Macrodimension: An Invariant of Local Dynamics

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Abstract

We define a Markov process on the set of countable graphs with spins. Transitions are local substitutions in the graph. It is proved that the scaling macrodimension is an invariant of such dynamics.

1 Introduction

Process with a local interaction on a fixed lattice, graph or continuous space, is now considered as one of the central objects in probability theory. It was recognized recently that there exists a natural generalization: where the graph itself is not fixed but randomly and locally changes in time. Such processes appear in computer science (stochastic grammars), in physics (quantum gravity) and in biology (DNA evolution).

General mathematical theory of such processes has started recently (see [2, 3]). Weak convergence is an important tool for study of such processes.

The plan of the paper is the following. Markov dynamics is defined on the set of finite graphs and countable graphs. For a countable graph the macrodimension is defined. Roughly speaking the macrodimension of the graph is d if in the N-neighborhood of each vertex there is approximately N^d vertices.

We prove that it is an invariant of any local dynamics (for any finite time) under the following assumptions: dynamics should be local, locally bounded and locally reversible. Note however that in the limit $t \to \infty$ the macrodimension can change.

2 Local substitutions in spin graphs

We consider non-directed connected graphs G with the set of vertices $V = V(G)$ (finite or countable) and the set of links $L = L(G)$. The following properties are always assumed: between each pair of vertices there is 1 or 0 edges; each node (vertex) has finite degree (the number of edges incident to it). Denote GF_n the set of all finite graphs with these properties and where each node has degree not more than n. Denote GC_n the set of all countable graphs with the same properties.

A *subgraph* of G is a subset $V_1 \subset V$ of vertices together with some links connecting pairs of vertices from V_1 and inherited from L. A *regular subgraph*

$G(V_1)$ of G is a subset $V_1 \subset V$ of vertices together with ALL links connecting pairs of vertices from V_1 and inherited from L.

The set V of nodes is a metric space with the following metrics: the distance $d(x, y)$ between vertices $x, y \in V$ is the minimum of the lengths of paths connecting these vertices. The lengths of all edges are assumed to be equal, say to some constant, assumed equal to 1. Vertices connected by a link are called neighbours. The neighbourhood (more exactly, N-neighbourhood) $O_N(v)$, of vertex v in G is the regular subgraph with the set of vertices consisting of v itself and of all vertices at distance not greater than N from v. Put $O_1(v) = O(v)$.

A *spin graph* (also coloured graph, marked graph, spin system etc.) is a pair $\alpha = (G, s)$, where $s = s(.)$ is a function $s : V \to S$ where S is the set of "spin values", the alphabet. An isomorphism of spin graphs is an isomorphism of graphs respecting the spins. The empty spin graph $\emptyset$ is the empty graph with no spin. With some abuse of notation we shall say also that G is a spin graph and will call α simply a graph.

A spin graphs dynamics is a random sequence (where the moments $0 < ... < t_k < t_{k+1} < ...$, are also random) of spin graphs

$$\alpha_0 = (G_0, s_0), \alpha_{t_1} = (G_{t_1}, s_{t_1}), ..., \alpha_{t_k} = (G_{t_k}, s_{t_k}), ...$$

to be defined below. The graph α_{t_k} is obtained from $\alpha_{t_{k-1}}$ by a simple transformation from some fixed class of transformations.

Definition 1 *The substitution rule (production)* $Sub = (\Gamma, \Gamma', V_0, \varphi)$ *is defined by two "small" spin graphs* Γ *and* Γ'*, subset* $V_0 \subset V = V(\Gamma)$ *and mapping* $\varphi : V_0 \to V' = V(\Gamma')$*;* V_0 *and* Γ' *can be empty. The transformation (substituion)* $T = T(Sub, \psi)$ *of a spin graph* α*, corresponding to a given substitution* Sub *and an isomorphism* $\psi : \Gamma \to \Gamma_1$ *onto a spin subgraph* Γ_1 *of* α*, is defined in the following way. Consider nonconnected union of* α *and* Γ'*, delete all links of* Γ_1*, delete all vertices of* $\psi(V) \setminus \psi(V_0)$ *together with all links incident to them, identify each* $\psi(v) \in \psi(V_0)$ *with* $v' = \varphi(v) \in \Gamma'$*. The function* s *on* $V(G) \setminus V(\Gamma_1)$ *is inherited from* α *and on* $V(\Gamma')$ *- from* Γ'*. We denote the resulting graph by* $T(Sub, \psi)\alpha$*.*

Examples of substitutions are: deleting a link, appending a link in a given vertex with another new vertex, joining a pair of vertices (with no link between them by a link, updating a spin value etc. Moreover the mere possibility of these substituions can depend on the neighborhood of the vertex where the substituion is to be done.

Definition 2 *Graph grammar is defined by a finite set of substitutions* $Sub_i = (\Gamma_i, \Gamma'_i, V_{i,0}, \varphi_i), i = 1, ..., r$*. We call a graph grammar local if for all* i *the* Γ_i*'s, corresponding to* Sub_i*, are connected. We call a graph grammar locally bounded if the subsets* GF_n *and* GC_n *are invariant with respect to all substitutions for sufficiently large* n*.*

Let positive rates $\lambda_i = \lambda(Sub_i), i = 1, ..., r$, be given. They define a continuous time Markov chain on GF_n, called random graph grammar, in the following obvious way: at a given time interval $(t, t+dt)$ each possible transformation $T(Sub_i, \psi)$ is produced with probability $\lambda_i dt + o(dt)$. It was proved in [2] that on GF a minimal Markov chain (non explosive) can be defined on all time interval $(0, \infty)$.

3 Macrodimension of a countable graph

If one wants to consider an infinite graph as a topological space, this can be done from local (micro) and global (macro) point of view. From micro point of view graph is a 1-dimensional complex, thus having dimension 1. But also a graph G can be considered as a 1-dimensional skeleton of some complex $C(G)$ of higher dimension. There are many ways to define $C(G)$ for a given G. One of them is to consider the "simplicial completion" of the 1-dimensional complex G. That is the simplicial complex with vertices $V(G)$, all edges are 1-dimensional simplices, a subset $S \subset V(G)$, consisting of 3 vertices, is a simplex iff all its two element subsets are simplices, etc. From macro point of view there are also many ways to define its topological dimension, called in this case a macrodimension. We give here one of the possible definitions. For any $x \in V(G)$ put

$$D_n(x) = \frac{\ln |O_n(x)|}{\ln n}, \overline{D}(x) = \lim\sup_{n\to\infty} D_n(x), \underline{D}(x) = \lim\inf_{n\to\infty} D_n(x)$$

Lemma 3 *([1]) For all x, y*

$$\overline{D}(x) = \overline{D}(y) = \overline{D}, \underline{D}(x) = \underline{D}(y) = \underline{D}$$

Proof. It follows easily from the fact that

$$O_{n-a}(y) \leq O_n(x) \leq O_{n+a}(y), a = \rho(x, y)$$

If $\overline{D} = \underline{D} = D_S$ then D_S is called the scaling macrodimension of G. For example, trees can have any positive scaling dimension $1 \leq D_S \leq \infty$. For the binary tree $D_S = \infty$. For any homogeneous lattice L in the euclidean space R^d the macrodimension $D_S = d$.

4 Dynamics

Let at time $t = 0$ we have a connected spin graph $\alpha(0) = (G = G(0), s(0)) \in GC_n$ for some $n > 0$ sufficiently large. We give now exact definition of the dynamics $\alpha(t) = (G(t), s(t))$ on a small time interval $(0, \varepsilon)$, see [3] for more details.

We need some notation. Fix some vertex in $G(0)$ and denote it 0. Let $O_N = O_N(0) = G^N(0)$ be the N-neighbourhood of 0 in G. Consider an increasing family O_N of finite spin graphs

$$O_1 \subset ... \subset O_N \subset ...$$

Denote $\xi_\beta(t)$ the Markov process on GF starting with finite spin graph β such that $\xi_\beta(0) = \beta$. We want to study the limit $N \to \infty$ for the family $\xi^N(t) = \xi_{O_N}(t)$ of Markov processes on the probability spaces Ω^N. Consider some graph G and a subset $V' \subset V = V(G)$. Introduce the external boundary

$$\partial V' = \{v \in V \setminus V' : \rho(v, V') = 1\}$$

A subset V_1 of vertices is connected if the regular subgraph with this set of vertices is connected. We assume that the radii of all Γ entering the definition of substitutions of our graph grammar do not exceed 1. The latter assumption is only to do the presentation shorter.

Fix now N and t. For the process $\xi^N(t)$ and its trajectory ω we say that the vertex $v \in O_N$ has been touched on the time interval $[0, t]$ if it belongs to one of the graphs $\psi(\Gamma)$ in the random sequence of transformations on this time interval. Denote $Q(N, t; \omega)$ the set of all vertices in $G^N(0) = O_N$ which were touched by transformations of the process ξ^N during this time interval. Introduce the random field $\eta^N(v)$ on $V(G(0))$: $\eta^N(v) = 1$ if $v \in Q(N, t; \omega)$ and $\eta^N(v) = 0$ otherwise. For any finite $T \subset V$ define the correlation functions

$$< \eta_T^N >=< \prod_{v \in T} \eta^N(v) >$$

Limiting correlation functions (their existence was proved in [3]) define, by Kolmogorov theorem, a probability measure on $\{0,1\}^{V(G(0))}$ (that is the limiting random field $\eta(v)$) or on the set of all subsets of $V(G(0))$. Define for any initial graph $\alpha = \alpha(0)$ the random point set $E(\omega, t, \alpha) = \{v : \eta(v) = 1\}$.

Theorem 4 *There exists $t_0 > 0$ such that for any initial spin graph α and any fixed $0 \le t < t_0$ the sequence of random fields $\eta^N(v)$ tends weakly to a random field $\eta(v) = \eta(v; \alpha)$ on $V(\alpha)$ when $N \to \infty$, that is $\eta_T^N >\to< \eta_T >=< \prod_{v \in T} \eta(v) >$. Moreover, the random set $E(\omega, t, \alpha)$ consists of countable number of finite connected components a.s.*

Proof see in [3].

We shall use the sets $E(\omega, t, \alpha)$ for constructing dynamics on the set of infinite graphs. For infinite dynamics the vertices of $V(G(0)) \setminus E$ are interpreted as vertices which were touched by substitutions on the time interval $[0, t]$. They stay unchanged together with their spins.

Let us consider the probability space $\Omega_1 = \{0,1\}^{G(0)}$ where the limiting random field $\eta(v)$ is defined. Denote μ_1 the corresponding probability measure on Ω_1.

For given subgraph B of $G(0)$ consider the process, starting with the spin graph $B \cup \partial B$, that is the process $\xi_{B \cup \partial B}(t)$ under the condition that all vertices in B are touched but the vertices on the boundary ∂B of B are not touched. Denote $\zeta(t, B)$ this conditional process and let μ_B be the measure of this process. Its trajectories on the time interval $[0, t]$ we denote $\omega(B)$. Denote its probability space by $(\Omega_B, \Sigma_B, \mu_B)$.

For given ω_1 let $B_k(\omega_1), k = 1, 2, ...,$ be all connected components of $E(\omega, t, \alpha)$. The limiting probability space for the time interval $[0, t]$ is the set Ω of all arrays

$$\omega = (\omega_1, \omega(B_k(\omega_1)), k = 1, 2, ...)$$

Take some connected subsets $D_1, ..., D_n$ of $G(0)$ and some measurable subsets $U_i \subset \Omega_{D_i}$. Let $C(U_1, D_1; ...; U_n, D_n)$ the set of all ω such that each D_i is equal to some of $B_k(\omega_1)$ and moreover $\omega(B_k(\omega_1)) \in U_k$. Consider the minimal σ-algebra Σ generated by all subsets $C(U_1, D_1; ...; U_n, D_n)$ of Ω.

The probability distribution μ is uniquely defined by the following conditions: the projection of μ on $\{0,1\}^V$ is equal to μ_1, and

$$\mu(C(U_1, D_1; ...; U_n, D_n)) = \mu_1(A(D_1, ..., D_n)) \prod_{i=1}^{n} \mu_{D_i}(U_i)$$

where $A(D_1, ..., D_n)$ is the set of all ω_1 such that D_i are connected components of $E(\omega_1, t, \alpha(0))$. Note that for given ω_1 the conditional distributions of trajectories on $B_k(\omega_1)$ are independent for different k.

Points of the probability space (Ω, Σ, μ) are, by definition, the trajectories of the dynamics on the set of countable spingraphs. The latter formula constitutes the cluster property. Similar representations can be obtained for each finite N. In this sense the infinite dynamics is the limit of the finite dynamics when $N \to \infty$. Note that the interval $(0, \varepsilon)$ does not depend on the initial spin graph. Thus by glueing together intervals $(n\varepsilon, n\varepsilon + 1)$ the dynamics can be constructed on all time interval $(0, \infty)$. We can formulate now the result.

Theorem 5 *For any t there exists dynamics on the set of countable spin graphs which is the thermodynamic limit of the sequence of countable Markov chains. Moreover it has the cluster property for $t < t_0$ uniformly in the initial spin graphs.*

5 Dynamics and Metrics

Here we define several classes of dynamics, accordingly to how they change the metrical properties of a graph.

Connectivity We assume that all substitutions of the fixed graph grammar respect connectivity, that is if α is connected then $T(Sub_i, \psi)\alpha$, for all possible i and ψ, are also connected. It is easy to give examples of local dynamics which does not respect connectivity. However, it respects connectivity if together with locality we assume the following state communicating condition on GF: if from some state i one can reach state j in some steps with positive probability, then i can be reached from j in some steps with positive probability (in particular one can assume irreducibility).

Metrical Boundedness

Lemma 6 *Assume that the graph grammar is local, locally bounded, respects connectivity and that $G(0) \in GC_n$ for n sufficiently large. Then it is metrically bounded from above, that is there exists a constant $C = C(n) > 0$ such that for each G and each pair $x, y \in V(G)$ of vertices*

$$d_{TG}(x,y) \leq d_G(x,y) + C$$

for each transformation $T = T(Sub_i, \psi)$, where we write TG instead of $T\alpha$.

Proof. Take some C sufficiently large and assume that there exist $G, T = T(Sub_i, \psi)$ and $x, y \in V(G)$ such that $d_{TG}(x,y) > d_G(x,y) + C$. We can assume that $x, y \in \partial\Gamma_1$. Take some $v \in \partial\Gamma_1 \subset V(G)$ and choose N so that $0 \ll n^N \ll C$. Then applying T to $O_N(v)$ we see that $T(O_N(v))$ will be nonconnected because there is no paths from x to y. In fact, the minimal path should be longer than the number n^N of vertices in $O_N(v)$.

Local reversibility Important class of evolutions which respect connectivity comes from physics. Consider a countable continuous time Markov chain with state space X and transition rates $\lambda_{ij}, i, j \in X$. We can consider X as the set of vertices of the directed graph where there is a link from i to j iff λ_{ij} is positive. It is called reversible (we do not assume recurrence) if from $\lambda_{ij} > 0$ it follows $\lambda_{ji} > 0$ and, for any cycle $\Gamma = (i_1, i_2), ..., (i_n, i_1)$, we have for $a_{ij} = \frac{\lambda_{ij}}{\lambda_{ji}}$

$$a(\Gamma) = a_{i_1 i_2} ... a_{i_n i_1} = 1, n \geq 2$$

Here a_{ij} is a function on the set of links with values in $R \setminus \{0\}$, n is the length of the cycle. Reversible chaoin is called locally reversible if, for some $n_0 < \infty$ and for any finite G, the relations $a(\Gamma) = 1$ for all n follow from these relations for all $n \leq n_0$. Let us give some examples of local reversibility:

1. If all λ_{ij} are positive then n_0 can be chosen equal to 3;
2. Simple random walks in Z^d: $n_0 = 4$;
3. Glauber dynamics for the Ising model in a finite volume: $n_0 = 4$.

6 Main Result

Theorem 7 *Assume the Markov chain on GF to be local, locally bounded and locally reversible. Then the scaling macrodimension is an invariant of the dynamics.*

Proof. Let at time $t = 0$ we have a connected spin graph $\alpha(0) = (G = G(0), s(0)) \in GC_n$ for some $n > 0$ sufficiently large, having the scaling macrodimension d. Consider the dynamics $\alpha(t) = (G(t), s(t))$ on a small time interval $(0, \varepsilon)$. It is sufficient to prove that all $G(t)$ have the same scaling macrodimension d.

We need the following estimates.

Lemma 8 *Consider the event that a cluster (connected component) D of $E(\omega, \varepsilon, \alpha(0))$, containing a fixed vertex, has size m. Let $p(m)$ be the probability of this event. Then $p(m) < C\delta^m$ for some $\delta < 1$. Moreover, $\delta = \delta(\varepsilon) \to 0$ when $\varepsilon \to 0$.*

This is the cluster estimate proved in [3].

Lemma 9 *Let D be a cluster of size m and $p(k, m, D), k > 1$, be the probability that the number of vertices in the graph $\zeta(\varepsilon, D)$ becomes greater than km. Then there exist constants $c > 0$ and $\delta_1 < 1$ such that $p(k, m, D) < c\delta_1^{km}$ uniformly in D. Also $\delta_1 = \delta_1(\varepsilon) \to 0$ if $\varepsilon \to 0$.*

Proof. Consider a kind of a pure growth process, more exactly the Markov process on Z_+ with rates λi of jumping from i to $i + r$. Let $q(r, m, k, t)$ be the probability that this process starting from m particles will have km particles at time t. Then

$$q(1, 1, k, t) = \exp(-\lambda t)(1 - \exp(-\lambda t))^{k-1}$$

It follows

$$q(1, m, k, t) = \sum_{k_1+k_2+\ldots+k_m=km} \prod_{i=1}^{m} \exp(-\lambda t)(1-\exp(-\lambda t))^{k_i-1} < 2^{km}(1-\exp(-\lambda t))^{m(k-1)}$$

Then for small t

$$q(r, m, k, t) < 2^{\frac{km}{r}}(1 - \exp(-\lambda t))^{m(\frac{k}{r}-1)} < C\beta^{km}$$

for $\beta = \beta(t) \to 0$ as $t \to 0$. At the same time, see [2],

$$p(k, m, D) \leq \sum_{j \geq k} q(r, m, j, t)$$

Consider the set $R(0) = V(G(0)) \setminus E(\omega, \varepsilon, \alpha(0))$ of vertices which were not touched by the transformations on the time interval $(0, \varepsilon)$, put $R_N(0) = R(0) \cap O_N(v_0)$ where v_0 is a fixed vertex in $V(G(0))$. Let $v(\omega) \in R(0)$ be some vertex on a minimal possible distance $r(\omega)$ from 0. With some ambiguity of notation we shall consider $v(\omega)$ and $R(0)$ belonging to $V(G(t))$ for all $t \in (0, \varepsilon)$. Let $O_N(t) = O_N(t, \omega)$ be the N-neighborhood of $v(\omega)$ in $V(G(t))$.

Lemma 10 *There exists $C > 1$ such that a.s. there exists $N_0 = N_0(\omega)$ such that for all $N > N_0$ we have*

$$|O_{NC^{-1}}(\varepsilon, \omega)| < |O_N(0)| < |O_{NC}(\varepsilon, \omega)|$$

..

Now the theorem easily follows. For example,

$$D_S(G) = \lim_{N\to\infty} \frac{\ln O_N(0)}{\ln N} = \lim \frac{\ln O_N(0)}{\ln(CN)} = \lim\inf\inf \frac{\ln O_N(0)}{\ln(CN)} \leq \lim\inf\inf \frac{\ln O_{NC}(\epsilon)}{\ln(NC)}$$

and thus $D_S(G) \leq \underline{D}(G(\varepsilon))$. Similarly, $D_S(G) \geq \overline{D}(G(\varepsilon))$, and thus $D_S(G) = D_S(G(\varepsilon))$.

Proof of the Lemma. Denote $d_t(x,y)$ the distance between points $x, y \in V(G(t))$ in $G(t)$. Take any point $x \in R_N(0)$. Fix some C sufficiently large and let $P(x,N) = P(x,N;C)$ be the probability of the event $A(x,N)$ that x does not belong to $O_{NC}(t,\omega)$, that is $d_t(x, v(\omega)) > NC$. Then

$$P(x,N) < \beta_1^{NC}$$

In fact, compare $d_0(0,x)$ and $d_t(v(\omega),x)$. Note first that

$$\Pr(|d_0(0,x) - d_t(v(\omega),x)| > B) < const \beta^B$$

for some β small. Take some path Υ inside $O_N(0)$ between x and 0 of minimal length $l(\Upsilon) = d_0(0,x) \leq N$. Let $B(\Upsilon_0), \Upsilon_0 \subset \Upsilon$, be the event that exactly the vertices of Υ_0 do not belong to $R(0)$. Let D_k be all clusters with which Υ intersects. Then the conditional probability

$$\Pr(|\cup_k D_k| > NC|\Upsilon_0) < \beta^{\sum_k |D_k \setminus \Upsilon_0|}$$

for some $\beta < 1$. Then

$$P(x,N) = \sum_{\Upsilon_0} \sum_{\{D_k\}} \Pr(\Upsilon_0) \Pr(|\cup_k D_k| > NC|\Upsilon_0) < \beta^{\frac{NC}{2}} = \beta_1^{NC}$$

The probability $P(N)$ that $|O_N(0)| \geq |O_{NC}(t,\omega)|$ is bounded by

$$P(N) \leq \sum_x P(x,N) \leq C(\gamma) N^{d+\gamma} \beta_1^{NC}$$

for any $\gamma > 0$ and some $C(\gamma) > 0$. Then the second inequality of the Lemma follows from Borel-Cantelli Lemma. To prove the first inequality we shall use the statement dual to Lemma 6.

Lemma 11 *Assume that the Markov chain on GF to be local, locally bounded, locally reversible and that $G(0) \in GC_n$ for n sufficiently large. Then it is metrically bounded from below, that is there exists a constant $C = C(n) > 0$ such that for each G and each pair $x, y \in V(G)$ of vertices*

$$d_{TG}(x,y) \geq d_G(x,y) - C$$

for each transformation $T = T(Sub_i, \psi)$, where we write TG instead of $T\alpha$.

The first inequality of the Lemma 10 follows from this Lemma quite similarly.

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