s_i - s_{i-1}| \geqslant 1 \quad \text{for all } i. \tag{6.15}$$

Let us note that there is not more than

$$\prod_i (m_{\max})^{|\Gamma_i|}(|L_{i,l}|!) \tag{6.16}$$

graphs G with the same quotient graph G_α.

We subdivide

$$\frac{1}{(1 + t_j - t_s)}^{\nu/2 - \nu/2m} = \frac{1}{(1 + t_j - t_s)}^{\nu/2 - \nu/2m - 1} \frac{1}{(1 + t_j - t_s)}$$

in (6.13) and use the second factor to compensate the combinatorial factor (6.16). For any vertex $j \in G_\alpha$ by (6.14) and (6.15),

$$\prod_p |s_j - s_p| \geqslant ((k/m_{\max})!)^{m_{\max}}, \tag{6.17}$$

where the product is taken over all left lines $(j, p) \in L_{j,l}$.

We are left with the bound

$$\sum_{G_\alpha} \int ds_1 \cdots ds_k \prod_{\text{lines of } G_\alpha} \frac{1}{|s_i - s_j|} a, \tag{6.18}$$

where integration is over the entire domain

$$s_k > s_{k-1} > \cdots > s_1 > s_0 = 0$$

under the conditions (6.15). We shall prove that if

$$a = \nu/2 - \nu/2m_{\min} - 1 > 1, \tag{6.19}$$

then (6.18) is bounded by $C^{nm_{\max}}$. Integration in variables $y_i = s_i - s_{i-1}$ yields

$$\sum_{G_\alpha} \sum_{y_k=1}^{\infty} \cdots \sum_{y_1=1}^{\infty} \cdot \tag{6.20}$$

This gives an estimate from above. Due to the connectedness of G_α, (6.20) is bounded by

$$\left(2 \sum_{y=1}^{\infty} \frac{1}{(1+y)} a\right)^{nm_{\max}}$$

(see a similar bound in [4]). This ends the proof of Theorem 2 for the simplest class (6.1) of interactions under condition (6.19).

The proof in the general case of the interactions (1.3a) is similar to that in the general case of Theorem 1.

7. Two-Point Interaction

Let us prove Theorem 3, first for the antisymmetric case in the space $\mathscr{H}$; this means that $\varepsilon_0 > 0$ does not depend on N.

The difference from the case above is that the interaction operator in 'second quantized' representation looks like this:

$$\varepsilon \int V(x - y) a^*(x) a(x) \, dx, \quad x \in R^\nu, \tag{7.1}$$

i.e. it has an additional δ-function.

However, the proof of this theorem repeats the proof of Theorem 1 with changes produced by the above remark. We focus only on these changes.

The difference is that in (2.2) we have $M = 1$, $m_i = l_i = 1$ and there is an additional δ-function

$$V = \int V(x_1 - y)\delta(x_2 - x_1)a^*(x_2)a(x_1)\,\mathrm{d}x_2\,\mathrm{d}x_1 \otimes \delta_y\,\mathrm{d}y,$$

where δ_y is δ-function in y.

Let $\tilde{V}(k)$ be the Fourier transform of $V(x)$, then in the k-representation the nth term of the series (2.1) for vectors (1.7) looks like

$$\begin{aligned}&\int_{Q_r^n}\mathrm{d}t_n\cdots\mathrm{d}t_1\prod_v\left\{\iint\tilde{V}(k_{v2}-k_{v1})\exp(it_v[h(k_{v1})-h(k_{v2})])\times\right.\\&\left.\times\, a^*(k_{v1})a(k_{v2})\,\mathrm{d}k_{v1}\,\mathrm{d}k_{v2}\right\}a^*(\psi_r)\cdots a^*(\psi_1)\Omega\otimes\\&\otimes\left[\prod_{v=1}^{n}\int \mathrm{e}^{i(t_{v-1}-t_v)h(k_v)}\delta(-k_{v1}+k_{v2}-k_v+k_{v+1})\,\mathrm{d}k_v\right]\mathrm{d}k_{n+1}\psi(k_1),\end{aligned}\tag{7.2}$$

where $h(k) = k^2$.

Let us note that in this case, all functions satisfy $f_v \equiv 1$, $g_v \equiv 1$ for $v = 1, \ldots, n$; $f_0 = \psi$.

We get rid of the δ-functions in (7.2), obtaining

$$\begin{aligned}&\int_{Q_r^n}\mathrm{d}t_n\cdots\mathrm{d}t_1\prod_v\left\{\iint\tilde{V}(k_{v2}-k_{v1})\exp(it_v[h(k_{v1})-h(k_{v2})])\times\right.\\&\left.\times\, a^*(k_{v1})a(k_{v2})\,\mathrm{d}k_{v1}\,\mathrm{d}k_{v2}\right\}a^*(\psi_r)\cdots a^*(\psi_1)\Omega\otimes\\&\otimes\left[\prod_{v=1}^{n}\mathrm{e}^{i(t_{v-1}-t_v)h(\bar{k}_v)}\right]\mathrm{d}k_{n+1}\psi(\bar{k}_1),\end{aligned}\tag{7.3}$$

where

$$\bar{k}_v = k_{v1} - k_{v2} + k_{n+1}.$$

We expand (7.3) in the Friedrichs diagrams. We denote by $'v(v)$ the number of the vertices which are coupled with the vertex v, $'v(v) > v$. Let us put $\psi \equiv 1$, if $v'(v) \neq 0$. Any term of this sum has the form

$$\begin{aligned}&\int F\left\{\prod_{v\in I_2}a^{\#}(k_{v2})\right\}\left\{\prod_{j\in I}a^*(\psi_j)\right\}\Omega\otimes\\&\otimes\mathrm{e}^{-it_nh(k_{n+1})}\psi(\bar{k}_1)\left\{\prod_{v=1}^{n}\mathrm{d}k_{v1}\right\}\left\{\prod_{v\in I_2}\mathrm{d}k_{v2}\right\},\end{aligned}\tag{7.4}$$

where

$$F = \left\{ \prod_{r \in \dot{\mathrm{I}}_1} \tilde{V}(k_{r(r)1} - k_{r1}) \mathrm{e}^{-i(t_r - t_{r(r)})h(k_{r1})} \psi_{j(r)}(k_{r1}) \right\} \times$$
$$\times \left\{ \prod_{r \in \dot{\mathrm{I}}_2} \tilde{V}(k_{r2} - k_{r1}) \mathrm{e}^{-i(t_r - t_{r(r)})h(k_{r1})} \psi_{j(r)}(k_{r1}) \times \right. \tag{7.5}$$
$$\times \exp\left(i \sum_{r=1}^{n} [(t_{r-1} - t_r) h(\bar{k}_r)] \right)$$

and $\dot{\mathrm{I}}_1$ is the set of vertices with which there are couplings, $\dot{\mathrm{I}}_2$ is the set of vertices with no couplings and $\dot{\mathrm{I}}$ is the set of noncoupled arms of vertex 0.

Let us consider functions F_1 in the variables $\{k_{r1}, r = 1, \ldots, n\}$, and $\{k_{r2}, r \in \dot{\mathrm{I}}_2\}$, k_{n+1}, where

$$F_1 = \left\{ \prod_{r \in \dot{\mathrm{I}}_1} \tilde{V}(k_{r(r)1} - k_{r1}) \psi_{j(r)}(k_{r1}) \right\} \left\{ \prod_{r \in \dot{\mathrm{I}}_2} \tilde{V}(k_{r2} - k_{r1}) \psi_{j(r)}(k_{r1}) \right\}.$$

Let us carry out the Fourier transform of (7.5). It is sufficient to prove the analog of Proposition 2 for the function

$$F_2(k_{r2}, r \in \dot{\mathrm{I}}_2, k_{n+1}) = \frac{1}{(2\pi)^{n\nu/2}} \int \left| \left(\int \exp\left(-i \sum_{r=1}^{n} (x_r, k_{r1}) \right) F_1(k) \prod_{r=1}^{n} \mathrm{d}k_{r1} \right) \right| \prod_{r=1}^{n} \mathrm{d}x_r. \tag{7.6}$$

PROPOSITION 3. *The function $F_2(k_{r2}, r \in \dot{\mathrm{I}}_2, k_{n+1})$ belongs to $L_2(R^{\nu N})$, where N is the number of variables $\{k_{r2}, r \in \dot{\mathrm{I}}_2\}$, k_{n+1} and its L_2-norm has the bound*

$$\|F_2\| \leqslant C^n C_\psi,$$

where $C > 0$ is some constant not depending on $n, \psi_1, \psi, \ldots, \psi_r$, $C_\psi > 0$ depends only on $\psi, \psi_1, \ldots, \psi_r$.

Proof. Let us put

$$k = (k_{11}, \ldots, k_{n1}), \qquad x = (x_1, \ldots, x_n).$$

Subdividing for any component of $k_{r1} = (k_{r1}^{(1)}, \ldots, k_{r1}^{(\nu)})$ the interior integral into zones $(|x^{(i)}| \geqslant 1)_r$ and $(|x^{(i)}| < 1)$ and integrating by parts in the first case twice over $k_{r1}^{(i)}$, we obtain

$$|F_2(k_{r2}, r \in \mathrm{I}_2, k_{n+1})| \leqslant C^n \int \left| \int \prod_{r=1}^{n} \frac{1}{(|x_r| + 1)^{2\nu}} \left| \frac{\partial^{2\nu n} F_1(k)}{\partial^2 k_{11}^{(1)} \cdots \partial^2 k_{n1}^{(\nu)}} \right| \prod_{r=1}^{n} \mathrm{d}k_{r1} \prod_{r=1}^{n} \mathrm{d}x_r \right. \tag{7.7}$$
$$\leqslant C^n \int \left| \frac{\partial^{2\nu n} F_1(k)}{\partial^2 k^{(1)} \cdots \partial^2 k^{(\nu)}} \right| \prod_{r=1}^{n} \mathrm{d}k_{r1},$$

where $|x_r| = |x_r^{(1)}| + \cdots + |x_r^{(\nu)}|$.

Let us note that $\psi(\bar{k}_1)$ depends on not more than $2r + 1$ variables, since

$$\bar{k}_1 = -\sum_{v \in \dot{I}_2} k_{v2} + k_{n+1} + \sum_{v \in \dot{I}_3} k_{v1},$$

where $\dot{I}_3$ is the set of vertices which have lines with vertex 0.

So we get no more than $C^n(2r)^{2r}$ terms for the derivative in (7.7), and, moreover, the derivatives of $V, \psi_1, \ldots, \psi_r$ will be of orders not exceeding 4ν and those of ψ of orders not greater than $2r\nu$.

Thus

$$|F_2(k_{v2}, v \in \dot{I}_1, k_{n+1})| \leqslant C^n \sum_j \int |F_1^{(j)}(k)| \prod_{v=1}^{n} \mathrm{d}k_{v1}, \tag{7.8}$$

where $F_1^{(j)}$ looks like F_1, but instead of V, ψ, ψ_i, their derivatives of the orders pointed out above appear.

It follows that

$$\|F_2\| \leqslant C^n \sum_j \sum_i \int \left(\int |F_1^{(j)}(k)| \prod_{v=1}^{n} \mathrm{d}k_{v1} \int |F_1^{(i)}(k')| \prod_{v=1}^{n} \mathrm{d}k'_{v1} \right) \times \\ \times \prod_{v \in \dot{I}_2} \mathrm{d}k_{v2}\, \mathrm{d}k_{n+1}. \tag{7.9}$$

Any integral in (7.9) can easily be estimated by Fubini's theorem and by using

$$|V^{(i)}(k_{v2} - k'_{v1})| \leqslant C, \quad \text{for } v \in \dot{I}_2,$$
$$|\psi(\bar{k}'_1)| \leqslant C_\psi.$$

After a change of variables, we get the required bound.

Remark 5. In the symmetric case, the Bose creation $a^*(f)$ and annihilation $a(f)$ operators are unbounded on $F(L_2(R^\nu))$ and bounded on any $F_s^{(r)}(L_2(R^\nu))$ by $\|f\|_2(r+1)^{r/2}$.

In our case, we always get exactly r noncoupled creation operators; this allows us to uniformly estimate the norm of their product for any r.

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