

# Introduction to stochastic models of transportation flows. Part I.

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## Abstract

We consider here probabilistic models of transportation flows. The main goal of this introduction is rather not to present various techniques for problem solving but to present some intuition to invent adequate and natural models having visual simplicity and simple (but rigorous) formulation, the main objects being cars not abstract flows. The papers consists of three parts. First part considers mainly linear flows on short time scale - dynamics of the flow due to changing driver behavior. Second part studies linear flow on longer time scales - individual car trajectory from entry to exit from the road. Part three considers collective car movement in complex transport networks.

## Contents

<b>1</b>	<b>Car flows</b>	<b>2</b>
1.1	Marked point fields . . . . .	2
1.2	Traffic capacity as a function of velocity . . . . .	4
1.3	Random dynamics with overtaking (random grammars) . . . . .	4
1.4	Growth of a jam . . . . .	5
1.5	Speeding up . . . . .	6
1.6	Short and long range order with time dependent velocities . . . . .	7
1.7	About relation of stochastic approach with kinetic equations . . . . .	8
<b>2</b>	<b>Calculation of mean velocity on the road</b>	<b>8</b>
2.1	Road as one-dimensional queuing network . . . . .	8
2.2	Mean speed slow down due to repair works . . . . .	9
2.3	Mean speed slow down due to slow cars . . . . .	11
<b>3</b>	<b>Analysis of complex transport system</b>	<b>12</b>
3.1	Closed networks . . . . .	12
3.2	Open networks . . . . .	14
3.3	Algorithm to find the critical load in closed networks . . . . .	15

Mathematical models of car traffic can be very different: from partial differential equations to modern computer graphics where points move on video along the edges of some graph. We consider here only probabilistic models. The main goal of this introduction is rather not to present various techniques for problem solving but to present some intuition to invent adequate and natural models having visual simplicity and simple (but rigorous) formulation, the main objects being cars not abstract flows. Moreover, the postulates and parameters in these models should allow statistical verification and estimation, and should not use doubtful physical analogies. Probabilistic models should be tightly related to psychology of drivers, if the drivers are not robots. There is no such theory, and we present an attempt to start it.

Probabilistic approach to transportation problems exists already more than 50 years, see [1, 2, 3], however in this paper we follow newer probability theories and consider more difficult problems. There are however some works which, for various reasons, we did not include in our introduction, see for example [12, 14]

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# 1 Car flows

## 1.1 Marked point fields

Current is normally the mean number  $J$  of cars crossing, per unit of time and in the given direction, some section of the road. Flow is normally either a static random configuration

$$\dots < x_i < x_{i-1} < \dots$$

of cars at a given time moment, or a probability measure on the set of car trajectories  $\{x_i(t)\}$ .

**What is a car configuration** In this paper the maximally detailed description for the configuration of cars at a given time moment will be as follows. The car is specified by some index  $\alpha$ . For example, a road (route) with  $k$  (traffic) lanes  $1, 2, \dots, k$ , is represented by  $k$  lines, parallel to  $x$ -axis. Then the index  $\alpha = (m, i)$  marks out the  $i$ -th car on the lane  $m$ . Index  $i$  enumerates cars on the lane so that  $i$ -th car follows car with index  $i - 1$ . Let  $d_\alpha$  the length of the car indexed by  $\alpha$ ,  $x_\alpha(t)$  - its coordinate (for example, of the front buffer). The cars move in the positive  $x$ -direction. Further on we omit the lane index (the reader can add it whenever necessary) and use only index  $i$ .

Denote

$$d_i^+(t) = x_{i-1}(t) - x_i(t) - d_{i-1}$$

the distance of the car  $i$  to the previous car at time  $t$ . The distance of the car to the following car

$$d_i^-(t) = d_{i+1}^+(t)$$

is also important for the driver.

**How probabilities on the car configurations are introduced** Formally, the point flow on the real line  $R$  is given by the probability measure on the set of all countable locally finite (that is finite on any bounded interval) subsets of  $R$ . Otherwise speaking, it is given by compatible systems of probabilities

$$P(I_1, k_1; \dots; I_n, k_n)$$

of the events that in the intervals  $I_j, j = 1, \dots, n$  there are exactly  $k_j$  particles.

The main question of course is how to specify this system more concretely. There are two big parts of probability theory which use different ways of doing this. These are renewal theory (see for example, [6]) and Gibbs point fields theory [23, 24]. The first is essentially simpler but works only in one-dimensional case. The second has deep relations with physics, is suitable also for multidimensional point configurations but is more complicated, and we shall not touch it here.

The simplest point flow is the Poisson flow, see [30]. Most natural way to understand it is as follows. Consider the interval  $[-N, N]$  and throw (choose) on it independently and randomly (more exactly, uniformly)  $M = [\rho N]$  points, where  $\rho > 0$  is some constant called density. It is easy to calculate the binomial probability  $P_{N,M}(k, I)$  that exactly  $k$  points will be inside the finite interval  $I$ . This probability tends, as  $N \rightarrow \infty$ , to the Poisson expression (Poisson distribution for given  $I$ )

$$P(k, I) = \frac{\{\rho|I|\}^k}{k!} e^{-\rho|I|}.$$

More general flows can be easily constructed on the half line  $[0, \infty)$ . Namely, random points

$$x_0 = 0, x_1, \dots, x_n, \dots$$

are defined as the sums

$$x_1 = \xi_1, x_2 = \xi_1 + \xi_2, \dots$$

of i.i.d. (independent identically distributed) random variables  $\xi_i > 0, i = 1, 2, \dots$ , with the distribution function  $G(x) = Pr(\xi_i < x)$ .

To define a translation invariant point flow on all real line one problem remains - how to choose point 0, from which one can consecutively place independent variables to the right and to the left. For this one should use the following (in fact, one of the main) assertion of the renewal theory. Let  $P(t, t + \Delta t)$  be the probability that the interval  $(t, t + \Delta t)$  will contain exactly one point. Then, if  $\xi_i$  have density, then on the half-line the limit  $\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} P(t, t + \Delta t)$  exists and tends (as  $t \rightarrow \infty$ ) to  $\rho = (E\xi)^{-1}$ . Then the limit (as  $t \rightarrow \infty$ ) probability

density of the event  $A(t, s)$ , that the distance from point  $t$  to the first random point  $x_i > t$  is greater than some  $s > 0$ , equals

$$\rho Pr(\xi_i = x_i - x_{i+1} > s) = \rho(1 - G(s)). \quad (1)$$

That is why the first (after 0) point  $x_1$  should be taken on the random distance from 0 with density (1). The following consecutive distances should be independent with distribution  $G(x)$ .

**Alternating flows** The distances in-between neighbor points are not necessarily identically distributed. The distributions can alternate. Take, for example, two sequences of i.i.d. random variables  $\xi_1, \xi_2, \dots$  and  $\eta_1, \eta_2, \dots$ , and put

$$x_{2n} = \xi_1 + \dots + \xi_n + \eta_1 + \dots + \eta_n, x_{2n-1} = \xi_1 + \dots + \xi_n + \eta_1 + \dots + \eta_{n-1}.$$

Then the construction of the flow on the whole line can be done as above.

In our case the (i.i.d.) lengths of cars  $d_i$  and (i.i.d.) functions  $d_i^+$  alternate.

**Marked flows** To each point  $x_i$  of the point process one can assign some variable  $\sigma_i$ , taking values in some set  $S$ . This variable is called a mark or a spin of the point (particle, car)  $i$ , then one can call the new object random marked point set (flow, process). It is defined by some probability measure on the set of sequences of pairs  $(x_i, \sigma_i)$ . Often the measure on the countable sets is given, that is the flow without marks being given, then  $\sigma_i$  are considered to be i.i.d. random variables. In section 1.5 we shall consider marked process where the marks will be the velocities of the cars, moreover with sufficiently complicated correlations of the velocities with the coordinates.

**About Markov processes** The models of time evolution of cars often use Markov processes and it is useful to remind some terminology. However, the mere definitions of the Markov process and its properties (for example, ergodicity) may differ. We give the definition we use for discrete time.

Consider on some phase space  $X$  a system of measures (transition probabilities)  $P(A|x)$ , meaning the probability that the process at time  $t + 1$  hits the set  $A \subset X$ , if at time  $t$  the process was at the state  $x \in X$ . If all measures  $P(A|x)$  are one-point, then it is equivalent to the deterministic map  $T : X \rightarrow X$ , that is  $P(\cdot|x)$  is the unit measure at the point  $T(x)$ . This deterministic map defines a dynamical system.

Note that the system of measures  $P(A|x)$  defines the mapping  $U$  of the set of all measures on  $X$  to itself

$$U\mu = \int P(\cdot|x)d\mu(x)$$

The notion of invariant (with respect to  $U$ ) measure is very important. Normally one studies its existence, uniqueness and other properties.

Using the system of transition probabilities one can construct various sequences of random variables  $\xi_n = x_n$  with values in  $X$ , or their distributions  $\mu_n$  on  $X$ , where  $\Pr(\xi_n \in A) = \mu_n(A)$ . The probability space is the set of trajectories  $\{x_n\}$ . For example, it can be stationary Markov process  $\xi_n, n \in \mathbb{Z}$ , where  $\mu_n$  are invariant measures on  $X$ , or non-stationary process  $\xi_n, n \in \mathbb{Z}_+$ , with given initial distribution  $\mu_0$  on  $X$ .

Markov process can be understood either as ONE of this sequences  $\xi_n$ , or as the whole family of such sequences. Correspondingly, the terminology (for example the notion of ergodicity) differs. Dynamical system with given invariant measure  $\mu$  is called ergodic if any invariant subset has  $\mu$  zero or one. Any stationary process can be considered as the dynamical system - time shift on the set of trajectories. Then the ergodicity can be defined as the ergodicity of this dynamical system. However more often, Markov process is understood more generally and the notion of ergodicity is defined differently. Namely, the process is called ergodic if there exists the only invariant measure  $\mu$  on  $X$ , and for any initial measure  $\mu_0$  as  $n \rightarrow \infty$  we have  $\mu_n \rightarrow \mu$  in the sense of weak convergence.

Note that Markov processes normally belong to one of two classes. The first class includes all processes such that there exists a positive measure on  $X$  (may be infinite), with respect to which all measures  $P(\cdot|x)$  are absolutely continuous. This includes all classical Markov process - finite and countable Markov chains, diffusion processes, etc. Such process are ergodic if there are no non-trivial invariant subsets (this property is called irreducibility), and there is a unique invariant measure. In most cases convergence to it from any initial state follows. For countable chains an equivalent condition is positive recurrence that is finiteness (for any pair  $x, y \in X$ ) of the mean hitting time of  $x$  starting from  $y$ .

The second class is characterized by the property that all measures  $P(\cdot|x)$  are mutually singular. All infinite particle processes are in this class. The theory of such processes is essentially more complicated.

## 1.2 Traffic capacity as a function of velocity

**Driver's psychology in the simplest flow** Detailed modeling of the driver psychology is impossible but some relation is obvious. Thus, the driver  $i$  sees several cars (often only one car in front of him) in the flow and chooses subjectively optimal distance to the preceding car. If the velocity  $v_{i-1}(t) = \frac{dx_{i-1}(t)}{dt}$  of the preceding car changes slowly, then one can assume that the driver's reaction is faster, and the chosen distance  $D_i^+$  depends of the velocity of the preceding car at the same time moment

$$D_i^+ = D_i^+(v_{i-1})$$

(index  $i$  says that the functions  $D_i^+(v)$  can be different for different drivers). Let us call the flow algorithmic at time  $t$ , if for all  $i$

$$d_i^+(t) = D_i^+(v_{i-1}(t)),$$

that is all velocities are consecutively defined by the velocity of the previous cars. Of course, individual functions can be interesting for policemen, but we are interested with their statistical characteristics. In probabilistic approach the functions  $D_i^+(v)$  become i.i.d. random variables, depending on the velocity  $v$  of the previous car as a parameter. The distribution of these functions cannot be deduced from mathematical or physical laws, but rather it depends on the individual and collective psychology of drivers and should be found experimentally, see [22].

**Deterministic dynamics without overtaking** If all cars, drivers and velocities  $v$  are identical, then many problems appear to be very simple. Let  $d$  be the length of cars and  $d^+ = D^+(v)$  be the distance to the previous car, which the driver strictly follows. Already such dynamics allows to understand many qualitative aspects.

Let us define the road capacity as maximally possible current through it

$$J_{max} = \max_v v\lambda(v),$$

where maximum is over the allowed interval of velocities, and where

$$\lambda(v) = \frac{k}{d + D^+(v)}$$

is the density of cars on  $k$ -lane road for a fixed velocity  $v$ . It is clear that the flow capacity can become smaller when the velocity increases. This simple conclusion says only that many drivers increase the distance to the previous car when the velocity becomes too big.

**Random dynamics without overtaking** We get the same conclusion if the velocities  $v$  are the same, but the functions  $d_i^+$  are random independent, and their expectations (for given  $v$ ) are equal to some number  $d^+(v)$ . We see that the fact of non-trivial dependence of the road capacity from the velocity is in fact the trivial consequence of the driver behavior and does not need any models. However, for nicer questions stochastic models are necessary. Now we will introduce a sufficiently general probabilistic model with a rich spectrum of phases. The well-known exclusion processes appear as a degenerate particular case. Other models see in [12, 10, 22].

## 1.3 Random dynamics with overtaking (random grammars)

Here we have an interesting connection with recently discovered object - random grammars, see [25]. We give short substantial description of one such model.

Let at time  $t = 0$  all cars are situated on the left half-axis, and the movement is one-lane. We subdivide the lane onto intervals (cells, enumerated by  $Z$ ) of fixed length and assume that there can be at most one car in the cell. Thus, finite sequence (group, cluster) of cars is identified with the pair  $(S, r)$ , where  $r \in Z$ , and  $S$  is a finite sequence (word) of three symbols 0, 1, 2

$$S = s_N \dots s_2 s_1.$$

Here 0 corresponds to empty cell, 1 - to active (fast) driver in the cell, 2 - to a quiet driver. The length  $N = N(t)$  of the word and all symbols  $s_k(t)$  change with time, with the restriction that always  $s_1(t) \neq 0$  for all  $t \geq 0$ . At any time  $t$  any symbol  $s_k(t)$  has coordinate  $x(s_k(t))$ . The coordinates

$$x(s_k(t)) = x(s_1(t)) - k + 1 \tag{2}$$

are uniquely defined by the coordinate  $x(s_1(t))$  of the first symbol, denoted by  $r = r(t)$ .

This dynamics models the process of acceleration and slowdown of different drivers. It is defined as continuous time Markov chain  $(S(t), r(t))$  on the set  $\{(S, r)\}$  of pairs. Jump intensities are defined as follows. Process  $r(t)$  models the movement of all group with common velocity  $v$ . Namely,  $r$  is increased by one with probability  $v dt$  for time  $dt$ , and all coordinates immediately change correspondingly to formula (2). Thus,  $S(t)$  describes replacements relative to some common motion, and is given by some random grammar, that is by the list of possible local substitutions (5 substitution types), that is a subword of  $S$  to another subword. Any substitutions of this list are independent and have different intensities (4 parameters). Here is the list:

1.  $10 \rightarrow 01$  - fast driver moves to empty place in front of him, simultaneously creating additional free place after him, with probability  $\lambda_0^+ dt$  for time  $dt$ ;
2.  $120 \rightarrow 021$  - fast driver overtakes quiet driver with intensity  $\lambda_1^+ dt$ ;
3.  $22 \rightarrow 202, 21 \rightarrow 201$  - a cautious driver brakes, increasing the distance in front of him with probability  $\lambda_2^- dt$ . Note that the length  $S$  increases here (extra empty cell appears), that causes the one-cell shift of all cars behind. This is a non-local jump, in fact such rearrangement takes some time, but we assumed that this time is on a shorter scale;
4.  $200 \rightarrow 020$  - a quiet driver accelerates with probability  $\lambda_2^+ dt$  (if he decides that in front of him there is too much free place).

It is necessary to say that for exact formulation of results which we only shortly describe and to get various qualitative types (phases) of movement, one needs to do various scalings of parameters  $t, N, \lambda$ . We shortly describe only 3 phases.

1. If  $\lambda_2^\pm$  are small relative to other two parameters, then cars of type 2 move synchronously with common velocity, and fast cars have additional relative velocity. If the number of fast drivers is small, then this relative velocity is defined by the movement of one car among static obstacles and depends on the density  $\rho_2$  of type 2 cars and density of holes  $\rho_0$ , it is approximately equal to

$$v_{rel} = \lambda_0^+ \rho_0 + 2\lambda_1^+ \rho_2$$

2. If  $\lambda_2^-$  is small with respect to other parameters (no non-local effects), and  $\lambda_2^+$  has the same order as  $\lambda_0^+, \lambda_1^+$ , then the difference between types disappears. We have a process close to the so called TASEP - totally asymmetric exclusion process. If

$$\lambda_0^+ = \lambda_1^+, \lambda_2^- = 0$$

then it coincides with TASEP (about TASEP see [12]).

3. If  $\lambda_2^+$  is small, and  $\lambda_2^-$  is large with respect to other two parameters, then the picture is different. Any overtaking  $120 \rightarrow 021$  immediately forces braking of the car 2 and, as a consequence, ALL following cars slow down. For the cars closer to the end of the word the slow down will be very essential, if the flow is sufficiently dense (few cells with zeros), as many type 2 cars will brake.

One can complicate the introduced dynamics, for example, to avoid discretization (see the end of this section), introducing positive real numbers instead of zeroes - distances between consecutive cars. This demands essential reformulation (see section 1.6), in particular for type 3 jumps, but rough qualitative effects will be the same.

## 1.4 Growth of a jam

If the incoming traffic flow to some fixed domain equals  $J_{in}$ , and outgoing flow is  $J_{out} < J_{in}$ , the number of cars in this domain increases in time  $t$  on

$$t(J_{in} - J_{out})$$

This is true, however, only if this domain is not situated on the road itself. For example, if the number of cars in the jam on the road increases, the answer is different. The problem is that the domain itself can grow because of the arriving cars. To make this more precise one should find an appropriate model.

Assume that the cars of the same length  $d$  move on one lane road with velocity  $v$  on the same distance  $d^+$  between consecutive cars. During some time  $t$  the movement has been stopped by some obstacle, for example by red traffic light. Assume that any car stops on the distance  $d_0^+ < d^+$  from the previous car.

**Exersize 1** Prove that during time  $t \rightarrow \infty$  the jam (that is maximal length  $L(t)$  of the part of road, where all cars stand still) before the obstacle will have the length asymptotically equal to

$$L(t) \sim_{t \rightarrow \infty} tv \frac{d + d_0^+}{d^+ - d_0^+}. \quad (3)$$

It seems that this result depends only on the mean values and remains true even if overtaking is possible. This was proved in [16] for independent movement of the cars (that is when the cars do not hinder each other), moreover the car velocities can fluctuate but have the same values and equal  $v$ . The proof is absolutely not trivial.

**Local widenings and contractions of the road** What occurs when a part of road with  $k$  lanes turns to the part with  $l$  lanes. Let this occur at the point  $x = 0$ .

Case  $k < l$ . Assume that maximally allowed velocity equals  $v_{max}$  and the drivers are assumed to be disciplined. All cars move along  $k$ -lane road with velocity  $v < v_{max}$ . And it is impossible to move faster due to the fundamental relation

$$d + D^+(v) = \rho^{-1}$$

between density  $\rho$  of cars and their velocity. Then along  $l$ -lane road the cars could conserve the density and move with the same speed, but  $\rho$  can change so that the cars will move with some greater speed  $v_1$ . The time gain is

$$\frac{L}{v} - \frac{L}{v_1}$$

Case  $k > l$ . Here three different situations are possible.

**Free flow** If the flow is very sparse, then the cars arrive at the point 0 alone and will not notice the decrease in the number of lanes.

**Growing jam** Denote  $J_k$  the incoming current and let  $J_{l,max}$  be maximally possible current along  $l$ -lane road. If  $J_k > J_{l,max}$ , then the jam will grow, and the number of cars in this jam will grow as  $t(J_k - J_{l,max})$ , more exactly as in formula (3).

**Delay** For the case  $J_k < J_{l,max}$  practical observations say that at point 0 jams of random lengths can appear, which however will not grow too much. There are no corresponding stochastic models. To create such model, it is useful to have a collection of models for shorter time phenomenon, Namely, when the standing cars at the traffic lights start their movement. We shall describe some models of this type now.

## 1.5 Speeding up

In [9] the cars are points

$$\dots < x_i(t) < x_{i-1}(t) < \dots$$

on the line. At initial time  $t = 0$  the cars stand still and are described by Poisson point field with density  $\rho < 1$ . The cars can have two velocities: 0 and 1, and overtaking is prohibited. Any incumbent car starts its movement with speed 1, independently of the others, in exponential time with mean 1. It can occur that the car with number  $i$  will reach the car  $i - 1$  when it did not start its movement. Then the car  $i$  stops and resumes its movement in exponential time after the moment when the car  $i - 1$  starts movement. This rule acts at any time. The model describes roughly how clusters of cars start to move.

The main result is that with probability 1 each car will stop finite number of times (if  $\rho < 1$ ). Let  $t_i$  be the minimal time moment after which the car  $i$  does not stop anymore. Then for any  $i$  and  $k$  and any time moments  $t_i, t_{i-1}, \dots, t_{i-k}$  the random variables

$$x_{i-1}(t_{i-1}) - x_i(t_i), x_{i-2}(t_{i-2}) - x_{i-1}(t_{i-1}), \dots, x_{i-k}(t_{i-k}) - x_{i-k+1}(t_{i-k+1})$$

will be independent and exponentially distributed. Otherwise speaking after leaving the jam, the cars will form Poisson configuration with the same density  $\rho$ .

Assume now that at time 0 all cars are situated on the left half-axis. The distribution is again Poisson with density  $\rho$ . Each point moves with velocity  $v > 0$ , if the distance to the next car from the right is not less than some  $d_{eff} > 0$ , and stands still otherwise. Here it is evident that any particle will not stop starting from some time moment. But in this model one can get more. Consider the following random variables: random time  $\tau_k^{(1)}$  when  $k$ -th point starts movement, random time  $\tau_k^{(2)}$  when this point does not stop anymore, the distance  $x_k$  between points  $k$  and 1, starting from time moment  $\tau_k^{(2)}$ .

**Problem 1** Find asymptotics of distributions of these random variables when  $k \rightarrow \infty$ .

The connection with delay problem is evident. Let we have two lanes and on any lane the flow density be  $\rho$ ; the joint flow thus have density  $2\rho$ . The cars from the first lane now want to squeeze into the second. The squeezing algorithms can be quite different. For example, any car can squeeze independently of the others, if its distances (along  $x$ -axis) to the cars from the second lane are not less than some  $d^+$ .

## 1.6 Short and long range order with time dependent velocities

Here the cars are represented by points  $x_i$ . To the car  $i$  a stationary random process  $w_i(t)$  is assigned, which defines time evolution of its velocity on empty road (that is when there are no obstacles in front of this car). This process implicitly defines the activity of the driver at given time  $t$ . The processes  $w_i(t)$  are mutually independent and are defined only by the psychology of the individual driver. Assume that there are constants  $0 < C_1 < C_2 < \infty$  such that for all  $t, i$

$$C_1 < w_i(t) < C_2.$$

The flow is given by initial coordinates  $x_i(0)$  of the cars, and their movement is defined as

$$x_i(t) = x_i(0) + \int_0^t v_i(s) ds,$$

where  $v_i(t)$  is the velocity of  $i$ -th car, defined below. Moreover, we assume that the initial coordinates are such that the distances  $d_i^+(0)$  are independent and, for example, are exponentially distributed with parameter  $\rho(0)$ .

We shall say that the car  $i$  has an obstacle in front of itself at time  $t$ , if

$$x_i(t-0) = x_{i-1}(t)$$

The process will be completely defined if for any  $t_1, \dots, t_n, i_1, \dots, i_n$  we could define finite-dimensional distribution of the vectors

$$(v_{i_1}(t_1), \dots, v_{i_n}(t_n))$$

where some  $i_k$  can coincide. The following rules do this job:

1. (rule for free road) if none of the cars  $i_1, \dots, i_k$  for  $k \leq n$  does not have in front of itself an obstacle, then the distribution of the vector  $v_{i_1}(t_1), \dots, v_{i_k}(t_k)$  coincides with the distribution of the vector  $w_{i_1}(t_1), \dots, w_{i_k}(t_k)$  and is independent of the distribution of the vector  $v_{i_{k+1}}(t_{k+1}), \dots, v_{i_n}(t_n)$ ;
2. (rule for an obstacle) if the car  $i$  has obstacle in front of it at time  $t$ , then  $v_i(t) = v_{i-1}(t)$ ;
3. (rule for overtaking) if the car  $i$  has obstacle in front of it at time  $t$ , then it overtakes this obstacle with some intensity  $\lambda$ , during the whole (random) time interval while  $w_i(t) > v_{i-1}(t)$ . The meaning of this condition is that the driver overtakes if his activity is higher.

Even for such simplest behavior of drivers and corresponding definition of the flow there are many problems. We formulate some of them.

Let us call **free phase** the case when the intensity of overtaking equals infinity. Then for any cars  $i, j$ , their velocities are independent and the covariance is zero, that is

$$Cov_{ij}(t) = Ev_i(t)v_j(t) - Ev_i(t)Ev_j(t) = 0$$

The **complete order phase** is when  $\lambda = 0$ . Then there is no overtaking at all and the cars move as a queue.

**Problem 2** Assume that all  $w_i(t)$  have the same distribution. Is it true that for small  $\lambda$  the movement will resemble complete order phase, and the large  $\lambda$  will resemble free phase? Does exist some third phase (cluster phase) for intermediate values of  $\lambda$ ?

Possibly to get more interesting qualitative picture one has to define the same process with the lengths  $d_i, d_i^+$ , and with additional indices corresponding to several lanes. Also the behavior of the driver can depend not only on the car in front but also of the car just behind.

## 1.7 About relation of stochastic approach with kinetic equations

For any particle  $x_i$  on the real line there is a mark - velocity  $v_i > 0$ . Then the flow of cars is defined by countable subset  $\{(x_i, v_i)\}$  of the phase space  $R \times R_+$ . And the random flow, at a given time, is a probability measure  $\mu$  on the set of such configurations. Let  $n(A) = n_\mu(A)$  be the mean number of particles in bounded set  $A$  of the phase space. If there exists a function  $f(x, v)$  such that for any  $A$

$$n(A) = \int_A f(x, v) dx dv,$$

then  $f$  is called one-particle (correlation) function of the measure  $\mu$ .

Although kinetic equations approach to transportation problems is known long ago [33] and was studied a lot in physical papers [34, 35] we think that rigorous mathematical deduction of kinetic equations (that is equations for  $f(t; x, v)$ ) from dynamics of individual cars stays open. Here we give only short comments.

Let at time  $t = 0$  be given a distribution  $\mu$  with one-point function  $f(0, x, v)$ . If particles move freely, that is each particle moves with fixed velocity, different for different cars, then

$$f(t + \delta; x, v) = f(t; x - v\delta, v).$$

Subtracting  $f(t; x, v)$  from both parts of this equality, dividing by  $\delta$  and taking limit  $\delta \rightarrow 0$ , we get

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = 0. \quad (4)$$

This is the simplest kinetic equation without collisions. Its solution is of course

$$f(t; x, v) = f(0; x - vt, v). \quad (5)$$

In general kinetic equations should contain collisional term. What is collision in traffic ? It could be close approaching of cars, that is the reason for the drivers to change their behavior, for example to perform overtaking. Also one can consider formation of a new compound particle - cluster of several cars with short lifetime, which decays shortly on smaller clusters or separate cars. Moreover, there can be clusters on various scales - such situation had never been formalized even in mathematical physics.

Another possible approach is to consider movement of fast cars in the media of slow cars. Let there exist two types of cars with velocities  $v$  and  $v_1 > v$  correspondingly. Slow cars move independently and their one-particle function looks like (5). In the coordinate system moving with speed  $v$ , slow cars will become standing obstacles. Rules of overtaking define velocity  $v_1(x)$  of fast cars, in the new coordinate system, in the vicinity of separate obstacle. This can give, under appropriate scaling, some kinetic picture. But as far as we know there were no efforts to write down rigorously even such simple models.

There are however papers where Burgers equation is deduced from stochastic particle picture. However, the corresponding particle process is far from one-dimensional car flow, see reviews on exclusion processes [31, 32].

## 2 Calculation of mean velocity on the road

We consider here simplest problems concerning diminishing of the road capacity because of random fixed obstacles (accidents, repair work) and because of slow cars. Our goal is to show (solving completely some model problems) that it is possible to get simple formulas which allow to understand main reasons why the mean speed can decrease. Our main assumptions is the homogeneity of the road, exits and gates and other parameters.

### 2.1 Road as one-dimensional queuing network

The following model is taken from [8], p. 117. Let we have infinite road and two types of cars, given by points of the real line, the points move in the right direction. First type (faster) cars move with constant speed  $v_1$ , cars of the second type (slow cars) have constant speed  $v_2 < v_1$ .

Assume that faster cars initially (at time  $t = 0$ ) have Poisson configuration on  $R$  with density  $\lambda_1$ . Slow cars initially are at the points

$$x_0 = 0 < x_1 < \dots < x_n < \dots,$$

where the distances  $x_k - x_{k-1}$  are identically distributed with mean  $\lambda_2^{-1}$  (not necessarily exponentially). Slow cars move completely independently - other cars do not influence them. Faster cars «interact» with any car when their coordinates coincide. Namely, they can overtake slow cars. When a faster car catches up with a



slow car, that is their coordinates coincide, then it follows slow car for some time, that is moves with the speed  $v_2$ . In exponential time with parameter  $\mu$  the fast car overtakes the slow car and moves again with the speed  $v_1$ . If the slow car catches up the group of fast cars still following some slow car, then the overtaking occurs as in a queue, in order how the fast cars follow the slow car.

Without loss of generality one can put  $v_2 = 0$ , and the speed of fast cars put equal to  $v = v_1 - v_2$ . That is why each slow car can be considered as a server, where the clients (fast cars) arrive and are waiting for the service (overtaking), they are served with intensity  $\mu$ .

Now the problem can be reduced to the linear queuing system which we shall describe. There is infinite sequence

$$S_0 \rightarrow \dots \rightarrow S_k \rightarrow S_{k+1} \rightarrow \dots$$

of nodes (servers) of two types. Each server  $S_k$  is a  $M/M/1$  type server with FIFO (first-in-first-out) discipline, that is in the natural order. These servers correspond to slow cars, and clients correspond to fast cars. For example  $S_0$  corresponds to the extreme left slow car. Second letter  $M$  means exponential service rate. First letter  $M$  means Poisson arrival stream. Thus the clients to  $S_0$  arrive as a stationary Poisson stream with intensity  $\lambda_1 v$ . From elementary queuing theory it has been known, firstly, that if  $\lambda_1 v < \mu$ , then the stationary regime will be established with the probabilities

$$P_n = (1 - r)r^n, r = \frac{\lambda_1 v}{\mu}.$$

that the length of the queue equals  $n$ . Secondly, (Burke theorem), that in the stationary regime the output flow from  $M/M/1$  will be Poisson with intensity equal to intensity of the input flow, which is  $\lambda_1 v$  in our case.

After the first node, the output flow, with random but the same for all clients time shift  $\frac{x_1 - x_0}{v}$ , arrives to node  $S_1$ , where also stationary regime will be established.

Let us find mean velocity of fast cars on the interval  $(x_0, x_N)$ ,  $N \rightarrow \infty$ . We shall assume that the stationary regime has already been installed. The time along this path is the sum from  $N$  overtaking and  $N$  paths between slow cars. Mean overtaking time is

$$\sum_{n=0}^{\infty} (1 - r)r^n \frac{(n + 1)}{\mu} = \frac{1}{(1 - r)\mu} = \frac{1}{\mu - \lambda_1 v},$$

and mean time to catch up the next slow car is

$$\frac{1}{\lambda_2 v}$$

Thus, the mean speed of fast cars is

$$v_{mean} = \frac{\lambda_2^{-1}}{(\mu - \lambda_1 v)^{-1} + (\lambda_2 v)^{-1}}.$$

In the next section we will consider more complicated situation with more general distributions.

## 2.2 Mean speed slow down due to repair works

On a long road the cars move with constant speed  $v$ , but encounter obstacles. The obstacles have small size comparative to distances between them, and we assume the obstacles are just points. They can appear on arbitrary part of the road  $(x, x + dx) \subset R$  during time interval  $(t, t + dt) \subset R$  with probability  $\lambda dx dt$ . More exactly, the pairs (coordinate and time moment of obstacle appearance)  $(x_j, t_j) \in R \times R_+$  form Poisson random field  $\Pi$  on  $R \times R_+$  with intensity  $\lambda$ . Another equivalent definition is that for any time interval  $I \subset R$  there is Poisson arrival stream with intensity  $\lambda|I|$ , moreover the arriving obstacle choose the point uniformly on the interval  $I$ .

Assume that  $j$ -th obstacle stays on the road for some random time  $\tau_j$ , after that it is cleared off from the road. Random variables  $\tau_j$  are assumed to be i.i.d. with distribution function  $Q(t)$  and independent from Poisson random field  $\Pi$ . Assume finiteness of  $m_Q = E\tau_j$  and  $m_Q^{(2)} = E\tau_j^2$ .

Further on we consider two cases. In the first case the bypass is prohibited and the car stands until the obstacle will be deleted, after this the car immediately moves with speed  $v$ . In the second case, assume the new car arrives to an obstacle and sees  $n$  cars, which have not yet bypassed this obstacle. The new car is allowed to bypass the obstacle and these  $n$  cars simultaneously. This takes takes some random time which does not depend on  $n$ . Denote  $\eta_{m,i}$  the random bypass time of  $m$ -th obstacle by  $i$ -th car. Random variables  $\eta_{m,i}$  are

assumed to be i.i.d. with distribution function  $F(u)$ . These assumptions are natural for small density of cars, with small number of cars accumulating after the obstacle. Below we consider the case of greater load.

First of all we shall find the mean speed of the car. Under above assumptions the cars do not impede each other, that is why it is sufficient to consider one car problem. Denote  $T(x)$  random time the car passes distance  $x$ . We will find the limit of the ratio  $\frac{x}{T(x)}$  as  $x \rightarrow \infty$ .

Let  $b = \lambda m_Q$ , and  $\zeta$  be a random variable with density

$$h(t) = m_Q^{-1}(1 - Q(t)). \quad (6)$$

Note that

$$E\zeta = \frac{1}{m_Q} \int_0^\infty t(1 - Q(t))dt = \frac{1}{m_Q} \int_0^\infty (1 - Q(t))d\left(\frac{t^2}{2}\right) = \frac{1}{2m_Q} \int_0^\infty t^2 dQ(t) = \frac{m_Q^{(2)}}{2m_Q},$$

where  $m_Q^{(2)}$  is the second moment of  $Q(t)$ .

Put  $\alpha = \min(\eta, \zeta)$ , where equality is in distribution. Moreover, the random variables  $\eta, \zeta$  are considered independent and  $\eta$  has distribution function  $F(u)$ . Put

$$a = E\alpha.$$

**Theorem 1** *With probability 1 as  $x \rightarrow \infty$*

$$\frac{x}{T(x)} \rightarrow \frac{v}{1 + abv}. \quad (7)$$

*Proof.* With no loss of generality assume that the car enters the road at the point  $x = 0$  at time  $t = 0$ . Let  $T_0(x)$  be the idle time of the car. Then obviously  $T(x) - T_0(x) = v^{-1}x$  and

$$\frac{x}{T(x)} = \frac{x}{T(x) - T_0(x) + T_0(x)} = \frac{1}{v^{-1} + x^{-1}T_0(x)}$$

Thus it is sufficient to find the limit of the ratio  $\frac{T_0(x)}{x}$  as  $x \rightarrow \infty$ . We want to show that

$$T_0(x) = \sum_{i=1}^{\pi(x)} \alpha_i, \quad (8)$$

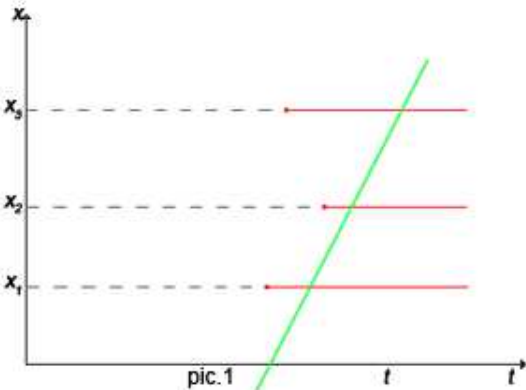
where  $\alpha_i$  are i.i.d. random variables distributed as  $\alpha$ ,  $\pi(x)$  is a random variable with Poisson distribution with parameter  $bx$ . Random variables  $\alpha_i$  and  $\pi(x)$  are assumed to be independent. The meaning of this formula, that the car on the distance  $x$  will meet  $\pi(x)$  obstacles and will loose random time  $\alpha_i$   $i$ -th obstacle.

From (8) and strong law of large numbers it follows easily that  $\frac{T_0(x)}{x} \rightarrow ab$  a.e. as  $x \rightarrow \infty$ .

Let us prove (8). Introduce marked Poisson point field  $\Pi_1$  on  $R \times R_+$  with configuration  $(x_j, t_j, \tau_j)$ , that is  $\tau_j$  is the mark at the point  $(x_j, t_j)$ . The following assertion can be found in [5]:

**Lemma 1** *Marked point field  $\Pi_1$  is equivalent (in probability) to the Poisson field on  $R \times R_+^2$  with intensity  $\lambda dx dt dQ(t)$ .*

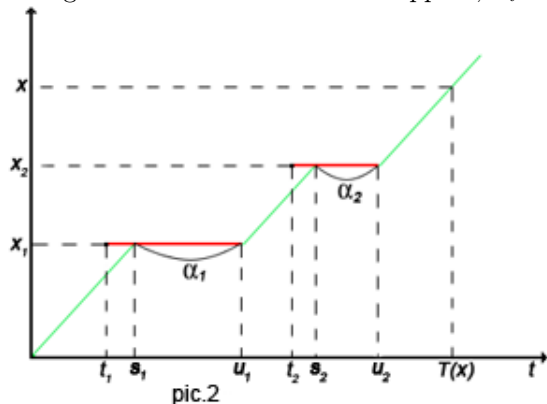
On picture 1 the obstacles are presented as horizontal segments. The coordinates of the initial point define place and time of its appearance (pair  $(x_j, t_j)$ ). The length of the segment is the time of obstacle stay on the road (mark  $\tau_j$ ).



Consider any straight line  $c_1t + c_2$  and consider its intersection points with horizontal segments. Denote  $\{x_i\}$  the space coordinates of these points as it is shown on picture 1. The next lemma is proved [7].

**Lemma 2** Configuration  $\{x_i\}$  is the Poisson process with intensity  $b = \lambda m_Q$ .

Picture 2 shows the trajectory of the car, which starts from point  $x = 0$  at time  $t = 0$ . Denote  $x_i$  the space coordinates of the obstacles appearing during movement of this car,  $t_i$  are the moments of their appearance,  $s_i$  – the moments when the car meets the obstacles,  $u_i$  – time moments when the car gets rid of them either bypassing or because the obstacle disappear;  $\alpha_i = u_i - s_i$  – car delay  $i$ -th obstacle.



From lemma 2 and from space-time homogeneity of Poisson field  $\Pi$  it follows that the points  $x_i$  make up Poisson process of intensity  $b$ .

The residual lifetime of an obstacle is its lifetime after the car catches it up. In other words, it is equal to the car delay on this obstacle.

**Lemma 3** The residual lifetime has the distribution with density  $h(s)$ , where  $h(s)$  is defined by (6).

In fact, from the properties of Poisson point field it follows that the conditional distribution of residual lifetime, under the condition that the full lifetime equals  $t$ , coincides with the uniform distribution on the interval  $[0, t]$ . By lemma 2 the probability of obstacle appearance on the interval  $dx$  equals  $\lambda m_Q dx + o(dx)$ , and the probability of appearance of an obstacle with full lifetime  $t$  in the interval  $dx$  is equal to  $\lambda t dQ(t) dx + o(dx)$ , that follows from lemma 1. As

$$\frac{\lambda t dQ(t) dx + o(dx)}{\lambda m_Q dx + o(dx)} = \frac{t dQ(t)}{m_Q}$$

is the conditional probability of appearance of an obstacle with lifetime  $t$ , then the density of residual lifetime is

$$\int_s^\infty \frac{t dQ(t)}{m_Q} \frac{ds}{t} = m_Q^{-1} (1 - Q(s)) ds = h(s) ds.$$

Lemma is proved.

If bypassing is possible, the car will lose time equal to minimum of bypass time and residual lifetime of the obstacle, that is  $\alpha_i = \min(\eta, \zeta)$ . Theorem is proved.

Let us discuss the result. We have seen already the meaning of the constant  $a$ , the constant  $b$  characterizes stationary density of obstacles in the space.

This result is rather exact for small density of cars, as there will not more than one car at the obstacle. For large density of cars the bypass time will increase dependent on the mean length of the queue at this obstacle.

### 2.3 Mean speed slow down due to slow cars

The road here is the real line  $R$ . Flows are not too dense, thus the length of cars does not play role, and at a given moment the position of the car is given by the point  $x_i(t) \in R$ , where  $i$  is the index enumerating the cars. Each car has predefined route: place and time of entrance  $x_{i,in}, t_{i,in}$ , and the exit coordinate  $x_{i,out}$ . But the exit time  $t_{i,out}$  depends on the road capacity. We define mean velocity of the car  $i$  as

$$V_i = \frac{x_{i,out} - x_{i,in}}{t_{i,out} - t_{i,in}}.$$

There are two types of cars - fast and slow, any car moves with constant speed in the right direction. Fast cars have speed  $v_1$ , slow cars have speed  $v_2$ , where  $v_1 > v_2 > 0$ . Put  $v = v_1 - v_2$ . Slow cars move along the road without stops, and fast cars - until they reach a slow car. After this the fast car  $i$  moves with slow car  $j$  for

some random time  $\tau_{i,j}$ , and then bypasses it immediately getting speed  $v_1$ . Main assumption is that random variables  $\tau_{i,j}$  are i.i.d. with distribution function  $F(s)$ .

This distribution function may be found statistically by two ways - by direct sampling and by estimating the density of obstacles, for example for one-lane road it can be the density of the opposite flow.

Arrivals of slow cars is given by the same Poisson point field  $\Pi$  with intensity  $\lambda$ , which was defined in the previous section. We will need more notation. To any slow car  $j$  we assign random distance  $\rho_j$  which it should pass, after that it will leave the road. Random variables  $\rho_i$  are assumed i.i.d. with distribution function  $G(r)$ . Moreover  $\rho_j$  does not depend on the random field  $\Pi$ . Assume existence of two first moments  $m_G = E\rho_1$ ,  $m_G^{(2)} = E\rho_1^2$ .

Slow car does not meet obstacles and passes all its route with speed  $v_2$ . Fast cars can be stopped by slow cars. Consider two cases. In the first case the overtaking is prohibited and fast car follows the slow car until the latter will leave the road, after that the fast car immediately gets its speed  $v_1$ . In the second case the overtaking is possible. More exactly, when  $i$ -th fast car catches up  $j$ -th slow car or a group of fast cars following  $j$ -th slow car, it takes random time  $\tau_{i,j}$  to bypass  $j$ -th slow car or all group of these cars. This time does not depend on the size of the group.  $\tau_{i,j}$  are assumed to be i.i.d. with distribution function  $F(u)$ .

Put  $d = \lambda m_G (v_2^{-1} - v_1^{-1})$ . Introduce random variable  $\beta$  with distribution density  $g(x) = m_G^{-1}(1 - G(x))$  and put  $\gamma = \min(v_2\tau_{1,1}, \beta)$ , where the equality is in distribution, and random variables  $\tau_{1,1}, \beta$  are assumed independent. Note that

$$E\beta = \frac{m_G^{(2)}}{2m_G}.$$

Put  $c = E\gamma$ .

**Theorem 2** *With probability 1 as  $x \rightarrow \infty$*

$$\frac{x}{T(x)} \rightarrow \bar{v}_1 = \frac{1 + dc}{1 + dc v_1 v_2^{-1}} v_1. \quad (9)$$

Proof. Let us note that this case can be reduced to the case  $v_2 = 0$ . Introduce new coordinate system moving with the speed  $v_2$  relative to the initial system. Find the mean speed of fast cars in the new coordinate system with the formula (7), substituting  $v = v_1 - v_2$ ,  $b = \frac{\lambda m_G}{v_2}$ ,  $a = \frac{c}{v_2}$ :

$$\frac{1}{(v_1 - v_2)^{-1} + \frac{\lambda m_G}{v_2^2}} = \frac{v_1 - v_2}{1 + dc v_1 v_2^{-1}}.$$

Then the mean speed of a fast car relative to the initial coordinate system is

$$\bar{v}_1 = \frac{v_1 - v_2}{1 + dc v_1 v_2^{-1}} + v_2 = \frac{1 + dc}{1 + dc v_1 v_2^{-1}} v_1.$$

### 3 Analysis of complex transport system

Graph is a natural tool to describe transport networks (for example, city streets), where the set  $V$  of vertices represent crossroads (nodes or service stations), and the set  $L = \{(i, j)\}$  of edges represent segments of streets without crosses. Let  $N$  be the number of vertices. We assume that there not more than one edge between two vertices.

Mostly the following two classes of networks were studied. First class includes (by the name of the authors and in the order of increasing generality) Jackson networks, BCMP and DB networks (see [8], [26]). Therein the client (communication, car, job), after being served in some vertex, chooses randomly the next vertex. The second class are the Kelly networks (see [8]), where each client has apriori fixed route. These classes have much in common - close results and techniques. For example, they may have very useful multiplicativity property, that is the stationary distribution has the so called product form. Here we consider only the first class of networks.

#### 3.1 Closed networks

If the cars do not come from without and do not leave outside, the network is called **closed**. Then the number of cars in the network is conserved and we denote it by  $M$ . The movement of a car is defined as follows. The car is waiting for some time on the cross  $i$  and is directed afterwards to some vertex  $j$ . The choice of  $j$  is defined

by some stochastic matrix, we shall call it the routing matrix,  $P = \{p_{ij}\}_{i,j=1,\dots,N}$ , where  $p_{ij}$  is the probability that from vertex  $i$  the car will go (after service time) to vertex  $j$  (for example, to the right, left or straight on), that is along the street  $(i, j)$ .

Stochastic matrix  $P$  defines finite discrete time Markov chain with state space  $V = \{1, \dots, N\}$ . We assume this Markov chain irreducible. Then the system of linear equations

$$\rho P = \rho, \rho = (\rho_1, \dots, \rho_N) \iff \sum_{i=1}^N \rho_i p_{ij} = \rho_j, j = 1, \dots, N \quad (10)$$

has a unique solution (up to a common factor). Normed solution gives its stationary distribution

$$\pi_i = \frac{\rho_i}{\sum_{i=1}^N \rho_i}, i = 1, \dots, N$$

For each vertex  $i \in V$  we define a function  $\mu_i(n_i)$  of the number  $n_i$  of cars in  $i$ -th vertex, where  $\mu_i(0) = 0$  and  $\mu_i(n_i) > 0$  for  $n_i > 0$ . This function characterizes capacity of this node and defines the intensity of output flow from this vertex. Namely, the probability that from node  $i$ , during time period  $t$ , exactly one car will leave the vertex is  $\mu_i(n_i)dt + o(dt)$ , under the condition that in the node there are  $n_i$  cars at time  $t$ . Using queuing theory terminology we call  $\mu_i(n_i)$  the service intensity in vertex  $i$ .

The order in which the arriving clients are served is defined by service discipline (protocol). The simplest possibility is FIFO - first come first served. If there are  $n_i$  cars in the vertex  $i$ , then the first car in the queue is served with intensity  $\mu_i(n_i)$ .

More general service discipline is the resource sharing discipline, the resource here is the capacity of the crossroad. This means that resource is divided in some proportion between cars in this crossroad. We assume that that  $k$ -th car in  $i$ -th node is served with intensity  $\mu_{i,k}(n_i) \leq \mu_i(n_i)$  so that

$$\sum_{k=1}^{n_i} \mu_{i,k}(n_i) = \mu_i(n_i).$$

For example, common resource can be divided equally between each car in the queue

$$\mu_{i,k}(n_i) = \frac{\mu_i(n_i)}{n_i}$$

If  $\mu_{i,1}(n_i) = \mu_i(n_i)$ , then we get FIFO discipline. Thus, the intensities  $\mu_{i,k}(n_i)$  completely define service discipline in the nodes.

Dynamics in the network is described by  $N$ -dimensional continuous time Markov chain  $\xi(t) = (\xi_i(t), i = 1, \dots, N)$ , where  $\xi_i(t)$  is the number of cars-at vertex  $i$  at time  $t$ . Thus, the state space  $S_M$  is the set of all vectors with nonnegative integer coordinates  $\bar{n} = (n_1, \dots, n_N)$ , such that  $n_1 + \dots + n_N = M$ .

Let  $e_i$  be a base vector, where  $i$ -th coordinate is 1, and the rest coordinates are 0. From state  $\bar{n}$  Markov chain  $\xi(t)$  can jump to the state  $T_{i,j}\bar{n} = \bar{n} - e_i + e_j, i \neq j$ , with intensity

$$\alpha(\bar{n}, T_{i,j}\bar{n}) = \mu_i(n_i)p_{i,j}, \quad (11)$$

under the condition that  $n_i \neq 0$ . The jump  $\bar{n} \rightarrow T_{i,j}\bar{n}$  corresponds to the fact that after leaving cross  $i$  the car arrives to cross  $j$ .

Note that Markov chain  $\xi(t)$  is uniquely defined by the routing matrix  $P$  and service intensities  $(\mu_i(n_i), i = 1, \dots, N)$ .

Let  $\rho = (\rho_1, \dots, \rho_N)$  be the solution of equations (10), which we consider as a formal equation for intensities  $\rho_i$  of input flows to the nodes (in the stationary regime these are equal to output intensities). Solving these equations, we find  $\rho_i$ . Then the stationary distribution  $\nu(n_1, \dots, n_N)$  of Markov chain  $\xi(t)$  looks like

$$\nu(n_1, \dots, n_N) = \frac{1}{Z_{N,M}} \prod_{i=1}^N \frac{\rho_i^{n_i}}{\mu_i(1)\mu_i(2)\dots\mu_i(n_i)}, \quad (12)$$

where the normalizing factor (the canonical partition function)

$$Z_{N,M} = \sum_{n_1+\dots+n_N=M} \prod_{i=1}^N \frac{\rho_i^{n_i}}{\mu_i(1)\mu_i(2)\dots\mu_i(n_i)},$$

This can be verified by direct substitution of (12) to Kolmogorov equations for stationary probabilities, see for example [8].

### 3.2 Open networks

Consider network consisting of  $N$  nodes. Contrary to close network, total number of cars is not fixed. Assume that to node  $i$  arrives (from outside) Poisson flow with intensity  $\lambda_i$ ,  $i \in \{1, \dots, N\}$ .

The routing matrix  $P = \{p_{ij}\}_{i,j=1,\dots,N}$  is assumed to be decomposable and

$$\forall i : \sum_{j=1}^N p_{ij} \leq 1, \quad \exists i_0 : \sum_{j=1}^N p_{i_0 j} < 1. \quad (13)$$

Similar to closed network,  $p_{i,j}$  is the probability that a car from node  $i$  goes to node  $j$ . The difference is that there is also probability

$$p_{i0} = 1 - \sum_{j=1}^N p_{ij}.$$

for a car at node  $i$  to leave the network.

As for closed network, let  $\mu_i(n_i)$  be the service intensity at node  $i$ . Then the car at node  $i$  leaves the network with intensity  $\mu_i(n_i)p_{i,0}$ .

The dynamics of an open network is described with  $N$ -dimensional continuous time random vectors  $\eta(t) = (\eta_i(t), i = 1, \dots, N)$ , where  $\eta_i(t)$  is the number of cars at node  $i$  at time  $t$ . Random process  $\eta(t)$  is a continuous time Markov chain with state space  $S$ , where  $S$  is the set of  $N$ -dimensional vectors with non-negative integer components  $\bar{n} = (n_1, \dots, n_N)$ . From the state  $\bar{n}$  Markov chain  $\xi(t)$  can jump to one of the states  $T_{i,j}\bar{n} = \bar{n} - e_i + e_j$ ,  $T_{i,0}\bar{n} = \bar{n} - e_i$ ,  $T_i\bar{n} = \bar{n} + e_i$  with intensities

$$\begin{aligned} \alpha(\bar{n}, T_{i,j}\bar{n}) &= \mu_i(n_i)p_{i,j}, \\ \alpha(\bar{n}, T_{i,0}\bar{n}) &= \mu_i(n_i)p_{i,0}, \\ \alpha(\bar{n}, T_i\bar{n}) &= \lambda_i, \end{aligned} \quad (14)$$

under condition that  $T_{i,j}\bar{n}, T_{i,0}\bar{n}, T_i\bar{n} \in S$ .

Thus, Markov chain  $\eta(t)$  is uniquely defined by the triplet  $(\lambda, \mu, P)$ , where  $\lambda = (\lambda_1, \dots, \lambda_N)$  is the vector of intensities of input flows,  $\mu = (\mu_i(n_i), i = 1, \dots, N)$  – the vector of service intensities in the nodes and  $P$  is the routing matrix.

Consider formal equation for input intensities to the nodes (in the stationary regime they are equal to output intensities)

$$\rho = \lambda + \rho P \iff \rho_i = \lambda_i + \sum_{k=1}^N \rho_k p_{ki}, \quad \forall i \quad (15)$$

Under the condition (13) and the irreducibility of  $P$ , this equation has a unique solution which can be written as follows

$$\rho = \lambda + \sum_{n=1}^{\infty} \lambda P^n.$$

Consider now the case when the service intensities  $\mu_i(n_i) \equiv \mu_i$  do not depend on the number of cars in nodes. Define the loads in the nodes by the formula

$$r_i = \frac{\rho_i}{\mu_i}, \quad i = 1, \dots, N.$$

The following theorem can be found, for example in [8], [20], it is called Gordon-Newell.

**Theorem 3** *Markov chain  $\eta(t)$  is ergodic if and only if for all  $i = 1, \dots, N$  we have  $r_i < 1$ . Then the stationary distribution is given by*

$$\sigma(n_1, \dots, n_N) = \prod_{i=1}^N (1 - r_i) r_i^{n_i}.$$

It easily follows from this theorem that the mean queue lengths in the stationary regime are-

$$m_i = \frac{r_i}{1 - r_i}.$$

If in some nodes  $i_1, \dots, i_k$  the load is strictly larger than 1, then Markov chain  $\eta(t)$  is transient. Then the mean queue lengths at nodes  $i_1, \dots, i_k$  tend to infinity as  $t \rightarrow \infty$ . A detailed analysis of open networks see in [19]. In particular, it is shown that in the nodes with the load exceeding 1, mean queue lengths grow linearly with time. There are also explicit formulas for the mean speed of growth.

### 3.3 Algorithm to find the critical load in closed networks

This section is based on the paper [18]. Consider a sequence of closed networks  $J_N$ ,  $N = 1, 2, \dots$ . Network  $J_N$  consists of  $N$  nodes and  $M = M(N)$  cars. Service intensities at the nodes of network  $J_N$  do not depend on the queue length :  $\mu_{i,N}(n_i) \equiv \mu_{i,N}$ . Let  $P_N = \{p_{i,j,N}\}$  be the routing matrix in the  $N$ -th network;  $P_N$  is assumed to be irreducible.

Let  $\rho_N = (\rho_{1,N}, \dots, \rho_{N,N})$  be the vector with positive components satisfying equation

$$\rho_N = \rho_N P_N. \quad (16)$$

Relative loads in the nodes are defined as

$$r_{i,N} = C_N^{-1} \rho_{i,N} \tau_{i,N},$$

where  $\tau_{i,N} = \mu_{i,N}^{-1}$  and  $C_N = \max_{i=1, \dots, N} \rho_{i,N} \tau_{i,N}$ . It is evident that  $r_{i,N} \in [0, 1]$ .

Correspondingly to (12) the stationary distribution of the number of cars  $\xi_{i,N,M}$  in the nodes of  $J_N$  is equal to

$$P_{N,M}(\xi_{i,N,M} = n_i, i = 1, \dots, N) = \frac{1}{Z_{N,M}} \prod_{i=1}^N r_{i,N}^{n_i},$$

where the normalizing factor (canonical partition function)

$$Z_{N,M} = \sum_{n_1 + \dots + n_N = M} \prod_{i=1}^N r_{i,N}^{n_i}. \quad (17)$$

Many important network characteristics can be expressed in terms of this partition function.

**Exersize 2** Show that the mean number of cars in the  $i$ -th node in the stationary regime is

$$m_{i,N,M} = E\xi_{i,N,M} = \frac{r_{i,N}}{Z_{N,M}} \frac{\partial Z_{N,M}}{\partial r_{i,N}} \quad (18)$$

We are interested in cases when  $N, M$  are large enough, more exactly  $N, M \rightarrow \infty$  and so that  $\frac{M}{N} \rightarrow \lambda = const$ , that is the number of cars per one node is constant. We shall see that  $\lambda$  defines existence or non-existence of jams in the network.

To formulate results as general as possible, we shall assume weak convergence of relative loads  $r_{i,N}$ . More exactly, define sample measure on the interval  $[0, 1]$

$$I_N(A) = \frac{1}{N} \sum_{i: r_{i,N} \in A} 1,$$

where  $A$  is arbitrary Borel subset of  $[0, 1]$ . Assume that as  $N \rightarrow \infty$  the measures  $I_N$  weakly converge to some probability measure  $I$  on  $[0, 1]$ .

**Remark 1** It could be interesting to find concrete sequences of growing networks for which the limiting measure  $I$  can be found explicitly. Some examples where measure  $I$  is one-point one can find in the bibliography to the paper [18].

In terms of limiting measure  $I$  we shall find critical value  $\lambda_{cr}$  of the density, so that for  $\lambda < \lambda_{cr}$  mean lengths of the queues are uniformly bounded. If  $\lambda \geq \lambda_{cr}$ , then at the node with maximal value of load the mean length of the queue tends to infinity, that means the existence of a jam at this node. Put

$$h(z) = \int_0^1 \frac{r}{1-zr} dI(r),$$

where  $z \in \mathbb{C} \setminus [1, +\infty)$ . The function  $h(z)$  is strictly increasing on  $[0, 1]$ . Put

$$\lambda_{cr} = \lim_{z \rightarrow 1^-} h(z).$$

We shall assume that  $\lambda_{cr} > 0$ .

**Theorem 4** • If  $\lambda < \lambda_{cr}$ , then mean lengths of queues are uniformly bounded: there exists constant  $B$ , such that  $m_{i,N} < B$ , uniformly in  $N \geq 1$  and  $1 \leq i \leq N$ .

- If  $\lambda \geq \lambda_{cr}$  and  $i(N)$  satisfies condition  $r_{i(N),N} = 1$ , then  $m_{i(N),N} \rightarrow \infty$ , as  $N \rightarrow \infty$ , that is jams will be in the nodes where the load is maximal.

For  $z \in \mathbb{C} \setminus [1, +\infty)$  put

$$S_N(z) = -\lambda(1 + \varepsilon_N) \ln z - \frac{1}{N} \sum_{i=1}^N \ln(1 - zr_{i,N}), \quad (19)$$

$$S(z) = -\lambda \ln z - \int_0^1 \ln(1 - zr) dI(r),$$

where  $\lambda(1 + \varepsilon_N) = \frac{M}{N}$ .

Introduce the generating function (the grand partition function)

$$\Xi_N(z) = \sum_{M=0}^{\infty} z^M Z_{N,M} = \prod_{i=1}^N \frac{1}{1 - zr_i}, \quad |z| < 1.$$

By Cauchy formula and (19) we have the following expression for the partition function (17):

$$Z_{N,M} = \frac{1}{2\pi i} \int_{\gamma} \frac{\Xi_N(z)}{z^{M+1}} dz = \frac{1}{2\pi i} \int_{\gamma} \frac{\exp(NS_N(z))}{z} dz, \quad (20)$$

where  $\gamma = \{z \in \mathbb{C} : |z| = \sigma < 1\}$ . For the means, accordingly to (18), we have

$$m_{i,N} = \frac{1}{2\pi i Z_N} \int_{\gamma} \frac{r_{i,N}}{1 - zr_{i,N}} \exp(NS_N(z)) dz. \quad (21)$$

One can show that for the stationary distribution of queue lengths the following formula holds

$$P_{N,M}(\xi_{1,N,M} = n_1, \dots, \xi_{K,N,M} = n_K) = \frac{1}{2\pi i Z_N} \int_{\gamma} z^{-1} \prod_{i=1}^K (1 - zr_{i,N})(zr_{i,N})^{n_i} \exp(NS_N(z)) dz. \quad (22)$$

In the proof of theorems of this section, essential role is played by the steepest descent method (see, [27]), more exactly, its generalization concerning the case when the function in the exponent depend on  $N$ . From equation

$$\frac{\partial S_N(z)}{\partial z} = 0 \quad (23)$$

we find steepest descent points. Let  $z_{0,N}$  be the root of this equation lying in  $(0, 1)$ .

**Exercise 3** Show that all roots of equation (23) are real and positive. Always there exists unique root, which belongs to the interval  $(0, 1)$ .

Let  $z_0$ -be the root of equation

$$h(z) = \frac{\lambda}{z} \Leftrightarrow \frac{\partial S(z)}{\partial z} = 0, \quad (24)$$

lying in the interval  $(0, 1)$ .

**Exercise 4** • Prove that for all  $\lambda$  there exists limit  $\lim_{N \rightarrow \infty} z_{0,N} = z_0 = z_0(\lambda) > 0$ .

- If  $\lambda < \lambda_{cr}$ , then  $z_0(\lambda)$  is the root of equation (24);  $z_0(\lambda)$  is strictly increasing in  $\lambda$ ,  $z_0(\lambda) \in (0, 1)$ ,  $\lim_{\lambda \rightarrow \lambda_{cr-}} z_0(\lambda) = 1$ .
- If  $\lambda \geq \lambda_{cr}$ , then  $z_0 = 1$ .

In the next theorem we find the asymptotics of the partition function and limiting distribution for the sequence of closed networks  $J_N$ .



**Theorem 5** Let  $\lambda < \lambda_{cr}$ , then

- As  $N \rightarrow \infty$  the partition function  $Z_N$  and the free energy  $F_N = \frac{1}{N} \ln Z_N$  have the following asymptotics

$$Z_N \sim \frac{\exp(NS_N(z_{0,N}))}{z_0 \sqrt{2\pi N S''(z_0)}}, \quad F_N = \frac{1}{N} \ln Z_N \sim S(z_0).$$

- If for  $i = 1, \dots, K$  there exist the limits  $r_i = \lim_{N \rightarrow \infty} r_{i,N}$ , then

$$\lim_{N \rightarrow \infty} m_{i,N} = \frac{z_0 r_i}{1 - z_0 r_i},$$

$$\lim_{N \rightarrow \infty} P_{N,M}(\xi_{1,N,M} = n_1, \dots, \xi_{K,N,M} = n_K) = \prod_{i=1}^K (1 - z_0 r_i) (z_0 r_i)^{n_i}.$$

Thus in the limit we get an open network consisting of independent queues.

**Proof of theorems 4 and 5** We give more general result from which the theorems 4 and 5 follow. Let  $U_d(v) = \{z \in \mathbb{C} : |z - v| < d\}$ . Consider the contour  $\gamma = \{z \in \mathbb{C} : |z| = z_0(\lambda)\}$ .

**Theorem 6** Let  $\lambda < \lambda_{cr}$  and  $f(\theta, z)$ ,  $\theta \in \Theta$ , be a family of functions holomorphic in the ring  $\{z \in \mathbb{C} : z_0(\lambda) - \delta_0 < |z| < z_0(\lambda) + \delta_0\}$  for some  $\delta_0 > 0$ , uniformly bounded in this ring and such that for sufficiently small  $\epsilon > 0$  there exists such  $\delta_u > 0$  and nonzero constant  $f_u$ , such that  $|f(\theta, z)/f_u - 1| < \epsilon$  for  $z \in U_{2\delta_u}(z_0)$ ,  $\theta \in \Theta$ .

Then, for  $N$  sufficiently large, uniformly in  $\theta \in \Theta$

$$\frac{1}{2\pi i} \int_{\gamma} f(\theta, z) \exp(NS_N(z)) dz = \frac{f_u \exp(NS_N(z_{0,N}))}{\sqrt{2\pi N S''(z_0)}} (1 + \zeta_N),$$

where  $|\zeta_N| < 25\epsilon$ .

Proof of this theorem is based on the saddle-point method, see [27]. Difference from the standard situation is that the function in the exponent depends on  $N$ . Detailed proof one can find in the original paper [18].

Proof of theorem 5. Using theorem 6 we prove the first part of the theorem 5. Using (20) we have

$$Z_{N,M} = \frac{1}{2\pi i} \int_{\gamma} \frac{\exp(NS_N(z))}{z} dz,$$

where  $\gamma = \{z \in \mathbb{C} : |z| = z_0(\lambda)\}$ . Putting  $f(\theta, z) = z^{-1}$ ,  $f_u = z_0^{-1}$  and using theorem 6, we get that for any sufficiently small  $\epsilon > 0$  for sufficiently large  $N$

$$Z_N = \frac{\exp(NS_N(z_{0,N}))}{z_0 \sqrt{2\pi N S''(z_0)}} (1 + \zeta_N), \quad |\zeta_N| < 25\epsilon. \quad (25)$$

The second part of theorem 5 can be proved similarly by using (21) for the mean queue length and (22) for joint distribution of the queue lengths.

**Exersize 5** Prove third part of the theorem 5, using theorem 6 and formulas (21), (22).

Proof of theorem 4. To prove the first part of theorem 4, consider the family of functions

$$f(\theta, z) = \frac{A}{z} + \frac{\theta}{1 - z\theta}, \quad \theta \in \Theta = [0, 1], \quad A > 0, \quad f_u = \frac{A}{z_0}.$$

Fix small  $\epsilon > 0$  and choose  $\sigma_u = \frac{\epsilon}{8}$ ,  $A = \frac{16z_0}{(1-z_0)\epsilon}$ . By theorem 6 we have for sufficiently large  $N$  and all  $\theta \in \Theta$

$$\frac{1}{2\pi i} \int_{\gamma} \left( \frac{A}{z} + \frac{\theta}{1 - z\theta} \right) \exp(NS_N(z)) dz = \frac{A \exp(NS_N(z_{0,N}))}{z_0 \sqrt{2\pi N S''(z_0)}} (1 + \zeta_N), \quad |\zeta_N| < 25\epsilon.$$

Dividing by  $Z_N$  and applying (25) to the right-hand part of the resulting equality, we get for sufficiently large  $N$

$$A + \frac{1}{Z_N} \frac{1}{2\pi i} \int_{\gamma} \frac{\theta}{1 - z\theta} \exp(NS_N(z)) dz = A(1 + \zeta'_N), \quad |\zeta'_N| < 30\epsilon.$$

From last equality and formula (21) we have uniform boundedness of  $m_{i,N}$ .

Prove the second part of theorem 4. For this we need the following monotonicity property: for any  $M_2 \geq M_1 > 0$  and any  $N \geq 1$  the inequality  $m_{i,M_2,N} \geq m_{i,M_1,N}$  holds.

As  $z_0(\lambda)$  is strictly increasing in  $\lambda$ ,  $z_0(\lambda) \in (0, 1)$  and  $\lim_{\lambda \rightarrow \lambda_{cr-}} z_0(\lambda) = 1$ , then the function

$$\frac{z_0(\lambda)}{1 - z_0(\lambda)}$$

is monotone increasing and tends to  $\infty$ , when  $\lambda \nearrow \lambda_{cr}$ . That is why for any  $m > 0$  there exists such  $\lambda' = \lambda'(m) < \lambda_{cr}$ , that

$$\frac{z_0(\lambda')}{1 - z_0(\lambda')} = m + 1.$$

Without loss of generality we can assume that  $i(N) \equiv 1$  and  $r_{1,N} = 1$ . If we put  $M'(N) = [\lambda'N]$ , then by theorem 5

$$\lim_{N \rightarrow \infty} m_{1,M'(N),N} = \frac{z_0(\lambda')}{1 - z_0(\lambda')}.$$

It follows that for sufficiently large  $N$

$$m_{1,M'(N),N} > \frac{z_0(\lambda')}{1 - z_0(\lambda')} - 1 = m.$$

But  $M/N \rightarrow \lambda \geq \lambda_{cr} > \lambda'$ , that is why for sufficiently large  $N$  we have  $M(N) \geq M'(N)$ . By monotonicity  $m_{1,N} = m_{1,M(N),N} \geq m_{1,M',N} > m$  for sufficiently large  $N$ . This proves that  $m_{1,N} \rightarrow \infty$ .

**Technical generalizations and mathematical problems** We assumed above instantaneous displacement between crosses. I.e. we did not take into account time spent along the streets. This can be easily overpassed by introducing more extended graph. Namely, introduce additional vertices  $u_{ij}$ , corresponding to the street between crosses  $i$  and  $j$ , and mean duration  $\tau_{ij} = \mu_{ij}^{-1}$  of movement along the streets. In the queuing terms this means that the streets are considered as new service nodes with infinite number of servers and exponential service time with mean  $\tau_{ij} = \mu_{ij}^{-1}$ .

Note that the results of section 3.1 can be generalized to the case, when the network contains nodes with infinite number of servers. Let, for example, the network contain one such node ( $i = 0$ ) and  $\mu_{0,N}(n) = n\nu_N$  is the service intensity in this node. Let  $\rho_N = (\rho_{0,N}, \dots, \rho_{N,N})$  be the solution of equation (16). Then relative loads can be defined by formula

$$r_{i,N} = \frac{\mu_{0,N} \rho_{i,N}}{\rho_{0,N} \mu_{i,N}},$$

so that  $r_{0,N} = 1$ . According to (12), the stationary distribution of the queue lengths is

$$P_{N,M}(\xi_{i,N,M} = n_i, i = 1, \dots, N) = \frac{1}{\widehat{Z}_{N,M}} \frac{1}{(M - \sum_{i=1}^M n_i)!} \prod_{i=1}^N r_{i,N}^{n_i},$$

where

$$\widehat{Z}_{N,M} = \sum_{n_1 + \dots + n_N \leq M} \frac{1}{(M - \sum_{i=1}^M n_i)!} \prod_{i=1}^N r_{i,N}^{n_i},$$

and the grand partition function is

$$\widehat{\Xi}_N(z) = e^z \prod_{i=1}^N \frac{1}{1 - zr_{i,N}}.$$

Put

$$q_{i,N} = \frac{r_{i,N}}{p_N}, \quad w = zp_N, \quad p_N = \max_{1 \leq i \leq N} r_{i,N}.$$

Then

$$\widehat{\Xi}_N(w) = e^{w/p_N} \prod_{i=1}^N \frac{1}{1 - wq_{i,N}}.$$

Under assumption that  $p_N N \rightarrow \alpha > 0$  as  $N \rightarrow \infty$  we can find critical value of density  $\lambda$  with the formula

$$\lambda_{cr} = \alpha^{-1} + \lim_{w \rightarrow 1^-} \int_0^1 \frac{q}{1 - wq} dI(q),$$

where, as earlier, measure  $I$  is the weak limit of sampling measures for  $N \rightarrow \infty$

$$I_N(A) = \frac{1}{N} \sum_{i: q_{i,N} \in A} 1,$$

where  $A$  is an arbitrary Borel subset of  $[0, 1]$ .

In [21] for closed networks similar results are obtained for more general dependence of intensities of the queue lengths.

The trick with the graph extension allows get rid of other restriction: that for given cross the mean duration of red light is the same for all directions. It is necessary then, instead of vertex  $i$ , corresponding to this cross, introduce several vertices  $(i, d)$ , where  $d$  enumerates possible directions on cross  $i$ . This poses some restrictions on the service times  $\tau_{i,d}$  in the new vertices, like

$$\sum_d \tau_{i,d} = \tau_i.$$

We limited ourselves to the problem, when in the system at least one jam appear. It is interesting to find number of jams and the mean number of cars standing in jams.

**Relation with real life** This model is convenient because all parameters can be statistically estimated. Namely, statistical estimates of the parameters  $p_{ij}, \mu_i$  look like (for example for constant  $\mu_i$ )

$$p_{ij} \sim \frac{N_{ij}(T)}{\sum_j N_{ij}(T)}, \mu_i = \frac{1}{T} \sum_j N_{ij}(T),$$

where  $N_{ij}(T)$  is the number of cars in the time interval  $[0, T]$ , choosing direction  $j$  on the cross  $i$ .

Practically interesting is the optimization of traffic lights, that can be achieved by choice of  $\tau_{i,d}$  and by traffic lights synchronization (that is totally ignored in any exponential model), and by changing matrix  $P$  by helping the choice of route.

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