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CONVERGENCE OF THE STOCHASTIC QUANTIZATION METHOD

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ABSTRACT

We consider classical and fermionic Gibbsian fields on the lattice. The main result of the present paper is the proof of convergence for infinite system of Langevin's equations in the case of weak interaction.

INTRODUCTION

Gibbsian random fields are defined and investigated usually either by means of the thermodynamic limit transition from finite volume, or by means of Dobrushin—Lanford—Ruelle equations (Dobrushin, 1968). The stochastic quantization method is based on the observation of the fact that Gibbsian fields can be obtained also as the limiting distribution as time goes to infinity of the evolution of systems of stochastic differential equations of special type, namely the Langevin equations associated with the potential of the corresponding Gibbsian field.
This method was first introduced by physicists (Parisi and Wu, 1981). It has not received yet a generally accepted physical interpretation, and at present it is considered either as a theoretical or a computational mean of investigation and approximation of Gibbsian fields. In Langevin equations, time is sometimes called the mechanical time, and it is supposed that the Monte-Carlo simulation of Langevin's equations is more simple than that of Gibbsian fields (Migdal, 1986).
We consider Gibbsian fields on the lattice. Convergence of evolution (ergodicity) of Langevin's equations in finite volume is a common fact from the theory of finite dimensional Ito's equations (Gikhman and Skorokhod, 1972), which can be proved, for instance, by the help of Liapunov's functions.
The main result of the present paper is the proof of convergence for infinite system of Langevin's equations in the case of weak interaction. The class of Gibbsian fields under consideration corresponds to the lattice $P(\varphi)$-models of Euclidean quantum field theory. We consider Grassman fields (in particular, a perturbation of Dirac's field). For Grassman fields we had to carry over all basic definitions of the theory of Ito's equations to the case of Grassman variables in a form which is convenient for us. We could not find appropriate formulations in the literature in spite of a large number of papers on Ito's equations on Clifford algebras.
We prefer to give a more constructive definition of stochastic differential equations

then the classical one. It is still not known whether a full analog of the L_2-definition is possible. Complete proofs will be published in (Ignatiuk et al., 1990).

1. THE CONVERGENCE OF THE METHOD OF STOCHASTIC QUANTIZATION IN THE CLASSICAL CASE

With any finite subset $\Lambda \subset \mathbb{Z}^d$ we associate a polynomial U_Λ depending on variables $\xi_x \in \mathbb{R}$, $x \in \Lambda$, such that

$$Z_\Lambda = \int_{\mathbb{R}^{|\Lambda|}} \exp\{-U_\Lambda\} \prod_{x\in\Lambda} d\xi_x < \infty,$$

and consider on $\mathbb{R}^{|\Lambda|}$ the Gibbsian measure μ_Λ with the potential U_Λ:

$$d\mu_\Lambda = \frac{1}{Z_\Lambda} \exp\{-U_\Lambda\} \prod_{x\in\Lambda} d\xi_x.$$

Definition 1. By the system of Langevin's equations for the measure μ_Λ we call the system of stochastic differential equations:

$$d\xi_{x,\Lambda} = -\frac{1}{2}\frac{\partial U_\Lambda}{\partial \xi_x}\left(\bar{\xi}(t)\right) dt + dw_x(t), \qquad x \in \Lambda, \tag{1}$$

where $\bar{\xi}_\Lambda(t) = \{\xi_{x,\Lambda}\,,\ x \in \Lambda\}$ and $w_x(t)$ are standart Wiener processes independent for different $x \in \Lambda$.
The measure μ_Λ is invariant for the corresponding system (1) of Langevin's equations. Let the limit $\lim_{\Lambda\uparrow\mathbb{Z}^d} \mu_\Lambda = \mu$ exists in the sense of the weak convergence of finite dimensional distributions. Assume also that for any $x \in \mathbb{Z}^d$, $F_x = \partial U_\Lambda/\partial\xi_x$ does not depend on Λ if Λ is sufficiently large.

Definition 2. By the system of Langevin's equations for the limiting measure μ we call the infinite system of stochastic differential equations:

$$d\xi_x = -\frac{1}{2}F_x(\,\bar{\xi}_t\,)\, dt + dw_x(t), \qquad x \in \mathbb{Z}^d\ , \tag{1a}$$

where $\bar{\xi}(t) = \{\xi_x(t),\ x \in \mathbb{Z}^d\}$, and $w_x(t),\ x \in \mathbb{Z}^d$, are standart Wiener processes.
In the sequel we will consider the case, when U_Λ can be represented as a quadratic part plus small perturbation.
Consider translation-invariant Gaussian random field $\{\xi_x,\ x \in \mathbb{Z}^d\}$ on the lattice $\mathbb{Z}^d$ with zero mean $\langle\xi_x\rangle = 0$, $x \in \mathbb{Z}^d$, and the spectral density

$$f(\lambda) = \left[2\sum_{y\in\mathbb{Z}^d} a_y\, e^{-i(\lambda,y)}\right]^{-1}, \qquad \lambda \in [-\pi,\pi]^d,$$

with coefficients a_y, $y \in \mathbb{Z}^d$, satisfying the following conditions:

a) $a_y \neq 0$ only for a finite number of $y \in \mathbb{Z}^d$ (we denote by Q the set of points $y \in \mathbb{Z}^d$ for which $a_y \neq 0$),

b) $a_y = a_{-y}$ for all $y \in \mathbb{Z}^d$,

c) $a_0 - \sum_{y\neq 0} | a_y |> 0$.

The measure μ_0 corresponding to this Gaussian field can be obtained as the limit $\Lambda \uparrow \mathbb{Z}^d$ in the sense of weak convergence of finite dimensional distributions of the Gibbsian measures $\mu_{\Lambda,0}$, $\Lambda = [-N_\Lambda, N_\Lambda)^d \cap \mathbb{Z}^d$ with the quadratic potential

$$U_\Lambda^0 = \sum_{x\in\Lambda} a_0 \xi_x^2 + \sum_{x\neq y;\, x,y\in\Lambda} a_{x-y}^\Lambda \, \xi_x \, \xi_y;$$

where $a_y^\Lambda = a_y$ for $y \in \Lambda$ and $a_{y+N_\Lambda e}^\Lambda = a_y^\Lambda$ for $y \in \Lambda$, $e \in \mathbb{Z}^d$, $| e |= 1$.
The system of Langevin's equations for μ_0 has the form:

$$d\xi_x(t) = -\sum_{y\in Q} a_y \xi_{x+y}(t)\, dt + dw_x(t). \tag{2}$$

Let the initial field $\xi_x(0)$, $x \in \mathbb{Z}^d$, be Gaussian and translation-invariant with $\langle \xi_x(0)\rangle \equiv 0$, $x \in \mathbb{Z}^d$, and

$$| \langle \xi_0(0)\, \xi_x(0)\rangle | \leqslant c\, e^{-\gamma|x|}, \qquad c, \gamma > 0, \tag{3}$$

for any $x \in \mathbb{Z}^d$. By $\mu_0(0)$ we denote the corresponding Gaussian measure on $\mathbb{R}^{\mathbb{Z}^d}$, and by $\langle\cdot\rangle_\mu$ the expectation corresponding to a measure μ.
It is easy to see that the solution of the system (2) is a Gaussian process $\{\xi_x(t), x \in \mathbb{Z}^d\}$, $t \in \mathbb{R}_+$, whose evolution is described by the first and the second moments $\langle \xi_x(t)\rangle$, $\langle \xi_x(t)\xi_y(s)\rangle$, $x, y \in \mathbb{Z}^d$, $t, s \in \mathbb{R}_+$. The proof of the convergence to the invariant distribution μ_0 in this case is not difficult, because the moments can be written in an explicit form.
Consider now a Gibbsian modification $\mu_\Lambda, \Lambda \subset \mathbb{Z}^d$, of the Gaussian measure μ_0. Let

$$\frac{d\mu_\Lambda}{d\mu_0} = \frac{1}{Z_\Lambda} \exp\left\{-\varepsilon \sum_{x\in\Lambda} P(\xi_x)\right\},$$

where $P(\xi)$ is a polynomial, which is bounded from below. The covariance of the measure μ_0 decreases exponentially:

$$| \langle \xi_x \, \xi_y\rangle_{\mu_0} | \leqslant C\, e^{-\gamma|x-y|}; \qquad C, \gamma > 0; \qquad x, y \in \mathbb{Z}^d. \tag{4}$$

This follows from the conditions on the coefficients a_y, $y \in \mathbb{Z}^d$.
It is known (Malyshev and Minlos, 1985) that in this case for sufficiently small $\varepsilon > 0$, $\varepsilon < \varepsilon_0$ there exists the limit $\lim_{\Lambda\uparrow\mathbb{Z}^d} \mu_\Lambda = \mu_\varepsilon$ in the sense of weak convergence of finite dimensional distributions. Let us write the system of Langevin's equations for the measure μ_ε:

$$d\xi_x(t) = -\sum_{y\in Q} a_y \, \xi_{x+y}(t)\, dt - \varepsilon\, p(\xi_x(t))\, dt + dw_x(t), \tag{5}$$

where $p(\xi) = (1/2)dP(\xi)/d\xi$, $x \in \mathbb{Z}^d$.
The system (5) has a solution in the real Hilbert space $H = l_2(\mathbb{Z}^d, e^{-|x|})$ with the scalar product $(f,g)_H = \sum_{x\in\mathbb{Z}^d} f(x)\,g(x)\,e^{-|x|}$ (Rozovsky, 1983).

Let $\{\xi_x(t),\ x \in \mathbb{Z}^d\}$, $t \in \mathbb{R}_+$, be the solution of the system (5). By $\mu_\varepsilon(t)$ we denote the probability measure on $\mathbb{R}^{\mathbb{Z}^d}$, which corresponds to the solution $\xi_x(t)$, $x \in \mathbb{Z}^d$, at the time moment $t \in \mathbb{R}_+$. The following theorem is our main result for the classical case.

THEOREM 1. *Let $\varepsilon > 0$ be sufficiently small. Then the measure μ_ε is invariant for the system (2). Moreover, if the measure corresponding to the initial field $\xi_x(0)$, $x \in \mathbb{Z}^d$, is the Gibbsian modification of the Gaussian measure $\mu_0(0)$ with the potential $\varepsilon \sum P_0\big(\xi_x(0)\big)$, where $P_0(\xi)$ is a polynomial bounded from below such that*

$$P_0(\xi) - \frac{1}{2}P(\xi) \geqslant c$$

for all $\xi \in \mathbb{R}$, then

$$\mu_\varepsilon(t) \to \mu_\varepsilon \quad (t \to \infty)$$

in the sense of the weak convergence of finite dimensional distributions.

2. ITO'S CALCULUS ON GRASSMAN ALGEBRAS

For any $\Lambda \subset \mathbb{Z}^d$ and $t \in \mathbb{R}_+$ we consider four Grassman algebras, namely the Grassman algebra $@_0^\Lambda$ with generators

$$\mathbf{1},\quad \xi^{(\alpha,x)},\quad \overline{\xi}^{(\alpha,x)},\quad (\alpha,x) \in \{1,\dots,N\} \times \Lambda;$$

the Grassman algebra W_t^Λ with generators

$$\mathbf{1},\quad \overline{w}^{(\alpha,x)}(t),\quad w^{(\alpha,x)}(t),\quad (\alpha,x) \in \{1,\dots,N\} \times \Lambda,$$

the Grassman algebra $W_{t]}^\Lambda$ with generators

$$\mathbf{1},\quad \overline{w}^{(\alpha,x)}(t'),\quad w^{(\alpha,x)}(t'),\quad (\alpha,x) \in \{1,\dots,N\} \times \Lambda,\quad t' \in [0,t],$$

and the Grassman algebra W_∞^Λ with generators

$$\mathbf{1},\quad \overline{w}^{(\alpha,x)}(t),\quad w^{(\alpha,x)}(t),\quad (\alpha,x) \in \{1,\dots,N\},\quad t \in \mathbb{R}_+,\quad t > 0.$$

Denote

$$@_t^\Lambda = @_0^\Lambda \otimes W_t^\Lambda;\quad @_{t]}^\Lambda = @_0^\Lambda \otimes W_{t]}^\Lambda;\quad @_\infty^\Lambda = @_0^\Lambda \otimes W_\infty^\Lambda.$$

All algebras over complex will be considered. For convenience, we use multiindices $\mu = (\mu^1,\dots,\mu^k)$, $\mu^j \in \{1,\dots,N\} \times \mathbb{Z}^d$, $k \in \mathbb{Z}_+$, and write

$$\overline{\xi}^{\mu} = \overline{\xi}^{\mu_1} \cdot \ldots \cdot \overline{\xi}^{\mu_k};\qquad \xi^{\mu} = \xi^{\mu_1} \cdot \ldots \cdot \xi^{\mu_k};\qquad \xi^{\varnothing} = \mathbf{1};$$

$$\overline{w}^{\mu}(t) = \overline{w}^{\mu_1}(t)\cdot\ldots\cdot\overline{w}^{\mu_k}(t); \quad w^{\mu}(t) = w^{\mu_1}(t)\cdot\ldots\cdot w^{\mu_k}(t).$$

We assume that $\{1,\ldots,N\}\times\mathbb{Z}^d$ is completely ordered, and for any multiindex $\mu = (\mu^1,\ldots,\mu^k)$, $\mu^1 < \ldots < \mu^k$. Let M_Λ denote the set of all multiindices $\mu = (\mu^1,\ldots,\mu^k)$, $k\in\mathbb{Z}_+$.

Definition 3. By the Wiener quasistate on W_∞^Λ we mean the Gaussian quasistate $\langle\cdot\rangle_w$ on W_∞^Λ satisfying the following conditions:

$$\langle\,\overline{w}^{(\alpha,x)}(t)\,\overline{w}^{(\beta,y)}(s)\,\rangle_w = \langle\,w^{(\alpha,x)}(t)\,w^{(\beta,y)}(s)\,\rangle_w = 0$$

and

$$\langle\,\overline{w}^{(\beta,y)}(t)\,w^{(\alpha,x)}(s)\,\rangle_w = \delta_{\alpha\beta}\,\delta_{xy}\min(t,s)$$

for any

$$(\alpha,\beta)\in\{1,\ldots,N\}; \quad x,y\in\Lambda; \quad t,s\in\mathbb{R}_+.$$

PROPOSITION 1. *For any $w\in W_\infty^\Lambda$ there exists $w'\in W_\infty^\Lambda$ such that*

$$\langle w\,w'\rangle \neq 0.$$

We shall consider only quasistates on $@_\infty^\Lambda = @_0^\Lambda\otimes W_\infty^\Lambda$ of the following form:

$$\omega(\cdot) = \langle\cdot\rangle_0\otimes\langle\cdot\rangle_w,$$

where $\langle\cdot\rangle_0$ is a quasistate on $@_0^\Lambda$. By σ_Λ we denote the set of all such quasistates on $@_0^\Lambda$.

We consider on $@_0^\Lambda$ the system of seminorms $\rho_{\zeta,\omega}(\cdot)$, $\zeta\in @_\infty^\Lambda$, $\omega\in\sigma_\Lambda$:

$$\rho_{\zeta,\omega}(\xi) = |\,\omega(\zeta\xi)\,|\,.$$

This system of seminorms separates points of $@_\infty^\Lambda$, i.e., the condition $\rho_{\zeta,\omega}(\xi) = 0$ for all $\xi\in @_\infty^\Lambda$, $\omega\in\sigma_\Lambda$ implies $\xi = 0$. The locally convex topology on $@_\infty^\Lambda$ with respect to which all $\rho_{\zeta,\omega}$, $\zeta\in @_\infty^\Lambda$, $\omega\in\sigma_\Lambda$, are continuous and the operation of addition is continuous, is a Hausdorff topology. By $\overline{@_\infty^\Lambda}$ we denote the closure of $@_\infty^\Lambda$ with respect to this topology.

Consider the following Grassman algebras, namely:

$\mathcal{A}_\infty^\Lambda$, with the generators

$$1, \quad \overline{\xi}^{(\alpha,x)}(t), \quad \xi^{(\alpha,x)}(t), \quad (\alpha,x)\in\{1,\ldots,N\}\times\Lambda; \quad t\in\mathbb{R}_+;$$

$\mathcal{A}_t^\Lambda$ with generators

$$1, \quad \overline{\xi}^{(\alpha,x)}(t), \quad \xi^{(\alpha,x)}(t), \quad (\alpha,x)\in\{1,\ldots,N\}\times\Lambda$$

and $\mathcal{A}_{t]}^\Lambda$, $t\in\mathbb{R}_+$, with the generators

$$1, \quad \overline{\xi}^{(\alpha,x)}(t'), \quad \xi^{(\alpha,x)}(t'), \quad (\alpha,x)\in\{1,\ldots,N\}\times\Lambda, \quad t'\in[0,t].$$

Definition 4. By the Grassman Wiener process we mean the homomorphism

$$w\,:\,\mathcal{A}_{\infty}^{\Lambda}\rightarrow @_{\infty}^{\Lambda}$$

such that

$$w\,:\,\overline{\xi}^{(\alpha,x)}(t)\rightarrow\overline{w}^{(\alpha,x)}(t),$$

$$w\,:\,\xi^{(\alpha,x)}(t)\rightarrow w^{(\alpha,x)}(t),\quad (\alpha,x)\in\{1,\ldots,N\},\quad t\in\mathbb{R}_{+}.$$

All other Grassman processes which we consider will be constructed from the Grassman Wiener process.
Now we shall define the stochastic integrals:

$$\mathfrak{I}^{\mu\nu}(T_1,T_2)=\int_{T_1}^{T_2}\overline{w}^{\,\mu}(t)w^{\nu}(t)\,dt,$$

$$\overline{\mathfrak{I}}^{\mu\nu}_{(\alpha,x)}(T_1,T_2)=\int_{T_1}^{T_2}\overline{w}^{\,\mu}(t)w^{\nu}(t)\,d\overline{w}^{\,(\alpha,x)}(t),$$

$$\mathfrak{I}^{\mu\nu}_{(\alpha,x)}(T_1,T_2)=\int_{T_1}^{T_2}\overline{w}^{\,\mu}(t)w^{\nu}(t)\,dw^{(\alpha,x)}(t),$$

for all $\mu,\nu\in M_{\Lambda}$, $(\alpha,x)\in\{1,\ldots,N\}\times\Lambda$, $T_1,T_2\in\mathbb{R}_{+}$, $T_1<T_2$ as limits of the corresponding partial sums:

$$S^{\mu\nu}(t_0,\ldots,t_n)=\sum_{j=0}^{n-1}\overline{w}^{\,\mu}(t_j)w^{\nu}(t_j)(t_{j+1}-t_j),$$

$$\overline{S}^{\mu\nu}_{(\alpha,x)}(t_0,\ldots,t_n)=\sum_{j=0}^{n-1}\overline{w}^{\,\mu}(t_j)w^{\nu}(t_j)\big(\overline{w}^{\,(\alpha,x)}(t_{j+1})-\overline{w}^{\,(\alpha,x)}(t_j)\big),$$

when the diameter of the partition $T_1=t_0<t_1<\ldots<t_n=T_2$ tends to zero.
It is easy to see that in the topology introduced above, all the limits of these partial sums exist, as well as the limits of the various multiproducts.
Next we define the stochastic integrals:

$$\int_{T_1}^{T_2}w(t)\,dt;\quad\int_{T_1}^{T_2}w(t)\,d\overline{w}^{\,(\alpha,x)}(t);\quad\int_{T_1}^{T_2}w(t)\,dw^{(\alpha,x)}(t);$$

$(\alpha,x)\in\{1,\ldots,N\}\times\Lambda$, for elements $w(t)\in @_t^{\Lambda}$ of the form

$$w(t)=\sum_{\mu,\nu\in M_{\Lambda}}c_{\mu\nu}\overline{w}^{\,\mu}(t)w^{\nu}(t),\quad c_{\mu\nu}\in\mathbb{C},$$

by linearity. Finally, we define

$$\int_{T_1}^{T_2} \xi w(t)\, dt = \xi \int_{T_1}^{T_2} w(t)\, dt;$$

$$\int_{T_1}^{T_2} \xi w(t)\, d\overline{w}^{(\alpha,x)}(t) = \xi \int_{T_1}^{T_2} w(t) d\overline{w}^{(\alpha,x)}(t);$$

$(\alpha, x) \in \{1,\ldots,N\} \times \Lambda$ for all $\xi \in @_0^\Lambda$.
We consider the superalgebra $@_\infty^{\Lambda,1}$ with the generators

$$\mathbf{1},\quad \overline{\xi}^{(\alpha,x)},\quad \xi^{(\alpha,x)},\quad \mathfrak{I}^{\mu\nu}(0,t),\quad \overline{\mathfrak{I}}^{\mu\nu}_{(\alpha,x)}(0,t),\quad \mathfrak{I}^{\mu\nu}_{(\alpha,x)}(0,t),$$

$$\mu,\nu \in M_\Lambda,\quad (\alpha,x) \in \{1,\ldots,N\} \times \Lambda,\quad t \in \mathbf{R}_+;$$

the superalgebra $@_t^{\Lambda,1}$ with the generators

$$\mathbf{1},\quad \overline{\xi}^{(\alpha,x)},\quad \xi^{(\alpha,x)},\quad \mathfrak{I}^{\mu\nu}(0,t),\quad \overline{\mathfrak{I}}^{\mu\nu}_{(\alpha,x)}(0,t),\quad \mathfrak{I}^{\mu\nu}_{(\alpha,x)}(0,t);$$

and the superalgebra $@_{t]}^{\Lambda,1}$ with the generators

$$\mathbf{1},\quad \overline{\xi}^{(\alpha,x)},\quad \xi^{(\alpha,x)},\quad \mathfrak{I}^{\mu\nu}(0,t'),\quad \overline{\mathfrak{I}}^{\mu\nu}_{(\alpha,x)}(0,t'),\quad \mathfrak{I}^{\mu\nu}_{(\alpha,x)}(0,t'),$$

$$(\alpha,x) \in \{1,\ldots,N\} \times \Lambda,\quad \mu,\nu \in M_\Lambda,\quad t' \in [0,t],$$

The following inclusion relations hold between the algebras introduced above:

$$@_\infty^\Lambda \subset @_\infty^{\Lambda,1};\quad @_t^\Lambda \subset @_t^{\Lambda,1};\quad @_{t]}^\Lambda \subset @_{t]}^{\Lambda,1},\quad t \in \mathbf{R}_+,$$

because

$$\overline{w}^{(\alpha,x)}(t) = \int_0^t \mathbf{1}\, d\overline{w}^{(\alpha,x)}(s) = \overline{\mathfrak{I}}^{\varnothing\varnothing}_{(\alpha,x)}(0,t),$$

$$w^{(\alpha,x)}(t) = \int_0^t \mathbf{1}\, dw^{(\alpha,x)}(s) = \mathfrak{I}^{\varnothing\varnothing}_{(\alpha,x)}(0,t),$$

$(\alpha,x) \in \{1,\ldots,N\} \times \Lambda, t \in \mathbf{R}_+$.
We extend any quasistate from σ_Λ by continuity on $@_\infty^{\Lambda,1}$ and identify all elements ζ, ξ from $@_\infty^{\Lambda,1}$ such that $\omega(\eta\zeta) = \omega(\eta,\xi)$ for all $\eta \in @_\infty^{\Lambda,1}$ and all $\omega \in \sigma_\Lambda$.
We shall consider on $@_t^{\Lambda,1}$, $t \in \mathbf{R}_+$, the family of homomorphisms $S_\tau^1 : @_t^{\Lambda,1} \to$ $\to @_{t+\tau}^{\Lambda,1}$ ($\tau \in \mathbf{R}_+$) such that

$$S_\tau^1\xi = \xi,\quad S_\tau^1\mathfrak{I}^{\mu\nu}(0,t) = \mathfrak{I}^{\mu\nu}(0,t+\tau),$$

$$S_\tau^1\overline{\mathfrak{I}}^{\mu\nu}_{(\alpha,x)}(0,t) = \overline{\mathfrak{I}}^{\mu\nu}_{(\alpha,x)}(0,t+\tau),\quad S_\tau^1\mathfrak{I}^{\mu\nu}_{(\alpha,x)}(0,t) = \mathfrak{I}^{\mu\nu}_{(\alpha,x)}(0,t+\tau).$$

for all $\xi \in @_0^\Lambda$, $\tau \in \mathbb{R}_+$, $(\alpha, x) \in \{1, \dots, N\} \times \Lambda$, $\mu, \nu \in M_\Lambda$, $t \in \mathbb{R}_+$, $t > 0$.

Definition 5. By Grassman random process on $@_\infty^{\Lambda,1}$ we mean a homomorphism $\Psi : @_\infty^\Lambda \to @_\infty^{\Lambda,1}$ such that

$$\Psi\,|_{\mathcal{A}_t^\Lambda} = \Psi_t : \mathcal{A}_t^\Lambda \to @_t^{\Lambda,1}, \quad t \in \mathbb{R}_+,$$

and

$$S_\tau \Psi_t = \Psi_{t+\tau}$$

for all $t, \tau \in \mathbb{R}_+$, $t > 0$. Let $\mathfrak{A}_1^\Lambda$ denote the set of all random processes on $@_\infty^{\Lambda,1}$. We shall construct by induction superalgebras $@_\infty^{\Lambda,n}$, $@_t^{\Lambda,n}$, $@_{t]}^{\Lambda,n}$, $t \in \mathbb{R}_+$, for any $n \in \mathbb{Z}_+$:

$$@_\infty^{\Lambda,0} = @_\infty^\Lambda \subset @_\infty^{\Lambda,1} \subset \dots \subset @_\infty^{\Lambda,n},$$

$$@_t^{\Lambda,0} = @_t^\Lambda \subset @_t^{\Lambda,1} \subset \dots \subset @_t^{\Lambda,n},$$

$$@_{t]}^{\Lambda,0} = @_{t]}^\Lambda \subset @_{t]}^{\Lambda,1} \subset \dots \subset @_{t]}^{\Lambda,n},$$

and sets of random processes $\mathfrak{A}_n^\Lambda$ on $@_\infty^{\Lambda,n}$, $n \in \mathbb{Z}_+$, as follows. We define consecutively for any $n \in \mathbb{Z}_+$ the stochastic integrals:

$$\int_{T_1}^{T_2} \Psi[\,\overline{\xi}^{\,\mu}(t)\xi^\nu(t)\,]\,dt;$$

$$\int_{T_1}^{T_2} \Psi[\,\overline{\xi}^{\,\mu}(t)\xi^\nu(t)\,]\,d\overline{w}^{(\alpha,x)}(t);$$

$$\int_{T_1}^{T_2} \Psi[\,\overline{\xi}^{\,\mu}(t)\xi^\nu(t)\,]\,dw^{(\alpha,x)}(t);$$

$\Psi \in \mathfrak{A}_{n-1}^\Lambda$, $\mu, \nu \in M_\Lambda$; $(\alpha, x) \in \{1, \dots, N\} \times \Lambda$; $T_1, T_2 \in \mathbb{R}_+$, and various multiproducts of these integrals as for $n = 1$ and extend by continuity on $@_\infty^{\Lambda,n}$ all quasistates from σ_Λ.

Definition 6. By Grassman random process on $@_\infty^{\Lambda,n}$ we mean a homomorphism

$$\Psi : \mathcal{A}_\infty^\Lambda \to @_\infty^{\Lambda,n}$$

such that

$$\Psi\,|_{\mathcal{A}_t^\Lambda} = \Psi_t : \mathcal{A}_t^\Lambda \to @_t^{\Lambda,n}, \quad t \in \mathbb{R}_+,$$

and

$$S_\tau^n \Psi_t = \Psi_{t+\tau},$$

for all $t, \tau \in \mathbb{R}_+$, $t > 0$, where S_τ^n, $\tau \in \mathbb{R}_+$, is the family of homomorphisms on $@_t^{\Lambda,n}$, $t \in \mathbb{R}_+$, such that

$$S_\tau^n\,|_{@_t^{\Lambda,n-1}} = S_\tau^{n-1}$$

for all $\tau, t \in \mathbb{R}_+$ and

$$S_\tau^n\left(\int_0^t \Psi'\left[\bar{\xi}^{\,\mu}(s)\xi^\nu(s)\right]ds\right) = \int_0^{t+\tau}\Psi'\left[\bar{\xi}^{\,\mu}(s)\xi^\nu(s)\right]ds,$$

$$S_\tau^n\left(\int_0^t \Psi'\left[\bar{\xi}^{\,\mu}(s)\xi^\nu(s)\right]d\widetilde{w}^{(\alpha,x)}(s)\right) = \int_0^{t+\tau}\Psi'\left[\bar{\xi}^{\,\mu}(s)\xi^\nu(s)\right]d\widetilde{w}^{(\alpha,x)}(s),$$

for all $(\alpha, x) \in \{1, \ldots, N\} \times \Lambda$; $\Psi' \in \mathfrak{A}_{n-1}^\Lambda$; $\mu, \nu \in M_\Lambda$; $t, \tau \in \mathbb{R}_+$; $t > 0$, where $\widetilde{w}$ means either $\overline{w}$ or w.
By $\mathfrak{A}_n^\Lambda$ we shall denote the set of random processes on $@_\infty^{\Lambda,n}$.
It is clear that

$$\mathfrak{A}_0^\Lambda \subset \mathfrak{A}_1^\Lambda \subset \ldots \subset \mathfrak{A}_n^\Lambda, \quad n \in \mathbb{Z}_+.$$

We shall consider the inductive limits:

$$@_\infty^{\Lambda,\infty} = \bigcup_n @_\infty^{\Lambda,n} \subset \overline{@_\infty^\Lambda};$$

$$@_t^{\Lambda,\infty} = \bigcup_n @_t^{\Lambda,n};$$

$$@_{t]}^{\Lambda,\infty} = \bigcup_n @_{t]}^{\Lambda,n};$$

$$\mathfrak{A}_\infty^\Lambda = \bigcup_n \mathfrak{A}_n^\Lambda.$$

Now we shall define the notion of generalized random process on $@_\infty^{\Lambda,\infty}$. For this we introduce the following topology on $\mathfrak{A}_\infty^\Lambda$:
Let $\Psi \in \mathfrak{A}_\infty^\Lambda$, $n \in \mathbb{Z}_+$ be given. We say that $\Psi_n \to \Psi \in \mathfrak{A}_\infty^\Lambda$ as $n \to \infty$ if for any $T \in \mathbb{R}_+$, $k \in \mathbb{Z}_+$

$$\sup_{t_0,\ldots,t_k \in [0,T]} \mid \langle \zeta \Psi_n \bar{\xi}^{\,\mu_0}(t_0)\ldots\bar{\xi}^{\,\mu_k}(t_k)\xi^{\nu_0}(t_0)\ldots\xi^{\nu_k}(t_k)\rangle -$$

$$-\langle \zeta \Psi \bar{\xi}^{\,\mu_0}(t_0)\ldots\bar{\xi}^{\,\mu_k}(t_k)\xi^{\nu_0}(t_0)\ldots\xi^{\nu_k}(t_k)\rangle \mid \to 0,$$

as $n \to \infty$.
We denote by $\overline{\mathfrak{A}_\infty^\Lambda}$ the closure of $\mathfrak{A}_\infty^\Lambda$ with respect to this topology.

Definition 7. Elements of the set $\overline{\mathfrak{A}_\infty^\Lambda}$ will be called generalized random processes on $@_\infty^{\Lambda,\infty}$, or random processes on $\overline{@_\infty^{\Lambda,\infty}}$.

For $\Psi \in \overline{\mathfrak{A}_\infty^\Lambda}$ we define stochastic integrals by continuity:

$$\int_{T_1}^{T_2} \Psi \bar{\xi}^\mu(t)\xi^\nu(t)\,dt,$$

$$\int_{T_1}^{T_2} \Psi \overline{\xi}^{\,\mu}(t)\xi^{\nu}(t)\, d\overline{w}^{\,(\alpha,x)}(t),$$

$$\int_{T_1}^{T_2} \Psi \overline{\xi}^{\,\mu}(t)\xi^{\nu}(t)\, dw^{(\alpha,x)}(t),$$

$(\alpha, x) \in \{1, \ldots, N\} \times \Lambda$.
Similarly as in the classical theory of stochastic integrals, we write

$$d\zeta(t) = \xi_1(t)\, dt + \xi_2(t)\, d\overline{w}^{\,(\alpha,x)}(t) + \xi_3(t)\, dw^{(\beta,y)}(t) +$$
$$+ d\overline{w}^{\,(\alpha',x')}(t)\, \xi_4(t) + dw^{(\beta',y')}(t)\, \xi_5(t)$$

if the equality

$$\zeta(t) = \zeta(0) + \int_0^t \xi_1(s)\, ds + \int_0^t \xi_2(s)\, d\overline{w}^{\,(\alpha,x)}(s) +$$
$$+ \int_0^t \xi_3(s)\, dw^{(\beta,y)}(s) + \int_0^t d\overline{w}^{\,(\alpha',x')}(s)\, \xi_4(s) + \int_0^t dw^{(\beta',y')}(s)\, \xi_5(s)$$

holds for any $t \in \mathbb{R}_+$.
The following property of Grassman stochastic integrals is the analogue of the classical Ito's formula. Let following two stochastic integrals be given:

$$d\zeta(t) = \eta_1(t)\, dt + \eta_2(t)\, d\overline{w}^{\,(\alpha,x)}(t) + \eta_3(t)\, dw^{(\beta,y)}(t),$$

$$d\zeta'(t) = \eta_1'(t)\, dt + d\overline{w}^{\,(\alpha',x')}\, \eta_2'(t) + dw^{(\beta',y')}(t)\, \eta_3'(t)$$

then

$$d\big(\zeta(t)\zeta'(t)\big) = \big(d\zeta(t)\big)\zeta'(t) + \zeta(t)\, d\zeta'(t) + d\zeta(t)\, d\zeta'(t),$$

where

$$d\zeta(t)\, d\zeta'(t) = \delta_{\alpha\beta'}\, \delta_{xy'}\, \eta_2(t)\, \eta_3'(t) - \delta_{\beta\alpha'}\, \delta_{yx'}\, \eta_3(t)\, \eta_2'(t).$$

The proof of Ito's formula follows easily from the definition of stochastic integrals.
Consider the following system of stochastic differential equations:

$$d\overline{\xi}^{\,(\alpha,x)}(t) = \overline{P}_{\alpha,x}\left(\overline{\xi}^{\,(\beta,y)}(t),\ \xi^{(\beta,y)}(t),\ (\beta,y) \in \{1, \ldots, N\} \times \Lambda\right) dt + d\overline{w}^{\,(\alpha,x)}(t),$$

$$d\xi^{(\alpha,x)}(t) = P_{\alpha,x}\left(\overline{\xi}^{\,(\beta,y)}(t),\ \xi^{(\beta,y)}(t),\ (\beta,y) \in \{1, \ldots, N\} \times \Lambda\right) dt + dw^{(\alpha,x)}(t),$$

$$\overline{\xi}^{\,(\alpha,x)}(0) = \overline{\xi}^{\,(\alpha,x)}, \quad \xi^{(\alpha,x)}(0) = \xi^{(\alpha,x)}, \quad (\alpha, x) \in \{1, \ldots, N\} \times \Lambda, \qquad (6)$$

where $\overline{P}_{\alpha,x}(\cdot)$, $P_{\alpha,x}(\cdot)$ are arbitrary polynomials of odd degree in variables $\overline{\xi}^{\,(\beta,y)}(t)$, $\xi^{(\beta,y)}(t)$, $(\beta, y) \in \{1, \ldots, N\} \times \Lambda$.

Definition 8. By a solution of the system (6) we mean the random process $\Psi \in \mathfrak{A}_{\infty}^{\Lambda}$ such that

$$\Psi\bar{\xi}^{(\alpha,x)}(t) = \bar{\xi}^{(\alpha,x)} + \int_0^t \Psi\overline{P}_{\alpha,x}\left(\bar{\xi}^{(\beta,y)}(s),\ \xi^{(\beta,y)}(s)\right) ds + \bar{w}^{(\alpha,x)}(t),$$

$$\Psi\xi^{(\alpha,x)}(t) = \xi^{(\alpha,x)} + \int_0^t \Psi P_{\alpha,x}\left(\bar{\xi}^{(\beta,y)}(s),\ \xi^{(\beta,y)}(s)\right) ds + w^{(\alpha,x)}(t)$$

holds for all $t \in \mathbb{R}_+$, $(\alpha, x) \in \{1, \ldots, N\} \times \Lambda$.

PROPOSITION 2. *The system* (6) *has a solution for any finite* $\Lambda \subset \mathbb{Z}^d$ $(\mid \Lambda \mid < < \infty)$.

3. GIBBSIAN QUASISTATES AND THE CORRESPONDING LANGEVIN'S EQUATIONS

Let $\Lambda \subset \mathbb{Z}^d$, $\mid \Lambda \mid < \infty$, be given.

Definition 9. A quasistate $\langle\cdot\rangle_\Lambda$ on $@_0^\Lambda$ is called Gibbsian if there exists a potential $U_\Lambda \in @_0^\Lambda$ such that for any $\xi \in @_0^\Lambda$

$$\langle\xi\rangle_\Lambda = \frac{\langle\xi\exp\{-U_\Lambda\}\rangle_B}{\langle\exp\{-U_\Lambda\}\rangle_B},$$

where $\langle V\rangle_B = \int V\, d\bar{\xi}^{\mu_\Lambda}\xi^{\mu_\Lambda}$ is the Berezin integral (Berezin, 1986) and μ_Λ is the maximal multiindex.

We always assume that the potential U_Λ is even. The Gibbsian quasistate with the potential U_Λ will be denoted by $\langle\cdot\rangle_{U_\Lambda}$.

Definition 10. By the system of Langevin's equations corresponding to the Gibbsian quasistate with potential U_Λ we mean the system of stochastic differential equations:

$$\begin{aligned} d\bar{\xi}^{(\alpha,x)}(t) &= -\frac{1}{2}\frac{\partial U_\Lambda(t)}{\partial_R\xi^{(\alpha,x)}(t)}\, dt + d\bar{w}^{(\alpha,x)}(t), \\ d\xi^{(\alpha,x)}(t) &= -\frac{1}{2}\frac{\partial U_\Lambda(t)}{\partial_L\bar{\xi}^{(\alpha,x)}(t)}\, dt + dw^{(\alpha,x)}(t), \end{aligned} \qquad (7)$$

where differentiation is the Berezin differentiation, ∂/∂_L and ∂/∂_R stand for the left and the right derivatives respectively, and $U_\Lambda(t)$ is obtained from U_Λ by changing the variables $\bar{\xi}^{(\alpha,x)}$, $\xi^{(\alpha,x)}$ by $\bar{\xi}^{(\alpha,x)}(t)$, $\xi^{(\alpha,x)}(t)$, $(\alpha,x) \in \{1,\ldots,N\} \times \Lambda$, respectively.

Let $\Psi \in \mathfrak{A}_{\infty}^{\Lambda}$ be a solution of the system (7), and let a quasistate $\langle\cdot\rangle \in \sigma^\Lambda$ on $@_\infty^{\Lambda,\infty}$ be given. Then Ψ defines a quasistate on the Grassman algebra $\mathcal{A}_\infty^\Lambda$ with the generators $\bar{\xi}^{(\alpha,x)}(t)$, $\xi^{(\alpha,x)}(t)$, $t \in \mathbb{R}_+$, $(\alpha,x) \in \{1,\ldots,N\} \times \Lambda$ by the relation

$$\langle\bar{\xi}^{\mu}(t)\,\xi^{\nu}(t)\rangle_\Lambda = \langle\Psi\bar{\xi}^{\mu}(t)\,\xi^{\nu}(t)\rangle, \quad \mu,\nu \in M_\Lambda.$$

The evolution of the quasistate $\langle\cdot\rangle_\Lambda$ on $@_0^\Lambda$ is defined by:

$$\langle\xi\rangle_{\Lambda,t} = \langle\xi(t)\rangle_\Lambda, \quad t \in \mathbf{R}_+, \quad \xi \in @_0^\Lambda.$$

PROPOSITION 3. *A Gibbsian quasistate on $@_0^\Lambda$ is invariant under the evolution determined by a solution of the corresponding system of Langevin's equations, i.e.,*

$$\langle\cdot\rangle_{\Lambda,t} = \langle\cdot\rangle_\Lambda = \langle\cdot\rangle_{U_\Lambda}$$

for any $t \in \mathbf{R}_+$.

We shall consider the case, when U_Λ is a small perturbation of a quadratic potential

$$U_\Lambda = U_\Lambda^\varepsilon = U_\Lambda^0 + \varepsilon V_\Lambda,$$

where

$$U_\Lambda^0 = \sum_{\alpha,\beta} \sum_{x,y\in\Lambda} a_\Lambda^{\alpha,\beta}(x-y)\overline{\xi}^{(\alpha,x)}\xi^{(\beta,y)},$$

$$V_\Lambda = \sum_{x\in\Lambda} P\Big(\overline{\xi}^{(\alpha,x)},\ \xi^{(\alpha,x)},\ \alpha\in\{1,\dots,N\}\Big),$$

$P\Big(\overline{\xi}^{(\alpha,x)},\ \xi^{(\alpha,x)},\ \alpha\in\{1,\dots,N\}\Big)$ is a polynomial of even degree in variables $\overline{\xi}^{(\alpha,x)}$, $\xi^{(\alpha,x)}$, $\alpha\in\{1,\dots,N\}$. The coefficients $a_\Lambda^{\alpha\beta}(y)$, $\alpha,\beta\in\{1,\dots,N\}$, $y\in\mathbf{Z}^d$, satisfy the following conditions. Let $\Lambda = [-N_\Lambda, N_\Lambda]^d \subset \mathbf{Z}^d$ then

a) for any $y\in\Lambda$ and any $\alpha,\beta\in\{1,\dots,N\}$, $a_\Lambda^{\alpha\beta}(y) = a^{\alpha\beta}(y)$, where $a^{\alpha\beta}(y)$ do not depend on Λ,

$$a^{\alpha\beta}(y)\neq 0, \text{ if } y\in Q, \quad a^{\alpha\beta}(y) = 0, \text{ if } y\notin Q,$$

where $Q\subset\Lambda$ and $|\,Q\,|<\infty$,

b) for any $\lambda\in[-\pi,\pi)^d$ the eigenvalues of the matrix $\widetilde{A}(\lambda)$ with entries

$$\widetilde{a}^{\alpha\beta}(\lambda) = \sum_{y\in\mathbf{Z}^d} a^{\alpha\beta}(y)\, e^{-i(\lambda,y)}, \quad \alpha\beta\in\{1,\dots,N\},$$

belong to the halfplane $\{z\in\mathbf{C} \mid \operatorname{Re} z > 0\}$.

It is known that for sufficiently small $\varepsilon > 0$ and for any $F\in @_0^{\mathbf{Z}^d}$ there exists the limit

$$\langle F\rangle_{U_\Lambda^\varepsilon} \to \langle F\rangle_\varepsilon$$

as $\Lambda\uparrow\mathbf{Z}^d$ (Malyshev, 1986).

For any finite cube $\Lambda\subset\mathbf{Z}^d$ the system of Langevin's equations corresponding to the Gibbsian quasistate with potential U_Λ defines the evolution $\langle\cdot\rangle_{t,\varepsilon,\Lambda}$ of a quasistate on $@_0^\Lambda$.

We define the evolution $\langle\cdot\rangle_{t,\varepsilon}$ of the quasistate $\langle\cdot\rangle_\varepsilon$ on $@_0^{\mathbb{Z}^d}$ as

$$\langle F\rangle_{t,\varepsilon} = \lim_{\Lambda\uparrow\mathbb{Z}^d}\langle F\rangle_{t,\varepsilon,\Lambda} \tag{8}$$

if this limit exists for any $F\in @_0^{\mathbb{Z}^d}$. It is easy to see that the quasistate $\langle\cdot\rangle_\varepsilon$ on $@_0^{\mathbb{Z}^d}$ is invariant under this evolution.

THEOREM 2. *Let the initial quasistate $\langle\cdot\rangle_{0,\varepsilon,\Lambda}$, $\Lambda = [-N_\Lambda, N_\Lambda)^d\cap\mathbb{Z}^d$, at $t=0$ be Gibbsian with the potential:*

$$\sum_{\alpha,\beta}\sum_{x,y} b_\Lambda^{\alpha,\beta}(x-y)\bar{\xi}^{(\alpha,x)}\xi^{(\beta,y)} + \varepsilon\sum_{x\in\Lambda} P_0\big(\bar{\xi}^{(\alpha,x)},\ \xi^{(\alpha,x)},\ \alpha\in\{1,\dots,N\}\big),$$

where $P_0\Big(\bar{\xi}^{(\alpha,x)},\ \xi^{(\alpha,x)},\ \alpha\in\{1,\dots,N\}\Big)$ is an even degree polynomial in variables $\bar{\xi}^{(\alpha,x)}$, $\xi^{(\alpha,x)}$, $\alpha\in\{1,\dots,N\}$, and the coefficients $b_\Lambda^{\alpha\beta}(y)$ satisfy the conditions a), b) *for the coefficients $a_\Lambda^{\alpha\beta}(y)$, $\alpha,\beta\in\{1,\dots,N\}$, $y\in\mathbb{Z}^d$, given above.*
Then for sufficiently small $0<\varepsilon<\varepsilon_0$ the limit (8) *exists for any $F\in @_0^{\mathbb{Z}^d}$, $t\in\mathbb{R}_+$ and*

$$\langle F\rangle_{t,\varepsilon}\to\langle F\rangle_\varepsilon \qquad (t\to\infty).$$

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