

## TWO-DIMENSIONAL LORENTZIAN MODELS

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*Dedicated to Robert Minlos on the occasion of his 70<sup>th</sup> birthday*

**ABSTRACT.** The goal of this paper is to present rigorous mathematical formulations and results for Lorentzian models, introduced in physical papers. Lorentzian models represent two dimensional models, where instead of a two-dimensional lattice one considers an ensemble of triangulations of a cylinder, and natural probability measure (Gibbs family) on this ensemble. It appears that correlation functions of this model can be found explicitly. Such models can be considered as an example of a new approach to quantum gravity, based on the notion of a causal set. Causal set is a partially ordered set, thus having a causal structure, similar to Minkowski space. We consider subcritical, critical and supercritical cases. In the critical case the scaling limit of the light cone can be restored.

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### 1. INTRODUCTION

Lorentzian models were introduced in physical papers, see [1, 2, 6]. The goal of this paper is to present rigorous mathematical formulations and results for such type of models. The presented models are explicitly solvable, and thus a complete control is possible.

Such models can be considered as an example of a new approach to quantum gravity, based on the notion of causal set, see [12, 4]. A causal set is a partially ordered set, thus having a causal structure, similar to the Minkowski space structure. There are two main approaches to the construction of random and quantum causal sets: the Gibbs and the Hamiltonian approaches. In the physical literature they define “complex amplitudes” and the definitions are not exact, thus the difference between them is not so easy to understand. The Gibbs approach corresponds to the Euclidean approach in quantum field theory, the Hamiltonian approach more resembles quantum spin systems. In this context, they give quite different models. In the critical Gibbs case, one can define the continuous limit but the unitarity

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is absent. In the Hamiltonian approach, on the contrary, one has the unitarity, however, the continuous limit gives unnatural results.

We consider Gibbs families  $\mu_N$  on a class  $\mathcal{C}_N$  of finite causal sets, depending on a thermodynamic parameter  $N$ , and prove that as  $N \rightarrow \infty$  the sequence  $\mu_N$  converges in probability to some (non-random) continuous surface, where at each point the future and past cones are defined. For other models such scaling problems were earlier considered in the physical literature, see [13, 14].

**1.1. Causal sets and graphs.** A causal set (not necessarily countable) is a partially ordered set with a binary (causal) relation  $\leq$ , satisfying the following properties: (transitivity) if  $x \leq y$  and  $y \leq z$  then  $x \leq z$ ; (reflexivity)  $x \leq x$ ; (antisymmetry) if  $x \leq y$  and  $y \leq x$  then  $x = y$ . For denumerable sets the following local finiteness property is assumed: for any  $x \leq y$  the Alexandrov set  $A(x, y) = \{z: x \leq z \leq y\}$  is finite.

For a given (countable or finite) causal set  $V$  one can define a directed graph  $G$  (called the Hasse diagram of  $V$ ), so that  $V$  becomes the set of vertices of  $G$  and there is a directed edge from  $x$  to  $y$ , if  $x \leq y$ ,  $x \neq y$ , and there is no other  $z$  such that  $x \leq z \leq y$ . Vice versa, with a directed graph with the set  $V$  of vertices without cycles (cycle is a closed directed path) one can associate a causal set  $V$ : for  $x, y \in V$  we put  $x \leq y$  if either  $y = x$  or there is a directed path from  $x$  to  $y$ .

We shall need some subsets of a causal set. The causal future of a point  $v$  is defined as  $F(v) = \{v': v' \geq v\}$ . A causal set is a subset  $A$  of  $V$  such that no pair  $v, v'$  of points in  $A$  are related by the causal relation.

We shall consider the following class of examples of causal sets. Let  $G$  be some connected (not directed) graph with the set  $V$  of vertices and the set  $L$  of edges. For example, it can be the one-dimensional skeleton of some triangulation of a manifold  $M$ . Choose some subset  $V_0 \subset V$ ; we call  $V_0$  the zero slice. Define the  $N$ -slices as  $R(V_0, N) = \{v: \rho(v, V_0) = N\}$ , where the distance  $\rho$  between two vertices is the minimal number of edges in a path connecting them. For  $N = 0, 1, \dots$ , we make directed any edge between the  $N$ -slice and the  $(N + 1)$ -slice. That is, we define its direction from  $v$  to  $v'$  if  $v$  belongs to the  $N$ -slice and  $v'$  belongs to the  $(N + 1)$ -slice. All other edges of  $G$  are not directed, we call them space-like edges. Thus, we get a causal set where all slices will be maximal acausal subsets, that is not belonging to any other acausal set.

For such examples one can define the future light cone  $FL(v)$  of a point  $v$ .  $FL(v)$  is the set of points  $v'$  such that  $v' \in F(v)$  and there is a point  $v''$ , which does not belong to  $F(v)$  but is connected with  $v'$  by a space-like edge of  $G$ .

**1.2. Planar Lorentzian Model.** Consider triangulations  $T$  of the strip  $S^1 \times [M, N]$ , where  $S^1$  is a circle,  $M < N$  are integers. Assume the following properties of  $T$ : each triangle belongs to some strip  $S^1 \times [j, j + 1]$ ,  $j = M, \dots, N - 1$ , and has all vertices and exactly one edge on the boundary  $(S^1 \times \{j\}) \cup (S^1 \times \{j + 1\})$  of the strip  $S^1 \times [j, j + 1]$ . Let  $k_j = k_j(T)$  be the number of edges on  $S^1 \times \{j\}$ . We assume  $k_j \geq 1$ . Then the number of triangles  $F = F(T)$  of  $T$  is equal to

$$F = 2 \sum_{j=M+1}^{N-1} k_j + k_M + k_N. \tag{1}$$

Assume that  $k_M = k$  and  $k_N = l$  are fixed. For technical reasons it will be convenient to consider triangulations with a fixed vertex  $v_0$  (the origin) on the slice  $S^1 \times \{M\}$ . We define combinatorial triangulations (further we call them triangulations) as equivalence classes of triangulations under the homeomorphisms  $\phi$  of  $S^1 \times [M, N]$  which transform slices, the sets of vertices, edges, triangles and the origin into themselves.

Introduce the Gibbs measure on the (countable) set  $\mathcal{A}_{[M,N]}(k, l)$  of all such triangulations

$$\mu_{[M,N],k,l}(T) = Z_{[M,N]}^{-1} \exp(-\mu F(T)). \tag{2}$$

Define correlation functions for random variables  $k_j$ ,  $j \in [M + 1, N - 1]$ , taking values in  $\mathcal{N} = \{1, 2, \dots\}$ , and for finite subsets  $J \subset [M + 1, N - 1]$

$$\mu_{[M,N],k,l}(k_j = n_j, j \in J). \tag{3}$$

**1.3. Main Results.** We define the subcritical, critical or supercritical region if  $2 \exp(-\mu) < 1$ ,  $2 \exp(-\mu) = 1$  and  $2 \exp(-\mu) > 1$  respectively.

In the subcritical region we get a limiting probability measure on  $\mathcal{N}^{\mathcal{Z}}$ .

**Theorem 1.** *If  $2 \exp(-\mu) < 1$  then the limiting correlation functions*

$$\lim_{N \rightarrow \infty} \mu_{[-N,N],k,l}(k_j = k_j^0, j \in J)$$

*exist for any finite subset  $J \subset \mathcal{Z}$  and any vector  $(k_j^0, j \in J)$ . The measure defined by these correlation functions is a stationary ergodic Markov chain on  $\mathcal{N}$  with the following stationary probabilities:*

$$\pi(n) = (1 - (\lambda_2(s))^2)^2 n (\lambda_2(s))^{2(n-1)},$$

where

$$\lambda_2(s) = \frac{1 - \sqrt{1 - s^2}}{s}, \quad s = 2 \exp(-\mu).$$

**Theorem 2.** *In the subcritical case, the asymptotics of the partition function is*

$$Z_{[-N,N]}(k, l) \sim k(1 - (\lambda_2(s))^2)^2 (\lambda_2(s))^{k+l-2-4N}.$$

We see that the latter expression is not symmetric with respect to  $k$  and  $l$ . This is because of the specified vertex on the  $(-N)$ -slice. Without this specified vertex, one should divide by  $k$ , after that the symmetry is restored.

**Theorem 3.** *In the critical case, the asymptotics of the partition function is*

$$Z_{[-N,N]}(k, l) \sim \frac{k}{4} N^{-2}, \quad \sum_{l=1}^{\infty} Z_{[-N,N]}(k, l) \sim \frac{k}{2} N^{-1}.$$

*For any  $J$  and any  $k_j^0$ , the correlation functions behave as*

$$\lim_{N \rightarrow \infty} \mu_{[-N,N],k,l}(k_j = k_j^0, j \in J) = 0.$$

In the critical case, there is no limiting measure but one can interpret the partition function as follows. Consider a Galton—Watson branching process with one particle type and the distribution of offsprings  $p_m = (1/2)^{m+1}$ ,  $m \geq 0$ . This branching process is critical. Let  $\eta_t$  be the number of particles in the branching process

of  $v$ . Then we add an edge connecting the vertex  $v$  to the vertex which is on the rightmost edge of the vertex  $u$  (see Fig. 2).

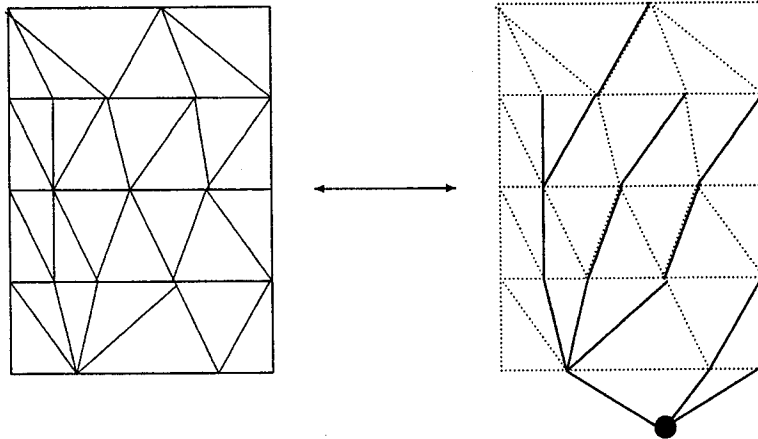


FIGURE 2. Tree parameterization

Let the tree  $Tr_N(k, l)$  correspond to the triangulation  $T_N(k, l)$  and  $D(Tr_N(k, l))$  be the number of vertices in the tree, except the root. Then there is a simple connection between the number of triangles  $F(T_N(k, l))$  in the triangulation  $T_N(k, l)$  and number of vertices  $D(Tr_N(k, l))$ :

$$F(T_N(k, l)) + k + l = 2D(Tr_N(k, l)). \tag{5}$$

Now the trick will be to establish a one-to-one correspondence between the trees and the trajectories of the simple random walk on the interval  $[0, N + 1]$  with reflecting barriers. Let  $\xi_t^{(N)}$  be a simple random walk with reflecting barriers at the points 0 and  $N + 1$ . More precisely, consider the random walk with the transition probabilities

$$P(\xi_{t+1}^{(N)} = n + 1 \mid \xi_t^{(N)} = n) = P(\xi_{t+1}^{(N)} = n - 1 \mid \xi_t^{(N)} = n) = \frac{1}{2},$$

if  $0 < n < N + 1$  and

$$P(\xi_{t+1}^{(N)} = 1 \mid \xi_t^{(N)} = 0) = P(\xi_{t+1}^{(N)} = N \mid \xi_t^{(N)} = N + 1) = 1.$$

Let  $\mathcal{TJ}_N(k, l; t)$  be the set of all trajectories of the length  $t$  which start at 0, reach 0 at the time  $t$ , and hit the bottom (top) barrier  $k(l)$  times.

It is well-known that there exists a one-to-one correspondence between the set  $\mathcal{TR}_N(k, l)$  and the set  $\mathcal{TJ}_N(k, l) = \bigcup_t \mathcal{TJ}_N(k, l; t)$ . Indeed, following the contour of the tree from the lower left edge, one can translate the sequence of ascents and descents into a trajectory of the random walk (this is shown on Fig. 3).

So, we have a one-to-one correspondence between  $\mathcal{TR}_N(k, l)$  and  $\mathcal{TJ}_N(k, l)$ . If the trajectory  $T_{jN}(k, l)$  corresponds to some triangulation  $T_N(k, l)$ , then the length of the trajectory  $t = F(T_N(k, l)) + k + l$ .

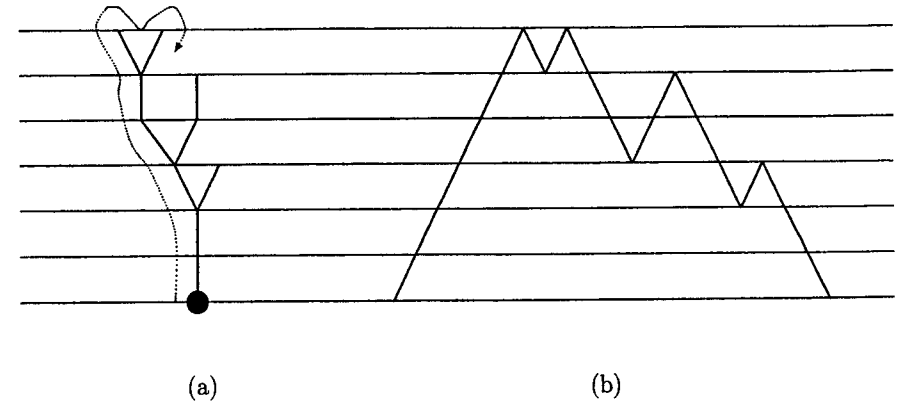


FIGURE 3. Random walk parameterization

Let  $P^{(N)}(t, k, l)$  be the probability for the random walk  $\xi_t^{(N)}$  starting at 0 to reach 0 in  $t$  steps hitting  $k$  times the bottom barrier and  $l$  times the top barrier.

**Lemma 6.** *The partition function can be written as follows:*

$$Z_{[0, N]}(k, l) = \frac{1}{(2 \exp(-\mu))^{k+l}} \sum_t (2 \exp(-\mu))^t P^{(N)}(t, k, l). \tag{6}$$

*Proof.* Thanks to the one-to-one correspondence between the sets  $\mathcal{TR}_N(k, l)$  and  $\mathcal{TJ}_N(k, l)$ , we have

$$\begin{aligned} Z_{[0, N]}(k, l) &= \sum_{T \in \mathcal{TR}_N(k, l)} (\exp(-\mu))^{F(T)} \\ &= \sum_{T_j \in \mathcal{TJ}_N(k, l)} (\exp(-\mu))^{t(T_j) - k - l}. \end{aligned}$$

Since the probability of trajectory with  $k + l$  reflections is  $(1/2)^{t-k-l}$ , we find

$$\begin{aligned} Z_{[0, N]}(k, l) &= \sum_{t=1}^{\infty} \sum_{\substack{T_j \in \mathcal{TJ}_N(k, l): \\ t(T_j)=t}} (\exp(-\mu))^{t-k-l} \\ &= \sum_{t=1}^{\infty} (2 \exp(-\mu))^{t-k-l} \sum_{\substack{T_j \in \mathcal{TJ}_N(k, l): \\ t(T_j)=t}} (1/2)^{t-k-l} \\ &= \sum_{t=1}^{\infty} (2 \exp(-\mu))^{t-k-l} P^{(N)}(t, k, l). \end{aligned}$$

The lemma is proved. □

**2.2. Partition Function and Transfer-matrix.** In the Hilbert space  $l_2(\mathcal{N})$  the partition function can be represented as follows:

$$Z_{[0,N]}(k, l) = \sum_{T \in \mathcal{A}_N(k,l)} \exp(-\mu F(T)) = (e_k, U^N e_l), \tag{7}$$

where  $e_k$  is the natural orthonormal basis in  $l_2(\mathcal{N})$ , and the elements of the transfer-matrix  $U$  are given by

$$u(n, m) = \frac{(n+m-1)!}{(n-1)!m!} \exp(-\mu(n+m)), \tag{8}$$

and  $n, m \geq 1$ . Indeed,

$$\begin{aligned} Z_{[0,N]}(k, l) &= \sum_{T \in \mathcal{A}_N(k,l)} \exp(-\mu F(T)) \\ &= \sum_{k_1, \dots, k_{N-1}} C_N(k, k_1, \dots, k_{N-1}, l) \exp\left(-\mu\left(k+l+\sum_{i=1}^{N-1} k_i\right)\right), \end{aligned} \tag{9}$$

where  $C_N(k, k_1, \dots, k_{N-1}, l)$  is the number of triangulations with  $M$  triangles,  $N$  strips,  $k$  points on the 0-slice and  $l$  points on the  $N$ -slice and  $k_i$  points on the  $i$ -slices. But

$$C_N(k, k_1, \dots, k_{N-1}, l) = c(k, k_1)c(k_1, k_2) \dots c(k_{N-1}, l), \tag{10}$$

where  $c(k_i, k_{i+1})$  is the number of triangulations of the strip  $S^1 \times [i, i+1]$ . Note that  $c(k_i, k_{i+1})$  is equal to the number of all arrangements of "up" and "down" triangles such that the first triangle is "up". Since the total number of triangles in the strip  $[i, i+1]$  is  $k_i + k_{i+1}$ , we have

$$c(k_i, k_{i+1}) = \frac{(k_i + k_{i+1} - 1)!}{(k_i - 1)!k_{i+1}!}. \tag{11}$$

Taking into account (9)–(11) we obtain (7).

We now find an explicit expression for the partition function.

Let  $P^{(N)}(t, l)$  be the probability for the random walk starting at 0 to reach 0 in  $t$  steps at the first time hitting the top barrier  $l$  times. Then

$$P^{(N)}(t, k, l) = \sum_{l_1 + \dots + l_k = l} \sum_{t_1 + \dots + t_k = t} P^{(N)}(t_1, l_1) \dots P^{(N)}(t_k, l_k). \tag{12}$$

Coming to the generating functions we have

$$F(s, k, l) = \sum_{l_1 + \dots + l_k = l} \tilde{F}(s, l_1) \dots \tilde{F}(s, l_k). \tag{13}$$

Let

$$\begin{aligned} \alpha_t^{(N+1)} &= P\left(\xi_t^{(N)} = 0, 0 < \xi_s^{(N)} < N+1, 0 < s < t \mid \xi_0^{(N)} = 1\right) \\ &= P\left(\xi_t^{(N)} = N+1, 0 < \xi_s^{(N)} < N+1, 0 < s < t \mid \xi_0^{(N)} = N\right) \end{aligned}$$

and

$$\begin{aligned} \beta_t^{(N+1)} &= P\left(\xi_t^{(N)} = N+1, 0 < \xi_s^{(N)} < N+1, 0 < s < t \mid \xi_0^{(N)} = 1\right) \\ &= P\left(\xi_t^{(N)} = 0, 0 < \xi_s^{(N)} < N+1, 0 < s < t \mid \xi_0^{(N)} = N\right). \end{aligned}$$

Define the corresponding generating functions

$$f_{N+1}(s) = \sum s^t \alpha_t^{(N+1)}, \quad g_{N+1}(s) = \sum s^t \beta_t^{(N+1)}.$$

It is well known that

$$f_{N+1}(s) = \frac{\lambda_1^N(s) - \lambda_2^N(s)}{\lambda_1^{N+1}(s) - \lambda_2^{N+1}(s)} \tag{14}$$

and

$$g_{N+1}(s) = \frac{\lambda_1(s) - \lambda_2(s)}{\lambda_1^{N+1}(s) - \lambda_2^{N+1}(s)}, \tag{15}$$

where

$$\lambda_{1,2}(s) = \frac{1 \pm \sqrt{1-s^2}}{s}, \quad s = 2 \exp(-\mu), \tag{16}$$

are the roots of the quadratic equation

$$s\lambda^2(s) - 2\lambda(s) + s = 0.$$

**Theorem 7.** *The partition function has the form*

$$Z_{[0,N]}(k, l) = \sum_{r=1}^{\min(k,l)} B(r, k, l) f_{N+1}^{l+k-2r}(s) g_{N+1}^{2r}(s), \tag{17}$$

where

$$B(r, k, l) = \binom{k}{r} \binom{l-1}{r-1}.$$

*Proof.* If  $l > 0$ ,

$$\tilde{F}(s, l) = s^{l+1} g_{N+1}^2(s) f_{N+1}^{l-1}(s),$$

and if  $l = 0$ ,

$$\tilde{F}(s, 0) = s f_{N+1}(s).$$

Inserting these expressions into (13), we find

$$F(s, k, l) = s^{k+l} \sum_{r=1}^{\min(k,l)} B(r, k, l) f_{N+1}^{l+k-2r}(s) g_{N+1}^{2r}(s),$$

where  $r$  is the number of nonzero  $l_i$  and  $B(r, k, l)$  is the number of all solutions to the equation  $l_1 + \dots + l_k = l$  such that the number of nonzero  $l_i$  equals  $r$ .  $\square$

**Corollary 8.** *If  $s = 2 \exp(-\mu) < 1$ , then, as  $N \rightarrow \infty$ ,*

$$Z_{[0,N]}(k, l) \sim k(1 - (\lambda_2(s))^2)^2 (\lambda_2(s))^{k+l-2+2N}.$$

*If  $s = 2 \exp(-\mu) = 1$ , then, as  $N \rightarrow \infty$ ,*

$$Z_{[0,N]}(k, l) \sim kN^{-2}.$$

**2.3. Subcritical case.** Put  $J = \{0, 1, \dots, i\}$ . By definition, we have

$$\mu_{[-N,N],k,l}(n_0, n_1, \dots, n_i) = \frac{Z_{[0,N]}(k, n_0) u(n_0, n_1) \dots u(n_{i-1}, n_i) Z_{[i,N]}(n_i, l)}{Z_{[-N,N]}(k, l)}$$

It follows from (17) that as  $N \rightarrow \infty$ ,

$$Z_{[-N,N]}(k, l) \sim k(1 - (\lambda_2(s))^2)^2 (\lambda_2(s))^{k+l-2+4N}.$$

Hence, the correlation functions  $\mu_{[-N,N],k,l}(n_0, n_1, \dots, n_i)$  are asymptotically equivalent to

$$\frac{k(1 - (\lambda_2(s))^2)^4 (\lambda_2(s))^{k+n_0-2+2N} u(n_0, n_1) \dots u(n_{i-1}, n_i) n_i (\lambda_2(s))^{n_i+l-2+2(N-i)}}{k(1 - (\lambda_2(s))^2)^2 (\lambda_2(s))^{k+l-2+4N}}.$$

It follows that as  $N \rightarrow \infty$ ,

$$\begin{aligned} \mu_{[-N,N],k,l}(n_0, n_1, \dots, n_i) \rightarrow \\ (1 - (\lambda_2(s))^2)^2 (\lambda_2(s))^{n_0-1} \frac{u(n_0, n_1)}{(\lambda_2(s))^2} \dots \frac{u(n_{i-1}, n_i)}{(\lambda_2(s))^2} n_i (\lambda_2(s))^{n_i-1}, \end{aligned}$$

where  $s = 2 \exp(-\mu) < 1$ .

Thus, in the limit we have an ergodic Markov chain on  $\mathcal{N}$  with the transition matrix

$$P = (p_{n_{i-1}, n_i}) = \left( \frac{u(n_{i-1}, n_i) n_i}{(\lambda_2(s))^{2n_{i-1}}} (\lambda_2(s))^{n_i - n_{i-1}} \right)$$

and the stationary distribution

$$\pi(n) = (1 - (\lambda_2(s))^2)^2 n (\lambda_2(s))^{2(n-1)}.$$

It is useful to know that

**Corollary 9.** *The transfer matrix  $U$  has the right eigenvector with components  $(\lambda_2(s))^{n-1}$  and the left eigenvector with components  $n(\lambda_2(s))^{n-1}$ . The corresponding eigenvalue equals  $(\lambda_2(s))^2$ .*

Now we calculate the mean drift for the limiting Markov chain.

$$\begin{aligned} M(n) &= \sum_{k=1}^{\infty} p_{nk}(k-n) = \sum_{k=1}^{\infty} \frac{u(n, k)k}{n} (\lambda_2(s))^{k-n-2}(k-n) \\ &= \sum_{k=1}^{\infty} \frac{(n+k-1)!}{n!(k-1)!} \exp(-\mu(n+k)) (\lambda_2(s))^{k-n-2}(k-n) \\ &= \exp(-\mu(n+2)) (\lambda_2(s))^{-n} \sum_{k=2}^{\infty} \frac{(n+k-1)!}{n!(k-1)!} \\ &\quad \times \exp(-\mu(k-2)) (\lambda_2(s))^{k-2}(k-1) + 1 - n. \end{aligned}$$

But

$$\begin{aligned} \sum_{k=2}^{\infty} \frac{(n+k-1)!}{n!(k-1)!} \exp(-\mu(k-2)) (\lambda_2(s))^{k-2}(k-1) \\ = \frac{d}{dx} \left( \sum_{k=2}^{\infty} \frac{(n+k-1)!}{n!(k-1)!} x^{k-1} \right)_{x=\exp(-\mu)\lambda_2(s)} = \frac{n+1}{(1 - \exp(-\mu)(\lambda_2(s)))^{n+2}}, \end{aligned}$$

because of

$$\sum_{k=1}^{\infty} \frac{(n+k-1)!}{n!(k-1)!} x^{k-1} = \frac{1}{(1-x)^{n+1}}.$$

Using this fact we come to the following expression:

$$M(n) = \exp(-\mu(n+2)) (\lambda_2(s))^{-n} \frac{n+1}{(1 - \exp(-\mu)(\lambda_2(s)))^{n+2}} + 1 - n.$$

Since

$$\frac{1}{(1 - \exp(-\mu)(\lambda_2(s)))^{n+2}} = \exp(\mu(n+2)) (\lambda_2(s))^{n+2},$$

we get

$$M(n) = -n(1 - \lambda_2^2(s)) + 1 + \lambda_2^2(s). \tag{18}$$

**2.4. Critical case.** In the critical region we do not have a stationary (translation invariant) probability measure for  $k_j$ , but instead we can get some scaling limit. Consider the partition function in the critical case. It follows from (14), (15) that

$$g_{N+1}(1) = \frac{1}{N+1}, \quad f_{N+1}(1) = \frac{N}{N+1}. \tag{19}$$

For some  $\alpha \in [0, 1]$  consider

$$LC_N(\alpha; k, l) = \frac{\sum_{n=1}^{\infty} n Z_{[0,\alpha N]}(k, n) Z_{[\alpha N, N]}(n, l)}{Z_{[0, N]}(k, l)}.$$

Denote

$$\alpha' = 1 - \alpha, \quad N_1 = \frac{\alpha N}{\alpha N + 1}, \quad N_2 = \frac{\alpha' N}{\alpha' N + 1}.$$

By (17) and (19) we have

$$\frac{LC_N(\alpha; k, l)}{N} = \frac{A}{B}, \tag{20}$$

where

$$B = N \sum_{r=1}^{\min(k,l)} B(r, k, l) N^{-2r} \left( \frac{N}{N+1} \right)^{k+l}$$

and

$$\begin{aligned}
 A &= \sum_{n=1}^{\infty} n \sum_{r_1=1}^{\min(k,n)} B(r_1, k, n) (\alpha N)^{-2r_1} N_1^{k+n} \sum_{r_2=1}^{\min(n,l)} B(r_2, n, l) (\alpha' N)^{-2r_2} N_2^{n+l} \\
 &= \sum_{n=1}^{\infty} \sum_{r_1=1}^{\min(k,n)} \sum_{r_2=1}^{\min(n,l)} \frac{\binom{k}{r_1} \binom{l-1}{r_2-1} N_1^{k+r_1} N_2^{l+r_2}}{(r_1-1)! r_2! (\alpha N)^{2r_1} (\alpha' N)^{2r_2}} \frac{\partial^{r_1+r_2}}{\partial x^{r_1} \partial y^{r_2}} (xy)^n \Big|_{\substack{x=N_1 \\ y=N_2}} \\
 &= \sum_{r_1=1}^k \sum_{r_2=1}^l \frac{\binom{k}{r_1} \binom{l-1}{r_2-1} N_1^{k+r_1} N_2^{l+r_2}}{(r_1-1)! r_2! (\alpha N)^{2r_1} (\alpha' N)^{2r_2}} \frac{\partial^{r_1+r_2}}{\partial x^{r_1} \partial y^{r_2}} \left( \frac{1}{1-xy} \right) \Big|_{\substack{x=N_1 \\ y=N_2}}. \tag{21}
 \end{aligned}$$

We split

$$\frac{LC_N(\alpha; k, l)}{N} = \frac{LC_N^{(1)}(\alpha; k, l)}{N} + \frac{LC_N^{(2)}(\alpha; k, l)}{N}, \tag{22}$$

where

$$\frac{LC_N^{(1)}}{N} = \frac{k N_1^{k+1} N_2^{l+1} (\alpha N)^{-2} (\alpha' N)^{-2} \frac{\partial^2}{\partial x \partial y} \left( \frac{1}{1-xy} \right) \Big|_{\substack{x=N_1 \\ y=N_2}}}{N \sum_{r=1}^{\min(k,l)} B(r, k, l) N^{-2r} \left( \frac{N}{N+1} \right)^{k+l}}, \tag{23}$$

$$\frac{LC_N^{(2)}}{N} = \frac{\sum_{r_1+r_2>2} \frac{\binom{k}{r_1} \binom{l-1}{r_2-1} N_1^{k+r_1} N_2^{l+r_2}}{(r_1-1)! r_2! (\alpha N)^{2r_1} (\alpha' N)^{2r_2}} \frac{\partial^{r_1+r_2}}{\partial x^{r_1} \partial y^{r_2}} \left( \frac{1}{1-xy} \right) \Big|_{\substack{x=N_1 \\ y=N_2}}}{N \sum_{r=1}^{\min(k,l)} \frac{B(r, k, l)}{N^{2r}} \left( \frac{N}{N+1} \right)^{k+l}}. \tag{24}$$

Since for any positive  $r_1$  and  $r_2$

$$\frac{\partial^{r_1+r_2}}{\partial x^{r_1} \partial y^{r_2}} \left( \frac{1}{1-xy} \right) \Big|_{\substack{x=N_1 \\ y=N_2}} = c N^{r_1+r_2+1}, \tag{25}$$

where  $c$  depends on  $r_1, r_2$  and  $\alpha$ . We have

$$\frac{LC_N^{(1)}}{N} = \frac{k \frac{N_1^{k+1} N_2^{l+1}}{(\alpha N)^2 (\alpha' N)^2} \frac{\partial^2}{\partial x \partial y} \left( \frac{xy}{(1-xy)^3} \right) \Big|_{\substack{x=N_1 \\ y=N_2}}}{\frac{1}{N} \left( k + o\left(\frac{1}{N}\right) \right)} \xrightarrow{N \rightarrow \infty} \alpha(1-\alpha).$$

For any  $r_1, r_2$  such that  $r_1 + r_2 > 2$ , we have

$$N \frac{N_1^{k+r_1} N_2^{l+r_2}}{(\alpha N)^{2r_1} (\alpha' N)^{2r_2}} \frac{\partial^{r_1+r_2}}{\partial x^{r_1} \partial y^{r_2}} \left( \frac{1}{1-xy} \right) \Big|_{\substack{x=N_1 \\ y=N_2}} \xrightarrow{N \rightarrow \infty} 0.$$

So

$$\frac{LC_N^{(2)}(\alpha; k, l)}{N} \xrightarrow{N \rightarrow \infty} 0. \tag{26}$$

The first assertion of Theorem 4 is proved.

To prove the second assertion of the theorem the following interpretation will be essential. Consider a discrete time branching process with the distribution of offsprings  $p_m = (1/2)^{m+1}$ ,  $m \geq 0$ . Let  $\eta_N$  be the number of offsprings at time  $N$ .

**Lemma 10.** *The partition function*

$$Z_{[0,N]}(k, l) = P(\eta_N = l \mid \eta_0 = k),$$

for all  $k, l > 0$ .

*Proof.* It is well known that the process  $\eta_N$  is a Markov chain with the transition probabilities

$$p(n, m) = \frac{(n+m-1)!}{m!(n-1)!} \left( \frac{1}{2} \right)^{m+n}.$$

In the critical case, it coincides with elements of the transfer matrix given by (8). The lemma is proved.  $\square$

By Lemma 10, the fraction

$$\frac{LC_N(\alpha; k)}{N} = \frac{\sum_{n=1}^{\infty} \sum_{l=1}^{\infty} n Z_{\alpha N}(k, n) Z_{(1-\alpha)N}(n, l)}{N \sum_{l=1}^{\infty} Z_N(k, l)},$$

can be rewritten as

$$\frac{LC_N(\alpha; k)}{N} = \frac{\sum_{n=1}^{\infty} n P(\eta_{\alpha N} = n \mid \eta_0 = k) \sum_{l=1}^{\infty} P(\eta_{(1-\alpha)N} = l \mid \eta_0 = n)}{N \sum_{l=1}^{\infty} P(\eta_N = l \mid \eta_0 = k)}. \tag{27}$$

It is well known for branching processes  $\eta_N$  that for any  $k, N$ ,

$$\begin{aligned}
 \sum_{l=1}^{\infty} P(\eta_N = l \mid \eta_0 = k) &= P(\eta_N \neq 0 \mid \eta_0 = k) \\
 &= 1 - (P(\eta_N = 0 \mid \eta_0 = 1))^k = 1 - \left( \frac{N}{N+1} \right)^k. \tag{28}
 \end{aligned}$$

Substituting (28) to (27) we get

$$\frac{LC_N(\alpha; k)}{N} = \frac{E(\eta_{\alpha N} \mid \eta_0 = k) - \sum_{n=1}^{\infty} n P(\eta_{\alpha N} = n \mid \eta_0 = k) N_2^n}{N \left( 1 - \left( \frac{N}{N+1} \right)^k \right)}.$$

We have

$$\begin{aligned}
 \sum_{n=1}^{\infty} n P(\eta_{\alpha N} = n \mid \eta_0 = k) N_2^n &= \sum_{n=1}^{\infty} n \sum_{r=1}^{\min(k,n)} \binom{k}{r} \binom{n-1}{r-1} \frac{1}{(\alpha N)^{2r}} N_1^{k+n} N_2^n \\
 &= N_1^k \sum_{r=1}^k \binom{k}{r} \frac{(N_1 N_2)^r}{(\alpha N)^{2r} (r-1)!} \frac{\partial^r}{\partial x^r} \left( \frac{1}{1-x} \right) \Big|_{x=N_1 N_2} \\
 &= \frac{N_1^{k+1} N_2}{(\alpha N)^2} \frac{k}{(1-N_1 N_2)^2} + N_1^k \sum_{r=2}^k \binom{k}{r} \frac{(N_1 N_2)^r}{(\alpha N)^{2r} (r-1)!} \frac{\partial^r}{\partial x^r} \left( \frac{1}{1-x} \right) \Big|_{x=N_1 N_2}
 \end{aligned}$$

Note that

$$\frac{\partial^r}{\partial x^r} \left( \frac{1}{1-x} \right) \Big|_{x=N_1 N_2} \sim c N^{1+r},$$

where  $c$  depends on  $r$  and  $\alpha$ . Hence

$$N_1^k \sum_{r=2}^k \binom{k}{r} \frac{(N_1 N_2)^r}{(\alpha N)^{2r} (r-1)!} \frac{\partial^r}{\partial x^r} \left( \frac{1}{1-x} \right) \Big|_{x=N_1 N_2} = O\left(\frac{1}{N}\right).$$

Finally

$$\frac{LC_N(\alpha; k)}{N} = \frac{k \left( 1 - \frac{N_1^{k+1} N_2}{(\alpha N)^2 (1 - N_1 N_2)^2} + O\left(\frac{1}{N}\right) \right)}{kN \left( \frac{1}{N+1} + o\left(\frac{1}{N}\right) \right)} \xrightarrow{N \rightarrow \infty} 1 - (1 - \alpha)^2.$$

Theorem 4 is proved.

*Remark 11 (Local distribution).* Consider the measure  $\mu_{[0, N], k, l}$ . In the critical case, we do not have a probability measure for  $k_j$  but instead we have a limiting probability measure for local variables. Then, for example, it can be shown that the probability  $\pi_{j(N)}(i, d)$  that the point on the slice  $j(N) \rightarrow \infty$  so that  $\frac{j(N)}{N} \rightarrow 0$ , and on the distance  $d$  from  $v_{j(N)}$ , has  $i$  outgoing links tends to  $(\frac{1}{2})^{i+1}$  for any  $d$ . One could define some unitary operator for this local distribution, but its physical sense is not clear.

**2.5. Supercritical case.** Here we prove Proposition 5. In the supercritical case  $s = 2 \exp(-\mu) > 1$ . Consider the functions  $f_{N+1}(s), g_{N+1}(s)$  given by (14), (15). Using the binomial formula and reducing the algebraic singularity  $\sqrt{1-s^2}$ , one can obtain that the functions  $f_{N+1}(s), g_{N+1}(s)$  are rational with a common denominator. It is well known that all roots of a polynomial in the denominator are real and the minimal root is given by:

$$s_1(N) = \frac{1}{\cos \frac{\pi}{N+1}}.$$

Note that the minimal root  $s_1(N)$  tends to 1 as  $N \rightarrow \infty$ . So  $Z_{[0, N]}(k, l)$  (as a function of  $\mu$ ) exists only if

$$2 \exp(-\mu) < s_1(N),$$

that is

$$\mu > \ln \left( 2 \cos \frac{\pi}{N+1} \right).$$

Note that

$$\ln \left( 2 \cos \frac{\pi}{N+1} \right) \xrightarrow{N \rightarrow \infty} \ln 2.$$

### 3. HAMILTONIAN APPROACH

**3.1. The Lorentzian Model as a Grammar.** In this section we identify some quantum grammar with the 2-dim unitary Lorentzian Model. Here triangulations will be created by quantum dynamics.

Let  $\Sigma = \{1, \dots, r\}$  be a finite set (the alphabet),  $\Sigma^*$  — the set of all finite words (including the empty one)  $\alpha = x_1 \dots x_n, x_i \in \Sigma$ , in this alphabet. The length  $n$

of the word  $\alpha$  is denoted by  $|\alpha|$ . Concatenation of two words  $\alpha = x_1 \dots x_n$  and  $\beta = y_1 \dots y_m$  is defined by

$$\alpha\beta = x_1 \dots x_n y_1 \dots y_m. \tag{29}$$

The word  $\beta$  is a factor of  $\alpha$  if there exist words  $\delta$  and  $\gamma$  such that  $\alpha = \delta\beta\gamma$ . A grammar over  $\Sigma$  is defined by a finite set  $Sub$  of substitutions (productions), that is the pairs  $\delta_i \rightarrow \gamma_i, i = 1, \dots, k = |Sub|, \delta_i, \gamma_i \in \Sigma^*$ . Further on we assume that all  $\delta_i, \gamma_i$  are not empty. The grammar is context-free if  $|\delta_i| = 1$  for all  $i = 1, \dots, k$ .

Let  $\mathcal{H} = l_2(\Sigma^*)$  be the Hilbert space with the orthonormal basis  $e_\alpha, \alpha \in \Sigma^*$ :  $(e_\alpha, e_\beta) = \delta_{\alpha\beta}$ , where the function  $e_\alpha(\beta) = \delta_{\alpha\beta}$ . Each vector  $\phi$  of  $\mathcal{H}$  is a function on the set of words and can be written as

$$\phi = \sum \phi(\alpha) e_\alpha \in \mathcal{H}, \quad \|\phi\|^2 = \sum |\phi(\alpha)|^2. \tag{30}$$

The states of the system are wave functions, that is vectors  $\phi$  with the unit norm  $\|\phi\|^2 = 1$ .

Consider one symbol alphabet  $S = \{a\}$  and the following substitutions  $a \rightarrow aa, aa \rightarrow a$ , with intensities  $\lambda(a \rightarrow aa) = \lambda(aa \rightarrow a) = \lambda$ . Define the substitution operators  $a_+(j)$  and  $a_-(j)$  which act as follows:

$$a_+(j)e_n = e_{n+1}, \quad j = 1, \dots, n, \quad a_+(j)e_n = 0, \quad j > n, \tag{31}$$

$$a_-(j)e_n = e_{n-1}, \quad j = 1, \dots, n-1, \quad a_-(j)e_n = 0, \quad j \geq n, \tag{32}$$

where  $e_n$  is an orthonormal basis of the Hilbert space  $l_2$ . Define the formal Hamiltonian by

$$H = \lambda \sum_{j=1}^{\infty} (a_+(j) + a_-(j)). \tag{33}$$

More exactly,  $H$  is essentially selfadjoint on the set  $D$  of finite linear combinations of  $e_n$ , see [8].

The word  $a^n = aa \dots a$  at time 0 is identified with a partition of the slice  $S^1 \times \{0\}$  of the cylinder  $S^1 \times [0, \infty]$ . This will be called the initial boundary. Another boundary (called frontal) will change at random time moments. The slice  $S^1 \times \{0\}$  (the boundary at time 0) is subdivided into  $n$  intervals, the length being irrelevant. One interval corresponds to one symbol  $a$  of the word  $a^n$ , a space quantum. Each substitution corresponds to appending a new triangle:  $a \rightarrow aa$  corresponds to constructing a triangle having one side — the interval, corresponding to  $a$ , two other sides are new space quanta. The substitution  $aa \rightarrow a$  constructs a triangle with two sides corresponding to  $aa$  and the third side over those two. See Fig. 4.

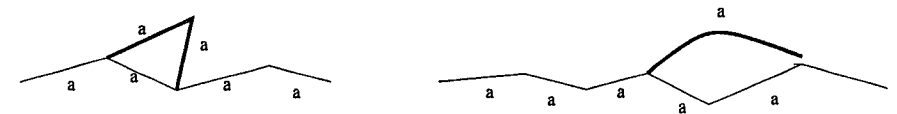


FIGURE 4. Lorentzian moves

3.1.1. *Generalized eigenfunctions.* Let us define the Hardy space  $\mathcal{H}^2$  as the set of analytic functions  $f(z)$  for  $|z| < 1$  such that the integrals

$$\int_0^{2\pi} |f(re^{i\phi})|^2 d\phi \leq C_f \tag{34}$$

are uniformly bounded for  $0 \leq r < 1$ , we refer here to the paper [15], where the spectrum of the operator  $H$  is obtained as well. For the Hardy space  $\mathcal{H}^2$  one can define the scalar product as

$$(f, g) = \lim_{r \rightarrow 1-0} \frac{1}{2\pi} \int_0^{2\pi} \overline{f(re^{i\phi})} g(re^{i\phi}) d\phi. \tag{35}$$

This Hilbert space is isomorphic to  $l_2(Z_+)$ . Introducing the generating functions

$$f(z) = \sum_{n=1}^{\infty} z^n f_n, \tag{36}$$

for vectors  $\{f_n, n \geq 1\}$  belonging to  $l_2$ , one can get the Hamiltonian in the following form:

$$Hf = \lambda_- z \left( \frac{f(z)}{z} \right)' + \lambda_+ z^2 (f(z))' \tag{37}$$

which acts in the Hardy space  $\mathcal{H}^2$ . Put  $f(z) = zp(z)$ . Then the equation  $Hf = \nu f$  is equivalent to

$$\lambda_- p(z)' + \lambda_+ z (zp(z))' = \nu p(z). \tag{38}$$

Solving this equation, we find the generalized eigenfunctions

$$p_\nu(z) = \frac{\exp\left(\frac{\nu}{\sqrt{\lambda_+ \lambda_-}} \arctan\left(\sqrt{\frac{\lambda_+}{\lambda_-}} z\right)\right)}{\sqrt{1 + \frac{\lambda_+}{\lambda_-} x^2}}. \tag{39}$$

3.1.2. *Time evolution of the mean length.* The action of the Hamiltonian  $H$  can be written in the form:

$$\begin{cases} (Hf)_1 = \lambda_- f_2, \\ (Hf)_n = \lambda_+(n-1)f_{n-1} + \lambda_- n f_{n+1}, \quad n \geq 2, \end{cases}$$

where  $f = \{f_n\} \in l_2(Z_+)$ .

Let  $f(t) = e^{itH} f(0)$ , where  $f_1(0) = 1$  and  $f_n(0) = 0$  for  $n \geq 2$ . Then  $f(t) = \{f_n(t), n \geq 1\}$  satisfies the following system of differential equations:

$$\begin{cases} \frac{df_1(t)}{dt} = i\lambda_- f_2(t), \\ \frac{df_n(t)}{dt} = i(\lambda_+(n-1)f_{n-1}(t) + \lambda_- n f_{n+1}(t)), \quad n \geq 2, \end{cases} \tag{40}$$

with the initial condition  $f_1(0) = 1, f_n(0) = 0, n \geq 2$ . We will seek a solution to (40) in the form  $f_n(t) = a(t)(b(t))^{n-1}$ . After the substitution we come to the

following system:

$$\begin{cases} \frac{\dot{a}(t)}{a(t)} = i\lambda_- b(t), \\ \frac{\dot{a}(t)}{a(t)} b(t) + (n-1)\dot{b}(t) = i(\lambda_+(n-1) + \lambda_- n b^2(t)), \\ a(0) = 1, b(0) = 0. \end{cases} \tag{41}$$

It follows that  $b(t)$  satisfies the equation

$$\dot{b}(t) = i(\lambda_+ + \lambda_- b^2(t)), \quad b(0) = 0.$$

The solution to this equation is

$$b(t) = \sqrt{\frac{\lambda_+}{\lambda_-}} \tan(i\sqrt{\lambda_+ \lambda_-} t) = i\sqrt{\frac{\lambda_+}{\lambda_-}} \tanh(\sqrt{\lambda_+ \lambda_-} t).$$

The function  $a(t)$  can be found from the equation:

$$\frac{\dot{a}(t)}{a(t)} = -\sqrt{\lambda_+ \lambda_-} \tanh(\sqrt{\lambda_+ \lambda_-} t).$$

Solving this equation, we find

$$a(t) = \frac{1}{\cosh(\sqrt{\lambda_+ \lambda_-} t)}.$$

So

$$f_n(t) = \frac{1}{\cosh(\sqrt{\lambda_+ \lambda_-} t)} \left( i\sqrt{\frac{\lambda_+}{\lambda_-}} \tanh(\sqrt{\lambda_+ \lambda_-} t) \right)^{n-1}.$$

We define the mean length of the quantum word at time  $t$  as

$$m(t) = \sum_{n=1}^{\infty} n |f_n(t)|^2.$$

**Proposition 12.** *If the initial condition has the form  $f_1(0) = 1, f_n(0) = 0, n > 1$ , then*

$$m(t) = \cosh^2(|\lambda|t). \tag{42}$$

*Proof.* If  $\lambda_+ = \bar{\lambda}_- = \lambda$  then

$$f_n(t) = \frac{1}{\cosh(|\lambda|t)} \left( i\frac{\lambda}{|\lambda|} \tanh(|\lambda|t) \right)^{n-1}.$$

Therefore

$$\begin{aligned} m(t) &= \sum_{n=1}^{\infty} n |f_n(t)|^2 = \frac{1}{\cosh^2(|\lambda|t)} \sum_{n=1}^{\infty} n (\tanh(|\lambda|t))^{2(n-1)} \\ &= \frac{1}{\cosh^2(|\lambda|t)} \frac{1}{(1 - \tanh^2(|\lambda|t))^2} = \cosh^2(|\lambda|t). \end{aligned}$$

We see that the volume of the Universe has nonlinear behaviour in  $t$ , thus the future cone does not have standard behavior.  $\square$



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