COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
27. RANDOM FIELDS, ESZTERGOM (HUNGARY), 1979.

SOME RESULTS AND PROBLEMS IN THE STUDY OF INFINITE-PARTICLE HAMILTONIANS

V.A. MALYSHEV — R.A. MINLOS

These lectures present a general approach and some recent results in the study of the structure of Hamiltonians of infinite systems of particles. We do not concern the problem in full extent and in particular we do not concern earlier work of Glimm, Jaffe, Spencer, and others on the subject.

Physical folklore tells us that most physical systems can be well described in terms of quasiparticles and their scattering. Many examples are elaborated in this direction from quantum field theory and statistical mechanics for zero and non-zero temperatures. One of the questions which arise in this connection is the following: given an operator H in some Hilbert space $\mathscr{H}$, what additional mathematical structure on H must we impose in order to get something like a quasiparticle structure for it? Surely this structure must generalize that of the finite particle Schrödinger operator

$$(0.1) \qquad H = - \sum_{i=1}^{n} \Delta_i + \sum_{1 \leq i < j \leq n} V(x_i - x_j)$$

to the infinite particle case. We give this definition for some cases and describe different possibities for its extension.

1. INTRODUCTION

Let a physical system near equilibrium be given. The physical Hilbert space $\mathscr{H}$ and the Hamiltonian H can be constructed in the following way.

Let $\mathfrak{A}$ be the C^*-algebra of observables, $1 \in \mathfrak{A}$, α_t, $t \in R^1$ be a strongly continuous group of automorphisms of $\mathfrak{A}$, ω be an invariant (with respect to α_t) state on $\mathfrak{A}$. Let us consider the GNS representation π of $\mathfrak{A}$ with respect to ω in the algebra of linear operators in some Hilbert space $\mathscr{H}$. This representation can be constructed by using a mapping $\widetilde{\pi}$ of $\mathfrak{A}$ into $\mathscr{H}$ such that

$$(\widetilde{\pi}(A), \widetilde{\pi}(B))_{\mathscr{H}} = \omega(A^*B), \qquad A, B \in \mathfrak{A}.$$

Then there exists a self-adjoint operator H in $\mathscr{H}$ such that for any $A \in \mathfrak{A}$, $t \in R^1$,

$$\exp\{itH\}\widetilde{\pi}(A) = \widetilde{\pi}(\alpha_t A)$$

and the vector $\Omega = \widetilde{\pi}(1) \in \mathscr{H}$ is the zero eigenvector for H (vacuum).

Let now also be given a representation α_g, $g \in G$ of some "symmetry" group G into the group of automorphisms of $\mathfrak{A}$, and suppose that G contains the time translations ($\boldsymbol{R}$ or $\boldsymbol{Z}$). For lattice systems we take $\boldsymbol{Z}^\nu \times \boldsymbol{R}$ or $\boldsymbol{Z}^{\nu+1}$ for G. Then there are unitary operators U_g, $g \in G$ in $\mathscr{H}$ such that for any $A \in \mathfrak{A}$,

$$U_g \widetilde{\pi}(A) = \widetilde{\pi}(\alpha_g A), \qquad g \in G.$$

The vacuum vector Ω is invariant with respect to U_g (see e.g. [1]).

So we have a vacuum state and now ask what a one-particle state (particle or quasiparticle) is. It is generally understood that such a state has as its support a subspace of $\mathscr{H}$ which is irreducible with respect to G (called one-particle space). However, we think that this definition must not be taken too literally. We now give a more exact definition for lattice systems at zero temperatures.

Let $\widetilde{\mathscr{H}}$ be the orthogonal complement to Ω. We consider the group $\widetilde{G} = Z^\nu \subseteq G$ of space translations and the decomposition of $\widetilde{\mathscr{H}}$ into the direct integral

$$(1.1) \qquad \widetilde{\mathscr{H}} = \dot{\oplus} \int_{S^\nu} \mathscr{H}(k) d\mu(k)$$

of Hilbert spaces $\mathscr{H}(k)$, where k, the momentum of the system in question, is a character of Z^ν which belongs to the ν-dimensional torus S^ν, and μ is some measure on S^ν. By the decomposition (1.1) we can write H and U_g, $g \in Z^\nu$, in the form

$$U_g = \dot{\oplus} \int_{S^\nu} U_g(k) d\mu(k)$$

$$H = \dot{\oplus} \int_{S^\nu} H(k) d\mu(k)$$

and

$$U_g(k) f = e^{i(g,k)} f, \qquad f \in \mathscr{H}(k),\ g \in Z^\nu.$$

By a particle (or quasiparticle) with momentum k and energy $\lambda(k)$ we mean an eigenvector $\psi(k) \in \mathscr{H}(k)$ of $H(k)$ such that

$$H(k)\psi(k) = \lambda(k)\psi(k).$$

The notions of elementary particle, bound state, quasiparticle, etc. are important in physics but we shall not discuss them here.

Let now Λ be a connected open domain in S^ν such that for any $k \in \Lambda$ there exists a $\psi(k)$ with energy $\lambda(k)$. Let $\psi(k)$ be analytic or continuous in k in some sense and Λ be maximal with respect to all these properties. Then we call the subspace

$$E(\Lambda) = \dot{\oplus} \int_{\Lambda} E(k) d\mu(k) \subseteq \mathscr{H}$$

a one-particle subspace where $E(k)$ is the one-dimensional subspace of $\mathscr{H}(k)$ spanned by $\psi(k)$. I.e. $E(\Lambda)$ corresponds to the same particle with different momentae $k \in \Lambda$. $E(\Lambda)$ is invariant with respect to H

and cyclic with respect to U_g, $g \in Z^\nu$. The spectrum of H in $E(\Lambda)$ is $\{\lambda(k),\ k \in \Lambda\}$.

There are at least two meanings of the words: n-particle subspace. First of all these are in and out subspaces in the Haag – Ruelle theory. For some results about the Haag – Ruelle theory for quantum lattice systems see [5].

The second point of view is more formal and is connected with the notion of clustering operator which will be defined in Section 2 of the present paper. The plan of this paper is the following:

In Section 2 we define the notion of clustering operator in an exact way and formulate our main results about them.

In Section 3 we sketch how the study of the Hamiltonian can be reduced to that of the transfer matrix for some Gibbs random field.

In Section 4 we formulate some results about the transfer matrix is the clustering operator.

In Section 5 we introduce complex Gibbs fields and consider some examples of them (one of them is connected with Θ-vacua).

2. CLUSTERING OPERATORS

Clustering operators are a natural generalization of the N-body Schrödinger operators in quantum mechanics. Here we consider the lattice version of them only.

Let $\mathfrak{B}_n = \mathfrak{B}_n(Z^\nu)$ be the set of all subsets of Z^ν with n elements,

$$\mathfrak{B} = \mathfrak{B}(Z^\nu) = \bigcup_{n=0}^{\infty} \mathfrak{B}_n(Z^\nu).$$

One can consider the set of functions $f(x) = f(z^1, \ldots, x^\nu) \in l_2(Z^\nu) = l_2(\mathfrak{B}_1)$, $x \in Z^\nu$. The antisymmetric tensor product of n copies of $l_2(\mathfrak{B}_1)$ can be easily identified with $l_2(\mathfrak{B}_n)$ and the antisymmetric Fock space over $l_2(Z^\nu)$ with $l_2(\mathfrak{B})$. An operator A in $l_2(\mathfrak{B})$:

(2.1) $$(Af)(T) = \sum_{T' \in \mathfrak{B}} a_{T,T'} f(T'),$$

$T \in \mathfrak{B}$, $f \in l_2(\mathfrak{B})$, is called clustering if:

1. A commutes with the translations U_t, $t \in Z^\nu$:

(2.2) $$(U_t f)(T) = f(T - t), \qquad T \in \mathfrak{B},\ f \in l_2(\mathfrak{B}).$$

2. $a_{\phi,\phi} = 1$, $a_{\phi,T} = a_{T,\phi} = 0$, $T \neq \phi$ and for $T, T' \neq \phi$ one has

(2.3) $$a_{T,T'} = \sum_{k=1,2,\dots} \sum_{\substack{(\tau_1,\dots,\tau_k): \cup \tau_i = (T,T') \\ \tau_i \cap \tau_j = \phi,\, i \neq j}} \omega_k(\tau_1, \dots, \tau_k)$$

where $\tau_i = (T_i, T_i') \in \mathfrak{B} \times \mathfrak{B}$, $\tau_1 \cup \tau_2 = (T_1 \cup T_2, T_1' \cup T_2')$ and the same for intersections, $\tilde{\phi} = (\phi, \phi)$. The summation in (2.3) is taken over all (non-ordered) partitions $(T, T') = \tau_1 \cup \tau_2 \cup \dots \cup \tau_k$, $\tau_i \cap \tau_j = \tilde{\phi}$ if $i \neq j$. It is assumed that the "cluster functions" $\omega_k(\tau_1, \dots, \tau_k)$ are symmetric with respect to permutations and satisfy the following conditions:

(1) For all $k = 1, 2, \dots$ and for any $s_1, \dots, s_k \in Z^\nu$,

(2.4) $$\omega_k(\tau_1 + s_1, \tau_2 + s_2, \dots, \tau_k + s_k) = \omega_k(\tau_1, \dots, \tau_k)$$

where $\tau + s = (T + s, T' + s)$ if $\tau = (T, T')$. Although the ω_k in (1.3) depend in fact only on the collections with $\tau_i \cap \tau_j = \tilde{\phi}$, $i \neq j$, one can assume that they are defined for all collections $(\tau_1, \dots, \tau_k)$.

(2) There exist β, $0 < \beta \leqslant 1$ (clustering parameter) and $M > 0$ such that for any $k = 1, 2, \dots$

(2.5) $$|\omega_k(\tau_1, \dots, \tau_k)| < M(\beta)^{\sum_1^k d_{\tau_i}}$$

where d_τ is defined in the following way.

$$Y_0 = \{t \in Z^{\nu+1}: t = (t^1, \dots, t^\nu, 0),\ t^i \in Z^1,\ i = 1, \dots, \nu\}$$

$$Y_k = \{t \in Z^{\nu+1}: t = (t^1, \dots, t^\nu, k),\ t^i \in Z^1,\ i = 1, \dots, \nu\}.$$

For any $t \in Z^{\nu+1}$ $(t \in Y_0)$ we set

$$t(k) = t + ke_{\nu+1}, \quad e_{\nu+1} = (0, \ldots, 0, 1) \in Z^{\nu+1}.$$

In a similar way we can define $T(k)$ for any $T \subset Z^{\nu+1}$.

Then d_B, $B \subset Z^{\nu+1}$, is defined to be the minimum length of the trees the vertices of which are the points of B. The length of a tree is the sum of the lengths of its lines, and the length of a line is taken in the metric

$$\rho(t_1, t_2) = \sum_{i=1}^{\nu+1} |t_1^i - t_2^i|,$$

$$t_j = (t_j^1, \ldots, t_j^{\nu+1}) \in Z^{\nu+1}, \quad j = 1, 2.$$

We define

$$d_\tau = d_{T \cup T'(1)}$$

where $\tau = (T, T')$, $T \times \{0\}, T' \times \{0\} \subset Y_0$, $(T' \times \{0\})(1) \subset Y_1$.

By the N-particle clustering operator we mean the restriction of the clustering operator onto $l_2(\mathfrak{B}_N)$, and the $(\leqslant N)$-particle clustering operator is the restriction of the clustering operator onto $\dot{\bigoplus}_{1 \leqslant n \leqslant N} l_2(\mathfrak{B}_n) \subset \subset l_2(\mathfrak{B})$.

Remark 1. Let us consider the lattice Schrödinger operator

$$A = \sum_{i=1}^{N} (-\Delta_i + m^2) + \sum_{1 \leqslant i < j \leqslant N} V(x_i - x_j)$$

where Δ_i is the lattice Laplacian in the variable $x_i \in Z^\nu$, $m > 0$, V has exponential decay. It is easy to verify that $\exp\{-tA\}$, $t > 0$, is clustering.

Remark 2. One can call the symbol of A the clustering operator A_0 which has the only non-zero matrix elements for $|T| = |T'|$ which are equal to $(a_0)_{T,T'} = \sum \omega_{|T|}(\tau_1, \ldots, \tau_{|T|})$ where the sum is taken over all $\tau_i = (\{t\}, \{t'\})$, $t \in T$, $t' \in T'$.

Remark 3. The set of all clustering operators is closed under multiplication. It is also an algebra in some sense (see II [2]).

Our main result is the following theorem [2]. Let a self-adjoint clustering operator A be given such that for each $k = 1, 2, \ldots$ and any $t_1, \ldots, t_k$, its symbol satisfies the following lower bound

$$A_0 > L(C\beta)^{\hat{N}} \tag{2.6}$$

where $\hat{N}$ is the operator "number particle": $\hat{N}f = Nf$, $f \in l_2(\mathfrak{B}_N)$, $N = 0, 1, 2, \ldots$, and $L, C > 0$ are constants.

Then for each $N \geqslant 1$ there exists a $\beta_0 = \beta_0(N, M, C, L)$ such that for $0 \leqslant \beta \leqslant \beta_0$ there are $N+1$ mutually orthogonal subspaces $\dot{\oplus}_{n>0} l_2(\mathfrak{B}_n) \supseteq \mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_N, \mathcal{H}_{N+1}$ which are invariant with respect to A and U_t and for $1 \leqslant s \leqslant N$ we have

$$\begin{aligned} &k_1 \beta^s \|x\|^2 \leqslant (Ax, x) \leqslant k_2 \beta^s \|x\|^2, && x \in \mathcal{H}_s \\ &|(Ax, x)| < k_2 \beta^{N+1} \|x\|^2, && x \in \mathcal{H}_{N+1} \end{aligned} \tag{2.7}$$

where $k_i = k_i(N, M, C, L)$ are constants, $i = 1, 2$.

Moreover, $\mathcal{H}_s$, $1 \leqslant s \leqslant N$, lies in the vicinity of $l_2(\mathfrak{B}_s)$ i.e.

$$\|x - P_s x\| \leqslant G\beta^{\frac{1}{2}} \|x\|, \qquad x \in \mathcal{H}_s \tag{2.8}$$

where P_s is the projection onto $l_2(\mathfrak{B}_s)$, $G = G(N, M, C, L)$ is a constant. For each $k = 1, \ldots, N$ there exists a unitary operator $V_k \colon \mathcal{H}_k \to l_2(\mathfrak{B}_k)$ such that

(1) $V_k(U_t |_{\mathcal{H}_k}) V_k^{-1} = U_t |_{l_2(\mathfrak{B}_k)}$

(2) the operator $A_k = V_k(A |_{\mathcal{H}_k}) V_k^{-1}$

in $l_2(\mathfrak{B}_k)$ is clustering with clustering parameter $\lambda(\beta) \to 0$ as $\beta \to 0$.

Now we list some problems about clustering operators.

1. Many results for the N-particle Schrödinger operator ought to be transferred to clustering operators: Faddeev – Jakubovsky equations, bound states, resonances, asymptotic completeness, and so on.

2. What is the structure of the resolvent for a general clustering

operator? We conjecture that for each $\gamma > 0$, outside a γ-neighbourhood of 0 this structure can be reduced in some sense to the study of an $(\leqslant N(\gamma))$-particle clustering operator.

3. Asymptotic completeness of Haag – Ruelle scattering theory for clustering operators.

4. It can be shown that the set of all $(\leqslant N)$-particle clustering operators generate, for N fixed, a $*$-algebra B_N of type I. What is the structure of $\liminf B_N$? Probably, it is not of type I.

3. REDUCTION TO MARKOV FIELDS

The only proof known up to now of the fact that the exponent of the Hamiltonian H is clustering, consists first of all in a reduction procedure to Markov fields. This reduction is made in several steps.

Let us suppose first of all that the space of the states of some quantum system is the Hilbert space $\mathscr{H} = L_2(X, \mu)$, where (X, μ) is a measurable space with nonnegative measure μ. The Hamiltonian of the quantum system is such that the semigroup of operators $\exp(-tH)$, $t > 0$, is defined by the nonnegative kernel $G(x_1, x_2; t)$:

$$\exp\{-tH\}(x_1, x_2) = G(x_1, x_2; t) \geqslant 0, \qquad x_1, x_2 \in X, \ t > 0.$$

Let us suppose that the vacuum Ω (see Section 1) is unique and that its eigenvalue λ_0 is the minimum of the spectrum $\sigma(H)$ of H, i.e. $\Omega = \psi_0(x)$ is the unique ground state of the system. Then we can define the stationary Markov process $\{x_t, \ t \in R^1\}$, $x_t \in X$. Its conditional probability density is equal to

$$\text{(3.1)} \qquad \pi_t(x \mid y) = e^{-\lambda_0 t} \psi_0(x) G(x, y; t) \psi_0^{-1}(y),$$

and the one-dimensional stationary density is

$$\text{(3.2)} \qquad \pi(x) = |\psi_0(x)|^2 .$$

(We do not discuss when (3.1) and (3.2) exist.) Under some general assumptions (e.g. if λ_0 is an isolated eigenvalue) it can be shown that the process x_t is the limit of the Markov processes $\{x_t^T, \ |t| < T\}$, $T > 0$ as $T \to \infty$.

The finite-dimensional distributions of x_t^T are given by the densities

$$
\begin{aligned}
(3.3)\qquad &\pi^T_{t_0, t_1, \ldots, t_{n+1}}(x_0, x_1, \ldots, x_{n+1}) = \\
&= Z^{-1}_{t_0, t_1, \ldots, t_{n+1}} \prod_0^n G(x_{i+1}, x_i; t_{i+1}, t_i)
\end{aligned}
$$

where $-T = t_0 < t_1 < \ldots < t_{n+1} = T$, and $Z_{t_0, \ldots, t_{n+1}}$ is the normalization constant. The limit as $T \to \infty$ is understood as a weak limit of finite-dimensional distributions.

The generator $\hat{H}$ of this limit Markov process x_t is an operator in $L_2(X, \pi d\mu)$ and is given by

$$(3.4)\qquad (\hat{H}f)(x) = \psi_0(x)[(H - \lambda_0 E)(\psi_0^{-1} f)](x).$$

It is unitarily equivalent to the operator $H_0 - \lambda_0 E$ acting in $L_2(X, d\mu)$. So we are reduced to the study of the spectrum of $P_t = \exp\{-t\hat{H}\}$ given by

$$(3.5)\qquad (P_t f)(y) = \int_X f(x)\pi_t(x \mid y) d\mu(x).$$

The most important case is

$$H = H_0 + V$$

where H_0 is the generator of a stationary Markov process $\{x_t^0,\ t \in R^1\}$ with values in X, and V is the multiplication by the function

$$(Vf)(x) = V(x)f(x), \qquad x \in X,\ f \in L_2(X, \mu).$$

Then the measure ν corresponding to the process x_t is locally absolutely continuous with respect to the measure ν_0 corresponding to x_t^0. To be more precise, for any T_1, T_2, $T_1 < T_2$, the processes x_t and x_t^0 generate the same σ-algebra $\mathfrak{B}_{T_1, T_2}$ and the restriction of ν onto $\mathfrak{B}_{T_1, T_2}$ is absolutely continuous with respect to $\nu_0 \big|_{\mathfrak{B}_{T_1, T_2}}$. Moreover, for μ-almost all $x_1, x_2 \in X$, the conditional distributions $\nu_{T_1, T_2}^{x_1 x_2}$ for $x_{T_1} = x_1$, $x_{T_2} = x_2$ are given by

$$\frac{d\nu^{x_1 x_2}_{T_1,T_2}}{d\nu_0{}^{x_1 x_2}_{T_1,T_2}} = (Z^{x_1 x_2}_{T_1,T_2})^{-1} \exp\left\{- \int_{T_1}^{T_2} V[x(\tau)]\,d\tau\right\}$$

where $Z^{x_1 x_2}_{T_1,T_2}$ is again the normalization constant. In this situation x_t^0 is often called "free process" and x_t its Gibbs modification (defined by the interaction V).

Example. Consider the so-called $(\nu + 1)$-dimensional Ising model with continuous time. First it can be considered as the quantum ν-dimensional model with Hamiltonian in volume Λ

$$H_\Lambda = \beta_1 \sum_{t\in\Lambda} S_t^x + \beta_2 \sum_{\substack{t,t'\in\Lambda \\ |t-t'|=1}} S_t^z S_{t'}^z$$

where S^x, S^z are Pauli matrices, β_1, β_2 are constants. All details about its reduction to a Markov field can be found in [9].

The same can be done for the Heisenberg ferromagnet.

4. THE TRANSFER-MATRIX AND ITS CLUSTER PROPERTIES

Instead of the Ising model with continuous time we shall consider the Ising model with discrete time. I.e. we consider the random field σ_t, $\sigma_t = \pm 1$ on the lattice Z^ν, $t = (t^0, \ldots, t^{\nu-1})$, which is the limit for $\Lambda \nearrow Z^\nu$ of the fields in volume Λ with probabilities

$$p_\Lambda(\{\sigma_t,\ t\in\Lambda\}) = Z^{-1} \exp\left(\beta \sum \sigma_{t_1}\sigma_{t_2}\right)$$

where the sum is taken over all pairs (t_1, t_2), $t_i \in \Lambda$, $|t_1 - t_2| = 1$. The random variables σ_t are defined on the probability space $\Omega = (-1, 1)^{Z^\nu}$ the underlying measure μ of which is the limit Gibbs distribution. We consider as our physical Hilbert space $L_2(\Omega_0, \mu)$ where $\Omega_0 = (-1, 1)^{Z^{\nu-1}}$ is the set of configurations for the time-slice $t^0 = 0$ with the restriction of μ on it. Evidently,

$$(4.1) \qquad \left\{\sigma_T = \prod_{t\in T} \sigma_t,\ T \in \mathfrak{B}(Z^{\nu-1})\right\}$$

forms a (non-orthogonal) basis in $L_2(\Omega_0, \mu)$.

One can define the elements of matrix $\mathscr{F} = \mathscr{F}_1$ in this basis as follows:

(4.2) $$(\sigma_{T_1}, \mathscr{F}\sigma_{T_2})_{L_2(\Omega_0,\mu)} = \langle \sigma_{T_1} \sigma_{T_2'} \rangle$$

where $\langle \cdot \rangle$ is the expectation with respect to μ and T_2' is the shift of T_2 by the vector $(1, 0, \ldots, 0) \in Z^\nu$. We shall define now another basis [7], [8], [3]:

$$\hat{\sigma}_\phi = 1, \quad \hat{\sigma}_T = \prod_{t \in T} \hat{\sigma}_t$$

$$\hat{\sigma}_t = \frac{(\sigma_t - P_t \sigma_t)}{\| \sigma_t - P_t \sigma_t \|},$$

where P_t is the orthogonal projection in $L_2(\Omega_0, \mu)$ onto the subspace generated by σ_T with T lying to the right of t, $t \in T$.

Assuming β small, the following facts can be proved.

1. $\{\hat{\sigma}_T, T \in \mathfrak{B}\}$ is complete. We have

(4.3) $$(\hat{\sigma}_{T_1}, \mathscr{F}\hat{\sigma}_{T_2})_{L_2(\Omega_0,\mu)} = \langle \hat{\sigma}_{T_1} \hat{\sigma}_{T_2'} \rangle = \sum \omega(\tau_1)\omega(\tau_2) \ldots \omega(\tau_k)$$

where all notations are the same as in Section 2 but $\omega(\tau)$ is a semiinvariant of the random variables $\hat{\sigma}_t$, $t \in T_1 \cup T_2'$. Due to orthogonality, all $\omega(\tau) = 0$, $\tau = (T_1, T_2)$ if either $T_1 = \phi$ or $T_2 = \phi$.

2. Then our main bound is

(4.4) $$|\omega(\tau)| < (C\beta)^{d_{T \cup T'}}, \qquad \tau = (T, T').$$

The proof consists in the proof of an estimate similar to (4.4) for an arbitrary semiinvariant $\langle \sigma_{T_1}, \sigma_{T_2}, \ldots, \sigma_{T_k} \rangle$, $T_i \in \mathfrak{B}(Z^\nu)$, $i = 1, \ldots, k$, $k = 1, 2, \ldots$; see [3], [4]. Then using this estimate and some auxiliary arguments we prove (4.4). Unfortunately, this proof is very elaborate; a simplification would be of great interest.

5. GIBBS QUASISTATES ON A LATTICE

In some physical systems the kernel $G(x_1, x_2; t)$ is not positive and therefore we cannot get a stochastic process. Nevertheless we shall not restrict ourselves to countably additive measures but shall consider only finitely additive ones, and then much of the theory of Gibbs states can be carried over to this case. Such "complex" Gibbs fields appear in many examples:

1. If one wants to study ground states of quantum spin systems, the semigroup $\exp\{-tH\}$ can be described by a nonpositive kernel. Here finitely additive measure appear naturally.

2. Resonances in spin systems can probably be described as complex eigenvalues of non-self-adjoint Hamiltonians. Here also the techniques of complex Gibbs fields seem to be applicable.

3. Much of high-temperature expansions can be generalized to the complex case (see below). There are some results also from low-temperature expansions [10].

4. Some interesting spin systems have a dual system (of Kramers – Wannier type) with complex potential.

5. Here we shall restrict ourselves only to classical spin systems mostly with compact spin space.

Definition of quasistates. Let the spin take its value at each point $t \in Z^\nu$ in a compact spin space S with Borel σ-algebra. For each finite $\Lambda \subset Z^\nu$, S^Λ has standard topology and σ-algebra. Let $\mathfrak{A}_\Lambda$ be the algebra of complex continuous functions on S^Λ; $\mathfrak{A}^0 = \bigcup_{\Lambda \subset Z^\nu; |\Lambda| < \infty} \mathfrak{A}_\Lambda$ be the algebra of local observables.

The quasistate of $\mathfrak{A}^0$ is a linear functional φ on $\mathfrak{A}^0$ such that its restriction to any $\mathfrak{A}_\Lambda$ is continuous and $\varphi(1_\Lambda) = 1$. This means a consistent family of countably additive complex measures μ_Λ on S^Λ such that $\mu_\Lambda(S^\Lambda) = 1$.

Generally speaking, a quasistate cannot be obtained as a restriction of some countably additive measure on S^{Z^ν} (i.e. Kolmogorov's theorem is not true). I.e. a quasistate cannot in general be continued to a continuous linear functional on the C^*-algebra $\mathfrak{A} = \bar{\mathfrak{A}}^0$ of quasilocal observables.

Simple examples of quasistates.

1. *Independent quasistates.* Let μ be some complex measure on S, $\mu(S) = 1$. Then μ_Λ can be defined as a Cartesian product of μ. The quasistate obtained in this way can be called independent.

2. *Markov one-dimensional quasistates.* Let $K(x, A)$, $x \in S$, A a Borel subset of S be a complex function such that for any given $x \in S$, $K(x, A)$ is a complex measure on S and for any Borel $A \subseteq S$, $K(x, A)$ is a measurable function on S. Let also

$$|K(x, A)| \leqslant \text{const},$$

and suppose that there exists an invariant complex measure μ,

$$\mu(A) = \int_S K(x, A) d\mu(x),$$

such that $\mu(S) = 1$. Then we can define measures of cylindrical sets in S^{Z^1} e.g.

$$\mu(\ldots \times S \times A \times B \times S \ldots) = \int_A K(x, B) d\mu(x).$$

So we obtain a quasistate which it is natural to call Markov quasistate.

3. *Gaussian quasistates.* Generalizing somewhat the definition of quasistate (to permit noncompact S) we can define a "Gaussian quasistate" on R^Λ by a "covariance" $A + iB$, where A and B are real covariances and A is positive definite.

Gibbs modifications of quasistates. A family of local functions $\Phi = \{\Phi_A, A \subset Z^\nu)$, $\Phi_A \in \mathfrak{A}_A$, $\Phi_A = 0$, where $\operatorname{diam} A > d \geqslant 0$, is called a potential. Let some quasistate $\langle \cdot \rangle$ be given. We shall define the Gibbs modification of the quasistate $\langle \cdot \rangle$ in a volume Λ:

$$\langle F \rangle_{\Lambda,\Phi,\Phi'} = Z_\Lambda^{-1} \left\langle F \exp \left\{ - \sum_{A \subseteq \Lambda} \Phi_A - \sum_{A \subset \partial_d \Lambda} \Phi'_A \right\} \right\rangle$$

$$Z_\Lambda = \left\langle \exp \left\{ - \sum_{A \subseteq \Lambda} \Phi_A - \sum_{A \subseteq \partial_d \Lambda} \Phi'_A \right\} \right\rangle,$$

and Φ'_A is a family of functions which are not equal to zero unless

$$A \subseteq \partial_d \Lambda = \{ t \in \Lambda :\ \rho(t, Z^\nu \setminus \Lambda) \leqslant d \}$$

(these boundary conditions "can be defined in a vast number of ways").

The limit Gibbs modification of $\langle \cdot \rangle$ by Φ is a weak limit point of the sequence $\langle \cdot \rangle_{\Lambda,\Phi,\Phi'}$ as $\Lambda \nearrow Z^\nu$ and Φ'_A is arbitrarily chosen for each Λ. Notice that the set of quasistates is compact in the weak topology (convergence on any Λ).

High-temperature expansions. Let now $\langle \cdot \rangle_0$ be an independent quasistate and $\beta\Phi_A$ be its potential. We shall consider the limit Gibbs modification of this quasistate:

$$\langle F \rangle = \lim_{\Lambda \nearrow Z^\nu} \langle F \rangle_{\Lambda,\Phi,0} = \lim_{\Lambda \nearrow Z^\nu} \frac{\left\langle F \exp \left\{ -\beta \sum_{A \subseteq \Lambda} \Phi_A \right\} \right\rangle_0}{\left\langle \exp \left\{ -\beta \sum_{A \subseteq \Lambda} \Phi_A \right\} \right\rangle_0}$$

(we put $\Phi'_A \equiv 0$).

We show that this limit exists for β sufficiently small and obtain $\langle F \rangle$ as a series in β. Many cluster expansions (if not all) are applicable. We give the most direct one. We have

$$\langle F \rangle_{\Lambda,\Phi,0} =$$

$$= \sum_n \frac{(-\beta)^n}{n!} \sum_{A_1 \subseteq \Lambda} \cdots \sum_{A_n \subseteq \Lambda} \langle F, \Phi_{A_1}, \Phi_{A_2}, \ldots, \Phi_{A_n} \rangle_0^C$$

where $\langle F, \Phi_{A_1}, \ldots, \Phi_{A_n} \rangle_0^C$ is the semiinvariant. This notion was defined in [4] for an arbitrary virtual field and it is applicable here. The following estimate was proved in [4]:

$$| \langle F, \Phi_{A_1}, \ldots, \Phi_{A_n} \rangle_0^C | \leqslant \sup |F| C^n n_1! \, n_2! \ldots n_m!$$

where m is the number of groups of identical A_j, $j = 1, \ldots, n$, n_j is the number of elements in these groups. Moreover, this semiinvariant is equal to 0 if the collection $\Gamma = (A, A_1, \ldots, A_n)$, A is the "support" of F, is not connected (see [4t). It follows that the limit exists and

$$\langle F \rangle = \sum \frac{(-\beta)^n}{n!} \sum_{A_1} \ldots \sum_{A_n} \langle F, \Phi_{A_1}, \ldots, \Phi_{A_n} \rangle_0^C$$

(it follows also that $Z_\Lambda \neq 0$ for β uniformly sufficiently small in Λ).

Below we give an example where the Gibbs modification is not unique even in the high-temperature region. This uniqueness takes place of course if $\langle \cdot \rangle_0$ is a state and β, Φ_A and Φ'_A are real. In fact, in this case an arbitrary limit point is a Gibbs state in the sense of DLR [1]. But then it is the limit point for some sequence of Gibbs states in finite volumes Λ with boundary conditions given by some configurations in $Z^\nu \setminus D$. I.e. the field Φ'_A is of order β and from this the uniqueness follows.

The above mentioned example is the two-dimensional gauge field with complex boundary conditions.* Let $\langle \cdot \rangle_0$ be the field defined on the edges t of the two-dimensional lattice Z^2; φ_t takes its values in the unit circle S^1.

Let $\langle \cdot \rangle_0$ be the state corresponding to the independent random field φ_t with measure $\frac{1}{d\pi} ds$ on each edge. Let 3 be any set of four edges which belong to the boundary of a unit square. Let us denote

$$\varphi_\square = \varphi_{t_1} + \varphi_{t_2} - \varphi_{t_3} - \varphi_{t_4} \pmod{2\pi}$$

where t_1, t_2 are the left and upper edges of the square $\square$, t_3, t_4 are its right and lower edges. Let $\mathscr{V}$ be an arbitrary real function on S^1.

*One of the authors thanks K. Gawendzki for this example and its relation to Θ-vacuums.

Let us consider a state in a square

$$\langle F \rangle_{\Lambda, \mathscr{V}, 0} = \frac{\left\langle F \exp\left\{-\sum_{\square \subset \Lambda} \mathscr{V}(\varphi_\square)\right\}\right\rangle_0}{\left\langle \exp\left\{-\sum_{\square \subset \Lambda} \mathscr{V}(\varphi_\square)\right\}\right\rangle_0}.$$

This model admits an exact solution, and it is not difficult to prove the uniqueness of DLR states for such models.

Let us consider now the boundary terms on the edges of the boundary of Λ, equal to $i\Theta\varphi_t$ on each edge t belonging to the boundary $\partial_0 \Lambda$ of Λ. We denote by φ_t the value of the angle on S^1, $\varphi_t \in [0, 2\pi]$, Θ is an integer.

The new quasistate is

$$\langle F \rangle_{\Lambda, \mathscr{V}, \Theta} = \frac{\left\langle F \exp\left\{-\sum_{\square \subset \Lambda} \mathscr{V}(\varphi_\square) - i\Theta \sum_{t \in \partial_0 \Lambda} (\pm \varphi_t)\right\}\right\rangle_0}{Z_\Lambda}$$

where we take the $+$ sign on the left and upper sides of the square Λ, the sign $-$ on the other sides. We shall prove that

$$\lim_{\Lambda \uparrow Z^\nu} \langle F \rangle_{\Lambda, \mathscr{V}, \Theta} = \langle F \rangle_{\mathscr{V}, \Theta}$$

exists and defines a quasistate. For different Θ these qausistates are different in general. We use Green's formula

$$\sum_{t \in \partial_0 \Lambda} (\pm \varphi_t) = \sum_{\square \subset \Lambda} \varphi_\square$$

and take for simplicity $F = F(\varphi_{\square_0})$ for some fixed square $\square_0$. We get

$$\langle F \rangle_{\Lambda, \mathscr{V}, \Theta} = \frac{\left\langle F \exp\left\{-\sum_{\square \subset \Lambda} [\mathscr{V}(\varphi_\square) + i\Theta\varphi_\square]\right\}\right\rangle_0}{Z_\Lambda}.$$

A change of variables (the same in the denominator and the numerator) gives

$$\langle F\rangle_{\Lambda,\mathscr{V},\Theta}=\frac{\int_0^{2\pi}F(\varphi)e^{-\mathscr{V}(\varphi)-i\Theta\varphi}d\varphi}{\int_0^{2\pi}e^{-\mathscr{V}(\varphi)-i\Theta\varphi}d\varphi}.$$

We suppose that the denominator is not 0. So $\langle F\rangle_{\mathscr{V},\Theta}$ trivially exists and is in general different for different Θ.

It may be of interest to describe those lattices for which such quasistates (Θ-vacua) can be defined, and to calculate the spectrum of the transfer matrix.

In a similar way one can define quasistates for fields with continuous time. For this it is sufficient to consider algebras $\mathfrak{A}_\Lambda$ for any $\Lambda = \{t_1, \dots, t_n\}$, $t_i \in R^1$ with a consistent family of continuous states.

Complex Markov chains with continuous time are the first examples. It is interesting to develop a general theory for such quasistates and for generalized complex fields, too.

REFERENCES

[1] D. Ruelle, *Statistical mechanics,* New York, 1969.

[2] V.A. Malyshev – R.A. Minlos, Invariant subspaces of clustering operators, I., *J. Stat. Phys.,* 22 (1979), 3; II., *Matem. Sbornik,* (in preparation).

[3] V.A. Malyshev, Complete cluster expansion for weakly coupled Gibbs random fields, in: *Many component random systems,* Springer, N.Y.

[4] V.A. Malyshev, *Comm. Math. Phys.,* 64 (1979), 131-157.

[5] V.A. Malyshev, *Lect. Notes Math.,* 653 (1978), 173-183.

[6] V.A. Malyshev, Cluster expansions in lattice models of statistical physics and quantum field theory, *Uspehi matem. nauk.,* (35) 2 (1980), 3-53.

[7] R.A. Minlos – Ya.G. Sinai, *Teor. matem. physika,* 2 (1970), 230-243.

[8] F. Abdulla-Zade – R.A. Minlos – S. Pogosyan, In: *Mnogokomp. sluch. syst.,* Nauka, Moskwa, 1978.

[[9] V.A. Malyshev, *Funk. analiz i Prim.,* 13 (1979), 31-41.

[10] S.A. Pirogov, (unpublished).

[11] S.N. Lakaev – R.A. Minlos, *Teor. Mat. Phys.,* 30 (1979), 83-93.

V.A. Malyshev – R.A. Minlos

Moscow State University, Faculty for Mechanics and Mathematics, USSR.