Multi-agent model of the price flow dynamics

Vadim Malyshev[1], Anatoli Manita[2] *, and Andrei Zamyatin[3]

[1] Lomonosov Moscow State University, Faculty of Mechanics and Mathematics, Russia, 119991, Moscow, GSP-1, Leninskiye Gory malyshev2@yahoo.com
[2] Lomonosov Moscow State University, Faculty of Mechanics and Mathematics, Russia, 119991, Moscow, GSP-1, Leninskiye Gory manita@mech.math.msu.su
[3] Lomonosov Moscow State University, Faculty of Mechanics and Mathematics, Russia, 119991, Moscow, GSP-1, Leninskiye Gory zamyatin@mech.math.msu.su

Summary. On the real line initially there are infinite number of particles on the positive half-line., each having one of K negative velocities $v_1^{(+)}, ..., v_K^{(+)}$. Similarly, there are infinite number of antiparticles on the negative half-line, each having one of L positive velocities $v_1^{(-)}, ..., v_L^{(-)}$. Each particle moves with constant speed, initially prescribed to it. When particle and antiparticle collide, they both disappear. It is the only interaction in the system. We find explicitly the large time asymptotics of $\beta(t)$ - the coordinate of the last collision before t between particle and antiparticle.

Keywords: phase boundary dynamics, random walks in cones, piece-wise linear dynamical systems, one instrument market.

1 Introduction

We consider one-dimensional dynamical model of the boundary between two phases (particles and antiparticles, bears and bulls) where the boundary moves due to reaction (annihilation, transaction) of pairs of particles of different phases.

Assume that at time $t = 0$ infinite number of $(+)$-particles and $(-)$-particles are situated correspondingly on R_+ and R_- and have one-point correlation functions

$$f_+(x,v) = \sum_{i=1}^{K} \rho_i^{(+)}(x)\delta(v - v_i^{(+)}), \qquad f_-(x,v) = \sum_{j=1}^{L} \rho_i^{(-)}(x)\delta(v - v_j^{(-)})$$

Moreover for any i, j

$$v_i^{(+)} < 0, \qquad v_j^{(-)} > 0$$

that is two phases move towards each other. Particles of the same phase do not see each other and move freely with the velocities prescribed initially.

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The only interaction in the system is the following. When two particles of different phases find themselves at the same point they immediately disappear (annihilate). It follows that the phases stay separated, and one might call any point in-between them the phase boundary (for example it could be the point of the last collision). Thus the boundary trajectory $\beta(t)$ is a random piece-wise constant function of time.

One of the possible interpretations is the simplest model of one instrument (for example, a stock) market. Particle initially at $x(0) \in R_+$ is the seller who wants to sell his stock for the price $x(0)$, which is higher than the existing price $\beta(0)$. There are K groups of sellers characterized by their activity to move towards more realistic price. Similarly the $(-)$-particles are buyers who would like to buy a stock for the price lower than $\beta(t)$. When seller and buyer meet each other, the transaction occurs and both leave the market.

The main result of the paper is the explicit formula for the asymptotic velocity of the boundary as the function of $2(K+L)$ parameters - densities and initial velocities. It appears to be continuous but at some hypersurface some first derivatives in the parameters do not exist. This kind of phase transition has very clear interpretation: the particles with smaller activities (velocities) cease to participate in the boundary movement - they are always behind the boundary, that is do not influence the market price $\beta(t)$. In this paper we consider only the case of constant densities $\rho_i^{(+)}, \rho_i^{(-)}$, that is the period of very small volatility in the market. This simplification allows us to get explicit formulas. In [1] much simpler case $K = L = 1$ was considered, however with non-constant densities and random dynamics.

Other one-dimensional models (hardly related to ours) of the boundary movement see in [2, 3].

Main technical tool of the proof may seem surprising (and may be of its own interest) - we reduce this infinite particle problem to the study of a special random walk of one particle in the orthant R_+^N with $N = KL$. The asymptotic behavior of this random walk is studied using the correspondence between random walks in R_+^N and dynamical systems introduced in [4].

2 Model and the main result

Initial conditions

At time $t = 0$ on the real axis there is a random configuration of particles, consisting of $(+)$-particles and $(-)$-particles. $(+)$-particles and $(-)$-particles differ also by the type: denote $I_+ = \{1, 2, ..., K\}$ the set of types of $(+)$-particles, and $I_- = \{1, 2, ..., L\}$ - the set of types of $(-)$-particles. Let

$$0 < x_{1,k} = x_{1,k}(0) < ... < x_{j,k} = x_{j,k}(0) < ... \qquad (1)$$

be the initial configuration of particles of type $k \in I_+$, and

$$.. < y_{j,i} = y_{j,i}(0) < ... < y_{1,i} = y_{1,i}(0) < 0 \tag{2}$$

be the initial configuration of particles of type $i \in I_-$, where the second index is the type of the particle in the configuration. Thus all (+)-particles are situated on R_+ and all (−)-particles on R_-. Distances between neighbor particles of the same type are denoted by

$$\begin{aligned} x_{j,k} - x_{j-1,k} &= u_{j,k}^{(+)}, \quad k \in I_+, \quad j = 1, 2, ... \\ y_{j-1,i} - y_{j,i} &= u_{j,i}^{(-)}, \quad i \in I_-, \quad j = 1, 2, ... \end{aligned} \tag{3}$$

where we put $x_{0,k} = y_{0,i} = 0$. The random configurations corresponding to the particles of different types are assumed to be independent. The random distances between neighbor particles of the same type are also assumed to be independent, and moreover identically distributed, that is random variables $u_{j,i}^{(-)}, u_{j,k}^{(+)}$ are independent and their distribution depends only on the upper and second lower indices. Our technical assumption is that all these distributions are absolutely continuous and have finite means. Denote $\mu_i^{(-)} = Eu_{j,i}^{(-)}$, $\rho_i^{(-)} = \left(\mu_i^{(-)}\right)^{-1}, i \in I_-$, $\mu_k^{(+)} = Eu_{j,k}^{(+)}$, $\rho_k^{(+)} = \left(\mu_k^{(+)}\right)^{-1}, k \in I_+$.

Dynamics

We assume that all (+)-particles of the type $k \in I_+$ move in the left direction with the same constant speed $v_k^{(+)}$, where $v_1^{(+)} < v_2^{(+)} < ... < v_K^{(+)} < 0$. The (−)-particles of type $i \in I_-$ move in the right direction with the same constant speed $v_i^{(-)}$, where $v_1^{(-)} > v_2^{(-)} > ... > v_L^{(-)} > 0$. If at some time t a (+)-particle and a (−)-particle are at the same point (we call this a collision or annihilation event), then both disappear. Collisions between particles of different phases is the only interaction, otherwise they do not see each other. Thus, for example, at time t the $j-$th particle of type $k \in I_+$ could be at the point

$$x_{j,k}(t) = x_{j,k}(0) + v_k^{(+)} t$$

if it will not collide with some (−)-particle before time t. Absolute continuity of the distributions of random variables $u_{j,i}^{(-)}$,$u_{j,k}^{(+)}$ guaranties that the events when more than two particles collide, have zero probability.

We denote this infinite particle process $\mathbf{D}(t)$.

We define the boundary $\beta(t)$ between plus and minus phases to be the coordinate of the last collision which occured at some time $t' < t$. For $t = 0$ we put $\beta(0) = 0$. Thus the trajectories of the random process $\beta(t)$ are piecewise constant functions, we shall assume them continuous from the left.

Main result

For any pair (J_-, J_+) of subsets , $J_- \subseteq I_-, J_+ \subseteq I_+$, define the numbers

$$V(J_-,J_+)=\frac{\sum_{i\in J_-}v_i^{(-)}\rho_i^{(-)}+\sum_{k\in J_+}v_k^{(+)}\rho_k^{(+)}}{\sum_{i\in J_-}\rho_i^{(-)}+\sum_{k\in J_+}\rho_k^{(+)}},V=V(I_-,I_+) \quad (4)$$

The following condition is assumed

$$\{V(J_-,J_+)\,:\ J_-\neq\varnothing,\ J_+\neq\varnothing\}\cap\{v_1^{(-)},...,v_L^{(-)},v_1^{(+)},...,v_K^{(+)}\}=\varnothing\,. \quad (5)$$

If the limit $W=\lim_{t\to\infty}\frac{\beta(t)}{t}$ exists a.e., we call it the asymptotic speed of the boundary. Our main result is the explicit formula for W.

Theorem 1. *The asymptotic velocity of the boundary exists and is equal to*

$$W=V(\{1,...,L_1\},\{1,...,K_1\})$$

where

$$L_1=\max\left\{l\in\{1,\dots,L\}:\ v_l^{(-)}>V(\{1,...,l\},I_+)\right\}, \quad (6)$$

$$K_1=\max\left\{k\in\{1,\dots,K\}:\ v_k^{(+)}<V(I_-,\{1,...,k\})\right\}. \quad (7)$$

Note that the definition of L_1 and K_1 is not ambiguous because $v_1^{(-)}>V(\{1\},I_+)$ and $v_1^{(+)}<V(I_-,\{1\})$.

Now we will explain this result in more detail. As $v_K^{(+)}<0<v_L^{(-)}$, there can be 3 possible orderings of the numbers $v_L^{(-)},v_K^{(+)},V$:

1. $v_K^{(+)}<V<v_L^{(-)}$. In this case

$$K_1=K,\quad L_1=L,\quad W=V$$

2. If $v_K^{(+)}>V$ then $V<0$ and $K_1<K,\quad L_1=L$. Moreover

$$W=V(\{1,...,L\},\{1,...,K_1\})=\min_{k\in I_+}V(\{1,...,L\},\{1,...,k\})\ <\ V\ <\ 0$$

If $v_L^{(-)}<V$ then $V>0$ and $K_1=K,\quad L_1<L$. Moreover

$$W=V(\{1,...,L_1\},I_+)=\max_{l\in I_-}V(\{1,...,l\},I_+)\ >\ V\ >0$$

Another scaling

Normally the minimal difference between consecutive prices (a tick) is very small. Moreover one customer can have many units of the commodity. That is why it is natural to consider the scaled densities

$$\rho_j^{(+),\epsilon}=\epsilon^{-1}\rho_j^{(+)},\quad \rho_j^{(-),\epsilon}=\epsilon^{-1}\rho_j^{(-)}$$

for some fixed constants $\rho_j^{(+)},\rho_j^{(-)}$. Then the phase boundary trajectory $\beta^{(\epsilon)}(t)$ will depend on ϵ. The results will look even more natural. Namely, it follows from the main theorem, that for any $t>0$ there exists the following limit in probability

$$\beta(t)=\lim_{\epsilon\to 0}\beta^{(\epsilon)}(t)$$

that is the limiting boundary trajectory.

Example of phase transition

The case $K = L = 1$, that is when the activities of (+)-particles are the same (and similarly for (−)-particles), is very simple. There is no phase transition in this case. The boundary velocity

$$W = \frac{v_1^{(+)}\rho_1^{(+)} + v_1^{(-)}\rho_1^{(-)}}{\rho_1^{(+)} + \rho_1^{(-)}} \tag{8}$$

depends analytically on the activities and densities. This is very easy to prove because the n-th collision time is given by the simple formula

$$t_n = \frac{x_n^{(+)}(0) - x_n^{(-)}(0)}{-v_1^{(+)} + v_1^{(-)}} \tag{9}$$

and n-th collision point is given by

$$x_n^{(+)}(0) + t_n v_1^{(+)} = x_n^{(-)}(0) + t_n v_1^{(-)}. \tag{10}$$

More complicated situation was considered in [1]. There the movement of (+)-particles has random jumps in both directions with constant drift $v_1^{(+)} \neq 0$ (and similarly for (−)-particles). In [1] *the order* of particles of the same type *can be changed* with time. There are no such simple formulas as (9) and (10) in this case. The result is however the same as in (8).

The phase transition appears already in case when $K = 2$, $L = 1$ and moreover the (−)-particles stand still, that is $v_1^{(-)} = 0$. Denote $\rho_1^{(-)} = \rho_0$, $v_i^{(+)} = v_i$, $\rho_i^{(+)} = \rho_i$, $i = 1, 2$. Consider the function

$$V_1(v_1, \rho_1) = \frac{\rho_1 v_1}{\rho_0 + \rho_1}\,.$$

It is the asymptotic speed of the boundary in the system where there is no (+)-particles of type 2 at all.

Then the asymptotic velocity is the function

$$W = V(v_1, v_2, \rho_1, \rho_2) = \frac{\rho_1 v_1 + \rho_2 v_2}{\rho_0 + \rho_1 + \rho_2}$$

if $v_2 < V_1$ and

$$W = V_1(v_1, \rho_1) = \frac{\rho_1 v_1}{\rho_0 + \rho_1}$$

if $v_2 > V_1$. We see that at the point $v_2 = V_1$ the function W is not differentiable in v_2.

3 Random walk in R_+^N and method of proof

Associated random walk

One can consider the phase boundary as a special kind of server where the customers (particles) arrive in pairs and are immediately served. However the situation is more involved than in standard queuing theory, because the server moves, and correlation between its movement and arrivals is sufficiently complicated. That is why this analogy does not help much. However we describe the crucial correspondence between random walks in R_+^N and the infinite particle problem defined above, that allows to get the solution.

Denote $b_i^{(-)}(t)$ ($b_k^{(+)}(t)$) the coordinate of the extreme right (left), and still existing at time t, that is not annihilated at some time $t' < t$, $(-)$-particle of type $i \in I_-$ ($(+)$-particle of type $k \in I_+$). Define the distances $d_{i,k}(t) = b_k^{(+)}(t) - b_i^{(-)}(t) \geq 0$, $i \in I_-, k \in I_+$. The trajectories of the random processes $b_i^{(-)}(t)$, $b_k^{(+)}(t)$, $d_{i,k}(t)$ are assumed left continuous. Consider the random process $D(t) = (d_{i,k}(t), (i,k) \in I) \in R_+^N$, where $N = KL$.

Denote $\mathcal{D} \in R_+^N$ the state space of $D(t)$. Note that the distances $d_{i,k}(t)$, for any t, satisfy the following conservation laws

$$d_{i,k}(t) + d_{n,m}(t) = d_{i,m}(t) + d_{n,k}(t)$$

where $i \neq n$ and $k \neq m$. That is why the state space $\mathcal{D}$ can be given as the set of non-negative solutions of the system of $(L-1)(K-1)$ linear equations

$$d_{1,1} + d_{n,m} = d_{1,m} + d_{n,1}$$

where $n, m \neq 1$. It follows that the dimension of $\mathcal{D}$ equals $K + L - 1$. However it is convenient to speak about random walk in R_+^N, taking into account that only subset of dimension $K + L - 1$ is visited by the random walk.

Now we describe the trajectories $D(t)$ in more detail. The coordinates $d_{i,k}(t)$ decrease linearly with the speeds $v_i^{(-)} - v_k^{(+)}$ correspondingly until one of the coordinates $d_{i,k}(t)$ becomes zero. Let $d_{i,k}(t_0) = 0$ at some time t_0. This means that $(-)$-particle of type i collided with $(+)$-particle of type k. Let them have numbers j and l correspondingly. Then the components of $D(t)$ become:

$$\begin{aligned} d_{i,k}(t_0 + 0) &= u_{j+1,i}^{(-)} + u_{l+1,k}^{(+)} \\ d_{i,m}(t_0 + 0) - d_{i,m}(t_0) &= u_{j+1,i}^{(-)}, \quad m \neq k \\ d_{n,k}(t_0 + 0) - d_{n,k}(t_0) &= u_{l+1,k}^{(+)}, \quad n \neq i \end{aligned}$$

and other components will not change at all, that is do not have jumps.

Note that the increments of the coordinates $d_{n,m}(t_0 + 0) - d_{n,m}(t_0)$ at the jump time do not depend on the history of the process before time t_0, as the random variables. $u_{j,i}^{(-)}$ ($u_{j,k}^{(+)}$) are independent and equally distributed for fixed type. It follows that $D(t)$ is a Markov process. However that this continuous time Markov process has singular transition probabilities (due to partly deterministic movement). This fact however does not prevent us from using the techniques from [4] where random walks in Z_+^N were considered.

Ergodic case

We call the process $D(t)$ ergodic, if there exists a neighborhood A of zero, such that the mean value $E\tau_x$ of the first hitting time τ_x of A from the point x is finite for any $x \in \mathcal{D}$. In the ergodic case the correspondence between boundary movement and random walks is completely described by the following theorem.

Theorem 2. *Two following two conditions are equivalent:*
1) The process $D(t)$ is ergodic; *2) $v_K^{(+)} < V < v_L^{(-)}$.*

All other cases of boundary movement correspond to non-ergodic random walks. Even more, we will see that in all other cases the process $D(t)$ is transient. Condition (5), which excludes the set of parameters of zero measure, excludes in fact null recurrent cases.

To understand the corresponding random walk dynamics introduce a new family of processes.

Faces

Let $\Lambda \subseteq I = I_- \times I_+$. The face of R_+^N associated with Λ is defined as

$$\mathcal{B}(\Lambda) = \{x \in R_+^N : x_{i,k} > 0, (i,k) \in \Lambda, x_{i,k} = 0, (i,k) \in \overline{\Lambda}\} \subseteq R_+^N \qquad (11)$$

If $\Lambda = \emptyset$, then $\mathcal{B}(\Lambda) = \{0\}$. For shortness, instead of $\mathcal{B}(\Lambda)$ we will sometimes write Λ. However, one should note that the inclusion like $\Lambda \subset \Lambda_1$ is ALWAYS understood for subsets of I, not for the faces themselves.

Define the following set of "appropriate" faces

$$\mathcal{G} = \left\{\Lambda : \overline{\Lambda} = J_- \times J_+, \; J_- \subseteq I_-, \; J_+ \subseteq I_+\right\}.$$

Lemma 1.

$$\mathcal{D} = \bigcup_{\Lambda_0 \in \mathcal{G}} (\mathcal{D} \cap \Lambda_0).$$

This lemma explains why in the study of the process $D(t)$ we can consider only "appropriate" faces.

Induced process

One can define a family $\mathbf{D}(t; J_-, J_+)$ of infinite particle processes, where $J_- \subseteq I_-$, $J_+ \subseteq I_+$. The process $\mathbf{D}(t; J_-, J_+)$ is the process $\mathbf{D}(t)$ with $\rho_j^{(+)} = 0, j \notin J_+$ and $\rho_j^{(-)} = 0, j \notin J_-$. All other parameters (that is the densities and velocities) are the same as for $\mathbf{D}(t)$. Note that these processes are in general defined on different probability spaces. Obviously $\mathbf{D}(t; I_-, I_+) = \mathbf{D}(t)$.

Similarly to $\mathbf{D}(t)$, the processes $\mathbf{D}(t; J_-, J_+)$ have associated random walks $D(t; J_-, J_+)$ in $R_+^{N_1}$ with $N_1 = |J_-||J_+|$. Usefulness of these processes is that they describe all possible types of asymptotic behavior of the main process $D(t)$.

Consider a face $\Lambda \in \mathcal{G}$, i.e., such face that its complement $\overline{\Lambda} = J_- \times J_+$ where $J_- \subseteq I_-$ and $J_+ \subseteq I_+$. The process $D_\Lambda(t) = D(t; J_-, J_+) = (d_{i,k}^\Lambda(t), (i,k) \in \overline{\Lambda})$ will be called an ***induced process***, associated with Λ. The coordinates $d_{i,k}^\Lambda(t)$ are defined in the same way as $d_{i,k}(t) = d_{i,k}^\Lambda(t)$, where $\overline{\Lambda} = \{\emptyset\}$. The state space of this process is $\mathcal{D}^{\overline{\Lambda}} = \mathcal{D}(R^{|\overline{\Lambda}|})$, where $|\overline{\Lambda}| = |J_-||J_+|$. Face Λ is called **ergodic** if the induced process $D_\Lambda(t)$ is ergodic.

Induced vectors

Introduce the plane

$$\mathcal{R}(\Lambda) = \{x \in R^N : x_{i,k} = 0, (i,k) \in \overline{\Lambda}\} \subseteq R^N$$

Lemma 2. *Let Λ be ergodic with $\overline{\Lambda} = J_- \times J_+$, and $D_y(t)$ be the process $D(t)$ with the initial point $y \in \mathcal{B}(\Lambda)$. Then there exists vector $v^\Lambda \in \mathcal{R}(\Lambda)$ such that for any $y \in \mathcal{B}(\Lambda)$ $t \geq 0$, such that $y + v^\Lambda t \in \mathcal{B}(\Lambda)$, we have as $M \to \infty$*

$$\frac{D_{yM}(tM)}{M} \to y + v^\Lambda t$$

This vector v^Λ will be called the **induced vector** for the ergodic face Λ. We will see other properties of the induced vector below.

Non-ergodic faces

Let Λ be the face which is not ergodic (non-ergodic face). Ergodic face Λ_1: $\Lambda_1 \supset \Lambda$ will be called outgoing for Λ, if $v_{i,k}^{\Lambda_1} > 0$ for $(i,k) \in \Lambda_1 \setminus \Lambda$. Let $\mathcal{E}(\Lambda)$ be the set of outgoing faces for the non-ergodic face Λ.

Lemma 3. *The set $\mathcal{E}(\Lambda)$ contains the minimal element Λ_1 in the sense that for any $\Lambda_2 \in \mathcal{E}(\Lambda)$ we have $\Lambda_2 \supseteq \Lambda_1$.*

Dynamical system

We define now the piece-wise constant **vector field** $v(x)$ in $\mathcal{D}$, consisting of induced vectors, as follows: $v(x) = v^{\Lambda}$ if x belongs to ergodic face Λ, and $v(x) = v^{\Lambda_1}$ if x belongs to non-ergodic face Λ, where Λ_1 is the minimal element of $\mathcal{E}(\Lambda)$. Let U^t be the **dynamical system** corresponding to this vector field.

It follows that the trajectories $\Gamma_x = \Gamma_x(t)$ of the dynamical system are piecewise linear. Moreover, if the trajectory hits a non-ergodic face, it leaves it immediately. It goes with constant speed along an ergodic face until it reaches its boundary.

We call the ergodic face $\Lambda = \mathcal{L}$ final, if either $\mathcal{L} = \emptyset$ or all coordinates of the induced vector $v^{\mathcal{L}}$ are positive. The central statement is that the dynamical system hits the final face, stays on it forever and goes along it to infinity, if $\mathcal{L} \neq \emptyset$.

The following theorem, together with theorem 2, is parallel to theorem 1. That is in all 3 cases of theorem 1, theorems 2 and 3 describe the properties of the corresponding random walks in the orthant.

Theorem 3.

1. *If $D(t)$ is ergodic then the origin is the fixed point of the dynamical system U^t. Moreover, all trajectories of the dynamical system U^t hit 0.*
2. *Assume $v_K^{(+)} > V$. Then the process $D(t)$ is transient and there exists a unique ergodic final face $\mathcal{L}$, such that $v_{i,k}^{\mathcal{L}} > 0$ for $(i,k) \in \mathcal{L}$. This face is*
$$\mathcal{L}(L, K_1) = \{(i,k):\ i = 1, ..., L,\ k = K_1 + 1, ..., K\}$$
where K_1 is defined by (7). Moreover, all trajectories of the dynamical system U^t hit $\mathcal{L}(L, K_1)$ and stay there forever.
3. *Assume $v_L^{(-)} < V$. Then the process $D(t)$ is transient and there exists a unique ergodic final face $\mathcal{L}$, such that $v_{i,k}^{\mathcal{L}} > 0$ for $(i,k) \in \mathcal{L}$. This face is*
$$\mathcal{L}(L_1, K) = \{(i,k):\ i = L_1 + 1, ..., L,\ k = 1, ..., K\}$$
where L_1 is defined by (6). Moreover, all trajectories of the dynamical system U^t hit $\mathcal{L}(L_1, K)$ and stay there forever.
4. *For any initial point x the trajectory $\Gamma_x(t)$ has finite number of transitions from one face to another, until it reaches $\{0\}$ or one of the final faces.*

Simple examples of random walks and dynamical systems

If $K = L = 1$ the process $D(t)$ is a random process on R_+. It is deterministic on $R_+ \setminus \{0\}$ - it moves with constant velocity $v^{(+)} - v^{(-)}$ towards the origin. When it reaches 0 at time t, it jumps backwards

$$D(t + 0) = \eta$$

where η has the same distribution as $u_1^{(+)} + u_1^{(-)}$. The dynamical system coincides with $D(t)$ inside R_+, and has the origin as its fixed point.

If $L = 1, K = 2$ and moreover $v_1^{(-)} = 0$ then the state space of the process is $R_+^2 = \{(d_{11}, d_{12})\}$. Inside the quarter plane the process is deterministic and moves with velocity $(v_1^{(+)}, v_2^{(+)})$. From any point x of the boundary $d_{12} = 0$ it jumps to the random point $x + \eta_1$, and from any point of the boundary $d_{11} = 0$ it jumps to the point $x + \eta_2$, where η_1, η_2 have the same distributions as $(u_{j,1}^{(-)}, u_{j,1}^{(-)} + u_{j,2}^{(+)})$ and $(u_{j,1}^{(-)} + u_{j,1}^{(+)}, u_{j,1}^{(-)})$ correspondingly. The classification results for random walks in Z_+^2 can be easily transfered to this case; the dynamical system is deterministic and has negative components of the velocity inside R_+^2. When it hits one of the axes it moves along it. The velocity is always negative along the first axis, however along second axis it can be either negative or positive. This is the phase transition we described above. Correspondingly the origin is the fixed point in the first case, and has positive value of the vector field along the second axis, in the second case.

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