Stability and Admissible Densities in Transportation Flow Models

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Abstract. One-way road traffic model with a local control is considered. For given density of transportation units we discover phase transitions in the control parameter space when there exists or not safe transportation. Also we discuss possible densities and instability in several road networks with crosses, if no control is imposed.

1 Introduction

Theoretical modelling and computer simulation of transportation systems is a very popular field, see very impressive review [3]. There are two main directions in this research - macro and micro models. Macro approach does not distinguish individual transportation units and uses analogy with the notion of flow in hydrodynamics, see [2]. Stochastic micro models are most popular and use almost all types of stochastic processes: mean field, queueing type and local interaction models. We consider here deterministic transportation flows. Although not as popular as stochastic traffic, there is also a big activity in this field, see [1,4–8].

In Sect. 2 we do not pursue maximal generality but rather consider the simplest flow, that is the following one-way road traffic model. Namely, at any time $t \geq 0$ there are many (even infinite number) of point particles (may be called also cars, units etc.) with coordinates $z_k(t)$ on the real line, enumerated as follows

$$... < z_N(t) < ... < z_1(t) < z_0(t) \tag{1}$$

For this infinite chain of cars we try to find control mechanism which garanties that the distance between any pair of neighbouring cars is greater or equal (on all time interval $(0, \infty)$) to some fixed number d, called safe distance, but at the same time is not too far from it (then we say that the density of cars is admissible).

This control mechanism is assumed to be local - any car has information only about the previous car. Moreover, this mechanism is of physical nature, like forces between molecules in crystals but our "forces" are not symmetric. The safety (stability) conditions appear to be similar to the dynamical phase transition in the model of the molecular chain rapture under the action of external force. However here we do not need the double scaling limit, used in [9].

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In Sect. 3 we consider idealistic models of transportation network as the collection of roads with crosses (intersections). Along each road the units (cars) move deterministically. We get admissible density of units on the given transportation network and show the dependence of this density on the number and multiplicity of crosses and number of cycles in the network.

2 Local Flow Control

In this section we consider the simplest flow (1), where it is assumed that the rightmost unit moves "as it wants". More exactly, the trajectory $z_0(t)$ is only assumed to be sufficiently smooth with positive velocity $v_0(t) = \dot{z}_0(t)$ and natural upper bounds on the velocity and acceleration

$$\sup_{t>0} v_0(t) \leqslant v_{max}, \quad \sup_{t>0} |\ddot{z}_0(t)| \leqslant a_{max} \tag{2}$$

We would like to organize such traffic so that for any t and k the distances $r_k(t) = z_{k-1}(t) - z_k(t)$ were greater or equal to some number $d > 0$, which is chosen to avoid collisions and keep maximal possible density of traffic. Moreover, the organization should use only (maximally) local control. More exactly, the k-th driver at any time t knows only its own coordinate and velocity and the coordinate $z_{k-1}(t)$ of the previous car. Thus, for any $k \geq 1$ the trajectory $z_k(t)$ is uniquely defined by the trajectory $z_{k-1}(t)$ of the previous particle.

Using physical terminology one could say that if, for example, $r_k(t)$ becomes larger than d, then some positive force F_k increases acceleration of the particle k. We will see that, besides F_k, for such stability, also friction force $-\alpha v_k(t)$, which, on the contrary, restrains the growth of the velocity $v_k(t)$, is necessary. The constant $\alpha > 0$ should be chosen appropriately.

Thus the trajectories are uniquely defined by the system of equations for $k \geq 1$

$$z_k^{''}(t) = F_k(t) - \alpha \frac{dz_k}{dt} = \omega^2 (z_{k-1}(t) - z_k(t) - d) - \alpha \frac{dz_k}{dt} \tag{3}$$

where F_k is taken to be simplest possible

$$F_k(t) = \omega^2 (z_{k-1}(t) - z_k(t) - d)$$

Stability Conditions. For given $\alpha, \omega, d, z_0(t)$ and initial conditions the trajectories are uniquely defined and we can denote

$$I = \inf_{k \geqslant 1} \inf_{t \geqslant 0} r_k(t), \quad S = \sup_{k \geqslant 1} \sup_{t \geqslant 0} r_k(t)$$

Put also

$$d^* = d^*(v_{max}, a_{max}) = \frac{1}{\omega^2}(a_{max} + \alpha v_{max})$$

Consider firstly the simplest initial conditions

$$z_k(0) = -kd, \frac{dz_k}{dt}(0) = v, k \geq 0 \tag{4}$$

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Theorem 1. *Assume (4) and $\alpha > 2\omega > 0$. Then for any chosen "safe distance" parameter $d > d^*$ in the Eq. (3) the following bounds hold*

$$I > (d - d^*) > 0, \quad S < 2d.$$

Now consider more general initial conditions satisfying

$$z_0(0) = 0, z_0'(0) = v, 0 < a \leqslant r_k(0) \leqslant b, \quad |\frac{dr_k}{dt}(0)| \leqslant c, \tag{5}$$

for any $k > 0$ and some non-negative a, b. Denote also

$$D^* = \max\{A, d^*\},$$

where

$$A = \frac{\alpha a^{'} + 2c}{2\gamma}, \gamma = \sqrt{\frac{\alpha^2}{4} - \omega^2}, a^{'} = \max\{|a-d|, |b-d|\}.$$

Theorem 2. *If $\alpha > 2\omega$, $\frac{\alpha a - 2c}{\alpha - 2\gamma} > \frac{a+b}{2}$, $\frac{a+b}{2} < d^* < \frac{\alpha a - 2c}{\alpha - 2\gamma}$ then for any initial conditions (5) and any smooth function $z_0(t)$, satisfying (2), there exists open subset $\mathcal{D} \subset \mathbb{R}$ such that for any choosen "safe distance" parameter $d \in \mathcal{D}$ in the Eq. (3), the following bounds hold*

$$I \geq (d - D^*) > 0, \quad S \leq d + D^*.$$

The proof of both theorems is based on the analysis of the chain of equations for $x_k = r_k - d, k > 0$

$$x_k^{''}(t) + \alpha x_k^{'}(t) + \omega^2 x_k(t) = \omega^2 x_{k-1}(t), \; k = 1, 2, ...$$

where

$$x_0(t) = \frac{1}{\omega^2}(z_0^{''}(t) + \alpha z_0^{'}(t))$$

Density and Currents. Let $n(t, I)$ be the number of units on the interval $I \subset R$ at time t and $n(T, x)$ be the number of units passing the point x in the time interval $(0, T)$. Then the density and the current through some point x are defined as follows ($|I|$ is the length of I)

$$\mu(t) = \lim_{|I| \to \infty} \frac{n(t, I)}{|I|}$$

$$J(x) = \lim_{T \to \infty} \frac{n(T, x)}{T}$$

if these limits exist. For fixed N define also mean length of the chain of cars $0, 1, ..., N$

$$L_N(t) = \frac{z_0(t) - z_N(t)}{N}.$$

Theorem 3. *Under the conditions of Theorem 2 assume also that the initial conditions are such that the following finite limits exist*

$$\lim_{N\to\infty} L_N(0) = L(0), \quad \lim_{N\to\infty} \dot{L}_N(0) = \dot{L}(0),$$

Then for any t there exist

$$\lim_{N\to\infty} L_N(t) = L(t),$$

and moreover

$$L(t) = L(0) + \frac{1}{\alpha}(1 - e^{-\alpha t})\frac{dL}{dt}(0)$$

Note that if moreover $\frac{dz_k(0)}{dt}$ are uniformly bounded then $\frac{dL}{dt}(0) = 0$, that is the mean length does not change with time.

Instability. First reason for the instability is the absence of dissipation.

Theorem 4. *Let $\alpha = 0$, then for the initial conditions (4) for $k > 0$ and for $z_0(t) = tv + \sin\omega' t,\ v > 0,\ \omega' \neq 0$ and $k \geqslant 2$ we have*

$$\inf_{t\geqslant 0} r_k(t) = -\infty$$

It is sufficiently easy to explain by the resonance effect: $x_1(t)$ is the harmonic movement with frequencies ω, ω', and the proper frequency of x_2 is ω.

Assume again (4). We will prove instability (even for rather simple behaviour of the leading unit $z_0(t)$) when k and t tend simultaneously to ∞ so that $t = \mu k,\ k \to \infty$ for some constant $\mu > 0$. The following theorem consists of two parts: the first one exhibits the possible zero density, the second one exhibits the possibility of collisions.

Theorem 5. *Let $\frac{dz_0}{dt}(t) = v$ for all $t \geqslant 0$. Then*

1. for any $\alpha > 0$ there exists $\omega > 0$ and constants $q_+ > 1,\ \mu_+ > 0, c_+ > 0$, so that for any $d > 0$

$$r_k(t) \sim \frac{c_+}{\sqrt{k}} q_+^k$$

as $t = \mu_+ k,\ k \to \infty$;

2. for any $\alpha > 0$ there exists $\omega > 0$ and constants $q_- > 1,\ \mu_- > 0, c_- < 0$, so that for any $d > 0$

$$r_k(t) \sim \frac{c_-}{\sqrt{k}} q_-^k$$

as $t = \mu_- k,\ k \to \infty$.

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3 Transport in Networks Without Control

Transportation without control means that any unit moves with its own velocity and does not know anything about the movement of other units. In other words the functions $z_k(t)$ are fixed. We will consider only the case when

$$z_k(t) = c_k + v_k t$$

for some constants $v_k > 0$ and

$$... < c_{k-1} < c_k = z_k(0) < ...$$

The transportation network is defined to be safe if at any time moment t the distance between any two units is greater or equal to some fixed number $d > 0$ called safety distance. It is clear that for the safe network all v_k should be equal to some $v > 0$, their joint constant velocity. It is also clear that $\mu(t) \geq d^{-1}$ and this density is attained in the case when

$$z_i(t) = z_i(0) + vt, \;\; z_{i-1}(0) - z_i(0) = d \implies z_{i-1}(t) - z_i(t) = d$$

It is easy to see that in our case at time t any density $\mu(t)$ is admissible iff

$$\mu(t) \leq d^{-1}$$

The network (of roads) is defined here as the one-dimensional topological space, which is the disjoint union of some number of real lines (roads) $k = 1, ..., M \leq \infty$ with coordinates z_k. It is assumed that any pair of real lines has finite number of identified points (crosses). Consider the graph G, which vertices are these identified points (intersections of the real lines) and the edges are all segments of the roads in-between the vertices. That is we delete all infinite intervals of the roads. The metrics is defined as usual: the distance $\rho(x_k, x_l)$ between two points x_k and x_l on the network is the minimal length of paths between these two points.

On any road k the transportation units are labeled as (k, i), where k is the road along which it moves and i is its order on this road. Here the sequences are assumed to be infinite to both sides

$$... < z_{k,i}(t) < z_{k,i-1}(t) < ...$$

and we again consider networks where the velocities and safe distance are the same for all roads

Proposition 1.

1. *Assume there are $M < \infty$ roads with only one common cross. In this case the graph G consists of one vertex only. Then at time t any density $\mu(t)$ is admissible iff*

$$\mu(t) \leq (Md)^{-1} \tag{6}$$

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2. *Assume there are M roads and let L be maximal multiplicity of their intersections. Assume also that the graph G has no cycles. Then at time t any density $\mu(t)$ is admissible iff*

$$\mu(t) \leq (Ld)^{-1}$$

3. *Consider 4 roads A, B, C, D with 4 different crosses $1 = A \cap B, 2 = B \cap C, 3 = C \cap D, 4 = D \cap A$, Then the graph G is a quadruple and has 4 edges $12, 23, 34, 41$. Then the admissible densities are*

$$\mu(t) \leq \frac{3}{8}(d)^{-1}$$

Note that if the number of cycles grows then the coefficient in front of d^{-1} decreases very quickly. It would be interesting to calculate it explicitely.

There is another reason showing that such transportation network is not only of small density but also is strongly unstable. Namely, consider the case when each road $k = 1, ..., M$ has its own safe distance d_k and its own velocity v_k, Thus

$$z_{k,i}(t) = z_{k,i}(0) + v_k t, \;\; z_{k,i-1}(0) - z_{k,i}(0) = d_k \Longrightarrow z_{k,i-1}(t) - z_{k,i}(t) = d_k$$

Then the corresponding currents are $J_k = v_k d_k^{-1}$. The following proposition shows extreme instability of the simplest network.

Proposition 2. *Consider two roads with one cross. Call the transportation stable if distances between any two units are greater or equal to some $D_0 > 0$ uniformly in t. Then:*

If $\frac{J_1}{J_2}$ is not rational, then such transportation cannot be stable.

If $\frac{J_1}{J_2} = \frac{n_1}{n_2}$ for some integers n_1, n_2 such that $(n_1, n_2) = 1$. Then stable transportation exists iff

$$\frac{n_1}{d_1} + \frac{n_2}{d_2} < \frac{1}{D_0}$$

Remark 1 (Control types). To avoid instability of Proposition 2 some control is necessary. There are two possibilities for the control. The first one we considered in Sect. 2 - local internal control, that is depending only on distances between cars. Second type is the control which forces velocities of cars to change in certain points of the network. The mostly used such control is the organization of traffic lights where the cars should stand still for some time. Other control types are also known, see [1,4–8], and it could be interesting to find general classification of control types.

References

1. Blank, M.: Ergodic properties of a simple deterministic traffic flow model. J. Stat. Phys. **111**, 903–930 (2003)
2. Prigogine, I., Herman, R.: Kinetic Theory of Vehicular Traffic. Elsevier, N.Y. (1971)

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3. Helbing, D.: Traffic and related self-driven many particle systems. Rev. Mod. Phys. **73**, 1067–1141 (2001)
4. Feintuch, A., Francis, B.: Infinite chains of kinematic points. Automatica **48**, 901–908 (2012)
5. Hui, Q., Berg, J.M.: Semistability theory for spatially distributed systems. In: Proceedings of the IEEE Conference on Decision and Control, January 2009
6. Jovanovic, M.R., Bamieh, B.: On the Ill-Posedness of certain vehicular platoon control problems. IEEE Trans. Autom. Control **50**(9), 1307–1321 (2005)
7. Melzer, S.M., Kuo, B.C.: Optimal regulation of systems described by a countably infinite number of objects. Automatica **7**, 359–366 (1971). Pergamon Press
8. Swaroop, D., Hedrick, J.K.: String stability of interconnected systems. IEEE Trans. Autom. Control **41**(3), 349–357 (1996)
9. Malyshev, V.A., Musychka, S.A.: Dynamical phase transition in the simplest molecular chain model. Theoret. Math. Phys. **179**(1), 123–133 (2014)

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