Resonances in Large Systems with Random External Force

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Abstract

Main goal of our presentation is a review of (earlier and new) results and problems concerning non-equilibrium statistical physics of large systems of particles (vertices of large metric graph) with arbitrary quadratic interaction and (deterministic and random) external influence. In particular, we consider conditions for resonance, time evolution of the distribution of potential and kinetic energies along the graph, and other problems.

Keywords: Piecewise deterministic Markov processes, Non-equlibrium statistical physics, Stochastic resonance, Many particle systems

1. Models

We observe many qualitative phenomena in physics and biology and it is natural to look for mathematical micro models with similar qualitative behavior. For example, biological organisms can exist only in very narrow temperature range. Moreover, the temperature of all parts of the organism should be in this range. Between these parts there are many interactions which still are not yet understood on the level of mathematical non-equilibrium statistical physics models. That is why it is important to find some many component models with external random forces and the following properties: 1) it should demonstrate various qualitative phenomena, like smooth (small fluctuations) temperature distribution along the components of large system, 2) some external forces can produce resonance, that can "kill" many component system, 3) most interesting is that some random forces also can help to avoid such resonance, 4) this model should not be too complicated, that is should allow to obtain various rigorous mathematical results.

First of all, we assume that internal forces are purely deterministic and only external can be random. We consider the following general linear systems of N_0 point particles in R^d with $N = dN_0$ coordinates $q_j \in \mathbb{R},\ j = 1, .., N$. We denote:

$$v_j = \frac{dq_j}{dt}, p_j = m_j v_j,\ j = 1, .., N,$$

$$q = (q_1, ..., q_N)^T, p = (p_1, ..., p_N)^T, \psi(t) = (q_1, ..., q_N, p_1, ..., p_N)^T,$$

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where m_j is the mass of the particle which has q_j as one of its coordinates. Potential (interaction) and kinetic energies are defined as follows:

$$U(\psi(t)) = \frac{1}{2} \sum_{1 \le j,l \le N} V_{j,l} q_j q_l = \frac{1}{2}(q, Vq), \; T(\psi(t)) = \sum_{j=1}^{N} \frac{p_j^2}{2} = \frac{1}{2}(p, p),$$

The time evolution is described by the general linear system

$$\ddot{q}_j = -\sum_k V_{jk} q_k + f_j(t, p_j), \; j = 1, .., N, \tag{1}$$

or, in vector notation,

$$F = (0, ..., 0, f_1(t, p_1), ..., f_N(t, p_N))^T$$

as

$$\dot{q}_j = p_j,$$
$$\dot{p}_j = -\sum_k V_{j,k} q_k + f_j(t, p_j),$$

or

$$\dot{\psi} = A\psi + F, \tag{2}$$

where

$$A = \begin{pmatrix} 0 & E \\ -V & 0 \end{pmatrix}$$

Here V is positive-definite matrix which defines the interaction between objects (particles), f_j are external forces acting on the coordinate j.

Two examples:

1. The simplest example (called harmonic oscillator) is $N = 1$ with the equation

$$\ddot{q} = -\omega_0^2 q + f(t, v),$$

where $f(t, v)$ is the sum of two terms: driving force $g_1(t)$ and dissipative force $g_2(t, v) = -\alpha(t)v + \beta(t)$, where $\alpha(t) \ge 0, \beta(t)$ can be random stationary processes.

2. Imagine that N coordinates q_i of particles are vertices of some metric graph without multiple edges $l = (i, j)$. The edges of such graphs are all $l = (i, j)$ such that $V_{ij} \ne 0$. Assume also that for each edge l some non-zero number a_l (its length) is given. We define potential:

$$U = \sum_{l=(i,j)} (q_i - q_j - a_l)^2,$$

which provides interaction force on the coordinate j:

$$Q_j = -\frac{\partial U}{\partial q_j}.$$

Particular case (very popular in the literature) is the linear graph with edges $(1, 2), (2, 3), ...,$ $(N - 1, N)$.

Most interesting case for us is when all $f_k = 0$ except one, for example, for $k = N$. For example, convergence to Gibbs equilibrium due to white noise force [12] or due to random collisions of particle N with external particles [9], convergence to Liouville measure due to energy conserving collisions [11]. But convergence to equilibrium is not yet a really non-equilibrium phenomena. For example, in biology such Gibbs equilibrium is death.

Simplest (but not "simple") non-equilibrium models are linear models where each particle does not walk randomly in space but is situated in its own potential well, however it can have complicated dynamics inside its well. Moreover, the well itself moves in space. For such model it is necessary that particles could not collide. Example is [10], where using such model, we proved the convergence (under some scaling) the macro Euler equations for Chaplygin gas, very popular now among physicists in (besides the gas itself) such fields as string theory, ball lightning, dark energy.

Adding nonlinear perturbations is more difficult problem and could demand cluster expansion techniques. But, seemingly, it will not add much in qualitative features of dynamics because any non-degenerate potential well is approximately quadratic.

The main question is to find parameters (V, F) such that $q_k(t)$ and $v_k(t)$ are bounded uniformly in t. Resonance is the contrary case - when as $T \to \infty$ for all or at least for some k:

$$\max_{0 \le t \le T} |v_k(t)| \to \infty, \max_{0 \le t \le T} |q_k(t)| \to \infty.$$

This can occur for many reasons:

1. External force is somehow synchronized with oscillations of the system itself. For the harmonic oscillator this is the case when $\omega = \omega_0$. Otherwise, there will be boundedness but the bound is proportional to $|\omega - \omega_0|^{-1}$. If the dissipation force $-\alpha v$ with constant $\alpha > 0$ is added, resonance does not occur. For general $\alpha(t)$ we give examples when resonance does not occur. We give also examples of other forces $f(t, v)$, see also [1], where there are examples when the deterministic dissipation force can be approximated by special random forces.

2. This could give impression that forces like

$$f(t) = \int a(\omega) \sin \omega t d\omega,$$

where the support of the function $a(\omega)$ contains ω_0, also provide resonance. But this is not always true. For example, if $a(\omega)$ is smooth and has finite support there will not be resonance. If $f(t)$ is random it should be in some sense symmetric with respect to ω_0. But now there is no general exact formulation.

3. For general case of N particles, if the external force $\sin \omega t$ acts on one specified particle, the result is similar: for boundedness ω should not belong to the spectrum of the matrix V. And in the boundedness case, the bound is also inverse proportional to its distance from the spectrum. Such results were obtained in [7] for the case of linear graph with periodic boundary conditions.

4. Resonance for one particle shows complete periodic transfer of kinetic (heat) energy to potential energy, and backwards. For large number of particles the situation is more complicated. The central question is to understand how both energies are distributed over parts of the system. In particular, the group of Petersburg physicists, see [7, 8], considered several linear models, similar to (1) for linear graph. They considered similar force, as in (1), acting on one particle (they call this "force loading"), and also the case called "kinematic loading" where one particle, say with number 1, has fixed dynamics $q_1(t) = C \sin \omega t$, which disturbes the whole chain.

5. Moreover, for the boundedness not only external forces but also the initial conditions (deterministic or random) $q_k(0), v_k(0), k = 1, ..., N$ are very important. However, here there are many asymptotics relative to large t and large N. In this case, without external forces, even if the initial conditions are uniformly bounded, it does not guaranty the boundedness at all. If the initial conditions are uniformly bounded but sufficiently random, then $q_k(t), v_k(t)$ essentially grow with time. That is, boundedness occurs relatively rare. But, for example, if the profile of initial conditions is sufficiently smooth in some exactly defined sense, there are many examples of boundedness in the series of papers [2, 3, 4, 5].

2. Conclusion

Main goal of our mathematical project is to find models of non-equilibrium mathematical physics which could model various qualitative phenomena in physical and biological media.

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