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Marginally closed processes with local interaction

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A simple probabilistic description of marginally closed locally interacting processes in discrete time is given. We find the invariant measures and prove the approach to equilibrium for a wide class of initial conditions.

1. Introduction

We consider discrete time Markov processes $\vec{\xi}_t = (\xi_t(x),\ x\in\mathbb{Z}^\nu)$, $t=0,1,\ldots$, with values in $S^{\mathbb{Z}^\nu}$ for a denumerable set S. The main assumptions are:

(1) local interaction (see Definition 1) which roughly means that $\xi_t(x)$ depends only on $\xi_{t-1}(x+y)$, $y\in Q$, for some fixed finite $Q\subset\mathbb{Z}^\nu$;

(2) conditional independence: $\xi_t(x)$, $x\in\mathbb{Z}^\nu$, are conditionally independent given $\vec{\xi}_{t-1}$;

(3) marginal closedness (see Definition 2). The meaning is that, if we write the finite dimensional distributions of order m of $\xi_t(x)$ for fixed t as linear combinations of finite dimensional distributions of $\xi_{t-1}(x)$, in a form similar to the BBGKY hierarchy (differential equations for the time evolution of correlation functions) in the physical literature, then there will appear only distributions of order $\leqslant m$.

It was remarked by Spitzer [16] that the simple exclusion process enjoys property (3). For the voter model this property was used in [2]. Property (3) for some processes was noted anew in [9], where the general class of processes with decoupled hierarchy

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of equations for correlation functions or decoupled moment hierarchy was first studied.

Here we continue that study and get a complete solution of some problems: necessary and sufficient conditions for marginal closedness and a simple probabilistic interpretation for it, as well as convergence of all correlation functions.

The main reason for a study of such processes is in the fact that there are very few examples of processes with local interactions admitting a sufficiently complete description. Among them there are small perturbations of independent processes, with finite [11, 13], compact [1, 18] or non-compact [8, 12] set of states, small perturbation of Gaussian processes [10], and also some where the low temperature region 'is controlled' [3, 17].

It is also very popular now [4, 5, 6] to study the hydrodynamical behaviour of processes of the type $(1-\varepsilon)L_0+\varepsilon L_1$, where L_0 is the stochastic operator for a marginally closed process and L_1 for a process which is not marginally closed. This process is usually studied at times of the order ε^{-1} where L_1 is applied 'a finite number of times'. The crucial property which permits one to treat processes of this kind is the marginal closedness of L_0. So it is of interest to study the general class of possible processes L_0.

We consider only processes in discrete time. Among conditionally independent processes, the voter model is the typical example. It appears to be degenerate, in some sense, but nevertheless, conditionally independent marginally closed processes have the same nature as the voter model. But non conditionally independent processes are richer: this class includes the simple exclusion model with several kinds of particles and 'chemical reactions' between them.

The main limitation of our study is the translation invariance of the initial distribution: however we study different types of conservation laws. Conservation laws govern the behaviour of the process with non translation invariant initial distribution (e.g., they govern the hydrodynamical behaviour [5, 14]).

To obtain the results, we used an exact 'path expansion' together with domination by a suitably chosen 'dual process'.

Results

In Section 3 the marginally closed conditionally independent processes are characterized. In Section 4 we examine those processes for dimension $\nu=1,2$ under any translation invariant initial distribution, study their ergodic behaviour and obtain the explicit formulas for the invariant measures. In the same section we also examine the cases where the convergence is exponential. We obtain sufficient conditions for this type of convergence in any dimension. In Section 5 we generalize the results of Section 4 with slightly stricter conditions (the decay of correlations is assumed to be not too slow) for dimension $\nu\geqslant 3$. We obtain invariant measures and ergodic properties. In Section 6 several examples of marginally closed non-conditionally independent processes are given.

2. Processes with local interaction

We shall consider Markov processes in discrete time

$$\vec{\xi}_t = (\xi_t(x)), \quad x \in \mathbb{Z}^\nu, \ t \in \mathbb{Z}_+,$$

where $\xi_t(x)$ takes values in a finite or countable set S. Let $\xi_t(A)$ be the configuration $(\xi_t(x), x \in A)$ on the set $A \in \mathbb{Z}^\nu$.

Definition 1. $\vec{\xi}_t$ is called a process with local interaction if there exists a finite set $Q \subset \mathbb{Z}^\nu$ such that, for all given values of $\vec{\xi}_t$, the conditional distribution of $\vec{\xi}_{t+1}$, $x \in \mathbb{Z}^\nu$, is such that its conditional finite dimensional distributions

$$P(\xi_{t+1}(x_1) = s_1, \xi_{t+1}(x_2) = s_2, \ldots, \xi_{t+1}(x_n) = s_n \mid \vec{\xi}_t)$$

depend only on $\xi_t(\bigcup(x_i + Q))$. We also assume that this conditional distribution is invariant with respect to translations, in x and t. The process $\vec{\xi}_t$ is called conditionally independent if for all $t = 0, 1, \ldots$, the random variables $\xi_{t+1}(x)$, $x \in \mathbb{Z}^\nu$ are conditionally independent given $\vec{\xi}_t$.

3. Equations for marginally closed processes

Let, for any finite $X \subset \mathbb{Z}^\nu$, $s_X \in S^X$ be a configuration on X with values in S. Let us put

$$P_X(s_X\,;\, t) = P(\xi_t(x) = s_x, x \in X).$$

From Definition 1, we have that

$$P(\xi_{t+1}(x) = s_x, x \in X \mid \vec{\xi}_t) = A_X(s_X\,;\, \xi_t(z), z \in X + Q).$$

Then,

$$\begin{aligned} P_X(s_X\,;\, t+1) &= \sum_{s_{X+Q}} A_X(s_X, s'_{X+Q}) \cdot \left\langle \prod_{z \in X+Q} \delta_{s'_z}(\xi_t(z)) \right\rangle \\ &= \sum_{s_{X+Q}} A_X(s_X, s'_{X+Q}) \cdot P_{X+Q}(s'_{X+Q}\,;\, t), \end{aligned} \tag{1}$$

where $\langle \cdot \rangle$ is the expectation of the random variable $(\cdot)$. This recurrent relation is similar to the BBGKY-hierarchy in statistical mechanics.

Remark 1. For a particle system the correlation functions are usually defined in a different way. Here the state space at a point is N^S, i.e., we specify the number of particles of types s for any $s \in S$. The correlation functions are written as $P(x_1, s_1; \ldots; x_n, s_n)$ where the x_i and (or) s_i can coincide. It is the probability that in the point x there are not less than $\sum_{i=1}^n \delta_{x,x_i}\delta_{s,s_i}$ particles of type s.

Sometimes, by using linear consistency conditions for $P_Y(s_Y; t)$ one can reduce (1) to

$$P_X(s_X; t+1) = \sum_Y \sum_{s'_Y} \hat{A}_{X,Y}(s_X, s'_Y) P_Y(s'_Y; t), \tag{2}$$

where the summation extends over all $Y \subset X + Q$ with $|Y| \leqslant |X|$.

This remark gives rise to the following,

Definition 2. A marginally closed process is a process for which there exist functions $\hat{A}_{X,Y}(s_X, s'_Y)$ on $S^X \times S^Y$ such that for any X, s_x, s'_y, t we have

$$P(\xi_{t+1}(x) = s_x, x \in X \mid \xi_t(x) = s'_z, z \in \mathbb{Z}^\nu) = \sum_Y \hat{A}_{XY}(s_X, s'_Y), \tag{3}$$

where the summation extends over all $Y \subset X + Q$ with $|Y| \leqslant |X|$. In particular, we have that $A_X(s_X; \xi_t(z), z \in X + Q)$, introduced above, is equal to $\sum_Y \hat{A}_{XY}(s_X; \xi_t(z), z \in Y)$.

Definition 3. A process $\vec{\xi}_t$ with local interaction is called conditionally linear if there exist real functions $a_y(\cdot, \cdot)$ on $S \times S$, $y \in Q$, such that

$$P(\delta_s(\xi_{t+1}(x)) = 1 \mid \vec{\xi}_t) = \sum_{s' \in S} \sum_{y \in Q} a_y(s, s') \delta_{s'}(\xi_t(x+y)), \tag{4}$$

where $\delta_s(s')$ is the Kronecker symbol.

It is easy to prove that a conditionally independent process is marginally closed if and only if it is conditionally linear.

Relations (4) are consistent iff for any $s \in S$ and any function $s'(y) \in S$, $y \in Q$,

$$\text{(i)} \quad \sum_{y \in Q} a_y(s, s'(y)) \geqslant 0,$$
$$\text{(ii)} \quad \sum_s \sum_y a_y(s, s'(y)) = 1. \tag{5}$$

So for any y and s',

$$\sum_s a_y(s, s') \doteq q_y \tag{6}$$

does not depend on s', and

$$\sum_y q_y = 1.$$

Let us note that the right-hand side of (4) is invariant with respect to transformations

$$a_y(s, s') \to a_y(s, s') + c_y(s)$$

with arbitrary $c_y(s)$ subject to the condition $\sum_y c_y(s) = 0$, and for any $s \in S$. Let us now show how, by using appropriately these transformations, we can force $a_y(s, s')$

to be non negative. Let us fix s and among the $|Q|$ vectors $a_y(s, s')$, let us find those having some negative components. Let $y_1, \dots, y_m$ be the indices of these vectors and let $d_{y_i} = \inf_{s'_i} a_{y_i}(s, s'_i)$ so that $d_{y_i} \leq a_{y_i}(s, s')$, for any $s' \in S$, $i = 1, \dots, m$. All other vectors $a_y(s, s')$, $y \notin \{y_1, \dots, y_m\}$ have only nonnegative components. Let us put for such y,

$$\tilde{d}_y = \inf_{s'} a_y(s, s') \geq 0.$$

Then by (i),

$$\sum \tilde{d}_y + \sum d_{y_i} \geq 0.$$

Then we can put

$$c_{y_i}(s) = |d_{y_i}|,$$

and choose for $y \notin \{y_1, \dots, y_m\}$,

$$-\tilde{d}_y \leq c_y(s) = \frac{(\sum d_{y_i}) \cdot \tilde{d}_y}{(\sum \tilde{d}_y)} \leq 0,$$

so that $\sum_{y \notin \{y_1, \dots, y_m\}} c_y(s) = \sum d_{y_i}$ and $\sum_{y \in Q} c_y(s) = 0$.

So we can assume from now on that $a_y(s, s') \geq 0$ for all y, s, s'.

4. Ergodic behaviour of conditionally linear processes

Below we study the invariant measures and large time behaviour of conditionally independent and marginally closed processes. In particular we show that this behaviour is completely determined by the behaviour of *the one point correlation functions.*

Let us denote

$$P_{x_1, \dots, x_n}(s_1, \dots, s_n; t) = P\{\xi_t(x_1) = s_1, \dots, \xi_t(x_n) = s_n\}$$
$$= \langle \delta_{s_1}(\xi_t(x_1)) \cdots \delta_{s_n}(\xi_t(x_n)) \rangle.$$

Then from (4) we have the closed relationship for one point functions

$$P_x(s; t+1) = \sum_{s', y} a_y(s, s') P_{x+y}(s'; t). \tag{7}$$

We assume the initial conditions to be translational invariant; then they will be so for any t and in particular $P_x(s; t) = p(s; t)$ does not depend on x and we get from (7),

$$p(s, t+1) = \sum_{s'} b(s, s') p(s'; t),$$

where

$$b(s, s') = \sum_{y} a_y(s, s').$$

By (5) $b(s, s')$ is the transition probability matrix (from s' to s) of a Markov chain with state space S which we denote by L_1. First of all let us note that, if L_1 has one class of essential states and is zero recurrent or transient, then $p(s; t) \to 0$ for $t \to \infty$ for any s. By domination it is clear that for any $x_1, \ldots, x_n, s_1, \ldots, s_n$,

$$\lim_{t\to\infty} P_{x_1,\ldots,x_n}(s_1, \ldots, s_n; t) = 0.$$

Further on we shall consider the case where L_1 has k classes of essential states and we shall assume that any of these classes is aperiodic and positive recurrent (see Theorem 5 about periodic case). All proofs will be based on the *path representation of n-point functions.*

Graphical representation for such problems were introduced by Harris [8] and have been widely used. But different situations need careful adjustments.

From (4) we get

$$P_{x_1,\ldots,x_n}(s_1, \ldots, s_n; t) = \sum_{s'_1,\ldots,s'_n} \sum_{y_1,\ldots,y_n} a_{y_1}(s_1, s'_1) \cdots a_{y_n}(s_n, s'_n) \cdot \langle \delta_{s'_1}(\xi_{t-1}(x_1+y_1)) \cdots \delta_{s'_n}(\xi_{t-1}(x_n+y_n))\rangle. \quad (8)$$

Some of the points $x_i + y_i$ can coincide and then the corresponding term is nonzero only if $s'_i = s'_j$ for coincident points $x_i + y_i = x_j + y_j$. We want to give graphical interpretation to (8) and to its iterations. For this reason, we consider 'space-time' points $(x, t) \in \mathbb{Z}^\nu \times \mathbb{Z}_+$. We shall call an ordered pair $((x, t), (x', t-1))$ of points an edge and call the two points the vertices of the edge.

If we iterate (8) until time zero, we shall have a number of terms. It is convenient to enumerate these terms by marked graphs. We begin with the definition of these graphs. Let us fix some set $X = \{x_1, \ldots, x_n\}$ and define the class $\mathcal{R} = \mathcal{R}(X; t)$ of graphs G. All vertices of these graphs lie in $\mathbb{Z}^\nu \times \{0, 1, \ldots, t\}$. Then $\mathcal{R} = \mathcal{R}(X; t)$ is uniquely defined by the following properties:

(1) the vertices of any graph $G \in \mathcal{R}(X; t)$ on the t-slice $\mathbb{Z}^\nu \times \{t\}$ are exactly $(x_1, t), \ldots, (x_n, t)$;

(2) for any vertex (x, t'), $0 < t' \leq t$, of G there is exactly one edge of G with the upper vertex (x, t'), i.e., an edge $((x, t'), (x', t'-1))$ for some x', such that $q_{x'-x} \neq 0$;

(3) for any vertex (x, t'), $0 \leq t' < t$, of G there is at least one edge with lower vertex (x, t'), i.e., an edge $((x', t'+1), (x, t'))$ such that $q_{x-x'} \neq 0$.

A marked graph is a graph G together with a function s_G assigning to any vertex v of G some $s(v) = s_G(v) \in S$. If not otherwise stated we consider only marks s_G such that $s_G((x_i, t)) = s_i$ for $i = 1, \ldots, n$. We define the contribution $I(G, s_G)$ of the marked graphs G, s_G by

$$I(G, s_G) = \prod_{\text{edges}} a_{x'-x}(s(v), s(v')) \cdot \left\langle \prod_v^{(0)} \delta_{s(v)}(\xi_0(x(v))) \right\rangle \doteq P(G, s_G) \cdot \Pi(G, s_G), \quad (9)$$

where the first product is over all edges $(v, v') = ((x, t'), (x', t'-1))$ of G and the second product is over all vertices $v = (x(v), 0)$ of G on the zero time slice. Iterating

(8) up to time zero and just looking at the result, we ea˂ ıy prove the following path summation formula.

Lemma 1.

$$P_{x_1,\dots,x_n}(s_1,\dots,s_n;\,t)=\sum_{G,s_G} I(G,s_G), \tag{10}$$

where the sum is over all marked graphs of $\mathcal{R}(X;\,t)$ *such that* $s_G(v)=s_i$ *for any vertex* $v=(x_i,\,t)$. □

We note that the number of vertices on the time slice $Z^\nu\times\{t'\}$ decreases when t' decreases. A graph G is connected if there is exactly one vertex on the zero time slice. Let us denote by $\mathcal{R}_c=\mathcal{R}_c(X;\,t)$ the set of all connected graphs in $\mathcal{R}=\mathcal{R}(X;\,t)$; let $\mathcal{R}_{nc}=\mathcal{R}-\mathcal{R}_c$ be all nonconnected graphs.

Domination by a 'dual' random walk

Our estimate will become clear with the use of the following random process $X(\tau)$, $\tau=0,\dots,t$. We call τ inverse time. At the moment $\tau=0$ n particles occupy initial positions at the points $x_1,\dots,x_n$. So $X(0)$ is the set $\{x_1,\dots,x_n\}$. Then particles begin to perform independently random walks $x_i(\tau)$ until two or more particles come to the same point. From this moment the particles glue together and continue to perform a random walk as one new particle from the point where they become glued. These random walks on $\mathbb{Z}^\nu$ are translation invariant and have one-step transition probabilities

$$p(x\to x+y)=q_y.$$

Let us fix $x_1,\dots,x_n$ and $s_1,\dots,s_n$ at the moment t (of direct time), the graph G and the values $s(v_1),\dots,s(v_m)$ of a mark at time zero, $v_1,\dots,v_m$ being all the vertices of G at time zero. Our main estimate is the following.

Lemma 2.

$$\sum_{s_G}^{f,i} P(G,s_G)\leqslant \prod_{\text{edges}} q_y, \tag{11}$$

where for an edge with upper vertex $(x,\,t')$ *and lower vertex* $(x',\,t'-1)$, y *denotes the difference* $x'-x$ *and the sum* $\sum^{f,i}$ *is over* s_G *with fixed final* $s_1,\dots,s_n$ *and initial* $s(v_1),\dots,s(v_m)$ *marks.*

Proof. It is clear that

$$\sum_{s_G}^{i} P(G,s_G)=\prod_{\text{edges}} q_y, \tag{12}$$

where $\sum^i$ is over s_G where only the initial marks $s(v_1),\dots,s(v_m)$ are fixed. Formula (12) is proved easily by induction $t-1\to t$ by using the fact that q_y do not depend on s' in (6). □

Dimension $\nu = 1, 2$

Theorem 1. *Let $\nu=1,2$ and let L_1 have exactly one class of essential states and be ergodic, $\pi(s)$ being its stationary probabilities. Then for any translation invariant initial distribution of the process $\vec{\xi}_t$, $t\geq 0$, the correlation functions $P_X(s_X; t)$ converge when $t\to\infty$ to a limit which is equal to*

$$P_X(s_X;\infty)=\sum_{\tau=1}^{\infty}\sum_{s}\sum_{G\in\bar{\mathscr{R}}_c(X;\tau)}\pi(s)\sum_{s_G}P(G,s_G), \tag{13}$$

where $\bar{\mathscr{R}}_c(X;\tau)\subset\mathscr{R}_c(X;\tau)$ is the set of graphs which become connected only on the zero slice (this means that their restriction to $\mathbb{Z}^\nu\times\{1,\ldots,\tau\}$ is not connected) and $\sum_{s_G}$ is over all marks with $s(x_i,\tau)=s_i$, $s_{v_0}=s$ where v_0 is the unique (due to the connectedness) vertex of G on the zero slice. So (13) is the unique invariant translation invariant distribution.

Proof. Let us separate in the right-hand side of (10) sums over connected and disconnected graphs. Let us observe that if $(x_1,\ldots,x_n;t)$ are fixed

$$P_{\rm nc}\doteq\sum_{G\in\mathscr{R}_{\rm nc}}\prod_{\rm edges}q_y \tag{14}$$

is exactly the probability that at least two of n particles beginning their random walks at $x_1,\ldots,x_n$ will not glue together during a time t. Let us choose two particles, e.g., the points x_1, x_2 at the initial moment $\tau=0$ of the inverse time τ.

Then the difference $x_1(\tau)-x_2(\tau)$, is the random walk of one particle in $\mathbb{Z}^\nu$ with one-step transition probabilities

$$p(x\to x')=\sum_{y_1,y_2:\,x'-x=y_1-y_2}q_{y_1}q_{y_2}. \tag{15}$$

This is a symmetric random walk on $\mathbb{Z}^\nu$.

So for $\nu=1,2$ it reaches the origin almost surely. As there are C_n^2 pairs of particles, then $P_{\rm nc}$ in (14) tends to zero when $t\to\infty$.

But the nonconnected part of (10) can be estimated by

$$\sum_{G\in\mathscr{R}_{\rm nc}}\sum_{s_G}I(G,s_G)$$
$$\leq P_{\rm nc}\sup_{v_1,\ldots,v_m}\sum_{s(v_1),\ldots,s(v_m)}P(\xi_0(x(v_1))=s(v_1),\ldots,\xi_0(x(v_m))=s(v_m))\equiv P_{\rm nc},$$

where the sup is over all positions of vertices on the zero-slice. So we have to deal only with connected graphs in (10). The series (13) is dominated by

$$\sum_{\tau=1}^{\infty}\sum_{s}\pi(s)P_c(\tau), \tag{16}$$

where $P_c(\tau)$ is the probability that all particles will become glued together exactly at the moment τ (some of them can glue earlier of course).

So (13) is evidently convergent.

But for fixed t the difference between $P_X(s_X, t)$ and the partial sum $\sum_{\tau=1}^{t}$ of (13) is dominated by

$$\sum_{\tau=1}^{t} P_c(\tau) \sum_s |\pi(s) - p_{t-\tau}(s)| + P_{nc}(t), \tag{17}$$

where $p_{t-\tau}(s)$ is the probability of the state s for the Markov chain L_1 at the moment $t-\tau$, for some fixed initial distribution $p_0(s)$. But (17) tends to zero for $t \to \infty$.

So the theorem is proved. □

Theorem 2. *Let ν be equal to 1 or 2 and L_1 have $k > 1$ classes of essential states each of them ergodic. Then the set of invariant translational invariant distributions is the convex envelope of k extreme distributions which are given by (13) with $\pi(s) = \pi_i(s)$, $i = 1, \ldots, k$, where $\pi_i(s)$ are the stationary probabilities of the ith class of L_1 (i.e., when $p_0(s) = 0$ except for s in the ith class). Moreover for any initial translation invariant distribution the correlation functions $P_X(s_X; t)$ converge to a limit completely defined by the one-point initial correlation function.*

Proof. The proof is the same as for Theorem 1. □

The voter model

This is the well known example where $S = \{0, 1\}$ and (4) can be written as

$$P(\xi_{t+1}(x) = 1 \mid \vec{\xi}_t) = \sum_{y \subset Q} a_y \xi_t(x+y),$$

with $a_y > 0$, $\sum_y a_y = 1$.

In a sense this example is degenerate as L_1 here has the unit matrix as transition probability matrix, i.e., $b(s, s') = \delta_{ss'}$. The extreme invariant measures are measures concentrated at the points $\xi(x) \equiv 1$ or $\xi(x) \equiv 0$. The extreme measures are not so trivial when some class of essential states of L_1 has more than one state. To understand their structure we would have to examine more closely the structure of the limit distribution described in Theorem 1, i.e., when there is one class of essential states in L_1.

Cases of exponential convergence

It appears that there is a large class of processes with exponential convergence (let us observe that this is not true for the voter model).

Theorem 3. *Let S be finite and L_1 be ergodic (here the dimension ν is arbitrary). If there exist y, s, such that $a_y(s, s') > 0$ for any s', then for some $0 < \alpha < 1$ not depending on n and for all $x_1, \ldots, x_n$, $s_1 \ldots, s_n$,*

$$|P_{x_1,\ldots,x_n}(s_1, \ldots, s_n; t) - P_{x_1,\ldots,x_n}(s_1, \ldots, s_n; \infty)| < C_n \alpha^t.$$

Proof. Let us first note that if there exist s_0 and y_0 such that $a_{y_0}(s_0, s')>0$ for all s', then putting

$$\delta = \min_{s'} a_{y_0}(s_0, s')$$

we can rewrite (3) as follows (using $\sum_{s'} \delta_{s'}(\xi_t(x+y)) \equiv 1$),

$$P(\delta_s(\xi_{t+1})=1 \mid \vec{\xi}_t) = \sum_{s'} \sum_{y} \tilde{a}_y(s, s')\delta_{s'}(\xi_t(x+y)) + \delta\delta_{s s_0}, \tag{18}$$

where $\tilde{a}_y = a_y$ for all $y \neq y_0$, $s \neq s_0$, s' and

$$\tilde{a}_{y_0}(s_0, s') = a_{y_0}(s_0, s') - \delta \quad \forall s'.$$

So if we put $\tilde{\delta}_y = \sum_s \tilde{a}_y(s, s')$ then

$$\sum_y \tilde{\delta}_y = 1-\delta. \tag{19}$$

After this we shall use a graphical representation similar to the previous one but with some modifications. The presence of δ in the right-hand side of (18) after iterations of (8) will give us some paths which end before $\tau = t$ with the δ term. More exactly in property (2) of the class $\mathcal{R}$ we shall change the word 'exactly' to 'at most'. If there is no edge satisfying property (2) then we shall say that the vertex is final and assign to it the extra factor δ. So for $t \to \infty$ the contribution of all graphs G will be finite with probability one and the limiting distribution will be given by

$$P_X(s_X; \infty) = \sum_{\tau=1}^{\infty} \sum_{G_\tau} \delta^{k(G_\tau)} \sum_{s_G} \prod_{\text{edges}} \tilde{a}_y(s, s'), \tag{20}$$

where G_τ runs over all graphs with last final point at the moment τ, $k(G_\tau)$ being the number of final points of G_τ.

So here we have a simpler situation. The last series is exponentially convergent due to (19). □

From the representation (20) the existence of exponential bounds for the limiting field follows easily. E.g., the following theorem holds.

Theorem 4. *Under the condition of Theorem* 3, *the limit random field exhibits exponential decay of* 2*-points correlation functions*

$$|P_{x_1,x_2}(s_1, s_2; \infty) - \pi(s_1)\pi(s_2)| < C\alpha^{|x_1-x_2|}$$

for some $0<\alpha<1$.

Proof. The proof is a standard exercise in cluster expansion. □

So one may think (with assumptions similar to those of Theorem 3) of the structure of invariant measures as follows: in the sense of statistical mechanics they are low-temperature expansions around the ground states which are given by the case where the k classes consist of one state each (so that the transition matrix is the unit matrix). It seems likely that Theorem 3 could be improved. E.g., if the conditions of Theorem 3 are not fulfilled, one can try to iterate (8) for a finite number of times to find y, s such that $a_y(s, s')>0$ for all s'. But the following example shows that this is not always possible. Let us take $|S|=2$, $Q=\{y_1, y_2\}$, $\nu=1$, y_1 be even and y_2 odd. Then let us put

$$a_{y_1}(0,0)=a_{y_1}(1,1)=\alpha>0, \qquad a_{y_1}(0,1)=a_{y_1}(1,0)=0,$$
$$a_{y_2}(0,0)=a_{y_2}(1,1)=0, \qquad a_{y_2}(0,1)=a_{y_2}(1,0)=\beta>0,$$
$$\alpha+\beta=1.$$

Then $y=ny_1+my_2$ (we consider the product of n matrices a_{y_1} and m matrices a_{y_2}) can be even only if m is even, but then the product could be only a diagonal matrix. If y is odd then m is odd too ($n\neq 0$) and in this case only matrices with zero diagonal terms can appear.

Periodic case

Let us first note that under the conditions of Theorem 3 L_1 can not be periodic. The following theorem gives examples of periodic behaviour of processes with local interaction.

Theorem 5. *Let the conditions of Theorem* 1 *be fulfilled except that we assume* L_1 *to be periodic with period* $d_0>1$. *Then for any* $0\leq d<d_0$ *the correlation functions* $P_X(s_X; d_0t+d)$ *converge when* $t\to\infty$ *and the limit is defined by the same formula* (13) *with* $\pi(s)=\pi_d(s)$ *being the limit probabilities for* $P_{d_0t+d}(s)$, *depending of course on the initial distribution on* L_1. *If at the initial time any periodic class of* L_1 *has stationary distribution, then our process exhibits the exact periodic behaviour.* □

5. Dimension $\nu\geq 3$

The main difference here from the case $\nu=1,2$ is that the non-connected graphs give non-zero contribution to the invariant measure. Some combinatorial machinery is necessary and we present it now. Let X be a finite subset of $\mathbb{Z}^\nu$ and s_X a configuration on it. Let $P_t(X; s_X)$ be the correlation functions at time t. Cumulants

$$\tilde{P}_t(X; s_X)=\langle\delta_{s_1}(\xi_t(x_1)), \ldots, \delta_{s_n}(\xi_t(x_n))\rangle$$

for $X=\{x_1, \ldots, x_n\}$, $s_X=(s_1, \ldots, s_n)$, are usually defined by the inductive formula

$$P_t(X; s_X)=\sum_\alpha \prod_{i=1}^{k} \tilde{P}_t(X_i, s_{X_i}), \tag{21}$$

where the sum is over all partitions $\alpha=\{X_1, \ldots, X_k\}$, $k=1, \ldots, n$, of X and s_{X_i} is the restriction of s_X to X_i.

Our main assumption here is more severe than for dimensions 1 and 2.

An l_1-assumption. For any n,

$$\sum_{X:0\in X,|X|=n}\sum_{s_X}|\tilde{P}_0(X, s_X)|<\infty. \tag{22}$$

We also assume that $\nu\geq 3$ and that the set of all linear combinations of vectors from Q with integer coefficients coincides with all the additive group $\mathbb{Z}^\nu$.

Using (21) we can write for Π in the right-hand side of (9) for any G, s_G,

$$\Pi(G, s_G)=\sum_{\alpha}\Pi_\alpha(G, s_G).$$

Let us denote $\alpha=\mathbf{0}$ the partition with $|\alpha|\doteq k=n$. Recall that $\Pi(\cdot,\cdot)$ is a correlation function and $\Pi_\alpha(\cdot,\cdot)$ is the corresponding cumulant.

Lemma 3. *With the same notation as in Lemma* 2, *for any* $\alpha\neq\mathbf{0}$,

$$\lim_{t\to\infty}\sum_{G,s_G}P(G, s_G)\cdot\Pi_\alpha(G, s_G)=0 \tag{23}$$

for any given $x_1,\ldots,x_n, s_1,\ldots,s_n$.

Proof. We use the following notation: if $\alpha=(X_1,\ldots,X_k)$, we fix some X_i^0 with $0\in X_i^0$ and such that there exist y_i with $X_i=X_i^0+y_i$, $i=1,\ldots,k$. Consider the following event $A_t=A_t(x_1,\ldots,x_n; X_1^0,\ldots,X_k^0)$ for the inverse random walk process, defined for fixed $t, x_1,\ldots,x_n$, and for fixed X_i^0, $i=1,\ldots,k$, $A_t=$ {there exist at least two particles, at inverse time zero, such that they fall at the moment t into some X_i}. We see that $P(A_t)\to 0$, as $t\to\infty$, because this probability is dominated by a finite sum of probabilities that two particles with initial difference x_i-x_j have at time t the fixed difference of a pair of points in X_l^0, for some l. Let us remark that this is true for any dimension. So the left-hand side in (23) is dominated by

$$\sum_{\{s_{X_1^0}\},X_1^0\ldots\{s_{X_k^0}\},X_k^0}P(A_t(x_1,\ldots,x_n; X_1^0,\ldots,X_k^0))\prod_{i=1}^{k}|\tilde{P}_0(X_i^0, s_{X_i^0})|, \tag{24}$$

which tends to zero by the l_1-assumption. Let us note that $\sum_{y_1,\ldots,y_k}$ is contained in $P(A_t)$. So we are left with

$$\sum_{G,s_G}P(G, s_G)\cdot\Pi_{\mathbf{0}}(G, s_G). \tag{25}$$

The class $\mathcal{R}$ is the disjoint union of classes $\mathcal{R}^k$ of graphs G with exactly k connected components; $\mathcal{R}^1\equiv\mathcal{R}_c$. Let $\mathcal{R}_t^n(X; s_X)$ be the subclass of $\mathcal{R}^n(X; t)$ characterized by the fixed mark s_X.

Let us consider first the class $\mathcal{R}^n$. Any $G\in\mathcal{R}^n$ is the union of n non-intersecting paths

$$\Gamma_i=\{(x_{i0}, 0), (x_{i1}, 1),\ldots,(x_{it}=x_i, t)\}, \tag{26}$$

$i=1,\ldots,n$. For $n=1$ we know that

$$\sum_{G,s_G} P\cdot \Pi_0=\sum_{s'} b^{(t)}(s_1,s')p(s';0)=p(s_1;t),$$

where $b^{(t)}$ is the matrix of t-step transition probabilities of L_1. For any n, we include the class $\mathscr{R}_t^n(X;s_X)$ into the new class $\tilde{\mathscr{R}}_t^n(X;s_X)$ of n-tuples of paths Γ_i with marks $s(x_{i_l},l)$ but with arbitrary intersections (so one point (x',t') may have several marks belonging to different paths). Let us define the contribution of the paths by

$$P(\Gamma_1,\ldots,\Gamma_n)=\prod_{i,l} a_{x_{il}-x_{i,l+1}}(s(x_{i,l+1},l+1),s(x_{il},l))\prod_i P(s(x_{i0},0);0).$$

Let $\tilde{\mathscr{R}}_t^{n,t-\tau}$ be the set of all $(\Gamma_1,\ldots,\Gamma_n)\in\tilde{\mathscr{R}}_t^n$ with the last intersection at the moment $t-\tau-1$; so at the moments $t-\tau,\ldots,t$ the paths Γ_i do not intersect. So

$$\begin{aligned}&\sum_{G\in\mathscr{R}_t^n} P(G,s_G)\Pi_0(G,s_G)\\ &=\prod_i p(s_i,t)-\sum_{\tau=0}^{t-1}\sum_{(\Gamma_1,\ldots,\Gamma_n)\in\tilde{\mathscr{R}}_t^{n,t-\tau}} P(\Gamma_1,\ldots,\Gamma_n)\\ &=\prod_i p(s_i,t)\\ &\quad-\sum_{\tau=0}^{t-1}\sum_{G_\tau,s(G_\tau)} P(G_\tau,s(G_\tau))\sum_{\{y_i,s_i'\}}\prod a_{y_i}(s(x_{i,t-\tau},t-\tau),s_i')p(s_i',t-\tau-1),\end{aligned}\tag{27}$$

where G_τ is the arbitrary graph in $\mathscr{R}_\tau^n(X;s_X)$ and $\sum_{\{y_i,s_i'\}}$ depends on G_τ and is over all $s_i'\in S$ and over all $y_i\in Q$ such that among the points $x_{i,t-\tau}+y_i$, $i=1,\ldots,n$, at least two coincide. We claim that (27) tends as $t\to\infty$ to

$$\prod_i \pi(s_i)-\sum_{\tau=0}^{\infty}\sum_{(G,s_G)\in\mathscr{R}_\tau^n(X,s_X)} P(G,s_G)\sum_{\{y_i,s_i'\}}\prod_{i=1}^n a_{y_i}(s_i(x_{i0},0),s_i')\pi(s_i').\tag{28}$$

Due to the last $\sum_{\{y_i,s_i'\}}$ the first $\sum_G$ is in fact over only such G that at least two of their initial points have distance not exceeding diam Q. So this sum $\sum_{\tau=0}^{\infty}\sum_{G\in\mathscr{R}_\tau^n} P_n(G,s_G)$ is dominated by $\frac{1}{2}n(n-1)$ times the mean number of visits that a random walk starting at point x_i-x_j makes to Q before hitting the origin. Let us recall that the mean number of visits to x beginning at x' before visiting the origin, $g_0(x,x')$, is bounded by $g_0(x,x)$ and that for $\nu\geqslant 3$ the random walk is transient; so $g_0(x,x')\leqslant g_0(x,x)\leqslant\sum p^{(t)}(0,0)<\infty$ (see [15, Proposition 1, Chapter III]). From this the convergence (27) $\to$ (28) follows.

Let us consider now, the class $\mathscr{R}_t^k$ for any $k<n$. Let $\mathscr{R}_t^k(\tau_1,\tau_2)$ be the class of all graphs with exactly k paths at the moments $t-\tau_2-\tau_1, t-\tau_2-\tau_1+1,\ldots,t-\tau_1$ and with more paths for $t-\tau_1+1,\ldots$. We assume also that at the moment $t-\tau_2-\tau_1$,

at least two vertices of these graphs have distance between them not exceeding diam Q. Then we can write,

$$\sum_{k=1}^{n-1} \sum_{(G,s_G)\in\mathscr{R}_t^k} P(G, s_G) \prod_0 (G, s_G)$$

$$= \sum_{k=1}^{n=1} \left\{ \sum_{\tau=1}^{t} \sum_{(G,s_G)\in\bar{\mathscr{R}}_\tau^k} P(G, s_G) \prod_{i=1}^{n} p(s_i', t-\tau) \right.$$

$$-(1-\delta_{k1}) \sum_{\tau_1=1}^{t} \sum_{\tau_2=0}^{t-\tau_1-1} \sum_{(G,s_G)\in\mathscr{R}_t^k(\tau_1,\tau_2)} P(G, s_G)$$

$$\left. \times \sum_{\{y_i,s_i'\}} \prod_{i=1}^{k} a_{y_i}(s_i(x_i, t-\tau_1-\tau_2), s_i')p(s_i', t-\tau_1-\tau_2-1) \right\}, \quad (29)$$

where in $\bar{\mathscr{R}}_\tau^k$ again as in (13) we take only graphs which have more than k components on $\mathbb{Z}^\nu \times \{1, \ldots, \tau\}$, $\delta_{k1} = 1$ if $k=1$ and zero otherwise. In a similar way as in (27) and (28) we have convergence to the following contribution:

$$\sum_{k=1}^{n-1} \left\{ \sum_{\tau_1=1}^{\infty} \sum_{(G,s_G)\in\bar{\mathscr{R}}_{\tau_1}^k} P(G, s_G) \prod \pi(s_i') \right.$$

$$-(1-\delta_{k1}) \sum_{\tau_1=1}^{\infty} \sum_{\tau_2=0}^{\infty} \sum_{(G,s_G)\in\mathscr{R}_{\tau_1+\tau_2}^k(\tau_1,\tau_2)} P(G, s_G)$$

$$\left. \times \sum_{\{y_i,s_i'\}} \prod a_{y_i}(s_i(0), s_i')\pi(s_i') \right\}. \qquad \square \quad (29')$$

Theorem 6. *Under the l_1-assumption for translation invariant initial conditions if L_1 has k classes of essential states which are aperiodic and ergodic, then the limit of correlation functions exists and is given by* (28) *and* (29′). *So for any invariant measure of L_1 there exists corresponding invariant distributions for $\vec{\xi}_t$ which are given by* (28) *and* (29′).

These are the only invariant translation invariant distributions, under the l_1-assumption.

Proof. The sum in (29′) can be majorized by $C\cdot g_0(0,0)$, where C can depend on n as for fixed τ_1, τ_2 we consider the product of probabilities that two particles met at τ_1 for the first time and that the difference of the positions of the two particles, lies in Q at time $\tau_1+\tau_2$. $\square$

6. Some remarks about non conditionally independent processes

First let us give a simple probabilistic interpretation for a conditionally independent conditionally linear process. Its evolution $t \to t+1$ is exactly the following one: first of all for any point x we choose randomly a point $y = y(x)$ with probability q_y and then choose randomly the value $\xi_{t+1}(x) = s$ with probability

$$\tilde{a}_y(s, \xi_t(x+y(x))) \doteq \frac{a_y(s, \xi_t(x+y(x)))}{q_y}.$$

Both these choices are made independently for different x.

Note that a marginally closed process (even non conditionally independent) is conditionally linear. So, using the same interpretation one can hope to get many examples of non conditionally independent processes which are marginally closed by making dependent choices of $y(x)$ for different x and dependent choices of s. We shall give now more exact description of such examples.

Let $(\Omega_0, \Sigma_0, \mu_0)$ be a probability space where the initial random field $\xi_0(x)$ is defined. Let us consider for any $t \geq 1$ probability spaces $(\Omega_t, \Sigma_t, \mu_t)$ which are all copies of the same probability space (Ω, Σ, μ). Moreover assume that on (Ω, Σ, μ) a group $U_x : \Omega \to \Omega$, $x \in \mathbb{Z}^\nu$, of measure preserving transformations is defined. Let us choose a function $y(\omega)$, $\omega \in \Omega$, $y(\omega) \in Q$, and for any $y \in Q$, $s' \in S$ choose some functions $s_{y,s'}(\omega) \in S$, $\omega \in \Omega$. Put

$$y(x; \omega) = y(U_{-x}\omega), \qquad s(x, y, s'; \omega) = s_{y,s'}(U_{-x}\omega).$$

We want that our process $\xi_t(x)$ be defined on the probability space

$$\bigotimes_{t=0}^{\infty} (\Omega_t, \Sigma_t, \mu_t).$$

We achieve this via the following inductive definition

$$\delta_s(\xi_t(x)) = \sum_{y,s'} \delta_{y,y(x;\omega_t)} \cdot \delta_{s,s(x,y,s';\omega_t)} \cdot \delta_{s'}(\xi_{t-1}(x+y(x;\omega_t))) \tag{30}$$

for $t \geq 1$, $\omega_t \in \Omega_t$. It is clear that the process defined by (30) is marginally closed. We shall not consider general processes of type (30). Instead we indicate some examples which have a clear intuitive interpretation.

Let $s = (i, k_i)$ where i denotes the kind of a particle, and k_i is the number of particles of type i, $k_i \in \mathbb{Z}_+$ or $\{0, \ldots, N\}$, $i \in \{0, \ldots, M\}$. Let also be given a map $\phi : S \to S$ which we shall call 'chemical reaction'. In any case it is given by a stochastic matrix $\tilde{a}(s, s')$, $s' \to s$. Then the evolution of $\vec{\xi}_t$ is the following one: we choose $y(x)$ for any x and put the product of the reaction in the point $x + y(x)$ at the point x, $\tilde{a}$ can depend on y.

We shall say that ϕ has conservation laws if there exists a partition $S_1 \cup \cdots \cup S_n$ of S such that $a(s, s') = 0$ if s, s' belong to different blocks of this partition. More generally we can say that a process with local interaction has conservation laws if L_1 has $k > 1$ classes of essential states. These classes will be called conservation laws.

Remark 2. In the definition of L_1 we did not use conditional independence. So the definition is valid for the general case too.

Usually (as in statistical mechanics) a conservation law has more strict 'trajectory-wise' meaning. We shall say that $\vec{\xi}_t$ has a strict conservation law, if there is a conservation law in the above sense and the function $y(x)$ is such that the map $x \to \{x+y(x)\}$ is one-to-one on $\mathbb{Z}^\nu$ with probability one. This means that the number of points x where $\vec{\xi}_t$ belongs to some class S_i is conserved with probability one. In the general case only the mean number of them is conserved. A conditionally independent process cannot have strict conservation laws.

Examples. 1. Voter model (see above).

2. Simple exclusion process. Here $S=\{0, 1\}$. There are many variants of this model, as we can arbitrarily choose a process $y(x)$. E.g. we can take i.i.d. random variables $\eta(x)$ with values $e_1, \ldots, e_\nu, -e_1, \ldots, -e_\nu$ and put $y(x)=e_\mu, y(x+e_\mu)=-e_\mu$ iff $\eta(x)=e_\mu$, $\eta(x+e_\mu)=-e_\mu$. In all other cases we put $y(x)=0$. We take $\tilde{a}(s, s')$ to be the unit matrix.

3. Streaming process and both steps of stirring processes in [4] where instead of $\mathbb{Z}^\nu$ a lattice $\mathbb{Z}^\nu \times N$ is considered where N is a finite set.

Remark 3. For general marginally closed non conditionally independent processes one can prove similar results as for conditionally independent case if $y(x)$ are weakly dependent at large distances, for example if its cumulants have the following decay

$$|\langle y(x_1), \ldots, y(x_n)\rangle_\mu| \leq C_n \alpha^{d(x_1,\ldots,x_n)}, \tag{31}$$

where $0<\alpha<1$ and $d(x_1, \ldots, x_n)$ is the minimal length of a connected tree with vertices $x_1, \ldots, x_n$. In the dominating inverse random walk n particles move dependently. If the probability of glueing is positive, then all results for conditionally independent case are valid here as well. On the contrary, if there is a strict conservation law then the results are similar to those known for the case of simple exclusion process (see [5]): limiting process is a Bernoulli process. We shall not formulate the corresponding results. The proof here follows the main lines of Theorems 2–5. The only difference is that particles cannot glue together and then with large probability distances $x_i(\tau)-x_j(\tau)$ are large for large τ.

It is of interest of course to study the cases when the process $y(x)$ has strong dependence. The examples of type 3 (see above) are the extreme cases of this situation.

Remark 4. Marginally closed systems do not exhaust all 'explicitly solvable' processes. E.g., there are moment closed ones (see some results in [9]).

References

[1] V. Basis, Stationarity and ergodicity of multicomponent Markov processes with local interaction, in: D.R. Sinai Ja., ed., Multicomponent Random Systems (Springer, Berlin, 1992).

[2] Yu. Belyaev, Yu. Gromack and V.A. Malyshev, Invariant random Boolean fields, Mat. Zametky 6(5) (1969) 555-566.

[3] S.A. Berezner, M. Krutina and V.A. Malyshev. Ergodicity of one-dimensional Toom's probabilistic cellular automata, to appear in: J. Statist. Phys.

[4] A. De Masi, R. Esposito, J.L. Lebowitz and E. Presutti, Hydrodynamics of stochastic cellular automata, Comm. Math. Phys. 125 (1989) 127-145.

[5] A. De Masi, N. Ianiro, A. Pellegrinotti and E. Presutti, A survey of hydrodynamical behaviour of many particles systems, in: Nonequilibrium Phenomena II (North-Holland, Amsterdam, 1984).

[6] A. De Masi, E. Presutti and E. Scacciatelli, The weakly asymmetric simple exclusion process, Ann. Inst. H. Poincaré, 25(1) (1989) 1-38.

[7] T.E. Harris, Additive set-valued Markov processes and graphical methods, Ann. Probab. 6(3) (1978) 355-378.

[8] I.A. Ignatyuk and V.A. Malyshev, Locally interacting processes with non-compact set of states, Vestnik Moscow Univ. Ser. I 2 (1987) 3-6, 5 (1988) 3-7 [in Russian].

[9] I.A. Ignatyuk, V.A. Malyshev and S.A. Molchanov, Moment closed processes with local interaction, Selecta Math. Soviet 8(4) (1989) 351-384.

[10] I.A. Ignatyuk, V.A. Malyshev and V. Sidoravicius, Convergence of stochastic quantisation method, Theory Probab. Appl. 37(2) (1992).

[11] T. Liggett, Interacting Particle Systems (Springer, New York, 1985).

[12] V.A. Malyshev, V.A. Podorolsky and T.S. Turova, Ergodicity of infinite system of stochastic equations, Mat. Zametki 45(4) (1989) 78-88. [In Russian.]

[13] S.A. Pirogov, Cluster expansion for cellular automata, Probl. Inform Transmission 22 (1986) 298-303.

[14] E. Presutti and H. Spohn, Hydrodynamics of the voter model, Ann. Probab. 11(4) (1983) 867-875.

[15] F. Spitzer, Principles of Random Walk (Van Nostrand, New York, 1964).

[16] F. Spitzer, Interaction of Markov processes, Adv. Math. 5 (1970) 246-290.

[17] A.L. Toom, Stable and attractive trajectories of multicomponent systems, in: Multicomponent Systems (Springer, Berlin, 1980).

[18] D. Wick, Convergence to equilibrium of the stochastic Heisenberg model, Comm. Math. Phys. 81 (1981).