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Non-relativistic classical mechanics of point particles: shortest elementary introduction

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Abstract. All physical laws are only approximation to reality in some range of parameters. However, this does not mean that rigorous mathematics is contraindicated to physics. On the contrary, inconsistencies and controversy of main physical laws give rise to dissatisfaction and urge inquiring minds to deeper plunging into the essence of things.

Standard mathematical approach also should tend to the following ideal:

1) there should be axioms, the less the better. However, for some values of parameters they should be as close as possible to the basics of theoretical physics,

2) the range of corollaries and coordination with theoretical physics should be as wide as possible.

We give a short mathematical introduction to the classical non-relativistic point particle physics. The goals are – to teach very beginner to do elementary calculations, to introduce him to most used terminology and to provide some global view on this science. Obviously this introduction is not suitable for deep concentration on some narrow part of particle mechanics. The necessary prerequisites – basics of the real analysis, linear algebra and ordinary differential equations.

Keywords: classical mechanics, point particles, introduction, Newton, Lagrange, Hamilton, Euler, BBGKY

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1.1. Space-time

Classical non-relativistic physics and philosophy assume not only that there exist space, topologically isomorphic to $\mathbf{R}^d$, and time, topologically isomorphic to $\mathbf{R}$, but also that in $\mathbf{R}\times\mathbf{R}^d$ there is a distinguished class of coordinate systems. One could describe this class as follows: there is some coordinate system (t, x) where the physical laws look like as we know it from school. Together with this system this class also includes coordinate systems (t', x') that can be obtained from (t, x) by shift transformations $x' = x + a, t' = t + s$, orthogonal coordinate transformations $x' = Ax$ and transformations

$$t' = t, x' = x - vt \tag{1.1}$$

for any "velocity vector" v. These transformations generate the transformation group called the **Galilean transformations**. These transformations conserve space and time distances between points of $\mathbf{R} \times \mathbf{R}^d$.

1.2. Main definitions

Everything below occurs in time $\mathbf{R}$ and space $\mathbf{R}^d$, where not only the case $d = 3$ is interesting. The main object is **point particle,** which at time t is situated at some point $x(t) = (x_1(t), \dots, x_d(t))$ of the space $\mathbf{R}^d$. It cannot disappear and moves along its **trajectory**, that is a function $x(t)$ on some time interval $I \subset \mathbf{R}$ with values in $\mathbf{R}^d$. We always assume $x(t)$ to be continuous and piece wise smooth.

If at time t the trajectory has corresponding derivatives, then the following vectors are defined: **velocity**

$$v(t) = (v_1(t), \dots, v_d(t)) = \frac{dx(t)}{dt} = \Big(\frac{dx_1(t)}{dt}, \dots, \frac{dx_d(t)}{dt}\Big)$$

and **acceleration**

$$a(t) = (a_1(t), \dots, a_d(t)) = \frac{d^2x(t)}{dt^2} = \Big(\frac{d^2x_1(t)}{dt^2}, \dots, \frac{d^2x_d(t)}{dt^2}\Big)$$

Besides its trajectory, the particle has two other real parameters, which do not change in time, – **mass** $m > 0$ and **charge** $q \in \mathbf{R}$.

The particle **dynamics**, that is its trajectory, is defined by the system of d **Newton equations**

$$m\frac{d^2x(t)}{dt^2} = F \tag{1.2}$$

where the vector

$$F = F\Big(t, x(t), \frac{dx(t)}{dt}\Big) = (F_1, \dots, F_d)$$

is called **force**, acting on the particle at the point $x = x(t)$ at time t. The force can depend on the time, on the space point, where the particle is at time t, and on the particle velocity at time t.

This Newton law can be understood of course as the definition of the force. In fact, if acceleration is intuitively well understood and is measurable in space-time terms, the mass can be measured on the weights, with the force the situation is more complicated. The force is often measured via the left part of (1.2). Moreover, often the definition of the force is given as follows – "the force is what creates the acceleration of the mass". Obviously, for any trajectory one can find the corresponding force. But there will not be tautology if the concrete function F is chosen from different considerations. Anyway, the following **linearity** property (called also **superposition principle**) holds: if two forces F_1 and F_2 act on the particle, then

$$F = F_1 + F_2$$

Existence and uniqueness of dynamics

To uniquely find the trajectory, say on the time interval $[0, \infty)$, besides the force one should also know initial data $x(0), v(0)$. These data uniquely define the future of the particle. That is why the classical physics is said to be **deterministic**. The **state** of the particle at time t is defined to be the pair $(x(t), v(t))$, and the set $\mathbf{R} \times \mathbf{R}^d$ of such pairs is called the **phase space**.

Remark 1.1. Note that if the Newton equations for the trajectory $x(t)$ were of the first order

$$\frac{dx(t)}{dt} = f(x(t))$$

with some "force" $f(x)$, which "moves" the coordinate of the particle, then determinism would have much stronger sense: only the initial coordinate of the particle completely defines its future. This contradicts the following simple observations: 1) if the force becomes zero, then the movement immediately stops, 2) particles with different initial conditions would never collide, see much more about this below. This follows from the uniqueness of solution.

Deterministic dynamics could be defined by the equations of the order higher than 2, but the nature uses simpler possibility.

If $F \equiv 0$, then the movement is called **free**. Then the movement is rectilinear with fixed velocity $v(0)$

$$x(t) = x(0) + v(0)t, \quad v(t) = v(0).$$

If the force F is constant, then

$$x(t) = x(0) + v(0)t + F\frac{t^2}{2}, \quad v(t) = v(0) + Ft$$

If the force $F(x) = ax, a > 0$, then the equation is linear, can be easily integrated and it is easy to see that the particle goes to infinity exponentially fast. If $F(x) = x^{1+\varepsilon}, \varepsilon > 0$ (in one dimension) then the phenomenon of **vertical asymptote of the trajectory** or the **explosion** – the particle escapes to infinity for finite time. It is easy to prove this because this equation is quadrature integrable, see (1.5) a bit lower.

In general, mathematical analysis of the model starts with the questions about existence and uniqueness of the dynamics, for given initial conditions. In our case the dynamics is defined by the Newton equation (1.2). From the course of ordinary differential equations it is known that, for the equation (1.2) with continuous force, the existence and uniqueness take place at least locally, that is for sufficiently small time interval. Then two problems can occur:

1) the particle reaches infinity for finite time. But if the force $F(x)$ is smooth and increases not faster than $C|x|$ for some $C > 0$, then this cannot occur;

2) if the force has singularities. for example is equal to infinity at some point, then the particle can reach this singularity for finite time.

Energy conservation law

The Newton law can be reformulated as follows: **momentum** $p = mv$ of the particle changes for small time Δt as

$$\Delta p = F\Delta t$$

Assume that the force is continuous in a simply connected domain O and does not depend on the velocity, then there exists real function

$$V(x,t) = -\int\limits_L F(x,t)dx,$$

where $L = L(x_0, x)$ is any path (for example, piece wise smooth) from arbitrary point x_0 to x, of the domain O, so that

$$F(x,t) = -gradV(x,t) = -\Big(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_d}\Big)$$

This function V is defined up to an additive constant and is called **potential energy (**or **potential** in short**)** of the particle, and the force is called **potential**. The graph of the potential energy is called **energy surface** or **energy landscape**. For example, in one-dimensional case the force is directed downwards landscape, to energy holes.

Kinetic energy of the particle is defined as $T = \frac{mv^2}{2} = \frac{p^2}{2m}$. The **total energy** $H = T+V$, that is the sum of kinetic and potential energies, is conserved if V does not depend on time. This conservation law is related to the fact that Newton equations can be rewritten in the **hamiltonian form** (and H is also

called the **Hamiltonian**)

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} \tag{1.3}$$

Then

$$\frac{dH}{dt} = \sum_i \Big(\frac{\partial H}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial H}{\partial x_i}\frac{dx_i}{dt} \Big) = 0$$

Note that the energy conservation holds for any function $H(x,p)$. More general assertion is for the case when the hamiltonian $H(x,p,t)$ depends also on time

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

Work of the force $F = (F_1, \dots, F_d)$ (or the energy, transferred by this force) on the part L of the trajectory between its end points $y_0.y_1$ is defined as

$$A = \int\limits_L \sum_{i=1}^d F_i dx_i = \int\limits_L \sum_{i=1}^d F_i v_i dt,$$

If the force is potential, then

$$A = V(y_0) - V(y_1) = T(y_1) - T(y_0).$$

Power of the force at time t (or the work per unit time) is defined as

$$W(t) = \frac{dA}{dt} = (F, v) = \sum_{i=1}^d F_i v_i$$

Then the work during time interval $[0,t]$ is

$$A = \int\limits_0^t W(t)dt$$

Time reversal operation and non-intersection of trajectories Newton equations have the following **time symmetry** property. For exact formulation note that Newton equation with the force $F(x)$, both in general form (1.2) and in hamiltonian form (1.3), is invariant with respect to time sign change. Namely, assume that the particle starts at time $t = 0$ with initial state (x_0, v_0) and at time T its state becomes (x_T, v_T). Then, if the initial state (with new time τ) is $(x_0', v_0') = (x_T, -v_T)$, the particle will move along the same trajectory in the opposite direction, and in time $\tau = T$ will be in the state $(x_0, -v_0)$.

It follows that if at any point of the phase space R^{2d} the existence and uniqueness hold, the trajectories in R^{2d} do not intersect. More exactly, from two different initial conditions at time $t = 0$ one cannot get to the same state at the same time.

1.3. Simplest examples

Finite smooth potential Assume the potential $V(x), x \in \mathbf{R}$, is smooth, has finite support and inside the support $V'(x) = 0$ in finite number of (critical) points, and moreover all critical points are non-degenerate, that is $V''(x) \neq 0$. Outside the support the particle moves as a free particle. All trajectories are defined on $(-\infty, \infty)$ and can be classified correspondingly to their large time behavior. Introduce the following definitions.

Bounded trajectory (we will call it **bound state**) on $(-\infty, \infty)$ is called any trajectory $x(t)$ such that for some $C > 0$, for any t on this interval $|x(t)| < C$.

Example is the particle between two maxima, at the points $a < b$, of the potential $U(x)$. Assume that initially $a < x(0) < b$, and moreover the initial total energy $H(0)$ is less than $U(a)$ and $U(b)$. Then, it follows from the energy conservation law that the particle cannot leave the interval (a, b).

Scattering trajectory on the time interval $I = (-\infty, \infty)$ is a trajectory $x(t)$ such that $|x(t)| \to \infty$ as $t \to \pm\infty$.

We say that the particle moves from $-\infty$, if on the time interval $(-\infty, t_0)$ it moves as a free particle, that is $x(t) = vt, v > 0$. If moreover, its kinetic energy $T_0 = \frac{mv^2}{2}$ is greater than $\max V(x)$, then it overcomes the potential support and escapes to $+\infty$. If it is less, then it will stop (will have zero velocity) at the first point x_1, where $V(x_1)$ becomes equal to $\frac{mv^2}{2}$. Moreover, there are two possibilities: either, if $V'(x_1) > 0$, it will return to $-\infty$ (scattering trajectory), or it will be stuck in this unstable state, if $V'(x_1) = 0$ and x_1 is a local maximum of the potential (this case is called **capture trajectory**). We see that the capture is a rare event – it can occur only for finite number of initial conditions.

If the potential is not of finite support, the situation is more difficult. Trajectory on the interval $(-\infty, t_0)$ can be defined either as the limit of trajectories on the intervals $(-T, t_0), T \to \infty$, or using time symmetry property.

Collision with the wall The velocity can become discontinuous when the particle reaches a surface where the potential equals infinity. This surface can be a plane (wall) or a sphere (spherical hard body). Let the particle reaches the smooth surface S at point x_0 at time t_0. Assume that at time $t_0 - 0$ the velocity equals v_-. After collision the velocity (at time $t + 0$) becomes equal v_+. The collision is called **elastic**, if $|v_+| = |v_-|$, that is the kinetic energy is conserved. Let $S(v_-)$ be the plane perpendicular to S and containing the velocity vector v_-. Elastic collision is called **mirror (specular) reflection** if v_+ belongs to $S(v_-)$ and the rule "the angle of incidence equals the angle of reflection" holds. In this case the momentum increment equals $2m(v_+, n)$, where n is the unit vector, perpendicular to S, assuming that $(v_-, n) < 0$. Then $-2m(v_+, n)$ is called the **transmitted momentum** (transmitted to the particles of the wall).

There are two approaches, if one wants to embed the situation to the particle mechanics:

1) Assume that the wall in $\mathbf{R}^3 = \{(x = x_1, x_2, x_3)\}$ coincides with the plane $x_1 = 0$. Near this plane there is smooth external potential $V_\varepsilon(x) = 0$ if $x_1 > \varepsilon$ and $V_\varepsilon(x) = \frac{x_1}{\varepsilon}$ if $0 < x_1 < \varepsilon$ with support $(0, \varepsilon)$. The reflection rules above can be obtained in the limit $\varepsilon \to 0$.

2) when it comes to a real wall, the particle interacts with the particles of the wall. Exact analysis of this situation uses statistical physics theory, is sufficiently difficult and we omit it here. But on this way one can explain also **non-elastic** collisions where either the kinetic energy of the particle decreases (**energy dissipation**) or increases (**energy pumping**), see below about particle collisions.

Harmonic oscillator Let the particle initially be at the very bottom of the potential close to the local minimum, say $x_0 = 0$, and has sufficiently small initial velocity. Then, as we already know, the particle will always stay at some small neighborhood of the minimum, and it seems natural to use the quadratic approximation of the potential (assuming $V''(x_0) \neq 0$)

$$V(x) = \frac{1}{2}\omega_0^2(x - x_0)^2 = \frac{1}{2}\omega_0^2 x^2$$

This potential is called **harmonic oscillator** potential. Obviously, all trajectories are bounded. The Newton equation is linear for quadratic potentials, and its solution in our case is

$$c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

where constants can be obtained from the initial conditions.

The Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2}\omega_0^2 x^2$$

For fixed energy $H = h$ we get that the trajectory is an ellipse in the phase space $R^2 = \{(x, v)\}$, defined by the equation

$$\frac{H}{h} = h^{-1}\Big(\frac{mv^2}{2} + \frac{1}{2}\omega_0^2 x^2\Big) = 1$$

Forced oscillations The oscillator under the external force is described by the equation (we assume here unit mass)

$$\frac{d^2x}{dt^2} = -\omega_0^2 x + f(t),$$

where for example $f(t) = a \cos \omega t$. Then for $\omega \neq \omega_0$ the general solution is the sum:

$$c_1 \cos \omega_0 + c_2 \sin \omega_0 t + \frac{a}{\omega_0^2 - \omega^2} \cos \omega t$$

of the general solution of homogeneous equation and particular solution of the non homogeneous.

The closer the frequency ω of the external force to its proper frequency ω_0, the greater the oscillation amplitude (**resonance** phenomenon). If $\omega = \omega_0 > 0$ the particular solution of the inhomogeneous equation is

$$\frac{a}{2\omega_0} t \sin \omega_0 t \tag{1.4}$$

that is the amplitude grows linearly with time. In the case of general force $f = f(x,t)$ terms like (1.4) also appear (for example in celestial mechanics) and are called **secular terms**.

Friction and dissipation In real life, due to "resistance of the medium", or energy dissipation, the amplitude cannot grow infinitely. The simplest possibility for such force is to take it proportional to the velocity of the particle and directed oppositely to the velocity, that is $F = -\alpha v, \alpha > 0$.

Such force is often used as the friction force. In the simplest equation as

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = -\alpha v$$

the velocity decreases exponentially fast.

Exersize 1.1. Solve the equation

$$\frac{d^2x}{dt^2} = -\omega_0^2 x - \alpha \frac{dx}{dt} + a \cos \omega t$$

describing harmonic oscillator under the action of both friction and external perturbation force.

Integrability of one-dimensional problems Assume that the particle initially (at time $t = 0$) is at the point $x(0) > 0$ and has velocity $v(0) > 0$. If it moves in the potential $V(x) = \frac{\gamma}{x}, \gamma > 0$, then the force repulses it from zero and its velocity increases to some value $v(\infty)$, that can be found from the energy conservation law

$$\frac{1}{2} m v^2(\infty) = \frac{1}{2} m v^2(0) + \frac{\gamma}{x(0)}$$

Moreover, one can get all trajectory from this law. In fact, at time $t > 0$

$$m\Big(\frac{v^2(t)}{2} - \frac{v^2(0)}{2}\Big) = U(x(0)) - U(x(t)) = \frac{\gamma}{2x(0)} - \frac{\gamma}{2x(t)} \Longrightarrow$$

$$\Longrightarrow v(t) = \frac{dx}{dt} = \sqrt{v^2(0) + \frac{\gamma}{mx(0)} - \frac{\gamma}{mx}} \Longrightarrow$$

$$\Longrightarrow dt = \frac{dx}{\sqrt{v^2(0) + \frac{\gamma}{mx(0)} - \frac{\gamma}{mx}}} = \frac{\sqrt{x(0)x}dx}{\sqrt{(v^2(0)x(0) + \frac{\gamma}{m})x - \frac{\gamma}{m}x(0)}} \Longrightarrow$$

$$t = \sqrt{x(0)} \int\limits_{x(0)}^{x(t)} \frac{\sqrt{x}dx}{\sqrt{(v^2(0)x(0) + \frac{\gamma}{m})x - \frac{\gamma}{m}x(0)}} \tag{1.5}$$

1.4. Particle in the central potential

The potential is called **central** if it depends only on the distance of the particle from some point (the centrum), one can take this point as the origin. Consider particle in $\mathbf{R}^3$ with the Hamiltonian

$$H = \frac{mv^2}{2} + U(r)$$

and central potential $U(r), r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Most interesting cases are the potential of celestial mechanics and Coulomb potential, they look quite similarly

$$U(r) = -\frac{c}{r} \tag{1.6}$$

We assume that $c > 0$, then the force

$$F(r) = -gradU = -c\frac{x}{|x|^3}$$

attracts the particle to the origin.

It is immediate that the trajectory lies in the plane, containing two vectors $x(0)$ and $v(0)$, as the vector F will always lie in this plane. We will assume this plane to be the plane (x_1, x_2). That is why the vector product of any two vectors $A = (A_1, A_2)$ and $B = (B_1, B_2)$ in this plane has one component, that we will denote by

$$A \times B = [A, B] = A_1B_2 - B_1A_2$$

Besides the energy conservation law there is conservation law of the **angular momentum** $L = [x, p]$ of the particle

$$\frac{dL}{dt} = [v, p] + [x, F(x)] = 0$$

Using only these two laws one can study qualitative character of the particle movement, without solving the Newton equations, although they are integrable in quadratures.

In polar coordinates (r, φ) the velocity of the vector $x = (r\cos\varphi, r\sin\varphi)$ is equal to the

$$v = (\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi, \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi)$$

Then

$$\frac{L}{m} = [x, v] = x_1 v_2 - x_2 v_1 = r\cos\varphi\dot{r}\sin\varphi + \\ + r^2\dot{\varphi}\cos^2\varphi - \dot{r}\cos\varphi r\sin\varphi + r^2\dot{\varphi}\sin^2\varphi = r^2\dot{\varphi},$$

and

$$\frac{m}{2}v^2 = \frac{m}{2}\Big(\Big(\frac{dr}{dt}\Big)^2 + r^2\Big(\frac{d\varphi}{dt}\Big)^2\Big)$$

It follows that

$$H = \frac{m}{2}v^2 + U = \frac{m}{2}\Big(\frac{dr}{dt}\Big)^2 + U_{eff} \tag{1.7}$$

where

$$U_{eff} = U + \frac{L^2}{2mr^2}$$

is called the effective potential energy. This gives the first order differential equation $H = const$ for the radius-vector $r(t)$.

Firstly, it is useful to consider the case when $v(0)$ is parallel to the vector $x(0)$ of the particle, that is the case $L(0) = 0$. Then the movement will be along $r(0)$. If $v(0) = 0$ or directed towards the centrum, then the velocity will always increase and the particles will reach (fall onto, collide with) the centrum in finite time. Let now the velocity be directed from the centrum. If $H = H(0) < 0$, then the particle will once stop (when $U(r(t)) = H$) and turn backwards, finally falling onto the centrum. If $H = 0$, then the particle will never stop and will escape to infinity with the velocity tending to zero. If $H > 0$, then the particle escapes to infinity with velocity tending to

$$v(\infty) = \frac{2}{m}H \neq 0$$

We will see that somewhat similar situation takes place also for any $v(0)$ (that is when the initial kinetic momentum is not zero). But first of all, let us show that the particle cannot collide with the centrum. For repulsive potential $(c < 0)$ this is clear. For the attracting potential it follows from (1.7) that

$$0 \leq \Big(\frac{dr}{dt}\Big)^2 = \frac{2}{m}(H - U_{eff}) \tag{1.8}$$

On the one hand, (1.8) is non negative, but on the other hand, if the particle comes close to the centrum, then (1.8) has the order $-r^{-2}$ for small r.

Another conclusion is that the velocity stays always bounded, and the solution always exists and is unique on all time interval $(-\infty, \infty)$.

Scattering trajectories For $H \geq 0$ any trajectory is a scattering trajectory. In fact, let r_{cr} be the unique root of the right-hand side of the equation

$$\frac{m}{2}\Big(\frac{dr}{dt}\Big)^2 = H + \frac{c}{r} - \frac{L^2}{2mr^2}$$

This root defines the point of trajectory nearest to the origin, as for $r < r_{cr}$ this right-hand part is negative, which is impossible. For $r > r_{cr}$, $(\frac{dr}{dt})^2$ is not zero. This means that the trajectory either goes from infinity to this point, or the contrary.

In particular, if $U(x) > 0$ (repulsive potential), then always $H > 0$, and there cannot be bounded trajectories.

Bounded trajectories If the initial energy $H < 0$, then the particle cannot be far from the centrum, because then the potential tends to zero and the kinetic energy is always non-negative. Thus, for $H < 0$ all trajectories are bounded. Moreover, $r(t)$ is a periodic function, as there are exactly two points (exception is the minimum of U_{eff}) with zero velocity $\frac{dr}{dt}$. Due to equation

$$\frac{d\varphi}{dt} = \frac{L}{mr^2}$$

also the function $\varphi(t)$ will be periodic. This means that the particle will really rotate around the centrum. In the mentioned exceptional case of minimum of V_{eff}, the rotation will be along the circle with constant velocity. It is easy to obtain these particular solutions – uniform movement along the circle in the plane (x, y)

$$x(t) = r\cos vt, \quad y(t) = r\sin vt$$

That is very easy to understand. The acceleration vector is

$$m\frac{d^2x}{dt^2} = -rv^2\cos vt, \quad \frac{d^2y}{dt^2} = -rv^2\sin vt$$

That is why the projection of the acceleration vector on the radius-vector in this point of the trajectory equals $-rv^2$, and the projection on the perpendicular direction is zero. It remains only to equate it to the constant force $-gradU(r)$, that is we get

$$v^2 = m\frac{gradU(r)}{r}$$

In all other cases the rotation will be along the ellipses. One can read more about this and about more general central forces, for example, in [1, 2].

Remark 1.2. We saw that the particle has only three possibilities: fall on the centrum, rotate around the centrum and go from infinity to infinity. The first possibility (capture trajectories) can be only collision with the centrum, and only if $v(0)$ is proportional to $x(0)$.

Perturbation of the velocity on the closed orbit If the particle on the bounded trajectories will get additional small velocity, then it will just change to the close orbit. In fact, for bounded trajectories $H < 0$, and any small velocity perturbation also will enjoy this property.

2. About N particle problem

2.1. Axioms

Classical mechanics of point particles is based on the following axioms. In the system of N particles $i = 1, \dots, N$ with masses m_i and trajectories $x_i(t) = (x_{i1}(t), \dots, x_{id}(t))$, any particle i at time t acts on the particle $j \neq i$ with force $F_{ij} = F_{ij}(x_i, x_j)$ depending only on the points x_i, x_j, where these particles are situated at the same moment t. Otherwise speaking, the following equations hold

$$m_j \frac{d^2 x_j}{dt^2} = \sum_{i:i\neq j} F_{ij} \tag{2.1}$$

The symmetry condition is assumed (one of the Newton's law)

$$F_{ij}(x_i, x_j) = -F_{ji}(x_j, x_i) \tag{2.2}$$

Often all forces F_{ij} are assumed to be potential with some (central) potential $V_{ij}(|x_i - x_j|)$, where real functions $V_{ij} = V_{ij}(r) = V_{ji} : R_+ \to R$, (called two particle interaction between particles i and j) depend only on the distance $r_{ij} = |x_i - x_j|$ between the particles i and j at the given moment. Then

$$F_{ij} = -\nabla_j V_{ij} = -\frac{dV_{ij}}{dr}(r_{ij})\frac{dr}{dx_j} = -\frac{dV_{ij}}{dr}(r_{ij})\frac{x_j - x_i}{|x_i - x_j|} \Longrightarrow$$

$$F_j = \sum_{i:i\neq j} F_{ij} = -\nabla_j U$$

where F_j is the total force on the particle j, and we introduced the potential energy of the system

$$U = \frac{1}{2}\sum_{i\neq j} V_{ij} = \sum_{i<j} V_{ij} \tag{2.3}$$

Then for all $k = 1, \dots, d$ and vectors $x_i = (x_{i1}, \dots, x_{id}), F_{ij} = (F_{ij,1}, \dots, F_{ij,d})$

$$F_{ij,k} = -\frac{\partial V_{ij}}{\partial x_{j,k}} = -\frac{dV_{ij}}{dr}(|x_i - x_j|)\frac{x_{j,k} - x_{i,k}}{|x_i - x_j|} \tag{2.4}$$

Then the symmetry condition (2.2) follows.

Remark 2.1. 1) If there are also external forces $F_i(x)$, acting on particles i, the equations (2.1) are as follows

$$m_j \frac{d^2 x_j}{dt^2} = \sum_{i:i\neq j} F_{ij} + F_j(x_j) \tag{2.5}$$

2) In other sciences (for example in sociophysics) one can encounter equations similar to (2.1), see [21], but the symmetry $F_{ij} = -F_{ji}$ is not assumed.

3) This N-particle problem in $\mathbf{R}^d$, which we are going to study, formally is a particular case of one particle problem in $\mathbf{R}^{dN}$ with very specific external force, where each coordinate sees other coordinates as the sources of its own time dependent external field.

2.1.1. What forces are more fundamental

One could restrict oneself to even more narrow class of forces. First of all, to the following two.

Gravitation and electrostatics Let in $\mathbf{R}^3$ (at some fixed time moment) two particles $i = 1, 2$, are the points x_i, moreover they have masses m_i and charges q_i. Then (in non-relativistic physics) particle 1 acts on the particle 2 with the force

$$F_2 = -G\frac{m_1 m_2}{|x_1 - x_2|^3}(x_2 - x_1) + \varepsilon_0^{-1}\frac{q_1 q_2}{|x_1 - x_2|^3}(x_2 - x_1)$$

The first of these forces is called gravitational force (universal gravitation law), the second is the electrostatic force (Coulomb law).

These two forces act on different scales and rarely meet together. For example, celestial mechanics uses only gravitational force, and in micro world (atoms and molecules) most important is the electrostatic force. In the system SI the constants and mass and charge of the electron are ($C = coulomb, V = volt$)

$$m_e = 10^{-31}kg,\ \ q_e = 10^{-19}C,\ \ \varepsilon_0 = 10^{-11}Cm^{-1}V^{-1},\ \ G = 10^{-10}m^3kg^{-1}s^{-2}$$

However, in physics many other forces are used. Example is Hooke's law (like in harmonic oscillator) with potential

$$U(x_1 - x_2) = \frac{w^2}{2}(x_1 - x_2)^2$$

and forces of Lennard – Jones type

$$\frac{c_1}{(x_1 - x_2)^p} + \frac{c_2}{(x_1 - x_2)^q}$$

The latter are similar to the fundamental forces but the potential has a minimum. The reader often will not understand where these forces come from: either they constitute fundamental laws or approximate experimental facts.

Thus there is a choice: 1) to obtain general results for wide classes of forces, 2) to develop axiomatic theory with minimal number of fundamental forces. Celestial mechanics is a famous example of such theory. It is strange, however, that Coulomb mechanics (with Coulomb force only) does not still exist.

The main problems however are the same as for one particle: to describe bounded and scattering trajectories.

2.1.2. Existence, uniqueness and collisions

If all V_{ij} are smooth (of course there are much weaker conditions) in a neighborhood of the initial data $\psi(0) = (x(0), v(0)$ then for some t_1 the solution, for this initial data, exists and is unique on $[0, t_1]$. Then one can proceed recurrently. Namely, take $\psi(t_1) = (x(t_1), v(t_1)$ as new initial data, then we get a solution in longer time interval $[0, t_2], 0 < t_1 < t_2$. Continuing similarly, two problem can appear:

1) for finite time we can escape to infinity. However, this is impossible if all V_{ij} are bounded from below. For example, if all forces F_{ij} are smooth and bounded, then the solution globally exists.

2) some of N particles can collide, that is for some t, i, j it can be that $x_i(t) = x_j(t)$. There are two types of such collisions:

a) if $V_{ij}(x) \to -\infty$ as $x \to 0$. Then if a collision occurred at time t_0, the solution cannot be continued for $t > t_0$. Let $B \in \mathbf{R}^{2d}$ be the set of initial data for which such collisions occur. And it is interesting whether $\lambda(B) = 0$ or $\lambda(B) > 0$, where λ is the Lebesgue measure on $\mathbf{R}^{2d}$. For example, in celestial mechanics with $N = 2$ $\lambda(B) = 0$. We saw this in the example the particle falls onto the centrum only for zero initial kinetic momentum. In case $N > 2$ this problem was deeply investigated, see [2] and references therein.

b) if $V_{ij}(0)$ is finite. Here the solution can be continued and often such collisions are not taken into account. However, the structure of such collisions is important for many problems, for example, for deriving Euler equations from particle mechanics, see below.

2.1.3. Conservation laws

Total mass M, energy H, momentum P, angular (kinetic) momentum L of the system of N particles $i = 1, \dots, N$ are defined as follows

$$M = \sum_i m_i, \;\; H = \sum_i \frac{m_i v_i^2}{2} + U, \;\; P = \sum_i m_i v_i = \sum_i p_i, \;\; L = \sum_i (x_i \times p_i)$$

Define also the center of mass X and its velocity

$$X = \frac{\sum_i m_i r_i}{M}, \quad V = \frac{dX}{dt}$$

then the kinetic energy can be rewritten as

$$T = \sum_{i=1}^{N} \frac{m_i v_i^2}{2} = \frac{1}{2} MV^2 + \frac{1}{2} \sum_i m_i (v_i - V)^2$$

The conservation laws

$$\frac{dH}{dt} = 0, \quad \frac{dP}{dt} = 0, \quad \frac{dL}{dt} = 0$$

are verified as follows. Energy conservation – as above for one particle. Momentum – also very simply

$$\frac{dP}{dt} = \sum_{i,j:i\neq j}^{N} F_{ij} = 0,$$

as $F_{ij} = -F_{ji}$. If there are also external fields $F_{\mathrm{i}}(x)$, exerted on particles i, then

$$\frac{dP}{dt} = \sum_i \frac{dp_i}{dt} = \sum_i (F_i^{(int)} + F_i^{(ext)}) = \sum_i F_i^{(ext)}.$$

For the kinetic momentum conservation we have

$$\frac{dL}{dt} = \sum_i ((v_i \times p_i) + \sum_j (x_i \times F_{ji})) = \sum_{i\neq j} (x_i \times F_{ji}) = \sum_{i<j} ((x_i \times F_{ji}) + (x_j \times F_{ij})) =$$

$$= \sum_{i<j} ((x_i - x_j) \times F_{ji}) = R = 0$$

as the vectors $x_i - x_j$ and $F_{ij}^{(int)}$ are parallel, if there are no external forces. In general case the **torque** (twisting moment) R of all forces, including external, is defined as follows

$$R = \sum_i R_i = \sum_i (x_i \times F_i^{(ext)}) + \sum_i \sum_{j:j\neq i} (x_i \times F_{ij}^{(int)}) =$$

$$= \frac{1}{2} \sum_{i\neq j} ((x_i - x_j) \times F_{ij}^{(int)}) + \sum_i (x_i \times F_i^{(ext)}) = \sum_i (x_i \times F_i^{(ext)})$$

2.2. Two body problem

If the potential $U(r) = U\{|x_1 - x_2|\}$ has finite support, and two particles are at sufficient distance from each other, then they move with constant velocities. What occurs when they become closer than the radius of the potential ? It appears that, under sufficiently general conditions, there is only one possibility: after spending some time on the bounded distance from each other they separate forever. This result of the scattering theory we shall prove here.

Another interesting observation is that, in many cases, the particle with smaller velocity robs the energy from the particle with greater velocity, justifying the observation "weaker takes the energy from the stronger". This is evident already from very simple example. If one particle stands still and another one bumps into it, then the first one obtains non zero velocity, thus taking kinetic power from the moving particle. Warming up close to the fire and the friction effect are based also on this fact. Let me mention also the following life observation "if one will get more someone will have less".

One dimensional case: energy redistribution Here we assume for simplicity that the potential $U(r)$ is smooth and that its support is the interval $[0, a], 0 < a < \infty$. In this case we have complete understanding and also typical picture of dynamics. First of all, for any initial data the solution exists on all time interval $(-\infty, \infty)$. Assume that at time $t = 0$ the particles are sufficiently far from each other, that is $|x_1(0) - x_2(0)| > a$, and have velocities $u_1 = v_1(0)$ and $u_2 = v_2(0)$ correspondingly.

In this case we call the trajectory $(x_1(t), x_2(t))$ **bounded**, if there is a constant $C > 0$ such that uniformly in t

$$|x_1(t) - x_2(t)| \leq C$$

For example, such are the trajectories with $v_1(t) = v_2(t), |x_1(t) - x_2(t)| = r > a$.

The trajectory $(x_1(t), x_2(t))$ is called **scattering** trajectory, if the following condition holds: the particles spend finite time on the distance not greater than a from each other. It follows that $v_k(t) = w_k$ for sufficiently large t, that is the velocities stabilize.

There are also **capture trajectories,** but for smooth potential such initial conditions have Lebesgue measure zero in the phase space, and we do not consider such cases. We saw this above in one dimensional case.

Lemma 2.1. *There are no other trajectories.*

Proof is contained in the following assertions.

Lemma 2.2. *For any scattering trajectory we have*

$$w_1 - u_1 = \frac{2m_2}{m_1 + m_2}(u_2 - u_1) \tag{2.6}$$

$$w_2 - u_2 = \frac{2m_1}{m_1 + m_2}(u_1 - u_2) \tag{2.7}$$

Proof. From conservation of momentum and energy we have

$$m_2u_2 + m_1u_1 = m_2w_2 + m_1w_1$$

$$m_2u_2^2 + m_1u_1^2 = m_2w_2^2 + m_1w_1^2$$

One can rewrite this

$$\begin{aligned} m_2(u_2 - w_2) &= m_1(w_1 - u_1) \\ m_2(w_2^2 - u_2^2) &= m_1(u_1^2 - w_1^2) \end{aligned} \tag{2.8}$$

Making cancellations in the latter formula we get

$$w_2 + u_2 = u_1 + w_1$$

or

$$-w = w_2 - w_1 = u = u_1 - u_2 \neq 0 \tag{2.9}$$

as we consider scattering trajectory. So, the relative velocity $v(t) = v_1(t) - v_2(t)$ changes sign. Moreover, substituting

$$w_2 = u_1 + w_1 - u_2$$

to (2.8), we have

$$2u_2 - u_1 - w_1 = \frac{m_1}{m_2}(w_1 - u_1)$$

or

$$2u_2 - 2u_1 + (-w_1 + u_1) = \frac{m_1}{m_2}(w_1 - u_1)$$

From this (2.6) follows, and (2.7) is symmetric. Lemma is proved. □

The following theorem easily follows from this lemma.

Theorem 2.1.

1. *Let $u_1 > u_2 > 0$ and $x_1(0) < x_2(0)$. Then $|w_1| < u_1$. This means that the particle with velocity u_1 looses kinetic energy, and the second, with less energy, increases its energy. Moreover, in this case always $w_2 > u_2 > 0$, and w_1 can change sign, dependently on the masses.*

2. *Let $u_1 > 0$ be the velocity of the particle going from $-\infty$, and $u_2 < 0$ be the velocity of the particle going from $+\infty$. Here $w_1 < u_1$ always, but $|w_1| < u_1$, holds iff*

$$\frac{|u_2|}{u_1} < \frac{m_2}{m_1}$$

The first assertion of the theorem follows from formula (2.6), as

$$-2u_1 < \frac{2m_2}{m_1+m_2}(u_2-u_1) < 0$$

The second assertion follows from the same formula (2.6). In fact,

$$|w_1| = \Big|u_1 + \frac{2m_2}{m_1+m_2}(u_2-u_1)\Big|$$

and, as $u_2 < 0$,

$$\frac{|w_1|}{u_1} = \Big|1 - \frac{2m_2}{m_1+m_2}\Big(\frac{|u_2|}{u_1}+1\Big)\Big|$$

Thus it should be

$$\frac{2m_2}{m_1+m_2}\Big(\frac{|u_2|}{u_1}+1\Big) < 2$$

that gives the result.

Formulae of the lemma can be rewritten as follows

$$w_1 = \frac{m_1-m_2}{m_1+m_2}u_1 + \frac{2m_2}{m_1+m_2}u_2$$

and symmetrically for w_2. Note that if $m_1 = m_2$, then $w_1 = u_2$, that is the particles exchange their velocities.

Reduction to one particle case in any dimension Consider two particles in $\mathbf{R}^2$ having masses m_1, m_2 and coordinates x_1, x_2 correspondingly. Consider smooth finite central potential $U(r), r = |x_1 - x_2|$

Denote by

$$x_1,\ x_2,\ X = \frac{m_1x_1+m_2x_2}{m_1+m_2},\ x = x_1 - x_2$$

the vectors of particle coordinates, the center of masses and the relative vector, respectively, and also

$$v_1,\ v_2,\ V = \frac{m_1v_1+m_2v_2}{m_1+m_2},\ v = v_1 - v_2$$

the corresponding velocities. Also denote the total mass $M = m_1 + m_2$, the reduced mass $\mu = m_1m_2/M$, and the momentae

$$p_1 = m_1v_1, \quad p_2 = m_2v_2, \quad P = p_1 + p_2 = MV, \quad p = \frac{m_2p_1 - m_1p_2}{M} = \mu v$$

where p is the momentum of "virtual relative" particle. Inverse formulas

$$x_1 = X + \frac{m_2}{m_1+m_2}x, \quad x_2 = X - \frac{m_1}{m_1+m_2}x$$

give explicit expressions for the velocities before (denoted by u_i) and after (w_i) scattering

$$v_1 = V + \frac{m_2}{m_1+m_2}v, \quad v_2 = V - \frac{m_1}{m_1+m_2}v$$
$$w_1 = V + \frac{m_2}{m_1+m_2}w, \quad w_2 = V - \frac{m_1}{m_1+m_2}w$$

It remains to find w given $v = v_1 - v_2$. From energy conservation we get that $|v| = |w|$.

It is easy to see that the Hamiltonian can be written as

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + U(|x_1 - x_2|)$$
$$= \frac{P^2}{2M} + \frac{p^2}{2\mu} + U(|x|)$$

This reduces the problem to that for the virtual particle with the Hamiltonian

$$H' = \frac{p^2}{2\mu} + U(|x|),$$

which was considered earlier in Section 1.

2.3. Types of trajectories for $N \geq 3$

We say that the trajectory is bounded if all N particles always remain in some neighborhood of their center of mass. Most part of the great science "Celestial mechanics" is devoted to the description of bounded trajectories, see [2, 5, 6, 11, 12].

Scattering theory describes other trajectory types. We call channel α any partition of the set $\{1, \ldots, N\}$ of particles onto disjoint subsets (fragments)

$$\{1, \ldots, N\} = J_1 \cup \ldots \cup J_k$$

If H is the Hamiltonian of the system, then the Hamiltonians $H(\alpha, J_i)$ are defined as follows: we delete from H all terms except kinetic energies of the particles from J_i and all interactions between particles in J_i. The channel Hamiltonian

$$H_\alpha = \sum_{i=1}^{k} H(\alpha, J_i),$$

is defined as the total Hamiltonian from which all interaction of particles from different fragments are deleted.

About scattering theory Assume again that all V_{ij} have finite support. We say that the trajectory is a scattering trajectory for the pair of channels $\alpha_- = J_1^- \cup \ldots \cup J_k^-$ and $\alpha_+ = J_1^+ \cup \ldots \cup J_n^+$ if the following conditions hold:

1. For some constants (depending on the trajectory) $t_- < t_+$ the trajectory of the system on the time interval $(-\infty, t_-)$ is defined by the Hamiltonian H_{α_-}. Moreover, as $t \to -\infty$ all $|x(t_i)| \to \infty$.

2. Similarly, the trajectory of the system on (t_+, ∞) is defined by the Hamiltonian H_{α_+}, and as $t \to \infty$ all $|x(t_i)| \to \infty$.

Hypothesis of asymptotic completeness in this case is the following assertion: for almost any trajectory there exists a pair of channels (α_-, α_+), with respect to which the trajectory is a scattering trajectory. "Almost any" means that the set of initial conditions $(x_1(0), v_1(0), \ldots, x_N(0), v_N(0))$, for which the corresponding trajectory is not a scattering trajectory, has Lebesgue measure 0 in $\mathbf{R}^{2dN}$.

For $N = 2$ this is easy to prove by reduction to one particle problem, see above. For $N = 3$ deep results were obtained by classics of celestial mechanics, see references in [2]. In more general case and in a bit different terms (scattering theory in Hilbert space) similar assertion was proved in [4].

3. Dynamics of sets, functions and measures

From now on, we consider the problems very different from those we considered earlier. For example, ergodicity problems.

Remind that the trajectories with initial conditions $x(0) = x, v(0) = v$ (denoted here as $u^t(x, v)$) of Hamiltonian dynamics on some set $X \subset \mathbf{R}^{2dN}$ of the phase space, do not intersect. That is, for any t, $u^t : X \to u_t(X)$ is one-to-one mapping.

More generally, let $u^t : X \to X$ be an arbitrary group of one-to-one transformations (diffeomorphisms) of some set $X \subset \mathbf{R}^{2dN}$. One can consider induced (raised, uplifted, elevated) transformation groups on different sets of "higher level". For example, on the set of subsets of $\Sigma(X)$, the set of real or complex functions on X and on the set of all measures on X.

Note that any nonlinear dynamics, being raised for example on the set of functions, becomes linear in infinite dimension. One can uplift the dynamics "even higher", for example on the set of differential forms or the set of operators in some function space. Such elevation gives some insight on the connection between classical and quantum physics, see references in [15].

3.1. Invariant tori

Consider the set $\Sigma(L)$ of all subsets $A \subset L$, and for given u^t define

$$U^t(A) = \{u^t\psi : \psi \in A\}$$

Subset $A \subset L$ is called **invariant** if $U^t(A) = A$ for any t. Structure of invariant subsets is useful in the problems of integrable systems and ergodicity. One of the strongest definitions of ergodicity is that all invariant subsets should have Lebesgue measure 0 or 1. In linear case the structure of invariant subsets and of the dynamics is the classical result, and we come to it now.

Consider the phase space

$$L = \mathbb{R}^{2N} = \Big\{\psi = \begin{pmatrix} q \\ p \end{pmatrix} : \ q = (q_1, \dots, q_N)^T, \ p = (p_1, \dots, p_N)^T, \ q_i, p_i \in \mathbb{R}\Big\},$$

where T is the transposition (that is ψ is considered as the column vector), Then L is a linear space with the standard scalar product

$$(\psi, \psi')_2 = (q, q')_2 + (p, p')_2 = \sum_{i=1}^{N} (q_i q_i' + p_i p_i')$$

and can be presented as the direct sum $L = l_N^{(q)} \oplus l_N^{(p)}$ of two orthogonal subspaces, each of dimension N. Consider the quadratic Hamiltonian

$$H(\psi) = \sum_{k=1}^{N} \frac{p_k^2}{2} + U(q), U(q) = \frac{1}{2}(q, Vq) \tag{3.1}$$

where matrix V, acting in $\mathbb{R}^N$, is assumed to be real and positive definite. This defines hamiltonian system of linear differential equations

$$\dot{q}_k = p_k, \dot{p}_k = -\sum_{l=1}^{N} V_{kl} q_l, k = 1, \dots, N \tag{3.2}$$

or

$$\frac{d\psi}{dt} = A\psi, \ \ A = \begin{pmatrix} 0 & E \\ -V & 0 \end{pmatrix}$$

For $h > 0$ the **energy surface** (of constant energy h)

$$\mathcal{M}_h = \{\psi \in L : H(\psi) = h\}$$

is a smooth manifold (ellipsoid) in L of codimension 1.

Let $v_1, \dots, v_N$ be the orthonormal eigenvectors of matrix V in $\mathbb{R}^N$, and the corresponding eigenvalues are denoted by $\omega_1^2, \dots, \omega_N^2$. Define $2N$-vectors in L:

$$Q_k = (v_k, 0)^T, \quad P_k = (0, v_k)^T, \ k = 1, \dots, N \tag{3.3}$$

Denote by $(\tilde{q_k}, \tilde{p_k})^T$ the coordinates of the vector $\psi \in L$ in the basis $Q_1, P_1, \dots, Q_N, P_N$, that is

$$\psi = \sum_{k=1}^{N} \tilde{q}_k Q_k + \sum_{k=1}^{N} \tilde{p}_k P_k \tag{3.4}$$

Dynamics For any $t \geqslant 0$ one can show that

$$e^{tA} = \begin{pmatrix} \cos(t\sqrt{V}) & (\sqrt{V})^{-1}\sin(t\sqrt{V}) \\ -\sqrt{V}\sin(t\sqrt{V}) & \cos(t\sqrt{V}) \end{pmatrix},$$

where $\sqrt{V}$ is the positive square root of the matrix V. Then for any $k = 1, \ldots, N$ and $t \geqslant 0$

$$e^{tA}Q_k = \cos(\omega_k t)Q_k - \omega_k \sin(\omega_k t)P_k, \tag{3.5}$$

$$e^{tA}P_k = \frac{\sin(\omega_k t)}{\omega_k}Q_k + \cos(\omega_k t)P_k. \tag{3.6}$$

Using (3.5) and (3.6) we have for any ψ

$$\psi(t) = e^{tA}\psi = \sum_{k=1}^{N}\Big(\cos(\omega_k t)\tilde{q}_k + \frac{\sin(\omega_k t)}{\omega_k}\tilde{p}_k\Big)Q_k +$$

$$+\sum_{k=1}^{N}\left(-\omega_k \sin(\omega_k t)\tilde{q}_k + \cos(\omega_k t)\tilde{p}_k\right)P_k$$

and it follows that

$$\tilde{q}_k(t) = \cos(\omega_k t)\tilde{q}_k(0) + \frac{\sin(\omega_k t)}{\omega_k}\tilde{p}_k(0)$$
$$\tilde{p}_k(t) = -\omega_k \sin(\omega_k t)\tilde{q}_k(0) + \cos(\omega_k t)\tilde{p}_k(0).$$

Action-angle variables From these formulas it is clear that any pair of functions $(\tilde{q}_k(t), \tilde{p}_k(t)), k = 1, \ldots, N$, corresponds to the dynamics of one dimensional oscillator with unit mass, frequency ω_k, coordinate $\tilde{q}_k(t)$ and momentum $\tilde{p}_k(t)$. Then the dynamics $e^{tA}\psi$ is isomorphic (unitary equivalent) to the uniform movement on the torus with velocity $(\omega_1^2, \ldots, \omega_N^2)$, that is to the dynamics of N independent one-dimensional oscillators with energies $r_k^2/2$, where

$$r_k^2(\psi) = \tilde{p}_k^2 + \omega_k^2\tilde{q}_k^2 = (\psi, P_k)_2^2 + \omega_k^2(\psi, Q_k)_2^2 = (p, P_k)_2^2 + \omega_k^2(q, Q_k)_2^2, \quad k = 1, \ldots, N$$

are the so called action coordinates of the point $\psi = (q, p)$. We assume that $r_k(\psi) = \sqrt{r_k^2(\psi)} \geqslant 0$. It is easy to see that r_k are integrals of the hamiltonian dynamics, that is for all $t \geqslant 0$

$$r_k^2(\psi) = r_k^2(e^{tA}\psi) \tag{3.7}$$

Together with the action coordinates also angle coordinates are introduced, that is the rotation angles of the corresponding oscillator on the ellipse in its phase plane.

It is not difficult to prove the following assertions.

Lemma 3.1.

1. *For all $r_1 \geqslant 0, \dots, r_N \geqslant 0$ the set*

$$T(r_1, \dots, r_N) = \{\psi \in L : r_k(\psi) = r_k, k = 1, \dots, N\}$$

is invariant and diffeomorphic to the torus of dimension $N - n$, where n is the number of zeros among $r_1, \dots, r_N$.

2. *For any point $\psi \in L$, the closure of its orbit (trajectory) coincides with the torus to which it belongs*

$$\overline{\{e^{tA}\psi : t \geqslant 0\}} = T(r_1(\psi), \dots, r_N(\psi)),$$

3. *Thus the torus is defined by the vector $\bar{r} = (r_1, r_2, \dots, r_N)$. Vice versa, any such vector with non-negative coordinates uniquely defines the torus. This torus lies on the energy surface $\mathcal{M}_h$ iff*

$$\sum_{k=1}^{N} r_k^2 = 2h$$

3.2. Dynamics of functions

We define now the dynamics

$$(U^t f)(x) = f(u^t x), \quad x \in X,\ f \in \mathbf{F}_X$$

on the set $\mathbf{F}_X$ of functions on X or on any invariant linear subspace of $\mathbf{F}_X$.

Remark 3.1. If the evolution u^t is one-to-one then sometimes it could be useful to define the evolution of functions differently:

$$(U^t f)(x) = f(u^{-t} x) \tag{3.8}$$

3.2.1. Symplectic formalism

Consider the set of functions $\mathbf{F}{=}C^\infty(R^{2N})$ on the phase space $\mathbf{R}^{2N} = ((q_1, p_1), \dots, (q_N, p_N))$. Define the bilinear map (**Poisson bracket**) $\mathbf{F} \times \mathbf{F} \to \mathbf{F}$

$$[f, g] = [f, g]_{q,p} = \sum_K \Big(\frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} \Big) \tag{3.9}$$

For example,

$$[q_k, p_l] = \delta_{kl}, \quad [q_k, q_l] = [p_k, p_l] = 0 \tag{3.10}$$

$$[q_k, q_l q_m] = 0, \quad [q_k, q_l p_m] = \delta_{km} q_k, \quad [q_k, p_l p_m] = \delta_{kl} p_m + \delta_{km} p_l$$

$$[p_k, p_l p_m] = 0, \quad [p_k, q_l p_m] = \delta_{kl} p_m, \quad [p_k, q_l q_m] = \delta_{kl} q_m + \delta_{km} q_l$$

It is easy to check that the mapping (3.9) is bilinear skew(anti)symmetric and that the **Jacobi identity** holds

$$[e, [f, g]] + [f, [g, e]] + [g, [e, f]] = 0$$

Thus, $\mathbf{F}$ becomes Lie algebra.

If the Hamiltonian H is given, then, in terms of Poisson brackets, the hamiltonian equations can be rewritten as follows

$$\frac{dq_k}{dt} = \frac{\partial H}{\partial p_k} = [q_k, H], \quad \frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k} = [p_k, H] \tag{3.11}$$

More generally, evolution of any observable (that is of some function $f = f(q_k(t), p_k(t))$) in the phase space is also defied by differential equation

$$\frac{df}{dt} = \frac{d(U^t f)}{dt} = \frac{df(q(t), p(t))}{dt} = \sum_K \Big(\frac{\partial f}{\partial q_k} \frac{dq_k}{dt} + \frac{\partial f}{\partial p_k} \frac{dp_k}{dt} \Big) =$$

$$= \sum_k \Big(\frac{\partial f}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial H}{\partial q_k} \Big) = [f, H] \tag{3.12}$$

If we introduce the linear operator $Lf = [f, H]$ on the set of functions then the dynamics can be formally written as

$$U^t f = e^{tL} f$$

3.2.2. Canonical transformations

Besides Hamiltonian dynamics uplifted on $\mathbf{F_X}$ one can consider various transformation of this space and of its linear subspaces.

1) Linear transformations $L\psi$ on linear functions on $\mathbf{X}$ which conserve symplectic form (3.10). They are considered in algebra;

2) arbitrary (nonlinear) one-to-one transformation $w : \mathbf{R}^{2N} \to \mathbf{R}^{2N}$ of the phase space, that is some change of variables

$$\{(q_k, p_k)\} \to w\{(q_k, p_k)\} = \{(Q_k, P_k)\} \tag{3.13}$$

Using

$$(Wf)(\xi) = f(w\xi)$$

this transformation can be raised to $\mathbf{F}$, that defines the linear operator $W : \mathbf{F} \to \mathbf{F}$. Such w (or W) is called **canonical** if it conserves Poisson brackets of any two functions f, g on $\mathbf{R}^{2dN}$

$$[f, g]_{p,q} = [f, g]_{Q,P} \tag{3.14}$$

that it is an automorphism of the Lie algebra.

3) groups of linear transformations of the infinite dimensional linear subspaces of $\mathbf{F_X}$, not necessarily induced by transformations defined in 2).

The important facts:

1) For any Hamiltonian $H = H(\{(q_k, p_k)\})$ the hamiltonian equations (3.11) look similarly in the new variables Q_k, P_k with Hamiltonian $H_w = H_w(Q_k, P_k) = H(q(Q,P), p(Q,P))$.

In fact, putting in (3.12) $f = Q_k$, we write down the evolution of Q_k as the consequence of the evolution $q_i(t)$ and $p_i(t)$. Namely, we use (3.9), (3.12) and (3.14) in the following equalities correspondingly

$$\frac{dQ_k}{dt} = \{Q_k, H\} = \{Q_k, H_w\} = \frac{\partial H_w}{\partial P_k}$$

and similarly for $\frac{dP_k}{dt}$.

2) In particular, Hamiltonian dynamics u_t defines canonical transformation for any t.

Canonical transformations (that is the corresponding change of variables) can simplify the equations and find invariants of the dynamics.

Examples

1) If some function G has the property $\{H, G\} = 0$ then G is an invariant of the dynamics defined by the Hamiltonian H. A particular case is when particles can be divided on two subsets $I_1, I_2 \subset \{1, \dots, N\}$ such that $I_1 \cap I_2 = \emptyset$ and there exists function $G(q_i, p_i : i \in I_1)$ such that $H = H(G(q_i, p_i : i \in I_1); q_j, p_j : j \in I_2)$;

2) For harmonic oscillator with the Hamiltonian

$$H = \frac{1}{2}(p^2 + \omega^2 q^2)$$

consider the canonical transformation

$$h = \frac{1}{2\omega}(p^2 + \omega^2 q^2), \quad \psi = \arctan\Big(\frac{\omega q}{p}\Big)$$

with the inverse transformation

$$q = \frac{f(h)}{\omega}\sin\psi, \quad p = f(h)\cos\psi$$

Then the new Hamiltonian

$$H = \frac{1}{2}f^2(h)$$

In these "action-angle" variables (note that h has the dimension of action, see below) the hamiltonian equations are

$$\frac{dh}{dt} = 0, \quad \frac{d\psi}{dt} = \omega \Longrightarrow h(t) = const, \quad \psi(t) = \psi(0) + \omega t$$

3.3. Dynamics of measures

The induced transformation on the set M_X of measures on X is defined as follows

$$(U^t\mu)(A) = \mu(U^{-t}A),\ \mu \in M_X$$

Intuitively it is the amount of mass which enters to the set A at time t. Further on we consider hamiltonian dynamics in the phase space. Measure μ is called invariant if for any A

$$(U^t\mu)(A) = \mu(A)$$

Liouville theorem Liouville theorem says that Lebesgue measure is invariant with respect to hamiltonian dynamics.

To prove this note first that if one-to-one smooth mapping $U : X \to Y = U(X)$ of the domain $X \subset \mathbf{R}^n$ to other domain $Y \subset \mathbf{R}^n$ is such that the module of its Jacobian is identically 1, then the Lebesgue measures λ of X and $U(X)$ are equal. In fact, one can use

$$\lambda(U(X)) = \int\limits_{U(X)} dx$$

and change of variables from Ux to x.

Now consider our group U^t of one-to-one mappings. Consider the Jacobian $J(t)$ for small t and its elements

$$I_{kl}(t) = \frac{\partial x_k(t)}{\partial x_l(0)}$$

where

$$(x_1, \ldots, x_{2dN}) = (q_1, \ldots, q_{dN}, p_1, \ldots, p_{dN})$$

For small t we have

$$p_k(t) = p_k(0) - \frac{\partial U}{\partial q_i(0)}t, \quad q_k(t) = q_k(0) + \frac{\partial U}{\partial p_i(0)}t$$

Then, up to $O(t^2)$, the off-diagonal elements are linear in t, and the diagonal elements $I_{kk}(t)$ are

$$\frac{\partial p_k(t)}{\partial p_k(0)} = 1 - t\frac{\partial^2 U}{\partial p_i(0)\partial q_i(0)} + O(t^2), \quad \frac{\partial q_k(t)}{\partial q_k(0)} = 1 + t\frac{\partial^2 U}{\partial p_i(0)\partial q_i(0)}) + O(t^2)$$

Thus the product of all diagonal terms does not contain terms linear in t, and then

$$J(t) = 1 + O(t^2) \Longrightarrow \frac{dJ(t)}{dt} = 0$$

whence $J(t) \equiv 1$.

Liouville measure on the energy surface Speaking roughly, it is defined as the restriction of Lebesgue measure on M_h. More exactly, it is defined as

$$\frac{dM_h}{|\nabla H|}$$

where dM_h is the volume of small piece of M_h, $|\nabla H|$ is the length of the energy gradient vector in some point of this piece. Even better definition: consider the volume of the domain $\{x : h - \frac{\varepsilon}{2} < x < h + \frac{\varepsilon}{2}\}$, divide it on ε and consider the limit $\varepsilon \to 0$.

4. Famous equations

4.1. Euler – Lagrange equations

Lagrangian Newton equations

$$m\frac{dv_k}{dt} = F_k = -\frac{\partial U}{\partial x_k},\ k = 1, \dots, N,$$

for N-particle system with potential energy $U(x_1, \dots, x_N)$ and kinetic energy $T(v_1, \dots, v_N)$ can be rewritten as follows

$$\frac{d}{dt}\Big(\frac{\partial T}{\partial v_k}\Big) = -\frac{\partial U}{\partial x_k} \implies \frac{d}{dt}\Big(\frac{\partial L}{\partial v_k}\Big) = \frac{\partial L}{\partial x_k} \tag{4.1}$$

where $L = T - U$ is called the **Lagrangian**, and the equations are called **Euler – Lagrange equations**. In terms of the total energy $H = T + U$

$$L = T - U = 2T - H = H - 2U$$

This shows that if the total energy is conserved, the change of Lagrangian is defined by any of its terms

$$\frac{1}{2}(L(t_2) - L(t_1)) = T(t_2) - T(t_1) = U(t_1) - U(t_2)$$

4.1.1. Legendre transform

Roughly speaking, two real functions on $\mathbf{R}$ are Legendre transforms of each other if their derivatives are mutually inverse.

Consider smooth function $F(x)$ on $\mathbf{R}$ having everywhere positive second derivative. Then $F'(x)$ increases, and thus $F(x)$ can be represented as the function of $y = F'(x)$. That is there exists function $x(y)$, inverse to $y = F'(x)$. Variables x and y are called conjugate. Define now **Legendre transform** G of F

$$G(y) = yx(y) - F(x(y))$$

Then

$$G'(y) = x(y) + yx'(y) - F'(x(y))x'(y) = x(y)$$

and so $G''(y) = \frac{dx}{dy} > 0$. One can check that F is the Legendre transform of G.

Example. Let $F(v) = H = \frac{mv^2}{2} + U$ be a Hamiltonian, where U is considered as constant. Then conjugate variable to v will be the momentum $p = F'(v) = mv$ and Legendre transform of the Hamiltonian H is the **Lagrangian**

$$G(p) = pv - H = T - U = L$$

Legendre transform can be defined also in multi-dimensional situation. In case of the function $F(x) : \mathbf{R}^d \to \mathbf{R}$, its Legendre transform is defined as the real function

$$G(y) = (y, x) - F(x), \quad x = (\nabla F)^{-1}(y)$$

if ∇F is one-to-one map of some open set $O \subset \mathbf{R}^d$ onto other open set $O' \subset \mathbf{R}^d$.

Or, in terms of differentials

$$F = F(x_1, \ldots, x_n), \quad dF = u_1 dx_1 + \ldots + u_n dx_n, \quad u_k = \frac{\partial f}{\partial x_k}$$

Then variables x_k, u_k are called conjugate. For any $m \leq n$ Legendre transform of F (dependently on m) can be defined as

$$G = F - \sum_{k=1}^{m} u_k x_k, dG = \sum_{k=m+1}^{n} u_k dx_k + \sum_{k=1}^{m} (-x_k) du_k$$

4.2. Hamilton – Jacobi equation

4.2.1. Action

Action (its differential) was defined (for one particle of unit mass) by Maupertuis as functional of the infinitesimally small) path

$$dW = vds = v^2 dt = 2Tdt = (T - U + H)dt = Ldt + Hdt$$

If the energy is conserved the path integral (up to $H(t - t_0)$)

$$S = S(x(s), 0 \leq s \leq t) = \int_{t_0}^{t} L(x(s), x'(s))ds,$$

over the trajectory $x(t)$ is called **action** (according to Hamilton), where dash is the time derivative. S can be considered also, more generally, as the functional

on the set of all sufficiently smooth curves $x(s)$ on finite time interval $[t_0, t]$, and its dimension is energy×time. If H is conserved then

$$W = \int_{t_0}^{t} 2T dt \tag{4.2}$$

is also called short action, or Maupertuis – Lagrange action.

4.2.2. Hamilton principle

Consider the real trajectory $x(s)$, of some hamiltonian system with lagrangian L, on the interval $[0, t]$, with initial data $x(0), v(0)$, and also another curve $y(s) = x(s) + \delta(s)$, defined on the same interval and close to $x(s)$, that is $\delta(s) = \delta x(s)$ is small uniformly in s together with necessary number of derivatives. Consider the difference (variation)

$$\delta S = S(y(s), 0 \le s \le t) - S(x(s), 0 \le s \le t)$$

up to first order terms

$$\delta S = \int_0^t \Big(\frac{\partial L(x(s), x'(s))}{\partial x} \delta(s) + \frac{\partial L(x(s), x'(s))}{\partial x'} \delta'(s) \Big) ds =$$

$$= \int_0^t \Big(\frac{\partial L}{\partial x} - \frac{d}{ds} \frac{\partial L}{\partial x'} \Big) \delta(s) ds + \Big[\frac{\partial L}{\partial x'}(s) \delta(s) \Big] \Big|_0^t \tag{4.3}$$

where we used integration by parts for the second term.

We get the following **principle of least action** of Hamilton (or **principle of stationary action**). Namely, the variation equals zero for all $\delta(s)$ such that $\delta(0) = \delta(t) = 0$, iff $x(t)$ satisfies the Euler – Lagrange equations (4.1).

4.2.3. Direct derivation of H-J equations

If the action is considered only on real trajectories (that is satisfying Euler – Lagrange equations), then one can consider the action not as the functional on the paths, but as the real function of 4 variables, as for example:

1) initial and final time moments t_0, t_1, and initial coordinate and velocity x_0, v_0;

2) t_0, t_1, and final coordinate and velocity $x_1 = x(t_1), v_1 = v(t_1)$;

3) t_0, t_1, x_0, x_1.

Moreover, for given t_0, x_0 one can consider the action as the function of two other variables. For example, for fixed $t_0 = 0, x_0$, the action will be uniquely

defined function $S(t,x)$ in a small neighborhood O_{t_1,x_1} of the pair $(t_1, x_1 = x(t_1))$. One could prove this as follows. Let $x(s)$ be the trajectory on $[t_0 = 0, t_1]$, with initial data $x(0), x'(0) = v(0)$. Let $x_1 = x(t_1)$. If we fix $x(0)$ and change only the initial velocity $v(0) \to v(0)+\delta v$ then under some conditions, this defines one-to-one map of small neighborhood $O_{v(0)}$ of $v(0)$ to some neighborhood $O_{x(t)}$ of $x(t) = x_1 + \delta x$, for any $t = t_1 + \delta t$, from some neighborhood of t_1. Then from (4.3) it follows, first of all, that in this neighborhood (further on, in all formulae we keep only terms of the first order in δt and δx)

$$\delta S(t,x) = \frac{\partial S}{\partial t}\delta t + \frac{\partial S}{\partial x}\delta x \tag{4.4}$$

And secondly, one can generalize (4.3) as follows.

We consider the real trajectory $x(t) + \delta x(t)$ on $[0,t]$, with $\delta(0) = 0$, but with initial velocity $v(0)+\delta v$, as a perturbation of the (also real) trajectory $x(t)$ with initial data $x(0), v(0)$. Here $\delta t = t - t_1$, and note that

$$\delta x = x(t)+\delta x(t)-x(t_1) = x(t)+\delta x(t)-x(t_1) = \delta x(t)+x'(t)\delta t = \delta x(t_1)+x'(t)\delta t \tag{4.5}$$

Then

$$\delta S(t,x) = \int_0^t L(x(t)+\delta(x(t)))dt - \int_0^{t_1} L(x(t))dt =$$

$$= \int_0^{t_1} (L(x(t)+\delta(x(t)))dt - L(x(t)))dt + \int_{t_1}^{t} L(x(t)+\delta(x(t)))dt =$$

$$= \int_0^{t_1} \Big(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial x'}\Big)\delta x(s)ds + \Big[\frac{\partial L}{\partial x'}(s)\delta x(s)\Big]\Big|_0^{t_1} + L\delta t =$$

$$= \frac{\partial L}{\partial x'}\delta x(t_1) + L\delta t = p\delta x(t_1) + L\delta t \tag{4.6}$$

where the integral is zero because the perturbed trajectory is real. Remind that, neglecting second order terms,

$$p(t) = p(t_1), x'(t) = x'(t_1)$$

But from (4.5) we get

$$\delta x(t_1) = \delta x - x'(t)\delta t$$

Substituting this to the right hand part of (4.6) we get

$$\delta S = p(\delta x - x'\delta t) = (L - px')\delta t + p\delta x \tag{4.7}$$

Comparing this with (4.4), we get

$$\frac{\partial S}{\partial x} = p \tag{4.8}$$

$$\frac{\partial S}{\partial t} = L - 2T = -H$$

This gives **Hamilton – Jacobi equations**

$$\frac{\partial S}{\partial t} + H\Big(x, \frac{\partial S}{\partial x}, t\Big) = 0 \tag{4.9}$$

for the function $S(x,t)$. Initial data here is $S(0,x) = 0$.

The short action (4.2) then satisfies the equation

$$H\Big(x, \frac{\partial W}{\partial x}\Big) = 0$$

Note that there is another derivation of the H-J equations – using generating functions of canonical transforms, see [1, 7].

Example of S Consider free particle case. The H-J equation is then

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\Big(\frac{\partial S}{\partial x}\Big)^2 = 0$$

If we put

$$S(x,t) = f(x) - Et,$$

then substituting, we get for some constant C

$$f(x) = \sqrt{2mE}x - C \Longrightarrow S(x,t) = \sqrt{2mE}x - Et - C$$

4.3. Liouville and BBGKY equations

4.3.1. Liouville equation

Let $\rho(\psi, 0)$ be the initial probability density (with respect to Lebesgue measure) of the N particle system with Hamiltonian H on the phase space $\mathbf{R}^{2dN}$. That is for any subset $\Lambda \subset \mathbf{R}^{2dN}$ the probability that initially $\psi \in \Lambda$ equals

$$\int\limits_{\Lambda} \rho(\psi, 0) d\lambda,$$

where λ is the Lebesgue measure. Then, similarly to (3.12) and (3.8), Liouville equation defines, as for any function on the phase space, the evolution of the density $\rho(\psi, t)$ on $\mathbf{R}^{2dN}$

$$\frac{d\rho}{dt} = [H, \rho] = \frac{\partial H}{\partial q_i}\frac{\partial \rho}{\partial p_i} - v_i \frac{\partial \rho}{\partial q_i} \tag{4.10}$$

see [20].

4.3.2. Point fields

Contrary to the Liouville equation on $\mathbf{R}^{2dN}$ equations of Boltzmann type are defined on $\mathbf{R}^{2d}$. And here the more probability theory should be involved. Here we give two different possibilities how point field can be defined.

Intuitively, this is finite (or countable) number of point particles $i = 1, \dots, N$, randomly thrown on $\mathbf{R}^{2d} = \{(q, p)\}$. Let the random variable $\xi_i(t, A) = 1$ if at time t the particle i is in the set $A \subset \mathbf{R}^{2d}$, and $\xi_i(t, A) = 0$ otherwise. Random field theory says that the point field is defined by the system of probabilities

$$P(\xi_{i_1}(t, A_1) = 1, \dots, \xi_{i_n}(t, A_n) = 1)$$

for any finite arrays of the particles $I = i_1, \dots, i_n$ and of disjoint subsets $A_1, \dots, A_n$. If there are functions $f_I(t, x_1, \dots, x_n)$ such that

$$P(\xi_{i_1}(t, A_1) = 1, \dots, \xi_{i_n}(t, A_n) = 1) = \int\limits_{A_1} \dots \int\limits_{A_n} f_I(t, x_1, \dots, x_n) dx_1 \dots dx_n$$

then f_I are called n-particle correlation functions.

If the particles are identical, then the number of particles can be finite or countable. We define random point field on some space X. For example, one can consider $X = \mathbf{R}^{2d}$. Then it is defined by the system of random variables $\xi(A)$ – random number of particles in $A \subset X$. This system is given by the probabilities

$$P(\xi(A_1) = k_1, \dots, \xi(A_n) = k_n)$$

for any $n = 1, 2, \dots$, any subsets $A_1, \dots, A_n$ and any integers $k_1, \dots, k_n = 0, 1, 2, \dots$.

For example for $X = \mathbf{R}^{2d}$ one particle correlation function $f(t, q, p)$ can be understood in various ways. For example, $f(t, q, p)dqdp$ gives the probability that at time t there is a particle in the volume $dqdp$, up to higher order terms. Also, it defines $n(t, A)$, the mean number of particles in the set A at time t is equal to

$$n(t, A) = \int\limits_A f(t, q, p) dqdp$$

4.3.3. Simplest linear Boltzmann equation

Here we consider one-particle correlation function $f(t, x, p)$ for the simplest system of particles. Assume also that the masses of all particles are the same and equal to m, that is the particles are identical. For free particles it is easy to see that

$$f(t + \delta; x, p) = f\Big(t; x - \frac{p}{m}\delta, p\Big) \tag{4.11}$$

Subtracting $f(t;x,p)$ from both sides of this equality, dividing by δ and taking the limit $\delta \to 0$, we get the equation

$$\frac{\partial f}{\partial t} + \frac{p}{m}\frac{\partial f}{\partial x} = \frac{\partial f}{\partial t} + \sum_{i=1}^{d} v_i \frac{\partial f}{\partial x_i} = 0$$

The unique solution of the Cauchy problem for this equation, for initial $f(0,x,p)$, is

$$f(t,x,p) = f\Big(0, x - \frac{p}{m}t, p\Big)$$

If there is also external force $F(x)$, the same for all particles, then the equation in dimension 1 will be

$$f\Big(t+\delta; x, \frac{p}{m}\Big) = f\Big(t; x - \frac{p}{m}\delta, \frac{p}{m} - \frac{F}{m}\delta\Big) + O(\delta^2)$$

$$\frac{\partial f}{\partial t} + \frac{p}{m}\frac{\partial f}{\partial x} + \frac{F}{m}\frac{\partial f}{\partial v} = 0$$

In dimension $d > 1$ it will look as

$$\frac{\partial f}{\partial t} + \sum_i \Big(\frac{p_i}{m}\frac{\partial f}{\partial x_i} + \frac{F_i}{m}\frac{\partial f}{\partial v_i}\Big) = 0 \tag{4.12}$$

or, if F is a potential force with Hamiltonian H,

$$\frac{\partial f}{\partial t} = -\frac{\partial H}{\partial p}\frac{\partial f}{\partial x} + \frac{\partial H}{\partial x}\frac{\partial f}{\partial p} = [H, f]$$

We see that in this case we get the closed equation for one-particle correlation function. In general it is not possible and one should consider the system of equation for n-particle correlation functions, for all n. That we do below.

4.3.4. BBGKY hierarchy

Assume that $N < \infty$ particles are enumerated and $\rho(q_1, p_1, \dots, q_N, p_N, t)$ is the density that particles i are in the points q_i, p_i of the phase space respectively. Then, for example,

$$f_1(q_1, p_1, t) = \int \rho(q_1, p_1, \dots, q_N, p_N, t) dq_2 dp_2 \dots dq_N dp_N$$

is called one-particle correlation function for the particle 1. It gives the density that particle 1 is at the point (q_1, p_1) of the phase space. Similarly, one can define two-particle correlation functions

$$f_{1k}(q_1, p_1, q_k, p_k, t) =$$

$$= \int \rho(q_1, p_1, \ldots, q_N, p_N, t) dq_2 dp_2 \ldots dq_{k-1} dp_{k-1} dq_{k+1} dp_{k+1} \ldots dq_N dp_N$$

for a pair of particles $1, k$, and so on – the n-particle correlation functions for any $n \leq N$.

These correlation functions satisfy the system of equations, called Bogolyubov – Born – Green – Kirkwood – Yvon hierarchy, that readily follows by integrating the Liouville equation (4.10). For example, integrating as above we get

$$\frac{df_1(p_1, q_1, t)}{dt} = \frac{\partial H}{\partial q_1} \frac{\partial f_1}{\partial p_1} - v_1 \frac{\partial f_1}{\partial q_1} + \sum_{k=2}^{N} \int \frac{\partial H}{\partial q_1} \frac{\partial f_{1,k}}{\partial p_1} dq_k dp_k \tag{4.13}$$

In fact, for the integration of the left part it is clear. To integrate the right-hand part, using its linearity with respect to H, we subdivide the Hamiltonian on two parts $H = H_1 + H'$, where

$$H' = \sum_{k=2}^{N} \frac{p_k^2}{2m_k} + \sum_{i,j=2}^{N} U_{ij}, H_1 = \frac{p_1^2}{2m_1} + \sum_{j=2}^{N} U_{1j}$$

Note that H' corresponds to the closed hamiltonian system of $N-1$ particles $2, \ldots, N$, and thus the integration of

$$\sum_{k=2}^{N} \Big(\frac{\partial H'}{\partial q_k} \frac{\partial f_k}{\partial p_k} - v_k \frac{\partial f_k}{\partial q_k} \Big)$$

gives zero. Moreover, H_1 does not contain $p_k, k > 1$, and can be considered as the external force, with which particles $2, \ldots, N$ act on the particle 1. This proves (4.13).

If there is interaction then the right hand side will be non-zero, and is often called "collisional" term

$$\frac{\partial f}{\partial t} + \frac{p}{m} \frac{\partial f}{\partial x} + \frac{F}{m} \frac{\partial f}{\partial p} = \Big(\frac{\partial f}{\partial t} \Big)_{coll}$$

4.3.5. Dynamics in the thermodynamic limit

All equations considered above were for arbitrary but fixed $N < \infty$. Now we say some words about the asymptotics $N \to \infty$. Often, other parameters are taken also as functions of N. The choice of these functions is called scaling (choice of the parameter scale).

The first most known limit is the thermodynamic limit where the number of particles becomes countable. It is of main interest in mathematical statistical physics, with a lot of excellent results in equilibrium situation, that is studying the invariant measures. Another useful limit is the ultralocal where the number of particles becomes continuum, see below.

First results in this direction and reviews see in [17–19, 23–28].

To get intuitive understanding of the thermodynamic limit of dynamics let us first consider the simplest system of particles in R^d. Assume that initially, at time 0, there are countable number of particles with coordinates $x_i = x_i(0)$. Moreover, particles have velocities $v_i(0)$. It is natural to define the dynamics of this system as

$$x_i(t) = x_i(0) + v_i(0)t$$

However, if the particles interact, there is a problem with the existence of this dynamics, even if the two-particle interaction potential $V(r)$ is smooth and has finite support. Fix one particle and consider its trajectory $x(t)$. It appears that this trajectory can be influenced by infinite number of particles. Intuitively it seems clear that the greater distance between two particles the less they influence each other. However, this has been proved in rare situations. Very important is the remark about cluster structure of the dynamics. This means that there can be situations when infinite number of particles is subdivided into finite number of groups, called clusters, so that for a short time each cluster moves independently of other particles. There are few results in this direction, some references see in [22].

The same problem exists also for finite but very large particle systems "in finite volume". This means that we consider N particles in the cube $\Lambda = \Lambda(N) \subset \mathbf{R}^3$ with centre at 0, let $|\Lambda|$ be its volume. Normally, the asymptotics (scaling)

$$0 < N|\Lambda(N)|^{-1} = \lambda_N \to \lambda = const < \infty$$

is considered. Particles move inside this cube with the same interaction $V(r)$, but on its boundary there is, for example, mirror reflection. Fix now the initial particle coordinates and velocities as $x_k(0), v_k(0)$. Denote the trajectories of particles in the cube as $x_k^{(N)}(t)$ this particle is initially in the cube, that is if all $x_k(0) \in \Lambda(N)$ and have the same velocities $v_k(0)$. Then the problem is to prove convergence for any k and any fixed t as $N \to \infty$

$$x_k(t) \to x(t)$$

There are of course other scalings, for example, the Boltzmann – Grad scaling, see [25].

4.4. Ultralocal limit and Euler equation

Contrary to Liouville, BBGKY and Boltzmann equations, the Euler equation is defined in the coordinate space $\mathbf{R}^d$. Here we want to show that to deduce Euler equation from newtonian dynamics the following regularity property plays most important role.

Regular continuum system $\mathbf{M}_T$ of point particles is the set of subsets $\Lambda_t \in \mathbf{R}^d$ enumerated by the time moments $t \in [0, T), 0 < T \leq \infty$. Moreover, Λ_0 is

assumed to be the closure of some open connected subset of $\mathbf{R}^d$ with piece-wise smooth boundary $\partial\Lambda_0$. Each point of this domain is considered as a "material particle" of infinitely small mass. The dynamics is defined by the system of one-to-one mappings (diffeomorphisms) $U_t = U_{0,t} : \Lambda_0 \to \Lambda_t, t \in [0, T)$. All these mappings are assumed to be sufficiently smooth in x and piece-wise smooth in t, and $U_0(x)$ is the identity map. Thus, each point (particle) $x \in \Lambda_0$ has its own trajectory in $\mathbf{R}^d$: $y(t, x) = U_t(x)$, where $y(0, x) = x$ is the initial coordinate of this particle. It follows from the definition, that the particles never collide, that is $y(t, x) \neq y(t, x')$ for any t and $x \neq x'$.

$\mathbf{M}_T$ is called a system without interaction, if $y(t, x)$ are the solutions of the following equations

$$\frac{d^2y(t,x)}{dt^2} = F_x(y(t,x)), \qquad y(0,x) = x, \ \frac{dy(0,x)}{dt} = v(x) \tag{4.14}$$

for some given functions: initial velocity $v(x)$ and external forces $F_x(y)$, possibly different for different particles. Further on we assume that $F_x(y) = F(y)$ does not depend on x. It is always assumed that $v(x)$ and $m(x)$ are sufficiently smooth in $x \in \Lambda_0$, and $F(y)$ is smooth or piece-wise smooth in y. Moreover, it is always assumed that any equation (4.14) has a unique solution on all considered interval $[0, T)$.

In the Euler approach the central object is the field $u(t, y), y \in \Lambda_t, t \in [0,\mathrm{T})$. It is defined as the velocity of the particle which is at the point y at time t. For this it is necessary that this particle should be unique, and this is exactly the regularity property. Otherwise speaking, for any $t, y \in \Lambda_t$, there is exactly **ONE POINT** $x \in \Lambda_0$ such that

$$u(t, y) = u(t, y(x)) = \frac{dy(t, x)}{dt}$$

Then it is easy to see, for non-interacting particle systems, that the so defined field of velocities, satisfies the **Euler equations**

$$\frac{\partial u(t,y)}{\partial t} + \sum_\alpha \frac{\partial u(t,y)}{\partial y_\alpha} u_\alpha(t, y) = F(y) \tag{4.15}$$

In fact, the acceleration of the particle having trajectory $y(t, x)$ is equal to

$$\frac{du_i(t, y(t,x))}{dt} = \frac{\partial u_i(t, y(t,x))}{\partial t} + \sum \frac{\partial u_i(t, y(t,x))}{\partial y_j} u_j(t, y(t,x)) \tag{4.16}$$

and equals to the force $F(y(t, x))$.

We give now the simplest example of regular system. Let $\Lambda_{\grave{a}} = [0, 1] \subset R$. Assume also that

1) the initial velocity function $\frac{dy(0,x)}{dt} = v(x)$ is a non decreasing function on $[0, 1]$,

2) the force $F(y)$ is non-decreasing function on $\mathbf{R}_+$.

Then this system is evidently regular on $[0,\infty)$ – particles cannot collide. Many other examples of non interacting systems see in [29]. Obviously, the conception of **continuum media** as consisting of the continuum number of particles of infinitely small mass, is well known in mathematics, see for example [31] p. 56.

But seemingly the first deduction of complete system of Euler equations for interacting particle system appeared in [30]. This system corresponds to one-dimensional Chaplygin gas. In this paper also the convergence of some N-particle system, as $N \to \infty$ and some scaling (that we call ultralocal limit). to these Euler equations was proved.

5. Short guide to literature

This is just for first orientation in this great science.

Everything [1–3]

N-body problems [8, 9, 11, 12]¸

Scattering [4, 10],

Celestial mechanics [5, 6]

Canonical transformations [32, 33]

H-J [7]

Infinite systems [17–19, 22, 25–28]

Convergence to equilibrium [15]

Continuum mechanics [13, 14, 16, 29–31]

Kinetic [20, 24]

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