Journal of Statistical Physics, Vol. 35, Nos. 3/4, 1984

Uniqueness of Gibbs Fields via Cluster Expansions

V. A. Malyshev[1] **and I. V. Nickolaev**[1]

Received October 11, 1983; revision received December 6, 1983

For the unbounded spin systems one cannot get cluster expansion if there exist large enough boundary values. A simple idea to avoid these difficulties is to prove that with probability $p_\Lambda \to 1$ when $\Lambda \uparrow \mathbb{Z}^\nu$ there is a large subvolume Λ' of Λ such that on $\partial\Lambda'$ all spin values do not exceed some fixed number. This gives a new method to prove uniqueness results for the unbounded spin systems generalizing some results of Refs. 1 and 2. The formulations of these results are in Section 1; the proofs are in Section 2.

KEY WORDS: Cluster expansion; Gibbs fields; random boundary conditions; unbounded spin system; Peierls argument; classes of uniqueness of Gibbs fields.

1. THE UNIQUENESS RESULTS

Let $\Lambda_{\omega N}$, $0 \leqslant \omega \leqslant 1$, be the family of cubes $\Lambda_{\omega N} = \{t \in \mathbb{Z}^\nu : |t^{(i)}| \leqslant \omega N, i = 1, \ldots, \nu\}$, $\Lambda_N = \Lambda$.

Let us define the boundaries of the set Λ

$$\partial\Lambda = \partial_e\Lambda = \left\{t \in \mathbb{Z}^\nu : t \overline{\in} \Lambda, \rho(t,\Lambda) = 1\right\}$$

$$\partial_i\Lambda = \left\{t : t \in \Lambda, \rho(t, \partial\Lambda) = 1\right\}$$

Let μ_0^Λ be the Lebesgue measure on $\mathbb{R}^\Lambda$. We consider the lattice model with values in $\mathbb{R}$ with the Gibbs measure $\mu_{\Lambda, x_{\partial\Lambda}}(dx_\Lambda)$ on $\mathbb{R}^\Lambda$ defined by

$$\frac{d\mu_{\Lambda,x_{\partial\Lambda}}}{d\mu_0^\Lambda} = Z_{\Lambda,x_{\partial\Lambda}}^{-1} \exp(-U_{\Lambda,x_{\partial\Lambda}}) \tag{1.1}$$

[1] Probability Chair, Mathematics Department, Moscow State University, Moscow, V-234, USSR.

0022-4715/84/0500-0375$03.50/0

where $x_{\partial\Lambda}$ is the configuration on $\partial\Lambda$ (boundary conditions) and

$$U_{\Lambda, x_{\partial\Lambda}} = m_0 \sum_{\substack{|t-t'|=1 \\ \{t,t'\}\cap\Lambda\neq\varnothing}} |x_t - x_{t'}|^\alpha + \sum_{t\in\Lambda} P(x_t)$$
$$P(x) = \sum_{j=1}^{k} m_j |x|^{\gamma_j} \tag{1.2}$$

where $\gamma_k > 1$, $\alpha \leqslant \gamma_k > \gamma_{k-1} > \cdots > \gamma_1 > 0$, $m_0 > 0$, $m_k > 0$. Let us fix a positive number B to be specified below and a configuration x_Λ on Λ and let us define the set of "B points" with respect to x_Λ:

$$D_B(x_\Lambda) = \{ t \in \Lambda : |x_t| > B \}$$

The sequence $L = (t_1, \ldots, t_n)$ which is contained in $(\Lambda - \Lambda_{N/4}) \cap D_B(x_\Lambda)$ is called the B path with respect to x_Λ if $t_1 \in \partial_i \Lambda$, $t_n \in \partial_e \Lambda_{N/4}$, $t_j \in (\Lambda - \Lambda_{N/4}) - \partial_i(\Lambda - \Lambda_{N/4})$, $j = 2, \ldots, n-1$, $|t_i - t_{i+1}| = 1$, $i = 1, \ldots, n-1$. Let $P_N = P_N(x_{\partial\Lambda})$ be the probability of the existence of at least one B path (with fixed boundary conditions).

The main technical result is as follows:

Lemma 1. If $\gamma_k > \alpha$ then there exist $B > 0$ and $A > 1$ such that if boundary conditions x_t for all $t \in \partial\Lambda_N$ satisfy the following estimation,

$$|x_t| \leqslant A^{A^N} \tag{1.3}$$

then

$$P_N \leqslant C_1 \exp(-C_2 N) \tag{1.4}$$

where constants $C_1, C_2 > 0$ depend only on B, $P(x)$, ν, m_0, and A. If $\gamma_k = \alpha$ then instead of (1.3) one must assume

$$|x_t| \leqslant A_1 A_2^N \tag{1.3$'$}$$

where $A_1 > 0$, $A_2 > 1$.

Let us consider the class $\mathfrak{U} = \mathfrak{U}(A)$, $A > 1$, of random fields on a lattice such that for any N and all $t \in S_N$ ($S_N = \{t : \rho(0,t) = N\}$)

$$P\left(|x_t| \geqslant A^{A^N}\right) = o\left(|S_N|^{-1}\right) \tag{1.5}$$

This class contains all translation-invariant fields for which $P(|x_t| \geqslant M)$ decreases roughly faster than $(\ln(\ln M))^{-(\nu-1)}$.

Let us consider also the class of Gibbs fields on the lattice with the interaction (1.2) for which either

(i) $\qquad \alpha = 2, \qquad P(x) = mx^2 + \lambda\tilde{P}(x)$

where $\tilde{P}$ is the polynomial of the finite degree $\geqslant 4$ bounded from below,

$\lambda > 0$ is sufficiently small, or

(ii) $\qquad m_0 > 0$ is sufficiently small

Theorem 1. Any Gibbs field with fixed interaction of the class (i) or (ii) is unique among the fields of the class $\mathfrak{U}(A)$ for some $A > 1$.

To prove this theorem let us note that as it follows from Lemma 1 there exists with probability $1 - P_N$ the (random) volume $\tilde{\Lambda}$, $\Lambda_{N/4} \subset \tilde{\Lambda} \subset \Lambda$, such that $|x_t| \leqslant B$ for $t \in \partial\tilde{\Lambda}$. Let us choose for the given Λ and x_Λ maximal such volume $\Lambda_{\max}$ (if Λ_1 and Λ_2 satisfy the above condition then their union $\Lambda_1 \cup \Lambda_2$ also satisfies this condition).

Let us consider the random event

$$Q(\tilde{\Lambda}, x_{\partial\tilde{\Lambda}}) = \{x_\Lambda : \tilde{\Lambda} = \Lambda_{\max} \text{ and } x_\Lambda|_{\partial\tilde{\Lambda}} = x_{\partial\tilde{\Lambda}}\}$$

This event is measurable with respect to the σ-algebra $\sum_{\Lambda-\tilde{\Lambda}}$ generated by random variables x_t, $t \in \Lambda - \tilde{\Lambda}$. Then the conditional distribution in $\tilde{\Lambda}$ with respect to $Q(\tilde{\Lambda}, x_{\partial\tilde{\Lambda}})$ is the Gibbs distribution in $\tilde{\Lambda}$ with boundary conditions $x_{\partial\tilde{\Lambda}}$.

Let us denote it by $\mu_{\tilde{\Lambda}, x_{\partial\tilde{\Lambda}}}$.

In cases (i) and (ii) one can obtain the cluster expansion (see Refs. 3, 4, and 5) for $\mu_{\tilde{\Lambda}, x_{\partial\tilde{\Lambda}}}$. Integrating this cluster expansion term by term with respect to the distribution P of $\tilde{\Lambda}$ and $x_{\partial\tilde{\Lambda}}$ we obtain the cluster expansion for the correlation functions with probability $1 - P_N$. We need not know the measure P exactly as it does not influence on the terms b_R of cluster expansion with $R \subset \Lambda_{N/4}$. Then if we tend $N \to \infty$ we prove uniqueness and obtain the cluster expansion coinciding with that for the empty boundary conditions. Theorem 1 is proved.

2. PROOF OF LEMMA 1

It is convenient to design the potential

$$\Phi(x_t, x_{t'}) = m_0|x_t - x_{t'}|^\alpha + \frac{1}{2\nu}(P(x_t) + P(x_{t'}))$$

and the transformation of $\mathbb{R}$ into itself

$$g(x) = g_{B,C}(x) = \begin{cases} x - C, & x > B \\ x, & |x| \leqslant B \\ x + C, & x < -B \end{cases} \tag{2.1}$$

where $0 < C < B$.

Lemma 2.1. We consider $C \ll B$. (a) If $|x_1|, |x_2| \geqslant B$ then

$$\Phi(x_1, x_2) \geqslant \Phi(g(x_1), g(x_2)) + \frac{1}{2\nu} \min_{x \geqslant B} (P(x) - P(g(x)))$$
$$\geqslant \Phi(g(x_1), g(x_2)) + d \qquad (2.2)$$

where $d = d(B, C) > 0$,

(b) $$\Phi(x_1, x_2) \geqslant \Phi(x_1, g(x_2)) \qquad (2.3)$$

if $|x_2| \geqslant B$ and $|x_1| \leqslant |x_2| + C_0 |x_2|^{1+\epsilon}$ for some $C_0 > 0$ and sufficiently large $B > 0$; moreover, $\epsilon > 0$ for $\gamma_k > \alpha$ and $\epsilon = 0$ for $\gamma_k = \alpha$.

Proof. (a) is evident.

To prove (b) let us write, e.g., for $x_2 \geqslant B$,

$$\Phi(x_1, x_2) - \Phi(x_1, g(x_2))$$
$$= |x_1 - x_2|^\alpha - |x_1 - x_2 + C|^\alpha + P(x_2) - P(x_2 - C)$$
$$\geqslant C_2 |x_2|^{\gamma_k - 1} - C_1 |x_1 - x_2|^{\alpha - 1}$$

for some positive constants C_1 and C_2 which do not depend on B. Then for $1 + \epsilon < (\gamma_k - 1)/(\alpha - 1)$ for $\gamma_k > \alpha$ we obtain the proof of Lemma 2.1.

For $\gamma_k = \alpha$ we must consider two cases. If $x_1 - x_2 \leqslant -C$, then $|x_1 - x_2|^\alpha > |x_1 - x_2 + C|^\alpha$ and $P(x_2) - P(x_2 - C) > 0$. So (2.3) is evident. If $x_1 - x_2 > -C$ then taking $\epsilon = 0$ and C_0 sufficiently small we obtain the proof of Lemma 2.1. ■

Now let a configuration x_Λ be given for which at least one B path exists. Let $\overline{T} = \overline{T}(x_\Lambda)$ be some 1-connected component of $D_B(x_\Lambda)$ containing at least one B path.

We define also n sphere $S_n = \{t : \tilde{\rho}(t, 0) = n\}$, where $\tilde{\rho}(t, t') = \max_i |t^{(i)} - t'^{(i)}|$, and define the sequence of numbers $B_1 = B + C_0 B^{1+\epsilon}$, $B_2 = B_1 + C_0 B_1^{1+\epsilon}, \ldots$.

We delete now some points from $\overline{T}$. Let us denote $\overline{T}_n$ the set of points $t \in \overline{T} \cap S_n$ such that $|x_t| \leqslant B_{n-N/2}$, $n = N/2 + 1, \ldots, N$. We denote the remaining subset by $T' = \overline{T} \backslash (\bigcup_{n=N/2+1}^{N} \overline{T}_n)$.

Definition. Any maximal 1-connected subset of T' containing a 1-connected path (B path) connecting $\partial_i \Lambda_{N/2}$ and $\partial_e \Lambda_{N/4}$ is called a regular B set.

There can be more than one regular B set R. We shall estimate the probability that there is at least one such R.

The main properties of R, which we shall use later, are the following:

(a) $$|R| \geqslant N/4 \qquad (2.4)$$

(b) If $t \in R$ and for some $m = 1, \ldots, N/2$ $|x_t| > B_m$ then for any t', such that $|t - t'| = 1$ and $t' \overline{\in} R$

$$|x_t| < B_{m+1} \tag{2.5}$$

Both (2.4) and (2.5) follow from the construction of R.

To prove the main estimation (1.4) we shall make use at the modification of "Peierls argument" for the comparison of the numerator and denominator in the Gibbs formula. Let us define the transformation $G = G(x_\Lambda, R)$ of the set X_Λ of configurations in Λ into itself

$$(Gx_\Lambda)_t = \begin{cases} x_t, & t \overline{\in} R \\ g(x_t), & t \in R \end{cases} \tag{2.6}$$

Let us fix the set $R \subset \Lambda$ and the set X_Λ^R of configurations x_Λ in Λ such that R is the regular B set for any $x_\Lambda \in X_\Lambda^R$. We are interested in the probability of such X_Λ^R:

$$P(X_\Lambda^R) = \frac{\int_{X_\Lambda^R} \exp(-U_{\Lambda, x_{\partial\Lambda}})\, d\mu_0^{(\Lambda)}}{\int_{X_\Lambda} \exp(-U_{\Lambda, x_{\partial\Lambda}})\, d\mu_0^{(\Lambda)}} \tag{2.7}$$

If we take into account that G leaves the Lebesgue measure $\mu_0^{(\Lambda)}$ invariant, then from (2.7) and estimations (2.2) and (2.3) it follows that

$$P(X_\Lambda^R) \leqslant C_1 \exp(-C_2|R|) \tag{2.8}$$

The constant C_2 may be taken large enough because B may also be taken large enough; and $d = d(B) \to \infty$ if $B \to \infty$ as $\gamma_k > 1$ and $C \ll B$. The number of regular B set R with $|R| = m$ does not exceed $|\partial\Lambda_{N/4}| C^m$. Recalling the property (2.4) we have the desired proof. ■

ACKNOWLEDGMENT

We thank J. Bricmont, R. L. Dobrushin, and S. Pirogov for valuable remarks.

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